On Mœglin's Parametrization of Arthur Packets for p-adic Quasisplit $Sp(N)$ and $SO(N)$

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Abstract. We give a survey on Mœglin's construction of representations in the Arthur packets for *p*-adic quasisplit symplectic and orthogonal groups. The emphasis is on comparing Mœglin's parametrization of elements in the Arthur packets with that of Arthur.

1 Introduction

Let F be a number field and G be a quasisplit connected reductive group over F. The local components of the automorphic representations of G belong to a very special class of irreducible smooth representations, which is usually referred to as the *Arthur class*. In the archimedean case, there is a geometric theory of irreducible smooth representations (see [ABV92]), which suggests a possible way to characterize the Arthur class. In the *p*-adic case, the general characterization of the Arthur class remains a mystery. Nonetheless, when G is a general linear group, the Arthur class is known in both cases due to Mœglin and Waldspurger's classification of the discrete spectrum of automorphic representations of general linear groups [MW89]. In this paper, we will only consider the *p*-adic case. So from now on, let us assume F is a *p*-adic field, and we will also denote $G(F)$ by G, which should not cause any confusion in the context. To describe the Arthur class for general linear groups, we need to introduce some notation first. If $G = GL(n)$, let us take B to be the group of upper-triangular matrices and T to be the group of diagonal matrices, then the standard Levi subgroup M can be identified with

$$GL(n_1) \times \cdots \times GL(n_r)$$

for any partition of $n = n_1 + \cdots + n_r$ as follows:

$$\begin{pmatrix} GL(n_1) & & \\ & \ddots & \\ & & GL(n_r) \end{pmatrix}$$

$$(g_1, \ldots, g_r) \longrightarrow \operatorname{diag}\{g_1, \ldots, g_r\}.$$

For $\pi = \pi_1 \otimes \cdots \otimes \pi_r$, where π_i is a finite-length smooth representation of $GL(n_i)$ for $1 \leqslant i \leqslant r$, we denote the normalized parabolic induction $\mathrm{Ind}_P^G(\pi)$ by $\pi_1 \times \cdots \times \pi_r$. Moreover, we denote the direct sum of its irreducible subrepresentations by $\langle \pi_1 \times \cdots \times \pi_r \rangle$. An irreducible supercuspidal representation of a general linear group can always be written in a unique way as $\rho\|^x := \rho \otimes |\det(\cdot)|^x$ for an irreducible unitary supercuspidal representation ρ and a real number x. To fix notation, we will always denote by ρ an irreducible unitary supercuspidal representation of $GL(d_\rho)$. Now for a finite length arithmetic progression $x, \ldots, y$ of real numbers of step size one and an irreducible unitary supercuspidal representation ρ of $GL(d_\rho)$, it is a general fact that

$$\rho\|^x \times \cdots \times \rho\|^y$$

has a unique irreducible subrepresentation, denoted by $\langle \rho; x, \ldots, y \rangle$ or $\langle x, \ldots, y \rangle$. If $x \geqslant y$, it is called a *Steinberg representation*; if $x < y$, it is called a *Speh representation*. Such sequence of ordered numbers is called a *segment*, and we denote it by $[x, y]$ or $\{x, \ldots, y\}$. In particular, when $x = -y > 0$, we can let $a = 2x + 1 \in \mathbb{Z}$ and write

$$\mathrm{St}(\rho, a) := \left\langle \frac{a-1}{2}, \ldots, -\frac{a-1}{2} \right\rangle,$$

which is an irreducible smooth representation of $GL(ad_\rho)$. It follows from Zelevinksy's classification theory that all discrete series of $GL(n)$ can be given by $\mathrm{St}(\rho, a)$ for pairs (ρ, a) satisfying $n = ad_\rho$, and this is a bijection. We define a *generalized segment* to be a matrix

$$\begin{bmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & & \vdots \\ x_{m1} & \cdots & x_{mn} \end{bmatrix}$$

such that each row is a decreasing (resp. increasing) segment and each column is an increasing (resp. decreasing) segment. The normalized induction

$$\times_{i \in [1,m]} \langle \rho; x_{i1}, \ldots, x_{in} \rangle$$

has a unique irreducible subrepresentation, and we denote it by $\langle \rho; \{x_{ij}\}_{m \times n} \rangle$. If there is no ambiguity with ρ, we will also write it as $\langle \{x_{ij}\}_{m \times n} \rangle$ or

$$\begin{pmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & & \vdots \\ x_{m1} & \cdots & x_{mn} \end{pmatrix}.$$

Moreover,

$$\langle \rho; \{x_{ij}\}_{m \times n} \rangle \cong \langle \rho; \{x_{ij}\}_{m \times n}^T \rangle,$$

where $\{x_{ij}\}_{m \times n}^T$ is the transpose of $\{x_{ij}\}_{m \times n}$. Let a, b be positive integers; we define $\mathrm{Sp}(\mathrm{St}(\rho, a), b)$ to be the unique irreducible subrepresentation of

$$\mathrm{St}(\rho, a)\|^{-(b-1)/2} \times \mathrm{St}(\rho, a)\|^{-(b-3)/2} \times \cdots \times \mathrm{St}(\rho, a)\|^{(b-1)/2}.$$

Then one can see that $\mathrm{Sp}(\mathrm{St}(\rho, a), b)$ is given by the generalized segment

$$\begin{bmatrix} (a-b)/2 & \cdots & 1-(a+b)/2 \\ \vdots & & \vdots \\ (a+b)/2 - 1 & \cdots & -(a-b)/2 \end{bmatrix}.$$

The Arthur class for $GL(n)$ consists of irreducible representations

$$(1.1) \qquad \overset{r}{\underset{i=1}{\times}} \big(\underbrace{\mathrm{Sp}(\mathrm{St}(\rho_i, a_i), b_i) \times \cdots \times \mathrm{Sp}(\mathrm{St}(\rho_i, a_i), b_i)}_{l_i} \big)$$

for any set of triples (ρ_i, a_i, b_i) with multiplicities l_i such that $\sum_{i=1}^r l_i a_i b_i d_{\rho_i} = n$. In particular, it contains all the discrete series. The local Langlands correspondence for general linear groups gives a bijection between the set of equivalence classes of irreducible unitary supercuspidal representations of $GL(d)$ with the equivalence classes of d-dimensional irreducible unitary representations of the Weil group W_F. If we identify ρ_i in (1.1) with the corresponding d_{ρ_i}-dimensional representations of W_F, then we get an equivalence class of n-dimensional representations of $W_F \times SL(2, \mathbb{C}) \times SL(2, \mathbb{C})$ by taking

$$\overset{r}{\underset{i=1}{\bigoplus}} l_i (\rho_i \otimes \nu_{a_i} \otimes \nu_{b_i}),$$

where ν_{a_i} (resp. ν_{b_i}) is the $(a_i - 1)$-th (resp. $(b_i - 1)$-th) symmetric power representation of $SL(2, \mathbb{C})$. So the Arthur class for $GL(n)$ can be parameterized by the set of equivalence classes of n-dimensional representations of

$$\psi \colon W_F \times SL(2, \mathbb{C}) \times SL(2, \mathbb{C}) \longrightarrow GL(n, \mathbb{C})$$

such that $\psi|_{W_F}$ is unitary and $\psi|_{SL(2,\mathbb{C}) \times SL(2,\mathbb{C})}$ is algebraic. We call such ψ an *Arthur parameter* for $GL(n)$. The two copies of $SL(2, \mathbb{C})$ in the definition of Arthur parameters have their own meanings. The first one, introduced by Deligne, corresponds to some monodromy operator, and is usually integrated with the Weil group as $L_F := W_F \times SL(2, \mathbb{C})$, named *Weil–Deligne group* (or *local Langlands group*). The second $SL(2, \mathbb{C})$ is introduced by Arthur, and it corresponds to the non-temperedness of the associated irreducible smooth representation of $GL(n)$ (*cf.* (1.1)).

For general G, we can define an *Arthur parameter* to be a $\widehat{G}$-conjugacy class of admissible homomorphisms from $L_F \times SL(2, \mathbb{C})$ to $^L G$ that are bounded on their restrictions to W_F. We denote the set of Arthur parameters by $\Psi(G)$. It is conjectured that the Arthur class for G should be parameterized by $\Psi(G)$. To be more precise, for any $\psi \in \Psi(G)$, we are expecting to be able to associate it with a finite set Π_ψ of irreducible smooth representations of G, which is called an *Arthur packet*. The structure of Π_ψ can be very delicate in general; for example, we would expect these packets to have nontrivial intersections with each other. When G is a classical group, Mœglin has developed a theory to characterize the elements in Π_ψ (*cf.* [Mœg06b, Mœg09], etc.). The main goal of this paper is to present her results in the case of quasisplit symplectic and orthogonal groups. First of all, we need to give the definition of Π_ψ in these cases.

To simplify the discussion in the introduction, we assume that $G = \mathrm{Sp}(2n)$ unless otherwise specified. We should point out all the theorems and propositions that we

state for symplectic groups below also have their analogues for orthogonal groups. For $\psi \in \Psi(G)$, there is a natural $GL(N, \mathbb{C})$-conjugacy class of embeddings $^LG \hookrightarrow GL(N, \mathbb{C})$ for $N = 2n + 1$. So we can view ψ as an equivalence class of representations of $L_F \times SL(2, \mathbb{C})$, or an Arthur parameter for $GL(N)$. Moreover, such ψ is necessarily self-dual. So by the previous discussion we can associate it with an irreducible smooth representation π_ψ of $GL(N)$ (*cf.* (1.1)) that is also self-dual. Arthur [Art13] showed that one can associate ψ with a "multi-set" Π_ψ of irreducible smooth representations of G such that the spectral transfer of some linear combination of characters in Π_ψ is the twisted character of π_ψ. If we define $\mathcal{S}_\psi$ to be the component group of the centralizer of the image of ψ in $\widehat{G}$ (which can be made independent of the choice of representatives of ψ, and shown to be abelian), then Arthur further showed that there is a "canonical" map from Π_ψ to the characters $\widehat{\mathcal{S}_\psi}$ of $\mathcal{S}_\psi$. So for any element $\varepsilon \in \widehat{\mathcal{S}_\psi}$, we can write $\pi(\psi, \varepsilon)$ for the direct sum of elements in Π_ψ that are associated with ε; then $\pi(\psi, \varepsilon)$ is a finite-length smooth representation of G. The possibility for Π_ψ being a multi-set rather than a set suggests the irreducible constituents in $\pi(\psi, \varepsilon)$ may have multiplicities, and also $\pi(\psi, \varepsilon)$ may have common irreducible constituents for different $\varepsilon \in \widehat{\mathcal{S}_\psi}$. But these possibilities are all ruled out by the following deep theorem of Mœglin.

Theorem 1.1 (Mœglin, [Mœg11c])   *For $G = \mathrm{Sp}(2n)$ and $\psi \in \Psi(G)$, Π_ψ is multiplicity free.*

In fact, for $\psi \in \Psi(G)$ and $\varepsilon \in \widehat{\mathcal{S}_\psi}$, Mœglin constructed a finite-length semisimple smooth representation $\pi_M(\psi, \varepsilon)$ of G. She showed that Π_ψ consists of $\pi_M(\psi, \varepsilon)$ for all $\varepsilon \in \widehat{\mathcal{S}_\psi}$, and by studying their properties she was able to conclude Theorem 1.1. A subtle point here is $\pi(\psi, \varepsilon)$ in Arthur's parametrization can be different from $\pi_M(\psi, \varepsilon)$. This point has been emphasized in various works of Mœglin, and she also gave the relation between these two. Our second goal in this paper is to make that relation more transparent, and in the meantime we are able to clarify the fact that the representations $\pi_M(\psi, \varepsilon)$ constructed by Mœglin are indeed elements in the Arthur packet Π_ψ. For this purpose, we would like to rewrite Arthur's parametrization $\pi(\psi, \varepsilon)$ by $\pi_W(\psi, \varepsilon)$ to emphasize its dependence on certain kind of Whittaker normalization (see Section 4). And the relation between $\pi_W(\psi, \varepsilon)$ and $\pi_M(\psi, \varepsilon)$ can be given in the following theorem.

Theorem 1.2   *For $G = \mathrm{Sp}(2n)$ and $\psi \in \Psi(G)$, there exists a character $\varepsilon_\psi^{M/W} \in \widehat{\mathcal{S}_\psi}$, such that for any $\varepsilon \in \widehat{\mathcal{S}_\psi}$, $\pi_M(\psi, \varepsilon) = \pi_W(\psi, \varepsilon\varepsilon_\psi^{M/W})$.*

For the statement in this theorem to be true, we have implicitly put some restrictions on Mœglin's parametrization $\pi_M(\psi, \varepsilon)$. In the most general setting, we will attach $\pi_M(\psi, \varepsilon)$ to characters ε in $\widehat{\mathcal{S}_{\psi^>}}$, which contains $\widehat{\mathcal{S}_\psi}$ (see Section 2), and we will also define $\varepsilon_\psi^{M/W}$ in $\widehat{\mathcal{S}_{\psi^>}}$. The starting point of this comparison theorem is in the case of discrete series. Let us define

$$\Phi_2(G) := \{\phi \in \Psi(G) : \phi = \bigoplus_{i=1}^{r} \rho_i \otimes \nu_{a_i} \otimes \nu_1, \text{ and } \rho_i^\vee = \rho_i\}.$$

Then the following theorem of Arthur showed $\Phi_2(G)$ parametrizes the discrete series of G.

Theorem 1.3 (Arthur)   *For $G = \mathrm{Sp}(2n)$, the set of irreducible discrete series representations of G admits a disjoint decomposition*

$$\Pi_2(G) = \bigsqcup_{\phi \in \Phi_2(G)} \Pi_\phi.$$

Moreover, for any $\phi \in \Phi_2(G)$ and $\varepsilon \in \widehat{S_\phi}$, $\pi_W(\phi, \varepsilon)$ is an irreducible representation.

For $\phi \in \Phi_2(G)$ and $\varepsilon \in \widehat{S_\phi}$, we can simply define

$$\pi_M(\phi, \varepsilon) := \pi_W(\phi, \varepsilon).$$

To justify this definition, we need to recall Mœglin's construction (joint with Tadić) of discrete series of G. We start by introducing some more notations, and here we will also include the case of special orthogonal groups.

If $G = \mathrm{Sp}(2n)$, let us define it with respect to $\left(\begin{smallmatrix} 0 & -J_n \\ J_n & 0 \end{smallmatrix} \right)$, where

$$J_n = \begin{pmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{pmatrix}.$$

Let us take B to be subgroup of upper-triangular matrices in G and T to be subgroup of diagonal matrices in G; then the standard Levi subgroup M can be identified with

$$GL(n_1) \times \cdots \times GL(n_r) \times G_-$$

for any partition $n = n_1 + \cdots + n_r + n_-$ and $G_- = \mathrm{Sp}(2n_-)$ as follows:

$$\begin{pmatrix} GL(n_1) & & & & & & 0 \\ & \ddots & & & & & \\ & & GL(n_r) & & & & \\ & & & G_- & & & \\ & & & & GL(n_r) & & \\ & & & & & \ddots & \\ 0 & & & & & & GL(n_1) \end{pmatrix}$$

(1.2) $(g_1, \cdots g_r, g) \longrightarrow \mathrm{diag}\{g_1, \ldots, g_r, g, {}_t g_r^{-1}, \ldots, {}_t g_1^{-1}\},$

where ${}_t g_i = J_{n_i}{}^t g_i J_{n_i}^{-1}$ for $1 \leqslant i \leqslant r$. Note n_- can be 0, in which case we simply write $\mathrm{Sp}(0) = 1$. For $\pi = \pi_1 \otimes \cdots \otimes \pi_r \otimes \sigma$, where π_i is a finite-length smooth representation of $GL(n_i)$ for $1 \leqslant i \leqslant r$ and σ is a finite-length smooth representation of G_-, we denote the normalized parabolic induction $\mathrm{Ind}_P^G(\pi)$ by $\pi_1 \times \cdots \times \pi_r \rtimes \sigma$. Moreover, we denote the direct sum of its irreducible subrepresentations by $\langle \pi_1 \times \cdots \times \pi_r \rtimes \sigma \rangle$. This notation can be easily extended to special orthogonal groups. If $G = SO(N)$ split, we define it with respect to J_N. When N is odd, the situation is exactly the same as the symplectic case. When $N = 2n$, there are two distinctions. First, the standard Levi subgroups given through the embedding (1.2) do not exhaust all standard Levi

subgroups of $SO(2n)$. To get all of them, we need to take the θ_0-conjugate of M given in (1.2), where

$$
\theta_0 = \begin{pmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & & 1 & & & \\ & & 1 & & & & \\ & & & & \ddots & & \\ & & & & & 1 \end{pmatrix}.
$$

Note that $M^{\theta_0} \ne M$ only when $n_- = 0$ and $n_r > 1$. In order to distinguish the θ_0-conjugate standard Levi subgroups of $SO(2n)$, we will only identify those Levi subgroups M in (1.2) with $GL(n_1) \times \cdots \times GL(n_r) \times G_-$, and we denote the other one simply by M^{θ_0}. Second, if the partition $n = n_1 + \cdots + n_r + n_-$ satisfies $n_r = 1$ and $n_- = 0$, then we can rewrite it as $n = n_1 + \cdots + n_{r-1} + n'_-$ with $n'_- = 1$, and the corresponding Levi subgroup is the same. This is because $GL(1) \cong SO(2)$. To fix notation, we will always write it as $SO(2)$. In this paper, we will also consider $G = SO(2n, \eta)$, which is the outer form of the split $SO(2n)$ with respect to a quadratic extension E/F and θ_0. Here η is the associated quadratic character of E/F by the local class field theory. Then the standard Levi subgroups of $SO(2n, \eta)$ will be the outer form of those θ_0-stable standard Levi subgroups of $SO(2n)$. In particular, they can be identified with $GL(n_1) \times \cdots \times GL(n_r) \times SO(n_-, \eta)$ and $n_- \ne 0$. Note that in the case of $SO(8)$, there is another outer form, but we will not consider it in this paper.

Now we are back to the case where $G = \mathrm{Sp}(2n)$. For $\phi = \bigoplus_{i=1}^r \rho_i \otimes \nu_{a_i} \otimes \nu_1 \in \Phi_2(G)$, we define

$$
\mathrm{Jord}(\phi) := \{(\rho_i, a_i) : 1 \le i \le q\}, \quad \text{and} \quad \mathrm{Jord}_\rho(\phi) := \{a_i : \rho = \rho_i\}.
$$

Then we can identify $\widehat{S_\phi}$ with the subspace of $\mathbb{Z}_2$-valued functions $\varepsilon(\cdot)$ on $\mathrm{Jord}(\phi)$ such that

$$
\prod_{(\rho,a)\in\mathrm{Jord}(\phi)} \varepsilon(\rho, a) = 1
$$

(see Section 2). The following theorem gives a parametrization of irreducible supercuspidal representations of G.

Theorem 1.4 ([Mœg11b, Theorem 1.5.1]) *For $G = \mathrm{Sp}(2n)$, the irreducible supercuspidal representations of G are parametrized by $\phi \in \Phi_2(G)$ and $\varepsilon \in \widehat{S_\phi}$, satisfying the following properties:*

(i) *if $(\rho, a) \in \mathrm{Jord}(\phi)$, then $(\rho, a - 2) \in \mathrm{Jord}(\phi)$ as long as $a - 2 > 0$;*
(ii) *if $(\rho, a), (\rho, a - 2) \in \mathrm{Jord}(\phi)$, then $\varepsilon(\rho, a)\varepsilon(\rho, a - 2) = -1$;*
(iii) *if $(\rho, 2) \in \mathrm{Jord}(\phi)$, then $\varepsilon(\rho, 2) = -1$.*

For non-supercuspidal irreducible representations of G, we can characterize their cuspidal supports by the following proposition.

Proposition 1.5 ([Xu15, Proposition 9.3]) *For $G = \mathrm{Sp}(2n)$, suppose $\phi \in \Phi_2(G)$, and $\varepsilon \in \widehat{S_\phi}$. For any $(\rho, a) \in \mathrm{Jord}(\phi)$, we denote by a_- the biggest positive integer*

smaller than a in $\mathrm{Jord}_\rho(\phi)$. And we also write $a_{\min}$ for the minimum of $\mathrm{Jord}_\rho(\phi)$. If $a = a_{\min}$, we let $a_- = 0$ if a is even, and -1 otherwise. In this case, we always assume $\varepsilon(\rho, a)\varepsilon(\rho, a_-) = -1$.

(i)   *If $\varepsilon(\rho, a)\varepsilon(\rho, a_-) = -1$ and $a_- < a - 2$, then*

(1.3) $\pi_W(\phi, \varepsilon) \hookrightarrow \langle (a-1)/2, \ldots, (a_- + 3)/2 \rangle \rtimes \pi_W(\phi', \varepsilon')$

   as the unique irreducible subrepresentation, where

   $$\mathrm{Jord}(\phi') = \mathrm{Jord}(\phi) \cup \{(\rho, a_- + 2)\} \backslash \{(\rho, a)\},$$

   and

   $$\varepsilon'(\cdot) = \varepsilon(\cdot) \text{ over } \mathrm{Jord}(\phi)\backslash\{(\rho, a)\}, \qquad \varepsilon'(\rho, a_- + 2) = \varepsilon(\rho, a).$$

(ii)   *If $\varepsilon(\rho, a)\varepsilon(\rho, a_-) = 1$, then*

(1.4) $\pi_W(\phi, \varepsilon) \hookrightarrow \langle (a-1)/2, \ldots, -(a_- - 1)/2 \rangle \rtimes \pi_W(\phi', \varepsilon'),$

   where

   $$\mathrm{Jord}(\phi') = \mathrm{Jord}(\phi)\backslash\{(\rho, a), (\rho, a_-)\},$$

   and $\varepsilon'(\cdot)$ is the restriction of $\varepsilon(\cdot)$. In particular, suppose $\varepsilon_1 \in \widehat{\mathcal{S}_\phi}$ satisfying $\varepsilon_1(\cdot) = \varepsilon(\cdot)$ over $\mathrm{Jord}(\phi')$ and

   $$\varepsilon_1(\rho, a) = -\varepsilon(\rho, a), \quad \varepsilon_1(\rho, a_-) = -\varepsilon(\rho, a_-).$$

   Then the induced representation in (1.4) has two irreducible subrepresentations, namely

   $$\pi_W(\phi, \varepsilon) \oplus \pi_W(\phi, \varepsilon_1).$$

(iii)   *If $\varepsilon(\rho, a_{\min}) = 1$ and $a_{\min}$ is even, then*

(1.5) $\pi_W(\phi, \varepsilon) \hookrightarrow \langle (a_{\min} - 1)/2, \ldots, a_0 \rangle \rtimes \pi_W(\phi', \varepsilon')$

   as the unique irreducible subrepresentation, where

   $$\mathrm{Jord}(\phi') = \mathrm{Jord}(\phi)\backslash\{(\rho, a_{\min})\},$$

   and $\varepsilon'(\cdot)$ is the restriction of $\varepsilon(\cdot)$.

The construction of discrete series by Mœglin and Tadić can be obtained by reversing steps (1.3), (1.4), and (1.5) in this proposition. Finally, in the general construction of $\pi_M(\psi, \varepsilon)$, one requires various reducibility results, which are all based on the following basic criterion.

Proposition 1.6 ([Xu15], Corollary 9.1)   *For $G = \mathrm{Sp}(2n)$, suppose π is a supercuspidal representation of G and $\pi \in \Pi_\phi$ for some $\phi \in \Phi_2(G)$. Then for any unitary irreducible supercuspidal representation ρ of $\mathrm{GL}(d_\rho)$, the parabolic induction*

$$\rho\|\|^{\pm(a_\rho+1)/2} \rtimes \pi$$

reduces exactly for

$$a_\rho = \begin{cases} \max \mathrm{Jord}_\rho(\phi), & \text{if } \mathrm{Jord}_\rho(\phi) \neq \varnothing, \\ 0, & \text{if } \mathrm{Jord}_\rho(\phi) = \varnothing, \rho \text{ is self-dual and is of opposite type to } \widehat{G}, \\ -1, & \text{otherwise.} \end{cases}$$

   The main tool in Mœglin's construction of elements in the Arthur packets of classical groups is the Jacquet module. Here we would like to summarize the relevant notation about Jacquet modules used in her work. For general G, we denote by $\mathrm{Rep}(G)$ the category of finite-length smooth representations of G. We include the zero space in $\mathrm{Rep}(G)$, and by an irreducible representation we always mean it is nonzero. Now let G be a quasisplit symplectic or special orthogonal group of $\overline{F}$-rank n. We fix a unitary irreducible supercuspidal representation ρ of $GL(d_\rho)$, and we assume $M = GL(d_\rho) \times G_-$ is the Levi component of a standard maximal parabolic subgroup P of G. Note that in case $G_- = 1$ and G is special even orthogonal, we require P to be contained in the standard parabolic subgroup of $GL(2n)$ by our convention. Then for $\pi \in \mathrm{Rep}(G)$, we can decompose the semisimplification of the Jacquet module

$$s.s.\,\mathrm{Jac}_P(\pi) = \bigoplus_i \tau_i \otimes \sigma_i,$$

where $\tau_i \in \mathrm{Rep}(GL(d_\rho))$ and $\sigma_i \in \mathrm{Rep}(G_-)$, both of which are irreducible. We define $\mathrm{Jac}_x \pi$ for any real number x to be

$$\mathrm{Jac}_x(\pi) = \bigoplus_{\tau_i = \rho \| ^x} \sigma_i.$$

If we have an ordered sequence of real numbers $\{x_1, \dots, x_s\}$, we can define

$$\mathrm{Jac}_{x_1,\dots,x_s} \pi = \mathrm{Jac}_{x_s} \circ \cdots \circ \mathrm{Jac}_{x_1} \pi.$$

Moreover, let

$$\overline{\mathrm{Jac}}_x = \begin{cases} \mathrm{Jac}_x + \mathrm{Jac}_x \circ \theta_0, & \text{if } G = SO(2n) \text{ and } n = d_\rho \neq 1, \\ \mathrm{Jac}_x, & \text{otherwise.} \end{cases}$$

Then Jac_x defines a functor on the category of $O(2n)$-conjugacy classes of finite-length smooth representations of $SO(2n)$. It is not hard to see Jac_x can be defined for $GL(n)$ in a similar way by replacing G_- by $GL(n_-)$. Furthermore, we can define Jac_x^{op} analogously to Jac_x but with respect to $\rho^\vee$ and the standard Levi subgroup $GL(n_-) \times GL(d_{\rho^\vee})$. So let us define $\mathrm{Jac}_x^\theta = \mathrm{Jac}_x \circ \mathrm{Jac}_{-x}^{op}$ for $GL(n)$. There are some explicit formulas for computing these Jacquet modules, and we refer the readers to [Xu15, Section 5].

2   Arthur Parameters

Let F be a p-adic field and let G be a quasisplit symplectic or special orthogonal group. We define the local Langlands group as $L_F = W_F \times SL(2, \mathbb{C})$, where W_F is the usual Weil group. We write $\Gamma_F = \Gamma_{\overline{F}/F}$ for the absolute Galois group over F. Let $\widehat{G}$ be the complex dual group of G, and let $^L G$ be the Langlands dual group of G. An Arthur parameter of G is a $\widehat{G}$-conjugacy class of admissible homomorphisms

$$\underline{\psi}: L_F \times SL(2, \mathbb{C}) \longrightarrow {}^L G,$$

such that $\underline{\psi}|_{W_F}$ is bounded. If $\underline{\psi}|_{SL(2,\mathbb{C})} = 1$, we say the parameter is tempered. We denote by $\Psi(G)$ the set of Arthur parameters of G. Here we can simplify the Langlands dual groups as in the following table:

G	$^L G$
$\mathrm{Sp}(2n)$	$SO(2n+1,\mathbb{C})$
$SO(2n+1)$	$\mathrm{Sp}(2n,\mathbb{C})$
$SO(2n,\eta)$	$SO(2n,\mathbb{C}) \rtimes \Gamma_{E/F}$

In the last case, η is a quadratic character associated with a quadratic extension E/F and $\Gamma_{E/F}$ is the associated Galois group. We fix an isomorphism $SO(2n,\mathbb{C}) \rtimes \Gamma_{E/F} \cong O(2n,\mathbb{C})$. So in either of these cases, there is a natural embedding ξ_N of $^L G$ into $GL(N,\mathbb{C})$ up to $GL(N,\mathbb{C})$-conjugacy, where $N = 2n+1$ if $G = \mathrm{Sp}(2n)$ or $N = 2n$ otherwise. We fix an outer automorphism θ_0 of G preserving an F-splitting. If G is symplectic or special odd orthogonal, we let $\theta_0 = \mathrm{id}$. If G is special even orthogonal, we let θ_0 be induced from the conjugate action of the nonconnected component of the full orthogonal group. Let $\widehat{\theta_0}$ be the dual automorphism of θ_0. We write $\Sigma_0 = \langle \theta_0 \rangle$, $G^{\Sigma_0} = G \rtimes \langle \theta_0 \rangle$, and $\widehat{G}^{\Sigma_0} = \widehat{G} \rtimes \langle \widehat{\theta_0} \rangle$. So in the special even orthogonal case, G^{Σ_0} (resp. $\widehat{G}^{\Sigma_0}$) is isomorphic to the full (resp. complex) orthogonal group. Let ω_0 be the character of G^{Σ_0}/G, which is nontrivial when G is special even orthogonal.

By composing ψ with ξ_N, we can view ψ as an equivalence class of N-dimensional self-dual representation of $L_F \times SL(2,\mathbb{C})$. So we can decompose ψ as follows:

$$(2.1) \qquad \psi = \bigoplus_{i=1}^{r} l_i \psi_i = \bigoplus_{i=1}^{r} l_i (\rho_i \otimes \nu_{a_i} \otimes \nu_{b_i}).$$

Here ρ_i are equivalence classes of irreducible unitary representations of W_F, which can be identified with irreducible unitary supercuspidal representations of $GL(d_{\rho_i})$ under the local Langlands correspondence (cf. [HT01, Hen00, Sch13]). And ν_{a_i} (resp. ν_{b_i}) are the $(a_i - 1)$-th (resp. $(b_i - 1)$-th) symmetric power representations of $SL(2,\mathbb{C})$. The irreducible constituent $\rho_i \otimes \nu_{a_i} \otimes \nu_{b_i}$ has dimension $n_i = n_{(\rho_i, a_i, b_i)}$ and multiplicity l_i. We define the multi-set of Jordan blocks for ψ as follows:

$$\mathrm{Jord}(\psi) := \{(\rho_i, a_i, b_i) \text{ with multiplicity } l_i : 1 \leqslant i \leqslant r\}.$$

For any ρ, let us define

$$\mathrm{Jord}_\rho(\psi) := \{(\rho', a', b') \in \mathrm{Jord}(\psi) : \rho' = \rho\}.$$

Fix a representative $\underline{\psi}$; we define for any subgroup $\Sigma \subseteq \Sigma_0$

$$S_{\underline{\psi}}^{\Sigma} = \mathrm{Cent}(\mathrm{Im}\,\underline{\psi}, \widehat{G}^{\Sigma}),$$

$$\overline{S}_{\underline{\psi}}^{\Sigma} = S_{\underline{\psi}}^{\Sigma} / Z(\widehat{G})^{\Gamma_F},$$

$$\mathcal{S}_{\underline{\psi}}^{\Sigma} = \overline{S}_{\underline{\psi}}^{\Sigma} / \overline{S}_{\underline{\psi}}^{0} = S_{\underline{\psi}}^{\Sigma} / S_{\underline{\psi}}^{0} Z(\widehat{G})^{\Gamma_F}.$$

We denote by s_ψ the image of the nontrivial central element of $SL(2,\mathbb{C})$ in $\mathcal{S}_{\underline{\psi}}$.

To characterize the centralizer groups $\mathcal{S}_{\underline{\psi}}$ and $\mathcal{S}_{\underline{\psi}}^{\Sigma_0}$, we need to introduce a parity condition on the set of Jordan blocks $\mathrm{Jord}(\underline{\psi})$. There is a common way to define the parity for self-dual irreducible representations ρ of W_F (see [Xu15, Section 3]). We say (ρ_i, a_i, b_i) is of *orthogonal type* if $\rho_i \otimes \nu_{a_i} \otimes \nu_{b_i}$ factors through an orthogonal

group, or equivalently $a_i + b_i$ is even when ρ_i is of orthogonal type and $a_i + b_i$ is odd when ρ_i is of symplectic type. Similarly we say (ρ_i, a_i, b_i) is of *symplectic type* if $\rho_i \otimes v_{a_i} \otimes v_{b_i}$ factors through a symplectic group, or equivalently $a_i + b_i$ is odd when ρ_i is of orthogonal type and $a_i + b_i$ is even when ρ_i is of symplectic type. Let ψ_p be the parameter whose Jordan blocks consists of those in $\mathrm{Jord}(\psi)$ with the same parity as $\widehat{G}$, and let ψ_{np} be any Arthur parameters of general linear group such that

$$\psi = \psi_{np} \oplus \psi_p \oplus \psi_{np}^{\vee}.$$

We denote by $\mathrm{Jord}(\psi)_p$ the set of Jordan blocks in $\mathrm{Jord}(\psi_p)$ without multiplicity. After this preparation, we can identify those centralizer groups above with certain quotient space of $\mathbb{Z}_2$-valued functions on $\mathrm{Jord}(\psi)_p$. To be more precise, let $s_0 = (s_{0,i}) \in \mathbb{Z}_2^{\mathrm{Jord}(\psi)_p}$ be defined as $s_{0,i} = 1$ if l_i is even and $s_{0,i} = -1$ if l_i is odd. Then

$$\mathcal{S}_{\underline{\psi}}^{\Sigma_0} \cong \left\{ s = (s_i) \in \mathbb{Z}_2^{\mathrm{Jord}(\psi)_p} \right\} / \langle s_0 \rangle$$

and

$$\mathcal{S}_{\underline{\psi}} \cong \left\{ s = (s_i) \in \mathbb{Z}_2^{\mathrm{Jord}(\psi)_p} : \prod_i (s_i)^{n_i} = 1 \right\} / \langle s_0 \rangle$$

if G is special even orthogonal. Under these identifications,

$$s_{\underline{\psi}} = s_{\psi} := (s_{\psi,i}) \in \mathbb{Z}_2^{\mathrm{Jord}(\psi)_p}$$

with $s_{\psi,i} = (-1)^{l_i}$ if b_i is even and $s_{\psi,i} = 1$ if b_i is odd. Let us denote by $\mathcal{S}_{\psi}$ (resp. $\mathcal{S}_{\psi}^{\Sigma_0}$) the corresponding quotient space of $\mathbb{Z}_2$-valued functions on $\mathrm{Jord}(\psi)_p$ such that $\mathcal{S}_{\psi} \cong \mathcal{S}_{\underline{\psi}}$ (resp. $\mathcal{S}_{\psi}^{\Sigma_0} \cong \mathcal{S}_{\underline{\psi}}^{\Sigma_0}$).

There is a natural inner product on $\mathbb{Z}_2^{\mathrm{Jord}(\psi)_p}$ that identifies its dual with itself. Let $\varepsilon = (\varepsilon_i)$ and $s = (s_i)$ be two elements in $\mathbb{Z}_2^{\mathrm{Jord}(\psi)_p}$; then their inner product is defined by $\varepsilon(s) = \prod_i (\varepsilon_i * s_i)$, where

$$\varepsilon_i * s_i = \begin{cases} -1, & \text{if } \varepsilon_i = s_i = -1, \\ 1, & \text{otherwise.} \end{cases}$$

So on the dual side,

$$\widehat{\mathcal{S}_{\psi}^{\Sigma_0}} = \left\{ \varepsilon = (\varepsilon_i) \in \mathbb{Z}_2^{\mathrm{Jord}(\psi)_p} : \prod_i \varepsilon_i^{l_i} = 1 \right\}.$$

When G is special even orthogonal, let $\varepsilon_0 = (\varepsilon_{0,i}) \in \mathbb{Z}_2^{\mathrm{Jord}(\psi)_p}$ be defined as $\varepsilon_{0,i} = 1$ if n_i is even, or $\varepsilon_{0,i} = -1$ if n_i is odd then $\varepsilon_0 \in \mathcal{S}_{\psi}^{\Sigma_0}$ is always trivial when restricted to $\mathcal{S}_{\psi}$, and

$$\widehat{\mathcal{S}_{\psi}} = \left\{ \varepsilon = (\varepsilon_i) \in \mathbb{Z}_2^{\mathrm{Jord}(\psi)_p} : \prod_i \varepsilon_i^{l_i} = 1 \right\} / \langle \varepsilon_0 \rangle.$$

In general, we can let $\varepsilon_0 = 1$ if G is not special even orthogonal. In this paper, we will always denote elements in $\widehat{\mathcal{S}_{\psi}^{\Sigma_0}}$ by ε and denote its image in $\widehat{\mathcal{S}_{\psi}}$ by $\overline{\varepsilon}$.

For computational purposes, it is more convenient to view $\mathcal{S}_{\psi}^{\Sigma_0}$ as functions on $\mathrm{Jord}(\psi_p)$. In fact there is a natural projection

$$\mathbb{Z}_2^{\mathrm{Jord}(\psi_p)} \xrightarrow{\mathrm{Cont}} \mathbb{Z}_2^{\mathrm{Jord}(\psi)_p}$$
$$s \longmapsto s'$$

such that

$$s'(\rho, a, b) = \prod_{\substack{(\rho', a', b') \in \mathrm{Jord}(\psi_p) \\ (\rho', a', b') = (\rho, a, b) \text{ in } \mathrm{Jord}(\psi)_p}} s(\rho', a', b')$$

for $(\rho, a, b) \in \mathrm{Jord}(\psi)_p$. In particular, s_0 has a natural representative $s_0^>$ in $\mathbb{Z}_2^{\mathrm{Jord}(\psi_p)}$ given by $s_0^>(\rho, a, b) = -1$ for all $(\rho, a, b) \in \mathrm{Jord}(\psi_p)$. When G is special even orthogonal, the determinant condition for defining $\mathcal{S}_\psi$ becomes

$$(2.2) \qquad \prod_{(\rho, a, b) \in \mathrm{Jord}(\psi_p)} s(\rho, a, b)^{n_{(\rho, a, b)}} = 1.$$

Moreover, s_ψ also has a natural representative $s_\psi^>$ in $\mathbb{Z}_2^{\mathrm{Jord}(\psi_p)}$ given by $s_\psi^>(\rho, a, b) = -1$ if b is even or 1 if b is odd. We define

$$\mathcal{S}_{\psi^>}^{\Sigma_0} = \left\{ s(\cdot) \in \mathbb{Z}_2^{\mathrm{Jord}(\psi_p)} \right\} / \langle s_0^> \rangle,$$

and

$$\mathcal{S}_{\psi^>} = \left\{ s(\cdot) \in \mathbb{Z}_2^{\mathrm{Jord}(\psi_p)} : \prod_{(\rho, a, b) \in \mathrm{Jord}(\psi_p)} s(\rho, a, b)^{n_{(\rho, a, b)}} = 1 \right\} / \langle s_0^> \rangle$$

if G is special even orthogonal. Then there are surjections $\mathcal{S}_{\psi^>}^{\Sigma_0} \to \mathcal{S}_\psi^{\Sigma_0}$ and $\mathcal{S}_{\psi^>} \to \mathcal{S}_\psi$.

On the dual side, we have a natural inclusion

$$\mathbb{Z}_2^{\mathrm{Jord}(\psi)_p} \xrightarrow{\mathrm{Ext}} \mathbb{Z}_2^{\mathrm{Jord}(\psi_p)}$$

$$\varepsilon \longmapsto \varepsilon'$$

such that $\varepsilon'(\rho, a, b) = \varepsilon(\rho, a, b)$ for $(\rho, a, b) \in \mathrm{Jord}(\psi_p)$. We can define an inner product on $\mathbb{Z}_2^{\mathrm{Jord}(\psi_p)}$ as for $\mathbb{Z}_2^{\mathrm{Jord}(\psi)_p}$. Then this inclusion is adjoint to the previous projection in the sense that

$$\varepsilon(\mathrm{Cont}(s)) = \mathrm{Ext}(\varepsilon)(s)$$

for $\varepsilon \in \mathbb{Z}_2^{\mathrm{Jord}(\psi)_p}$ and $s \in \mathbb{Z}_2^{\mathrm{Jord}(\psi_p)}$. Therefore, ε_0 can also be viewed as a function on $\mathrm{Jord}(\psi_p)$ through the inclusion map, and the condition imposed on defining $\widehat{\mathcal{S}_\psi^{\Sigma_0}}$ becomes

$$\prod_{(\rho, a, b) \in \mathrm{Jord}(\psi_p)} \varepsilon(\rho, a, b) = 1.$$

We also define

$$\widehat{\mathcal{S}_{\psi^>}^{\Sigma_0}} = \left\{ \varepsilon(\cdot) \in \mathbb{Z}_2^{\mathrm{Jord}(\psi_p)} : \prod_{(\rho, a, b) \in \mathrm{Jord}(\psi_p)} \varepsilon(\rho, a, b) = 1 \right\},$$

$$\widehat{\mathcal{S}_{\psi^>}} = \left\{ \varepsilon(\cdot) \in \mathbb{Z}_2^{\mathrm{Jord}(\psi_p)} : \prod_{(\rho, a, b) \in \mathrm{Jord}(\psi_p)} \varepsilon(\rho, a, b) = 1 \right\} / \langle \varepsilon_0 \rangle$$

if G is special even orthogonal. Then there are inclusions $\widehat{\mathcal{S}_\psi^{\Sigma_0}} \hookrightarrow \widehat{\mathcal{S}_{\psi^>}^{\Sigma_0}}$ and $\widehat{\mathcal{S}_\psi} \hookrightarrow \widehat{\mathcal{S}_{\psi^>}}$. For $\varepsilon \in \widehat{\mathcal{S}_{\psi^>}^{\Sigma_0}}$, we denote its image in $\widehat{\mathcal{S}_{\psi^>}}$ by $\bar\varepsilon$.

In the end, we are going to associate any Arthur parameter $\psi \in \Psi(G)$ with two Langlands parameters of G naturally. For the first one, we define

$$\phi_\psi(u) = \psi\left(u, \begin{pmatrix} |u|^{\frac{1}{2}} & 0 \\ 0 & |u|^{-\frac{1}{2}} \end{pmatrix}\right), \quad u \in L_F.$$

Note that ϕ_ψ is nontempered in general, and in the notation of (2.1) we can write it as

$$\phi_\psi = \bigoplus_{i=1}^r l_i \Big(\bigoplus_{j=0}^{b_i-1} (\rho_i \| \|^{(b_i-1)/2-j} \otimes v_{a_i}) \Big).$$

For the second one, we can compose ψ with

$$\Delta \colon W_F \times SL(2,\mathbb{C}) \longrightarrow W_F \times SL(2,\mathbb{C}) \times SL(2,\mathbb{C}),$$

which is the diagonal embedding of $SL(2,\mathbb{C})$ into $SL(2,\mathbb{C}) \times SL(2,\mathbb{C})$ when restricted to $SL(2,\mathbb{C})$, and is the identity on W_F. Note that the composition $\psi_d := \psi \circ \Delta$ is tempered. To expand ψ_d, we need to introduce some more notation. For $(\rho, a, b) \in \mathrm{Jord}(\psi)$, let us write $A = (a+b)/2 - 1$, $B = |a-b|/2$, and set $\zeta = \zeta_{a,b} = \mathrm{Sign}(a-b)$ if $a \neq b$ and arbitrary otherwise. Then we can replace (ρ, a, b) by (ρ, A, B, ζ). Under this new notation, we have

$$\psi_d = \bigoplus_{i=1}^r l_i \Big(\bigoplus_{j \in [A_i, B_i]} \rho_i \otimes v_{2j+1} \Big),$$

where j is taken over half-integers in the segment $[A_i, B_i]$.

Finally, Σ_0 acts on $\Psi(G)$ through $\widehat{\theta}_0$, and we denote the corresponding set of Σ_0-orbits by $\overline{\Psi}(G)$. It is clear that for $\psi \in \Psi(G)$, $\mathrm{Jord}(\psi)$ only depends on its image in $\overline{\Psi}(G)$. It is for this reason that we will also denote the elements in $\overline{\Psi}(G)$ by ψ. Moreover, through the natural embedding ξ_N, we can view $\overline{\Psi}(G)$ as a subset of equivalence classes of N-dimensional self-dual representations of $L_F \times SL(2,\mathbb{C})$.

3 Endoscopy

Before we can introduce the Arthur packets, we need to talk about the relevant cases of endoscopy in this paper. The discussion here will be parallel to that in [Xu15, Section 4]. Suppose $\psi \in \Psi(G)$ and $s \in \overline{S}_\psi$ is semisimple. In our case, there is a quasisplit reductive group H with the property that

$$\widehat{H} \cong \mathrm{Cent}(s, \widehat{G})^0,$$

and the isomorphism extends to an embedding

$$\xi \colon {}^L H \longrightarrow {}^L G$$

such that $\xi({}^L H) \subseteq \mathrm{Cent}(s, {}^L G)$ and ψ factors through ${}^L H$. So from ψ we get a parameter $\psi_H \in \Psi(H)$. We say $(H, \underline{\psi}_H)$ corresponds to $(\underline{\psi}, s)$ through ξ, and denote this relation by $(H, \underline{\psi}_H) \to (\psi, s)$. Such H is called an *endoscopic group* of G. In the following examples we will always assume $\psi = \psi_p$.

Example 3.1 (i) If $G = \mathrm{Sp}(2n)$, then ${}^L G = SO(2n+1, \mathbb{C})$. For $s \in S_{\psi^>}$, it gives a partition on $\mathrm{Jord}(\psi)$ depending on $s(\rho, a, b) = 1$ or -1, *i.e.,*

$$\mathrm{Jord}(\psi) = \mathrm{Jord}_+ \sqcup \mathrm{Jord}_- .$$

Without loss of generality, let us assume

$$\sum_{(\rho,a,b) \in \mathrm{Jord}_+} n_{(\rho,a,b)} = 2n_I + 1 = N_I \quad \text{and} \quad \sum_{(\rho,a,b) \in \mathrm{Jord}_-} n_{(\rho,a,b)} = 2n_{II} = N_{II}.$$

Define
$$\eta_I = \eta_{II} = \prod_{(\rho,a,b)\in\mathrm{Jord}_-} \eta_{(\rho,a,b)},$$
where $\eta_{(\rho,a,b)}$ is the quadratic character dual to $\det(\rho \otimes \nu_a \otimes \nu_b)$. Let
$$G_I = \mathrm{Sp}(2n_I) \quad \text{and} \quad G_{II} = SO(2n_{II}, \eta_{II}).$$
Then we have
$$H = G_I \times G_{II} \quad \text{and} \quad {}^L H = (\widehat{G}_I \times \widehat{G}_{II}) \rtimes \Gamma_{E_{II}/F},$$
where E_{II} is the quadratic extension of F associated with η_{II}. Let
$$\xi_i \colon {}^L G_i \longrightarrow GL(N_i, \mathbb{C})$$
be the natural embedding for $i = I, II$. Then $\xi := (\xi_I \otimes \eta_I) \oplus \xi_{II}$ factors through ${}^L G$ and defines an embedding ${}^L H \hookrightarrow {}^L G$. We define $\psi_I \in \overline{\Psi}(G_I)$ by
$$\mathrm{Jord}(\psi_I) := \big\{ (\rho \otimes \eta_I, a, b) : (\rho, a, b) \in \mathrm{Jord}_+ \big\},$$
and $\psi_{II} \in \overline{\Psi}(G_{II})$ by
$$\mathrm{Jord}(\psi_{II}) := \big\{ (\rho, a, b) \in \mathrm{Jord}_- \big\}.$$
Let $\psi_H = \psi_I \times \psi_{II}$.

(ii) If $G = SO(2n + 1)$, then ${}^L G = \mathrm{Sp}(2n, \mathbb{C})$. For $s \in \mathcal{S}_{\psi^>}$, it gives a partition on $\mathrm{Jord}(\psi)$ depending on $s(\rho, a, b) = 1$ or -1, i.e.,
$$\mathrm{Jord}(\psi) = \mathrm{Jord}_+ \sqcup \mathrm{Jord}_- .$$
We can assume
$$\sum_{(\rho,a,b)\in\mathrm{Jord}_+} n_{(\rho,a,b)} = 2n_I = N_I \quad \text{and} \quad \sum_{(\rho,a,b)\in\mathrm{Jord}_-} n_{(\rho,a,b)} = 2n_{II} = N_{II}.$$
Define $\eta_I = \eta_{II} = 1$. Let
$$G_I = SO(2n_I + 1) \quad \text{and} \quad G_{II} = SO(2n_{II} + 1).$$
Then we have
$$H = G_I \times G_{II} \quad \text{and} \quad {}^L H = \widehat{G}_I \times \widehat{G}_{II}$$
Let $\xi_i \colon {}^L G_i \hookrightarrow GL(N_i, \mathbb{C})$ be the natural embedding for $i = I, II$. Then $\xi := \xi_I \oplus \xi_{II}$ factors through ${}^L G$ and defines an embedding ${}^L H \hookrightarrow {}^L G$. We define $\psi_I \in \overline{\Psi}(G_I)$ by
$$\mathrm{Jord}(\psi_I) := \big\{ (\rho, a, b) \in \mathrm{Jord}_+ \big\},$$
and $\psi_{II} \in \overline{\Psi}(G_{II})$ by
$$\mathrm{Jord}(\psi_{II}) := \big\{ (\rho, a, b) \in \mathrm{Jord}_- \big\}.$$
Let $\psi_H = \psi_I \times \psi_{II}$.

(iii) If $G = SO(2n, \eta)$, then ${}^L G = SO(2n, \mathbb{C}) \rtimes \Gamma_{E/F}$. For $s \in \mathcal{S}_{\psi^>}$, there is a partition on $\mathrm{Jord}(\psi)$ depending on $s(\rho, a, b) = 1$ or -1, i.e.,
$$\mathrm{Jord}(\psi) = \mathrm{Jord}_+ \sqcup \mathrm{Jord}_- .$$
By condition (2.2), we can assume
$$\sum_{(\rho,a,b)\in\mathrm{Jord}_+} n_{(\rho,a,b)} = 2n_I = N_I \quad \text{and} \quad \sum_{(\rho,a,b)\in\mathrm{Jord}_-} n_{(\rho,a,b)} = 2n_{II} = N_{II}.$$

Define
$$\eta_I = \prod_{(\rho,a,b)\in\mathrm{Jord}_+} \eta_{(\rho,a,b)} \quad\text{and}\quad \eta_{II} = \prod_{(\rho,a,b)\in\mathrm{Jord}_-} \eta_{(\rho,a,b)},$$

where $\eta_{(\rho,a,b)}$ is the quadratic character dual to $\det(\rho \otimes v_a \otimes v_b)$. We also denote by E_i the quadratic extension of F associated with η_i for $i = I, II$. Let

$$G_I = SO(2n_I, \eta_I) \quad\text{and}\quad G_{II} = SO(2n_{II}, \eta_{II}).$$

Then we have

$$H = G_I \times G_{II} \quad\text{and}\quad {}^L H = (\widehat{G}_I \times \widehat{G}_{II}) \rtimes \Gamma_{L/F},$$

where $L = E_I E_{II}$. Let

$$\xi_i : {}^L G_i \hookrightarrow GL(N_i, \mathbb{C})$$

be the natural embedding for $i = I, II$. Then $\xi := \xi_I \oplus \xi_{II}$ factors through ${}^L G$ and defines an embedding ${}^L H \hookrightarrow {}^L G$. We define $\psi_I \in \overline{\Psi}(G_I)$ by

$$\mathrm{Jord}(\psi_I) := \big\{ (\rho, a, b) \in \mathrm{Jord}_+ \big\},$$

and $\psi_{II} \in \overline{\Psi}(G_{II})$ by

$$\mathrm{Jord}(\psi_{II}) := \big\{ (\rho, a, b) \in \mathrm{Jord}_- \big\}.$$

Let $\psi_H = \psi_I \times \psi_{II}$.

In the examples above, H is called an *elliptic endoscopic group* of G. We can define $\overline{\Psi}(H) = \overline{\Psi}(G_I) \times \overline{\Psi}(G_{II})$; then $\psi_H \in \overline{\Psi}(H)$. For $s \in \mathcal{S}_{\psi^>}$, we still say (H, ψ_H) correspond to (ψ, s) through ξ, and denote this relation by $(H, \psi_H) \to (\psi, s)$.

In part (iii), it is possible to also choose $s \in \mathcal{S}_{\psi^>}^{\Sigma_0}$ but not in $\mathcal{S}_{\psi^>}$, and then we get a partition on $\mathrm{Jord}(\psi)$, *i.e.*,

$$\mathrm{Jord}(\psi) = \mathrm{Jord}_+ \sqcup \mathrm{Jord}_-$$

so that

$$\sum_{(\rho,a,b)\in\mathrm{Jord}_+} n_{(\rho,a,b)} = 2n_I + 1 = N_I \quad\text{and}\quad \sum_{(\rho,a,b)\in\mathrm{Jord}_-} n_{(\rho,a,b)} = 2n_{II} + 1 = N_{II}.$$

Define
$$\eta_I = \prod_{(\rho,a,b)\in\mathrm{Jord}_+} \eta_{(\rho,a,b)} \quad\text{and}\quad \eta_{II} = \prod_{(\rho,a,b)\in\mathrm{Jord}_-} \eta_{(\rho,a,b)},$$

where $\eta_{(\rho,a,b)}$ is the quadratic character dual to $\det(\rho \otimes v_a \otimes v_b)$. Let

$$G_I = Sp(2n_I) \quad\text{and}\quad G_{II} = Sp(2n_{II}).$$

Then we can define $\psi_I \in \overline{\Psi}(G_I)$ by

$$\mathrm{Jord}(\psi_I) := \big\{ (\rho \otimes \eta_I, a, b) \in \mathrm{Jord}_+ \big\},$$

and $\psi_{II} \in \overline{\Psi}(G_{II})$ by

$$\mathrm{Jord}(\psi_{II}) := \big\{ (\rho \otimes \eta_{II}, a, b) \in \mathrm{Jord}_- \big\}.$$

Let

$$H = G_I \times G_{II} \quad\text{and}\quad {}^L H = \widehat{G}_I \times \widehat{G}_{II}.$$

In this case, H is called a *twisted elliptic endoscopic group* of G. Let

$$\xi_i : {}^L G_i \hookrightarrow GL(N_i, \mathbb{C})$$

be the natural embedding for $i = I, II$. Then

$$\xi := (\xi_I \otimes \eta_I) \oplus (\xi_{II} \otimes \eta_{II})$$

factors through $^L G$ and defines an embedding $^L H \hookrightarrow {}^L G$. Let $\psi_H = \psi_I \times \psi_{II}$. We say (H, ψ_H) corresponds to (ψ, s) through ξ, and write $(H, \psi_H) \to (\psi, s)$.

In this paper, we also want to consider the twisted elliptic endoscopic groups of $GL(N)$, but we will only need the simplest case here. Recall that for $\psi \in \Psi(G)$, we can view $\underline{\psi}$ as a self-dual N-dimensional representation through the natural embedding

$$\xi_N \colon {}^L G \longrightarrow GL(N, \mathbb{C}),$$

and in this way we get a self-dual Arthur parameter for $GL(N)$. We fix an outer automorphism θ_N of $GL(N)$ preserving an F-splitting, and let $\widehat{\theta}_N$ be the dual automorphism on $GL(N, \mathbb{C})$; then

$$\xi_N(^L G) \subseteq \mathrm{Cent}(s, GL(N, \mathbb{C})) \quad \text{and} \quad \widehat{G} = \mathrm{Cent}(s, GL(N, \mathbb{C}))^0$$

for some semisimple $s \in GL(N, \mathbb{C}) \rtimes \widehat{\theta}_N$. So we call G a *twisted elliptic endoscopic group* of $GL(N)$.

What lies at the heart of endoscopy theory is a (twisted) transfer map on the spaces of smooth compactly supported functions from G to its (twisted) elliptic endoscopic group H (similarly from $GL(N)$ to its twisted elliptic endoscopic group G). The existence of the (twisted) transfer map is quite deep, and it was conjectured by Langlands, Shelstad, and Kottwitz. In a series of papers Waldspurger [Wal95, Wal97, Wal06, Wal08] was able to reduce it to the *Fundamental Lemma* for Lie algebras over the function fields. Finally, it is in this particular form of the fundamental lemma, Ngo [Ngô10] gave his celebrated proof. Let us denote such transfers by

$$(3.1) \qquad\qquad\qquad C_c^\infty(G) \longrightarrow C_c^\infty(H)$$
$$f \longmapsto f^H,$$

and similarly

$$(3.2) \qquad\qquad\qquad C_c^\infty(GL(N)) \longrightarrow C_c^\infty(G)$$
$$f \longmapsto f^G.$$

In the definition of the (twisted) transfer maps, there is a normalization issue. To resolve that, we will always fix a Σ_0-stable (resp. θ_N-stable) Whittaker datum for G (resp. $GL(N)$) in this paper, and we will take the so-called Whittaker normalization on the transfer maps. We should also point out these transfer maps are only well defined after we pass to the space of (twisted) *orbital integrals* on the source and the space *stable orbital integrals* on the target. Note that the space of (twisted) (resp. stable) orbital integrals are dual to the space of (twisted) (resp. stable) invariant distributions on G; *i.e.*, one can view the (twisted) (resp. stable) invariant distributions of G as linear functionals of the space of (twisted) (resp. stable) orbital integrals. So, dual to these transfer maps, the stable distributions on H (resp. G) will map to (twisted) invariant distributions on G (resp. $GL(N)$). We call this map the (twisted) *spectral endoscopic transfer*. Since we can identify $C_c^\infty(G \rtimes \theta_0)$ (resp. $C_c^\infty(GL(N) \rtimes \theta_N)$) with $C_c^\infty(G)$ (resp. $C_c^\infty(GL(N))$) by sending $g \rtimes \theta_0$ (resp. $g_N \rtimes \theta_N$) to g (resp. g_N), we can define the twisted transfer map also for $C_c^\infty(G \rtimes \theta_0)$ (resp. $C_c^\infty(GL(N) \rtimes \theta_N)$).

If π is an irreducible smooth representation of G, then it defines an invariant distribution on G by the trace of

$$\pi(f) = \int_G f(g)\pi(g)dg$$

for $f \in C_c^\infty(G)$. We call this the *character* of π and denote it by $f_G(\pi)$. For any irreducible representation π^{Σ_0} of G^{Σ_0}, which contains π in its restriction to G, we define a twisted invariant distribution on G by the trace of

$$\pi^{\Sigma_0}(f) = \int_{G \rtimes \theta_0} f(g)\pi^{\Sigma_0}(g)dg$$

for $f \in C_c^\infty(G \rtimes \theta_0)$. We call this the twisted character of G, and denote it by $f_G(\pi^{\Sigma_0})$. We can also define the twisted characters for $GL(N)$ similarly, but we will write it in a slightly different way. Let π be a self-dual irreducible smooth representation of $GL(N)$; we can define a twisted invariant distribution on $GL(N)$ by taking the trace of $\pi(f) \circ A_\pi(\theta_N)$ for $f \in C_c^\infty(GL(N))$, where $A_\pi(\theta_N)$ is an intertwining operator between π and π^{θ_N}. We call this the twisted character of π and denote it by $f_{N^\theta}(\pi)$. Since the (twisted) elliptic endoscopic groups H in our case are all products of quasisplit symplectic and special orthogonal groups, we can define a group of automorphisms of H by taking the product of Σ_0 on each factor, and we denote this group again by Σ_0. Let $\overline{\mathcal{H}}(G)$ (resp. $\overline{\mathcal{H}}(H)$) be the subspace of Σ_0-invariant functions in $C_c^\infty(G)$ (resp. $C_c^\infty(H)$). Then it follows from a simple property of the transfer map (which we will not explain here) that we can restrict both (3.1) and (3.2) to $\overline{\mathcal{H}}(G)$ and $\overline{\mathcal{H}}(H)$.

4 Arthur Packets

For $\psi \in \overline{\Psi}(G)$, we define

$$\pi_\psi = \times_{(\rho,a,b)\in\mathrm{Jord}(\psi)} \mathrm{Sp}\big(\mathrm{St}(\rho,a),b\big).$$

From [Tad86], we know π_ψ is a unitary self-dual irreducible representation of $GL(N)$, and there is a Whittaker normalization of the intertwining operator $A_{\pi_\psi}(\theta_N)$ on π_ψ (see [Art13, 2.2]). Now we can state Arthur's local theory for G.

Theorem 4.1 (Arthur) *For any $\psi \in \overline{\Psi}(G)$ and $\overline{\varepsilon} \in \widehat{\mathcal{S}_\psi}$, there is a canonical way to associate a finite-length semisimple unitary representation viewed as $\overline{\mathcal{H}}(G)$-module $\pi(\psi,\overline{\varepsilon})$ (which can be zero), satisfying the following properties.*
  (i)

$$f(\psi) := \sum_{\overline{\varepsilon}\in\widehat{\mathcal{S}_\psi}} \overline{\varepsilon}(s_\psi)f_G(\pi(\psi,\overline{\varepsilon}))$$

defines a stable distribution for $f \in \overline{\mathcal{H}}(G)$. Moreover,

(4.1)
$$f^G(\psi) = f_{N^\theta}(\pi_\psi), \quad f \in C_c^\infty(GL(N)),$$

after we normalize the Haar measures on G and $GL(N)$ in a compatible way.

(ii) *Suppose $\psi = \psi_p$ and $s \in \mathcal{S}_{\psi^>}$. Let $(H, \psi_H) \to (\psi, s)$, and we define a stable distribution $f(\psi_H)$ for $f \in \overline{\mathcal{H}}(H)$ as in* (i). *Then, after we normalize the Haar measures on G and H in a compatible way, the following identity holds*

$$(4.2) \qquad f^H(\psi_H) = \sum_{\overline{\varepsilon} \in \widehat{\mathcal{S}_\psi}} \overline{\varepsilon}(ss_\psi) f_G(\pi(\psi, \overline{\varepsilon})), \quad f \in \overline{\mathcal{H}}(G),$$

where we denote the image of s in $\mathcal{S}_\psi$ again by s.

When G is special even orthogonal, we have an additional character relation.

Theorem 4.2 (Arthur) *Suppose G is special even orthogonal, $\psi = \psi_p \in \overline{\Psi}(G)$ and $\varepsilon \in \widehat{\mathcal{S}_\psi^{\Sigma_0}}$, for any irreducible representation π viewed as $\overline{\mathcal{H}}(G)$-module $[\pi]$ in $\pi(\psi, \overline{\varepsilon})$ such that $\pi^{\theta_0} \cong \pi$, one can associate it with an extension π^{Σ_0} to G^{Σ_0}. Then for any $s \in \mathcal{S}_{\psi^>}^{\Sigma_0}$ but not in $\mathcal{S}_{\psi^>}$ and $(H, \psi_H) \to (\psi, s)$ the following identity holds:*

$$f^H(\psi_H) = \sum_{\substack{\overline{\varepsilon} \in \widehat{\mathcal{S}_\psi}, \, [\pi] \in \pi(\psi, \overline{\varepsilon}): \\ \pi^{\theta_0} \cong \pi}} \varepsilon(ss_\psi) f_G(\pi^{\Sigma_0}), \quad f \in C_c^\infty(G \rtimes \theta_0),$$

where $\varepsilon \in \widehat{\mathcal{S}_\psi^{\Sigma_0}}$ is in the preimage of $\overline{\varepsilon}$, and it depends on the extension π^{Σ_0}. We denote the image of s in $\mathcal{S}_\psi^{\Sigma_0}$ again by s, and we normalize the Haar measures on G and H in a compatible way.

We denote the set of $\overline{\mathcal{H}}(G)$-modules $\pi(\psi, \overline{\varepsilon})$ for fixed $\psi \in \overline{\Psi}(G)$ and all $\overline{\varepsilon} \in \widehat{\mathcal{S}_\psi}$ by $\overline{\Pi}_\psi$. One can see from both (4.1) and (4.2) that the parametrization inside $\overline{\Pi}_\psi$ by $\widehat{\mathcal{S}_\psi}$ depends on the normalization of $A_{\pi_\psi}(\theta_N)$ and as well as those of intertwining operators related to ψ_H (*i.e.*, $A_{\pi_{\psi_i}}(\theta_{N_i})$ for $i = I, II$). In Arthur's theory, we always use the Whittaker normalization, as it is the most natural normalization from the global point of view, and it is in this sense that we say the association of $\pi(\psi, \overline{\varepsilon})$ with $\overline{\varepsilon} \in \widehat{\mathcal{S}_\psi}$ is canonical. But as it has been pointed out in [MW06], locally there is no reason to privilege the Whittaker normalization. Later on we will discuss another normalization used by Mœglin and Waldspurger in [MW06], which is critical for studying the structure of $\pi(\psi, \overline{\varepsilon})$. So in order to distinguish different parametrizations with respect to various normalizations, we will denote $\pi(\psi, \overline{\varepsilon})$ in Arthur's theory by $\pi_W(\psi, \overline{\varepsilon})$, and similarly denote $f(\psi)$ by $f_W(\psi)$ and denote $f_{N^\theta}(\pi_\psi)$ by $f_{N^\theta, W}(\pi_\psi)$.

Unlike the tempered case where all $\pi_W(\psi, \overline{\varepsilon})$ are distinct and irreducible (see Theorem 1.3 and [Xu15, Theorem 2.2]), Arthur's theory says little about $\pi_W(\psi, \overline{\varepsilon})$ except for its unitarity. In fact, $\pi_W(\psi, \overline{\varepsilon})$ can be reducible or even zero in general, and it is the main goal of this paper to explore the inner structure of $\pi_W(\psi, \overline{\varepsilon})$. To do so, we will mainly follow [Mœg06b, Mœg09, MW06].

As a consequence of Mœglin's results about $\pi_W(\psi, \overline{\varepsilon})$, we will be able to define the Arthur packet for G^{Σ_0} and describe its structure (see Section 8). To begin with, we define $\Pi_\phi^{\Sigma_0}$ for $\phi \in \overline{\Phi}_2(G)$ to be set of irreducible representations of G^{Σ_0}, whose restriction to G belong to $\overline{\Pi}_\phi$. Then Theorem 4.2 allows us to parametrize $\Pi_\phi^{\Sigma_0}$ by $\widehat{\mathcal{S}_\phi^{\Sigma_0}}$, and we have the following result.

Theorem 4.3 (Arthur) *Suppose $\phi \in \overline{\Phi}_2(G)$, there is a canonical bijection between $\Pi_\phi^{\Sigma_0}$ and $\widehat{S_\phi^{\Sigma_0}}$:*

$$\widehat{S_\phi^{\Sigma_0}} \longrightarrow \Pi_\phi^{\Sigma_0}$$
$$\varepsilon \longmapsto \pi_W^{\Sigma_0}(\phi, \varepsilon),$$

such that

- $\pi_W^{\Sigma_0}(\phi, \varepsilon\varepsilon_0) \cong \pi_W^{\Sigma_0}(\phi, \varepsilon) \otimes \omega_0$,
- $\left[\pi_W^{\Sigma_0}(\phi, \varepsilon)|_G \right] = 2\pi_W(\phi, \overline{\varepsilon})$ *if G is special even orthogonal and $S_\phi^{\Sigma_0} = S_\phi$, or $\pi_W(\phi, \overline{\varepsilon})$ otherwise,*
- *for any $s \in S_\phi^{\Sigma_0}$ but not in S_ϕ and $(H, \phi_H) \to (\phi, s)$, the following identity holds*

$$f_W^H(\phi_H) = \sum_{\overline{\varepsilon} \in \widehat{S_\phi}} \varepsilon(s) f_G\left(\pi_W^{\Sigma_0}(\phi, \varepsilon) \right), \quad f \in C_c^\infty(G \rtimes \theta_0).$$

5 The Mœglin–Waldspurger Normalization

The main reference for this section is [MW06]. Suppose $\psi \in \overline{\Psi}(G)$; we denote the normalized action of θ_N on π_ψ by $\theta(\psi)$ for simplicity. If it is the Whittaker normalization, we denote it by $\theta_W(\psi)$. Our aim is to introduce the normalization used by Mœglin and Waldspurger, which we denote by $\theta_{MW}(\psi)$, and to calculate explicitly the difference $\theta_{MW}(\psi)/\theta_W(\psi)$.

To give the definition, we need to specify a class of parameters in $\overline{\Psi}(G)$ called parameters with "discrete diagonal restriction". To be more precise, $\psi \in \overline{\Psi}(G)$ is said to have *discrete diagonal restriction* if $\psi_d \in \overline{\Phi}_2(G)$. It is an easy exercise to see that this is equivalent to requiring $\psi = \psi_p$ and that for any fixed ρ, the segments $[A, B]$ for $(\rho, A, B, \zeta) \in \text{Jord}_\rho(\psi)$ are disjoint. In particular this implies $\text{Jord}(\psi)$ is multiplicity free. Among this class of parameters, we call ψ is *elementary* if $A = B$ for all $(\rho, A, B, \zeta) \in \text{Jord}(\psi)$, or equivalently $\inf(a, b) = 1$ for all $(\rho, a, b) \in \text{Jord}(\psi)$. Note that in the original terminology of Mœglin and Waldspurger, elementary parameters are not required to have discrete diagonal restriction; nevertheless, whenever they treat the elementary parameters, they include the condition of discrete diagonal restriction. This is the reason that we include the condition of discrete diagonal restriction in our definition of elementary parameters. For simplicity, if ψ is elementary we also denote by $\text{Jord}_\rho(\psi_d)$ the set of integers α such that $(\rho, \alpha, 1) \in \text{Jord}(\psi_d)$, and we write $(\rho, \alpha, \delta_\alpha)$ for $(\rho, (\alpha-1)/2, (\alpha-1)/2, \delta_\alpha) \in \text{Jord}(\psi)$.

We first give the definition of $\theta_{MW}(\psi)$ for those elementary parameters. Suppose for all $(\rho, B, B, \zeta) \in \text{Jord}(\psi)$, we have $B = 0$; then simply let $\theta_{MW}(\psi) = \theta_W(\psi)$. Otherwise, we fix ρ and let B_0 be the smallest number with $(\rho, B_0, B_0, \zeta_0) \in \text{Jord}(\psi)$. If $B_0 \neq 0$, we have

$$\pi_\psi \hookrightarrow \rho \|^{\zeta_0 B_0} \times \pi_{\psi'} \times \rho \|^{-\zeta_0 B_0}$$

as the unique irreducible subrepresentation, where $\text{Jord}(\psi')$ is obtained from $\text{Jord}(\psi)$ by changing $(\rho, B_0, B_0, \zeta_0)$ to $(\rho, B_0 - 1, B_0 - 1, \zeta_0)$ when $B_0 \geqslant 1$, or removing $(\rho, B_0, B_0, \zeta_0)$ otherwise. Then we take $\theta_{MW}(\psi)$ to be induced from $\theta_{MW}(\psi')$. If

$B_0 = 0$, let B_1 be the next smallest number with $(\rho, B_1, B_1, \zeta_1) \in \mathrm{Jord}(\psi)$, and we have

$$\pi_\psi \hookrightarrow \langle \zeta_1 B_1, \ldots, 0 \rangle \times \pi_{\psi'} \times \langle 0, \ldots, -\zeta_1 B_1 \rangle,$$

where we get $\mathrm{Jord}(\psi')$ by removing $(\rho, B_0, B_0, \zeta_0)$ and $(\rho, B_1, B_1, \zeta_1)$ from $\mathrm{Jord}(\psi)$. Note that π_ψ appears with multiplicity one in the induced representation; then again we take $\theta_{MW}(\psi)$ to be induced from $\theta_{MW}(\psi')$. This finishes the case of elementary parameters.

Next we consider the case of parameters with discrete diagonal restriction. We choose (ρ, A, B, ζ) with $A > B$, then

$$\pi_\psi \hookrightarrow \langle \zeta B, \ldots, -\zeta A \rangle \times \pi_{\psi'} \times \langle \zeta A, \ldots, -\zeta B \rangle,$$

as the unique irreducible subrepresentation, where

$$\mathrm{Jord}(\psi') = \mathrm{Jord}(\psi) \cup \left\{ (\rho, A - 1, B + 1, \zeta) \right\} \setminus \left\{ (\rho, A, B, \zeta) \right\}.$$

Then we take $\theta_{MW}(\psi)$ to be induced from $\theta_{MW}(\psi')$.

Lemma 5.1   *In the set-up above, $\theta_{MW}(\psi)$ is independent of the choice of (ρ, A, B, ζ).*

The proof of this Lemma can be found in [MW06, Lemmas 1.12.1 and 1.12.2].
Now we can consider the general case. If $\psi \neq \psi_p$, we can write

$$\pi_\psi \cong \Big(\underset{(\rho, a, b)}{\times} \mathrm{Sp}(\mathrm{St}(\rho, a), b) \Big) \times \pi_{\psi_p} \times \Big(\underset{(\rho, a, b)}{\times} \mathrm{Sp}(\mathrm{St}(\rho, a), b)^\vee \Big),$$

where (ρ, a, b) are taken over $\mathrm{Jord}(\psi_{np})$, and hence define $\theta_{MW}(\psi)$ to be induced from $\theta_{MW}(\psi_p)$. So without loss of generality, we assume $\psi = \psi_p$. The general case requires us to put some total order $>_\psi$ on $\mathrm{Jord}(\psi_p)$ satisfying the following condition.

($\mathcal{P}$): If $(\rho, A, B, \zeta), (\rho, A', B', \zeta') \in \mathrm{Jord}(\psi_p)$ with $A > A', B > B'$ and $\zeta = \zeta'$, then $(\rho, A, B, \zeta) >_\psi (\rho, A', B', \zeta')$.

The necessity of this condition will be discussed in a moment. The point is there are many orders satisfying this condition, and we do not have a privileged one in general. Nonetheless, for parameters with discrete diagonal restriction, we can always choose an order such that for any ρ, $(\rho, A, B, \zeta) >_\psi (\rho, A', B', \zeta')$ if and only if $A > A'$. We call such orders the *natural orders* for parameters with discrete diagonal restriction. For $\psi \in \overline{\Psi}(G)$ with order $>_\psi$, we say that $\psi_\gg \in \overline{\Psi}(G_\gg)$ with order $>_{\psi_\gg}$ *dominates* ψ with respect to $>_\psi$ if there is an order preserving bijection between $\mathrm{Jord}(\psi_\gg)$ and $\mathrm{Jord}(\psi)$ that sends $(\rho, A_\gg, B_\gg, \zeta_\gg)$ to (ρ, A, B, ζ) satisfying $A_\gg - A = B_\gg - B \geqslant 0$ and $\zeta_\gg = \zeta$.

Suppose $(\psi_\gg, >_{\psi_\gg})$ dominates $(\psi, >_\psi)$ with both orders satisfying condition ($\mathcal{P}$), and $\psi_\gg$ has discrete diagonal restriction. Then we have

(5.1) $$\pi_\psi = \circ_{(\rho, A, B, \zeta) \in \mathrm{Jord}(\psi)} \mathrm{Jac}^\theta_{(\rho, A_\gg, B_\gg, \zeta) \mapsto (\rho, A, B, \zeta)} \, \pi_{\psi_\gg},$$

where the composition is taken in the decreasing order with respect to $>_\psi$. Note that if the condition ($\mathcal{P}$) is not satisfied, this may not be true. To describe the Jacquet functor

in (5.1), we consider the following generalized segment:

$$(5.2) \qquad X^{\gg}_{(\rho,A,B,\zeta)} = \begin{bmatrix} \zeta B_{\gg} & \cdots & \zeta(B+1) \\ \vdots & & \vdots \\ \zeta A_{\gg} & \cdots & \zeta(A+1) \end{bmatrix}.$$

Then the Jacquet functor in (5.1) means applying Jac^{θ}_x consecutively for x ranges over $X^{\gg}_{(\rho,A,B,\zeta)}$ from top to bottom and from left to right. Now it is clear how to define $\theta_{MW}(\psi)$. We first choose an order $>_{\psi}$ satisfying condition ($\mathcal{P}$), and then we choose a dominating parameter $\psi_{\gg}$ with discrete diagonal restriction and natural order. We define $\theta_{MW}(\psi)$ to be the one induced from $\theta_{MW}(\psi_{\gg})$ through the Jacquet module. The upshot is that $\theta_{MW}(\psi)$ only depends on the order $>_{\psi}$, and not on the dominating parameter $\psi_{\gg}$. This is explained in [MW06], and one can also see this when we derive the formula for $\theta_{MW}(\psi)/\theta_W(\psi)$.

Suppose $\psi \in \overline{\Psi}(G)$ and we fix an order $>_{\psi}$ on $\mathrm{Jord}(\psi_p)$ satisfying ($\mathcal{P}$); then we can define a set $\mathcal{Z}_{MW/W}(\psi)$ of *unordered pairs* of Jordan blocks from $\mathrm{Jord}(\psi_p)$ as follows.

Definition 5.2   A pair $\{(\rho,a,b),(\rho',a',b') \in \mathrm{Jord}(\psi_p)\}$ is contained in $\mathcal{Z}_{MW/W}(\psi)$ if and only if $\rho = \rho'$, and it is in one of the following situations.

(i)   Case: a,b are even and a',b' are odd.

   (a) If $\zeta_{a,b} = -1$ and
   $$\begin{cases} \zeta_{a',b'} = -1 \Rightarrow (\rho,a,b) >_{\psi} (\rho,a',b'), a > a'. \\ \zeta_{a',b'} = +1 \Rightarrow a > a'. \end{cases}$$

   (b) If $\zeta_{a,b} = \zeta_{a',b'} = +1$ and
   $$\begin{cases} (\rho,a,b) >_{\psi} (\rho,a',b') \Rightarrow a' > a, b > b'. \\ (\rho,a,b) <_{\psi} (\rho,a',b') \Rightarrow a > a', b > b'. \end{cases}$$

(ii)   Case : a is odd, b is even and a' is even, b' is odd.

   (a) If $\zeta_{a,b} = -1$ and
   $$\begin{cases} \zeta_{a',b'} = -1 \Rightarrow (\rho,a,b) >_{\psi} (\rho,a',b'), a < a'. \\ \zeta_{a',b'} = +1 \text{ and } \begin{cases} (\rho,a,b) >_{\psi} (\rho,a',b') \Rightarrow a < a'. \\ (\rho,a,b) <_{\psi} (\rho,a',b') \Rightarrow a > a'. \end{cases} \end{cases}$$

   (b) If $\zeta_{a,b} = \zeta_{a',b'} = +1$ and
   $$\begin{cases} (\rho,a,b) >_{\psi} (\rho,a',b') \Rightarrow a < a', b > b'. \\ (\rho,a,b) <_{\psi} (\rho,a',b') \Rightarrow a > a', b > b'. \end{cases}$$

Theorem 5.3   For $\psi \in \overline{\Psi}(G)$, $\theta_{MW}(\psi)/\theta_W(\psi) = (-1)^{|\mathcal{Z}_{MW/W}(\psi)|}$.

Proof   By our definition it suffices to prove the theorem for $\psi = \psi_p$, so we will assume $\psi = \psi_p$ from now on. The proof we give here is incomplete, because we will need to refer to ([MW06], Section 5) for several ingredients. First, we would like to assume this theorem for ψ having discrete diagonal restriction and natural order, and we refer

interested readers to [MW06, Theorem 5.6.1]). Secondly, we need to use the "unipotent normalization" $\theta_U(\psi)$ introduced in [MW06, Section 5], and we will recall two of its most important properties. The first property of $\theta_U(\psi)$ is parallel with a similar property for the Whittaker normalization $\theta_W(\psi)$. Let $(\rho, A, B, \zeta) \in \mathrm{Jord}(\psi)$, and we get $\psi_\gg$ simply by changing (ρ, A, B, ζ) to $(\rho, A_\gg, B_\gg, \zeta)$ with $A_\gg - A = B_\gg - B \geqslant 0$ and $\zeta_\gg = \zeta$. Suppose

$$\pi_\psi = \mathrm{Jac}^\theta_{(\rho, A_\gg, B_\gg, \zeta) \mapsto (\rho, A, B, \zeta)} \pi_{\psi_\gg}$$

with an action $\theta(\psi)$ induced from some $\theta(\psi_\gg)$. If $\zeta = -1$ and $\theta(\psi_\gg) = \theta_W(\psi_\gg)$, then $\theta(\psi) = \theta_W(\psi)$; if $\zeta = +1$ and $\theta(\psi_\gg) = \theta_U(\psi_\gg)$, then $\theta(\psi) = \theta_U(\psi)$ (see [MW06, Proposition 5.4.1]).

To state the second property, let us define $\mathcal{Z}(\psi)$ to be the set of unordered pairs $\{(\rho, a, b), (\rho, a', b')\}$ in $\mathrm{Jord}(\psi_p)$ such that $\sup(b, b')$ and $\sup(a, a')$ are both even, and $\inf(b, b')$ and $\inf(a, a')$ are both odd. Then we have $\theta_W(\psi)/\theta_U(\psi) = (-1)^{|\mathcal{Z}(\psi)|}$ (see [MW06, Theorem 5.5.7]).

Now we can start the proof. Let us index the Jordan blocks in $\mathrm{Jord}(\psi)$ according to the order $>_\psi$, i.e., $(\rho_i, a_i, b_i) >_\psi (\rho_{i-1}, a_{i-1}, b_{i-1})$. And we assume $\mathrm{Jord}(\psi) = \{(\rho_i, a_i, b_i)\}_{i=1}^l$. Let $\psi_\gg$ be a dominating parameter with discrete diagonal restriction and natural order. Then we can also obtain ψ^k from $\psi_\gg$ by changing $(\rho_i, a_{\gg, i}, b_{\gg, i})$ to (ρ_i, a_i, b_i) for $1 \leqslant i \leqslant k$. In particular, we can set $\psi^0 = \psi_\gg$. Let

$$\mathrm{Jac}^k := \mathrm{Jac}^\theta_{(\rho_k, a_{\gg, k}, b_{\gg, k}) \longmapsto (\rho_k, a_k, b_k)}.$$

Then we have the following sequence:

$$\pi_{\psi_\gg} = \pi_{\psi^0} \xrightarrow{\mathrm{Jac}^1} \cdots \xrightarrow{\mathrm{Jac}^k} \pi_{\psi^k} \xrightarrow{\mathrm{Jac}^{k+1}} \cdots \xrightarrow{\mathrm{Jac}^l} \pi_{\psi^l} = \pi_\psi.$$

From the properties of $\theta_W(\psi)$ and $\theta_U(\psi)$ that we have recalled above, we can compute $\theta_{MW}(\psi^k)/\theta_W(\psi^k)$. If $\zeta_k = -1$, we have

$$\theta_{MW}(\psi^k)/\theta_W(\psi^k) = \theta_{MW}(\psi^{k-1})/\theta_W(\psi^{k-1}).$$

If $\zeta_k = +1$, we have

$$\begin{aligned}
&\theta_{MW}(\psi^k)/\theta_W(\psi^k) \\
&= \theta_{MW}(\psi^k)/\theta_U(\psi^k) \cdot \theta_U(\psi^k)/\theta_W(\psi^k) \\
&= \theta_{MW}(\psi^{k-1})/\theta_U(\psi^{k-1}) \cdot \theta_U(\psi^k)/\theta_W(\psi^k) \\
&= \theta_{MW}(\psi^{k-1})/\theta_W(\psi^{k-1}) \cdot \theta_W(\psi^{k-1})/\theta_U(\psi^{k-1}) \cdot \theta_U(\psi^k)/\theta_W(\psi^k) \\
&= \theta_{MW}(\psi^{k-1})/\theta_W(\psi^{k-1}) \cdot (-1)^{|\mathcal{Z}(\psi^{k-1})|} \cdot (-1)^{|\mathcal{Z}(\psi^k)|}.
\end{aligned}$$

Moreover, let $\mathcal{Z}_k(\psi^{k-1})$ (resp. $\mathcal{Z}_k(\psi^k)$) be the subset of pairs in $\mathcal{Z}(\psi^{k-1})$ (resp. $\mathcal{Z}(\psi^k)$) containing $(\rho_k, a_{\gg, k}, b_{\gg, k})$ (resp. (ρ_k, a_k, b_k)); then

$$\begin{aligned}
&\theta_{MW}(\psi^k)/\theta_W(\psi^k) \\
&= \theta_{MW}(\psi^{k-1})/\theta_W(\psi^{k-1}) \cdot (-1)^{|\mathcal{Z}_k(\psi^{k-1})| + |\mathcal{Z}_k(\psi^k)|} \\
&= \theta_{MW}(\psi^{k-1})/\theta_W(\psi^{k-1}) \cdot (-1)^{|(\mathcal{Z}_k(\psi^{k-1}) \cup \mathcal{Z}_k(\psi^k)) \setminus (\mathcal{Z}_k(\psi^{k-1}) \cap \mathcal{Z}_k(\psi^k))|},
\end{aligned}$$

where we identify $(\rho_k, a_{\gg,k}, b_{\gg,k})$ with (ρ_k, a_k, b_k) in taking the intersection and union. To simplify the formula above, let us denote by $\mathcal{Z}_k(\psi^{k-1}, \psi^k)$ the set

$$\left(\mathcal{Z}_k(\psi^{k-1}) \cup \mathcal{Z}_k(\psi^k) \right) \setminus \left(\mathcal{Z}_k(\psi^{k-1}) \cap \mathcal{Z}_k(\psi^k) \right).$$

The proof is given by induction on k. So let us assume the theorem is valid for $\theta_{MW}(\psi_k)/\theta_W(\psi_k)$ with $0 \leqslant k \leqslant s$. Note that when $k = 0$, this is our assumption at the beginning. We need to prove the theorem for $k = s + 1$. According to our formula, we need to divide into two cases with respect to the parity of $a_{s+1} + b_{s+1}$. Here we will only treat the case when $a_{s+1} + b_{s+1}$ is even, while the other case is similar. Let $\rho = \rho_{s+1}$. From our previous discussion, we have

$$\theta_{MW}(\psi^{s+1})/\theta_W(\psi^{s+1}) = \begin{cases} \theta_{MW}(\psi^s)/\theta_W(\psi^s), & \text{if } \zeta_{s+1} = -1, \\ \theta_{MW}(\psi^s)/\theta_W(\psi^s) \cdot (-1)^{|\mathcal{Z}_{s+1}(\psi^s, \psi^{s+1})|}, & \text{if } \zeta_{s+1} = +1. \end{cases}$$

We first consider the case when $\zeta_{s+1} = -1$. Suppose $\{(\rho, a_{\gg,s+1}, b_{\gg,s+1}), (\rho, a, b)\}$ belongs to $\mathcal{Z}_{MW/W}(\psi^s)$; then by our definition we are in one of the following situations.

(a) If $(\rho, a_{\gg,s+1}, b_{\gg,s+1}) >_{\psi^s} (\rho, a, b)$, $\begin{cases} a_{\gg,s+1} \text{ even}; a, b \text{ odd} \Rightarrow a_{\gg,s+1} > a. \\ a_{\gg,s+1} \text{ odd}; a, b \text{ even} \Rightarrow \text{impossible.} \end{cases}$

(b) If $(\rho, a_{\gg,s+1}, b_{\gg,s+1}) <_{\psi^s} (\rho, a, b)$,

$$\begin{cases} a_{\gg,s+1} \text{ even}; a, b \text{ odd} \Rightarrow a_{\gg,s+1} > a, \zeta_{a,b} = +1. \\ a_{\gg,s+1} \text{ odd}; a, b \text{ even} \Rightarrow a_{\gg,s+1} < a, \zeta_{a,b} = -1. \end{cases}$$

Note that $a_{\gg,s+1} = a_{s+1}$, so in all the situations we have $\{(\rho, a_{s+1}, b_{s+1}), (\rho, a, b)\}$ belonging to $\mathcal{Z}_{MW/W}(\psi^{s+1})$ as well. In the same way, one can show

$$\{(\rho, a_{s+1}, b_{s+1}), (\rho, a, b)\} \in \mathcal{Z}_{MW/W}(\psi^{s+1}) \Longrightarrow$$
$$\{(\rho, a_{\gg,s+1}, b_{\gg,s+1}), (\rho, a, b)\} \in \mathcal{Z}_{MW/W}(\psi^s).$$

This means that our formula is valid for $k = s + 1$ in this case.

Next we come to the more difficult case $\zeta_{s+1} = +1$. Similarly, we first suppose $\{(\rho, a_{\gg,s+1}, b_{\gg,s+1}), (\rho, a, b)\}$ belongs to $\mathcal{Z}_{MW/W}(\psi^s)$, and we will be in one of the following situations.

(i)   If $(\rho, a_{\gg,s+1}, b_{\gg,s+1}) >_{\psi^s} (\rho, a, b)$,

$$\begin{cases} a_{\gg,s+1} \text{ even}; a, b \text{ odd} \Rightarrow a_{\gg,s+1} < a, b_{\gg,s+1} > b. \\ a_{\gg,s+1} \text{ odd}; a, b \text{ even} \Rightarrow \begin{cases} a_{\gg,s+1} < a, \zeta_{a,b} = -1. \\ a_{\gg,s+1} < a, b_{\gg,s+1} < b, \zeta_{a,b} = +1. \end{cases} \end{cases}$$

(ii) If $(\rho, a_{\gg,s+1}, b_{\gg,s+1}) <_{\psi^s} (\rho, a, b)$,

$$\begin{cases} a_{\gg,s+1} \text{ even}; a, b \text{ odd} \Rightarrow a_{\gg,s+1} > a, b_{\gg,s+1} > b, \zeta_{a,b} = +1. \quad \boxed{\star\text{-}1} \\ a_{\gg,s+1} \text{ odd}; a, b \text{ even} \Rightarrow \begin{cases} a_{\gg,s+1} < a, \zeta_{a,b} = -1. \\ a_{\gg,s+1} > a, b_{\gg,s+1} < b, \zeta_{a,b} = +1. \end{cases} \quad \boxed{\star\text{-}2} \end{cases}$$

Note that $a_{s+1} < a_{\gg,s+1}$ and $b_{s+1} = b_{\gg,s+1}$, so

$$\{(\rho, a_{s+1}, b_{s+1}), (\rho, a, b)\} \in \mathcal{Z}_{MW/W}(\psi^{s+1})$$

in all the situations except for $\boxed{\star\text{-}1}$ and $\boxed{\star\text{-}2}$ with the additional condition $a_{s+1} < a$. It is easy to check in the exceptional cases, either $\{(\rho, a_{\gg,s+1}, b_{\gg,s+1}), (\rho, a, b)\}$ or $\{(\rho, a_{s+1}, b_{s+1}), (\rho, a, b)\}$ belongs to $\mathcal{Z}_{s+1}(\psi^s, \psi^{s+1})$.

Conversely, if we suppose $\{(\rho, a_{s+1}, b_{s+1}), (\rho, a, b)\}$ belongs to $\mathcal{Z}_{MW/W}(\psi^{s+1})$, then we will be in one of the following situations.

(a) If $(\rho, a_{s+1}, b_{s+1}) >_{\psi^{s+1}} (\rho, a, b)$,

$$\begin{cases} a_{s+1} \text{ even}; a, b \text{ odd} \Rightarrow a_{s+1} < a, b_{s+1} > b. & \boxed{\star\text{-}3} \\ a_{s+1} \text{ odd}; a, b \text{ even} \Rightarrow \begin{cases} a_{s+1} < a, \zeta_{a,b} = -1. & \boxed{\star\text{-}4} \\ a_{s+1} < a, b_{s+1} < b, \zeta_{a,b} = +1. & \boxed{\star\text{-}5} \end{cases} \end{cases}$$

(b) If $(\rho, a_{s+1}, b_{s+1}) <_{\psi^{s+1}} (\rho, a, b)$,

$$\begin{cases} a_{s+1} \text{ even}; a, b \text{ odd} \Rightarrow a_{s+1} > a, b_{s+1} > b, \zeta_{a,b} = +1. \\ a_{s+1} \text{ odd}; a, b \text{ even} \Rightarrow \begin{cases} a_{s+1} < a, \zeta_{a,b} = -1. & \boxed{\star\text{-}6} \\ a_{s+1} > a, b_{s+1} < b, \zeta_{a,b} = +1. \end{cases} \end{cases}$$

We find $\{(\rho, a_{\gg,s+1}, b_{\gg,s+1}), (\rho, a, b)\} \notin \mathcal{Z}_{MW/W}(\psi^{s+1})$ only for $\boxed{\star\text{-}3}$, $\boxed{\star\text{-}4}$, $\boxed{\star\text{-}5}$, $\boxed{\star\text{-}6}$ with the additional condition $a_{\gg,s+1} > a$. Again, it is easy to check in these cases that either $\{(\rho, a_{\gg,s+1}, b_{\gg,s+1}), (\rho, a, b)\}$ or $\{(\rho, a_{s+1}, b_{s+1}), (\rho, a, b)\}$ belongs to $\mathcal{Z}_{s+1}(\psi^s, \psi^{s+1})$.

Finally, it suffices to figure out the set $\mathcal{Z}_{s+1}(\psi^s, \psi^{s+1})$ and show that it consists of exactly those pairs that we have encountered in $\boxed{\star\text{-}1} - \boxed{\star\text{-}6}$ with their additional conditions respectively. So let us suppose that either $\{(\rho, a_{\gg,s+1}, b_{\gg,s+1}), (\rho, a, b)\}$ or $\{(\rho, a_{s+1}, b_{s+1}), (\rho, a, b)\}$ belongs to $\mathcal{Z}_{s+1}(\psi^s, \psi^{s+1})$, and we list all the possibilities.

(i)   If $(\rho, a_{\gg,s+1}, b_{\gg,s+1}) >_{\psi^s} (\rho, a, b)$,

$$\begin{cases} a_{\gg,s+1} \text{ even}; a, b \text{ odd} \Rightarrow a_{\gg,s+1} > a > a_{s+1}, b_{\gg,s+1} > b. \quad \boxed{\star\text{-}3} \text{ with } a_{\gg,s+1} > a \\ a_{\gg,s+1} \text{ odd}; a, b \text{ even} \Rightarrow \\ \qquad \begin{cases} a_{\gg,s+1} > a > a_{s+1}, b_{\gg,s+1} < b, \zeta_{a,b} = +1. & \boxed{\star\text{-}5} \text{ with } a_{\gg,s+1} > a \\ a_{\gg,s+1} > a > a_{s+1}, b_{\gg,s+1} < b, \zeta_{a,b} = -1. & \boxed{\star\text{-}4} \text{ with } a_{\gg,s+1} > a \end{cases} \end{cases}$$

(ii)   If $(\rho, a_{\gg,s+1}, b_{\gg,s+1}) <_{\psi^s} (\rho, a, b)$,

$$\begin{cases} a_{\gg,s+1} \text{ even}; a, b \text{ odd} \Rightarrow a_{\gg,s+1} > a > a_{s+1}, b_{\gg,s+1} > b. \quad \boxed{\star\text{-}1} \text{ with } a_{s+1} < a \\ a_{\gg,s+1} \text{ odd}; a, b \text{ even} \Rightarrow \\ \qquad \begin{cases} a_{\gg,s+1} > a > a_{s+1}, b_{\gg,s+1} < b, \zeta_{a,b} = +1. & \boxed{\star\text{-}2} \text{ with } a_{s+1} < a \\ a_{\gg,s+1} > a > a_{s+1}, b_{\gg,s+1} < b, \zeta_{a,b} = -1. & \boxed{\star\text{-}6} \text{ with } a_{\gg,s+1} > a \end{cases} \end{cases}$$

Note that each case here corresponds exactly to one of $\boxed{\star\text{-}1} - \boxed{\star\text{-}6}$ with the required additional conditions, as we indicate on their right. This finishes the proof. ∎

Remark 5.4 There is a slight difference between our definition of $\mathcal{Z}_{MW/W}(\psi)$ (also $\mathcal{Z}(\psi)$) and that in [MW06]; namely, they use ordered pairs rather than unordered pairs. Moreover, this theorem slightly generalizes the formula in [MW06] in the sense that we only require that $>_\psi$ satisfies ($\mathcal{P}$).

We would also like to see the effect of Mœglin and Waldspurger's normalization on the parametrizations of representations inside Arthur packets. To do so, we need the following definition.

Definition 5.5 For $\psi \in \overline{\Psi}(G)$ and $(\rho, a, b) \in \mathrm{Jord}(\psi_p)$,

$$\mathcal{Z}_{MW/W}(\psi)_{(\rho,a,b)} :=$$
$$\{(\rho', a', b') \in \mathrm{Jord}(\psi_p) : \text{the pair of } (\rho, a, b) \text{ and } (\rho', a', b') \text{ lies in } \mathcal{Z}_{MW/W}(\psi)\},$$

and $\varepsilon_\psi^{MW/W}(\rho, a, b) := (-1)^{|\mathcal{Z}_{MW/W}(\psi)_{(\rho,a,b)}|}$.

Proposition 5.6 *Suppose $\psi \in \overline{\Psi}(G)$ has discrete diagonal restriction.*

(i) $\varepsilon_\psi^{MW/W} \in \widehat{\mathcal{S}_\psi^{\Sigma_0}}$ *and* $\varepsilon_\psi^{MW/W}(s_\psi) = \theta_{MW}(\psi)/\theta_W(\psi)$.

(ii) *If we write* $\pi_{MW}(\psi, \overline{\varepsilon}) := \pi_W(\psi, \overline{\varepsilon}\varepsilon_\psi^{MW/W})$ *for* $\overline{\varepsilon} \in \widehat{\mathcal{S}_\psi}$, *then the character identities in Theorem* 4.1 *can be rewritten as follows.*

   (a) *Let*

$$f_{MW}(\psi) := \sum_{\overline{\varepsilon} \in \widehat{\mathcal{S}_\psi}} \overline{\varepsilon}(s_\psi) f_G(\pi_{MW}(\psi, \overline{\varepsilon})), \quad f \in \overline{\mathcal{H}}(G).$$

   Then

(5.3) $$f_{MW}^G(\psi) = f_{N^\theta, MW}(\pi_\psi), \quad f \in C_c^\infty(GL(N)).$$

   (b) *If* $s \in \mathcal{S}_\psi$ *and* $(H, \psi_H) \to (\psi, s)$, *then we can define a stable distribution* $f_{MW}(\psi_H)$ *on H as in (a), and the following identity holds:*

(5.4) $$f_{MW}^H(\psi_H) = \sum_{\overline{\varepsilon} \in \widehat{\mathcal{S}_\psi}} \overline{\varepsilon}(s s_\psi) f_G(\pi_{MW}(\psi, \overline{\varepsilon})), \quad f \in \overline{\mathcal{H}}(G).$$

Proof For part (i), we have

$$\prod_{(\rho,a,b) \in \mathrm{Jord}(\psi)} \varepsilon_\psi^{MW/W}(\rho, a, b) = \prod_{(\rho,a,b) \in \mathrm{Jord}(\psi)} (-1)^{|\mathcal{Z}_{MW/W}(\psi)_{(\rho,a,b)}|}$$

$$= (-1)^{\sum_{(\rho,a,b) \in \mathrm{Jord}(\psi)} |\mathcal{Z}_{MW/W}(\psi)_{(\rho,a,b)}|}$$

$$= (-1)^{2|\mathcal{Z}_{MW/W}(\psi)|} = 1,$$

and hence $\varepsilon_\psi^{MW/W}$ defines a character of $\mathcal{S}_\psi^{\Sigma_0}$. To compute $\varepsilon_\psi^{MW/W}(s_\psi)$, let us recall that

$$s_\psi(\rho, a, b) = \begin{cases} -1, & \text{if } b \text{ is even,} \\ 1, & \text{if } b \text{ is odd,} \end{cases}$$

for $(\rho, a, b) \in \mathrm{Jord}(\psi)$. Then

$$\varepsilon_\psi^{MW/W}(s_\psi) = \prod_{\substack{(\rho,a,b)\in\mathrm{Jord}(\psi) \\ b \text{ is even}}} \varepsilon_\psi^{MW/W}(\rho, a, b) = (-1)^{\sum_{\substack{(\rho,a,b)\in\mathrm{Jord}(\psi) \\ b \text{ is even}}} |\mathcal{Z}_{MW/W}(\psi)_{(\rho,a,b)}|}$$

$$= (-1)^{|\mathcal{Z}_{MW/W}(\psi)|} = \theta_{MW}(\psi)/\theta_W(\psi).$$

Now we consider part (ii). First, by definition, we have for $f \in \overline{\mathcal{H}}(G)$

$$f_{MW}(\psi) = \sum_{\overline{\varepsilon}\in\widehat{\mathcal{S}_\psi}} \overline{\varepsilon}(s_\psi) f_G(\pi_{MW}(\psi, \overline{\varepsilon})) = \sum_{\overline{\varepsilon}\in\widehat{\mathcal{S}_\psi}} \overline{\varepsilon}(s_\psi) f_G(\pi_W(\psi, \overline{\varepsilon}\overline{\varepsilon}_\psi^{MW/W}))$$

$$= \sum_{\overline{\varepsilon}\in\widehat{\mathcal{S}_\psi}} \overline{\varepsilon\varepsilon}_\psi^{MW/W}(s_\psi) f_G(\pi_W(\psi, \overline{\varepsilon})) = \sum_{\overline{\varepsilon}\in\widehat{\mathcal{S}_\psi}} \overline{\varepsilon}(s_\psi) \overline{\varepsilon}_\psi^{MW/W}(s_\psi) f_G(\pi_W(\psi, \overline{\varepsilon}))$$

$$= \overline{\varepsilon}_\psi^{MW/W}(s_\psi) \sum_{\overline{\varepsilon}\in\widehat{\mathcal{S}_\psi}} \overline{\varepsilon}(s_\psi) f_G(\pi_W(\psi, \overline{\varepsilon})) = \overline{\varepsilon}_\psi^{MW/W}(s_\psi) f_W(\psi).$$

Combined with part (i) and (4.1), we then get

$$f_{MW}^G(\psi) = \theta_{MW}(\psi)/\theta_W(\psi) f_{N^\theta, W}(\pi_\psi) = f_{N^\theta, MW}(\pi_\psi)$$

for $f \in C_c^\infty(GL(N))$. Next, for any $s \in \mathcal{S}_\psi$ and $(H, \psi_H) \to (\psi, s)$, let $\psi_H = \psi_I \times \psi_{II}$ (see Example 3.1). Then by (4.2) we have

$$f_W^H(\psi_H) = \sum_{\overline{\varepsilon}\in\widehat{\mathcal{S}_\psi}} \overline{\varepsilon}(ss_\psi) f(\pi_W(\psi, \overline{\varepsilon})).$$

Also note the right-hand side of (5.4) is

$$\mathrm{RHS} = \sum_{\overline{\varepsilon}\in\widehat{\mathcal{S}_\psi}} \overline{\varepsilon}(ss_\psi) f(\pi_W(\psi, \overline{\varepsilon}\overline{\varepsilon}_\psi^{MW/W}))$$

$$= \sum_{\overline{\varepsilon}\in\widehat{\mathcal{S}_\psi}} \overline{\varepsilon\varepsilon}_\psi^{MW/W}(ss_\psi) f(\pi_W(\psi, \overline{\varepsilon}))$$

$$= \overline{\varepsilon}_\psi^{MW/W}(ss_\psi) \sum_{\overline{\varepsilon}\in\widehat{\mathcal{S}_\psi}} \overline{\varepsilon}(ss_\psi) f(\pi_W(\psi, \overline{\varepsilon})),$$

and the left-hand side of (5.4) is

$$\mathrm{LHS} = \overline{\varepsilon}_{\psi_H}^{MW/W}(s_{\psi_H}) f_W^H(\psi_H),$$

where $s_{\psi_H} = s_{\psi_I} \times s_{\psi_{II}}$ and $\varepsilon_{\psi_H}^{MW/W} = \varepsilon_{\psi_I}^{MW/W} \otimes \varepsilon_{\psi_{II}}^{MW/W}$. So it suffices to show that

$$\overline{\varepsilon}_{\psi_H}^{MW/W}(s_{\psi_H}) = \overline{\varepsilon}_\psi^{MW/W}(ss_\psi).$$

Moreover, by using part (i), this equality can be reduced to

$$(5.5) \qquad \overline{\varepsilon}_\psi^{MW/W}(s) = \theta_{MW}(\psi_H)/\theta_W(\psi_H) \cdot \theta_{MW}(\psi)/\theta_W(\psi),$$

where

$$\theta_{MW}(\psi_H)/\theta_W(\psi_H) = \theta_{MW}(\psi_I)/\theta_W(\psi_I) \cdot \theta_{MW}(\psi_{II})/\theta_W(\psi_{II}).$$

To show (5.5), one considers the partition $\mathrm{Jord}(\psi) = \mathrm{Jord}_+ \sqcup \mathrm{Jord}_-$ (see Example 3.1). Then $\bar{\varepsilon}_\psi^{MW/W}(s) = (-1)^m$, where

$$m = \sharp\left\{ \{(\rho,a,b),(\rho',a',b')\} \in \mathcal{Z}_{MW/W}(\psi) : (\rho,a,b) \in \mathrm{Jord}_+, (\rho',a',b') \in \mathrm{Jord}_- \right\}.$$

By Theorem 5.3, we can write the other side of (5.5) as

$$(-1)^{|\mathcal{Z}_{MW/W}(\psi)|-|\mathcal{Z}_{MW/W}(\psi_I)|-|\mathcal{Z}_{MW/W}(\psi_{II})|},$$

and hence the validity of (5.5) is clear. ∎

For $\psi = \psi_p \in \overline{\Psi}(G)$, we fix an order $>_\psi$ on $\mathrm{Jord}(\psi)$ satisfying condition ($\mathcal{P}$). We also choose $\psi_\gg$ dominating ψ with discrete diagonal restriction and natural order. We identify $\mathcal{S}_{\psi_\gg}$ with $\mathcal{S}_{\psi^>}$ and then $s_\psi^> = s_{\psi_\gg}$. For $\bar{\varepsilon} \in \widehat{\mathcal{S}_{\psi^>}}$, we define

$$(5.6) \qquad \pi_{MW}(\psi,\bar{\varepsilon}) := \circ_{(\rho,A,B,\zeta)\in\mathrm{Jord}(\psi)} \overline{\mathrm{Jac}}_{(\rho,A_\gg,B_\gg,\zeta)\mapsto(\rho,A,B,\zeta)} \pi_{MW}(\psi_\gg,\bar{\varepsilon}),$$

where the Jacquet functor is defined as in (5.2) and the composition is taken in the decreasing order. For these $\overline{\mathcal{H}}(G)$-modules, we have the following proposition.

Proposition 5.7 *Suppose $\psi = \psi_p \in \overline{\Psi}(G)$, and $>_\psi$ is an order on $\mathrm{Jord}(\psi)$ satisfying condition ($\mathcal{P}$). Suppose $\psi_\gg$ has discrete diagonal restriction and dominates ψ with natural order.*

(i) *Then $\varepsilon_\psi^{MW/W} \in \widehat{\mathcal{S}_{\psi^>}^{\Sigma_0}}$ and $\varepsilon_\psi^{MW/W}(s_\psi^>) = \theta_{MW}(\psi)/\theta_W(\psi)$.*

(ii) *For $\bar{\varepsilon} \in \widehat{\mathcal{S}_{\psi^>}}$,*

$$\pi_{MW}(\psi,\bar{\varepsilon}) = \begin{cases} \pi_W(\psi, \overline{\varepsilon\varepsilon_\psi^{MW/W}}), & \text{if } \overline{\varepsilon\varepsilon_\psi^{MW/W}} \in \widehat{\mathcal{S}_\psi}, \\ 0, & \text{otherwise.} \end{cases}$$

Proof The proof of part (i) is the same as that in Proposition 5.6, so we will only show part (ii) here. For $s \in \mathcal{S}_{\psi^>}$, we denote its image in $\mathcal{S}_\psi$ again by s. Let

$$\overline{\Pi}_{MW,s}(\psi_\gg) = \sum_{\bar{\varepsilon}\in\widehat{\mathcal{S}_{\psi_\gg}}} \bar{\varepsilon}(ss_{\psi_\gg})\pi_{MW}(\psi_\gg,\bar{\varepsilon}),$$

$$\overline{\Pi}_{W,s}(\psi) = \sum_{\bar{\varepsilon}\in\widehat{\mathcal{S}_\psi}} \bar{\varepsilon}(ss_\psi)\pi_W(\psi,\bar{\varepsilon}).$$

It follows for $\bar{\varepsilon} \in \widehat{\mathcal{S}_{\psi^>}}$ that

$$\pi_{MW}(\psi_\gg,\bar{\varepsilon}) = \frac{\bar{\varepsilon}(s_{\psi_\gg})}{|\mathcal{S}_{\psi_\gg}|} \sum_{s\in\mathcal{S}_{\psi_\gg}} \bar{\varepsilon}(s)\overline{\Pi}_{MW,s}(\psi_\gg).$$

Suppose that $(H_\gg,\psi_{H_\gg}) \to (\psi_\gg, s)$ and $(H,\psi_H) \to (\psi, s)$; then $\psi_{H_\gg}$ dominates ψ_H. By (4.2) and (5.4) we have

$$(5.7) \qquad \circ_{(\rho,A,B,\zeta)\in\mathrm{Jord}(\psi)} \overline{\mathrm{Jac}}_{(\rho,A_\gg,B_\gg,\zeta)\mapsto(\rho,A,B,\zeta)} \overline{\Pi}_{MW,s}(\psi_\gg) =$$
$$\theta_{MW}(\psi_H)/\theta_W(\psi_H)\overline{\Pi}_{W,s}(\psi).$$

Analogous to (5.5), one can show

$$\theta_{MW}(\psi_H)/\theta_W(\psi_H) = \bar{\varepsilon}_\psi^{MW/W}(ss_\psi^>).$$

Therefore,

$$\pi_{MW}(\psi, \overline{\varepsilon}) = \frac{\overline{\varepsilon}(s_\psi^>)}{|\mathcal{S}_{\psi^>}|} \sum_{s \in \mathcal{S}_{\psi^>}} \overline{\varepsilon}(s) \overline{\varepsilon}_\psi^{MW/W}(ss_\psi^>) \overline{\Pi}_{W,s}(\psi).$$

We rewrite this as

$$\pi_{MW}(\psi, \overline{\varepsilon\varepsilon}_\psi^{MW/W}) = \frac{\overline{\varepsilon\varepsilon}_\psi^{MW/W}(s_\psi^>)}{|\mathcal{S}_{\psi^>}|} \sum_{s \in \mathcal{S}_{\psi^>}} \overline{\varepsilon\varepsilon}_\psi^{MW/W}(s) \overline{\varepsilon}_\psi^{MW/W}(ss_\psi^>) \overline{\Pi}_{W,s}(\psi)$$

$$= \frac{\overline{\varepsilon}(s_\psi^>)}{|\mathcal{S}_{\psi^>}|} \sum_{s \in \mathcal{S}_{\psi^>}} \overline{\varepsilon}(s) \overline{\Pi}_{W,s}(\psi).$$

Note that $\overline{\Pi}_{W,s}(\psi)$ only depends on the image of s in $\mathcal{S}_\psi$, so

$$\sum_{s \in \mathcal{S}_{\psi^>}} \overline{\varepsilon}(s) \overline{\Pi}_{W,s}(\psi) = \begin{cases} \frac{|\mathcal{S}_{\psi^>}|}{|\mathcal{S}_\psi|} \sum_{s \in \mathcal{S}_\psi} \overline{\varepsilon}(s) \overline{\Pi}_{W,s}(\psi), & \text{if } \overline{\varepsilon} \in \widehat{\mathcal{S}_\psi}, \\ 0, & \text{otherwise.} \end{cases}$$

If $\overline{\varepsilon} \in \widehat{\mathcal{S}_\psi}$, then $\overline{\varepsilon}(s_\psi) = \overline{\varepsilon}(s_\psi^>)$, and it follows that

$$\pi_{MW}(\psi, \overline{\varepsilon\varepsilon}_\psi^{MW/W}) = \frac{\overline{\varepsilon}(s_\psi)}{|\mathcal{S}_\psi|} \sum_{s \in \mathcal{S}_\psi} \overline{\varepsilon}(s) \overline{\Pi}_{W,s}(\psi) = \pi_W(\psi, \overline{\varepsilon}).$$

If $\overline{\varepsilon} \notin \widehat{\mathcal{S}_\psi}$, then $\pi_{MW}(\psi, \overline{\varepsilon\varepsilon}_\psi^{MW/W}) = 0$. This finishes the proof. ∎

In general, for $\psi \in \overline{\Psi}(G)$, we define

$$\pi_{MW}(\psi, \overline{\varepsilon}) = \pi_{\psi_{np}} \rtimes \pi_{MW}(\psi_p, \overline{\varepsilon}),$$

for $\overline{\varepsilon} \in \widehat{\mathcal{S}_{\psi^>}}$. Since $\pi_W(\psi, \overline{\varepsilon}) = \pi_{\psi_{np}} \rtimes \pi_W(\psi_p, \overline{\varepsilon})$ for $\overline{\varepsilon} \in \widehat{\mathcal{S}_\psi}$, we again have

$$\pi_{MW}(\psi, \overline{\varepsilon}) = \begin{cases} \pi_W(\psi, \overline{\varepsilon\varepsilon}_\psi^{MW/W}), & \text{if } \overline{\varepsilon\varepsilon}_\psi^{MW/W} \in \widehat{\mathcal{S}_\psi}, \\ 0, & \text{otherwise.} \end{cases}$$

The main purpose of introducing the Mœglin–Waldspurger normalization is that one will have a recursive formula for $\overline{f}_{N^\theta, MW}(\pi_\psi)$ with $\psi \in \overline{\Psi}(G)$ having discrete diagonal restriction. Here we will occasionally write $\pi(\psi)$ for π_ψ. To introduce the formula, let us fix $(\rho, a, b) \in \mathrm{Jord}(\psi)$ such that $\inf(a, b) > 1$. Recall we also put $A = (a + b)/2 - 1$, $B = |a - b|/2$, and $\zeta = \zeta_{a,b} = \mathrm{Sign}(a - b)$ if $a \neq b$ and arbitrary otherwise. Then it is the same to require $A \neq B$ for the fixed Jordan block. Let ψ' be obtained from ψ by removing (ρ, a, b). Then we can define an element in the Grothendieck group of representations of $GL(N)$ as follows:

$$\pi(\psi)_{(\rho, A, B, \zeta)} := \bigoplus_{C \in]B, A]} (-1)^{A-C} \Big(\langle \zeta B, \ldots, -\zeta C \rangle$$

$$\times \mathrm{Jac}_{\zeta(B+2), \ldots, \zeta C}^\theta \pi(\psi', (\rho, A, B + 2, \zeta)) \times \langle \zeta C, \ldots, -\zeta B \rangle \Big)$$

$$\oplus (-1)^{[(A-B+1)/2]} \pi\big(\psi', (\rho, A, B + 1, \zeta), (\rho, B, B, \zeta)\big).$$

We impose the normalized actions of Mœglin–Waldspurger on $\pi(\psi', (\rho, A, B + 2, \zeta))$ and $\pi(\psi', (\rho, A, B + 1, \zeta), (\rho, B, B, \zeta))$, and we denote the resulting action on

$\pi(\psi)_{(\rho,A,B,\zeta)}$ by $\theta_{MW}(\psi)_{(\rho,A,B,\zeta)}$. The next theorem shows the relation between $\pi(\psi)_{(\rho,A,B,\zeta)}$ and $\pi(\psi)$.

Theorem 5.8 *Suppose $\psi \in \overline{\Psi}(G)$ has discrete diagonal restriction; then*

$$f_{N^\theta,MW}(\pi_\psi) = f_{N^\theta,MW}(\pi(\psi)_{(\rho,A,B,\zeta)}).$$

The proof of this theorem (see [MW06]) involves some complicated computations of Jacquet modules, and it is fair to say that the Mœglin–Waldspurger normalization is somehow artificially made for this theorem. This theorem has an immediate consequence on the Arthur packets for G.

For $\psi \in \overline{\Psi}(G)$ having discrete diagonal restriction, we write

$$(5.8) \qquad \overline{\Pi}_{MW}(\psi) := \sum_{\overline{\varepsilon} \in \widehat{\mathcal{S}_\psi}} \overline{\varepsilon}(s_\psi) \pi_{MW}(\psi, \overline{\varepsilon}).$$

Then we have the following proposition.

Proposition 5.9 *Suppose $\psi \in \overline{\Psi}(G)$ has discrete diagonal restriction and we fix $(\rho, A, B, \zeta) \in \mathrm{Jord}(\psi)$ such that $A > B$; then*

$$\overline{\Pi}_{MW}(\psi) = \bigoplus_{C \in]B,A]} (-1)^{A-C} \langle \zeta B, \dots, -\zeta C \rangle \rtimes \overline{\mathrm{Jac}}_{\zeta(B+2),\dots,\zeta C} \overline{\Pi}_{MW}(\psi', (\rho, A, B+2, \zeta))$$

$$\oplus (-1)^{[(A-B+1)/2]} \overline{\Pi}_{MW}(\psi', (\rho, A, B+1, \zeta), (\rho, B, B, \zeta)),$$

where ψ' is obtained from ψ by removing (ρ, A, B, ζ).

Proof This proposition follows easily from Theorem 5.8 and the twisted character relation (5.3), together with the compatibility of the twisted endoscopic transfer with parabolic inductions and Jacquet modules (see [Xu15], Section 6). ∎

From this formula, one can see that the case of parameters with discrete diagonal restriction can be reduced to the case of elementary parameters. Later on, we will give a recursive formula of Mœglin for $\pi_{MW}(\psi, \varepsilon)$, or more precisely for $\pi_M(\psi, \varepsilon)$ (see Section 7 for its definition), in the case of discrete diagonal restriction again, which is clearly motivated by the formula here. But in order to give Mœglin's formula, we need to first study the Arthur packets for elementary parameters.

6 Elementary Arthur Packets

Let us recall that $\psi \in \overline{\Psi}(G)$ is elementary if $\psi \circ \Delta \in \overline{\Phi}_2(G)$ and $A = B$ for all $(\rho, A, B, \zeta) \in \mathrm{Jord}(\psi)$. And we have the following theorem about elementary Arthur packets due to Mœglin [Mœg06b].

Theorem 6.1 (Mœglin) *Suppose $\psi \in \overline{\Psi}(G)$ is elementary; then $\pi_W(\psi, \overline{\varepsilon})$ is always nonzero and irreducible. Moreover, $\pi_W(\psi, \overline{\varepsilon}) \neq \pi_W(\psi, \overline{\varepsilon}')$ if $\overline{\varepsilon} \neq \overline{\varepsilon}'$.*

The main difficulty of this theorem remains proving that certain generalized Aubert involution (see Section 6.2) would take irreducible representations viewed as $\overline{\mathcal{H}}(G)$-modules in elementary Arthur packets to irreducible representations viewed

as $\overline{\mathcal{H}}(G)$-modules up to a sign in the corresponding Grothendieck group. But this does not admit a direct approach. So instead, we will follow [Mœg06b] to systematically construct a class of representations which generalizes the construction of discrete series representations of Mœglin and Tadić (see [MT02] and also [Xu15, Section 10]). This class of representations will form the candidates for elements in the elementary Arthur packets. In fact, what Mœglin constructed are representations of G^{Σ_0}, but we can then take the irreducible representations of G viewed as $\overline{\mathcal{H}}(G)$-modules defined by their restriction to G. The point is that it is easier to show the generalized Aubert involution preserve this class of representations of G^{Σ_0} and also their irreducibility. In the end, we are going to show the corresponding $\overline{\mathcal{H}}(G)$-modules are really elements in the elementary Arthur packets.

First we need to define parabolic induction and Jacquet module on the category $\mathrm{Rep}(G^{\Sigma_0})$ of finite-length smooth representations of G^{Σ_0}. Let $P = MN$ be a standard parabolic subgroup of G. If M is θ_0-stable, we write $M^{\Sigma_0} := M \rtimes \Sigma_0$. Otherwise, we let $M^{\Sigma_0} = M$. Suppose $\sigma^{\Sigma_0} \in \mathrm{Rep}(M^{\Sigma_0})$, $\pi^{\Sigma_0} \in \mathrm{Rep}(G^{\Sigma_0})$.

(a) If $M^{\theta_0} = M$, we define the normalized parabolic induction $\mathrm{Ind}_{P^{\Sigma_0}}^{G^{\Sigma_0}} \sigma^{\Sigma_0}$ to be the extension of the representation $\mathrm{Ind}_P^G(\sigma^{\Sigma_0}|_M)$ by an induced action of Σ_0, and we define the normalized Jacquet module $\mathrm{Jac}_{P^{\Sigma_0}} \pi^{\Sigma_0}$ to be the extension of the representation $\mathrm{Jac}_P(\pi^{\Sigma_0}|_G)$ by an induced action of Σ_0.

(b) If $M^{\theta_0} \neq M$, we define the normalized parabolic induction $\mathrm{Ind}_{P^{\Sigma_0}}^{G^{\Sigma_0}} \sigma^{\Sigma_0}$ to be $\mathrm{Ind}_G^{G^{\Sigma_0}} \mathrm{Ind}_P^G(\sigma^{\Sigma_0}|_M)$, and we define the normalized Jacquet module $\mathrm{Jac}_{P^{\Sigma_0}} \pi^{\Sigma_0}$ to be $\mathrm{Jac}_P(\pi^{\Sigma_0}|_G)$.

It follows from the definition that

$$\left(\mathrm{Jac}_{P^{\Sigma_0}} \pi^{\Sigma_0} \right)|_M = \mathrm{Jac}_P(\pi^{\Sigma_0}|_G).$$

Moreover,

$$\left(\mathrm{Ind}_{P^{\Sigma_0}}^{G^{\Sigma_0}} \sigma^{\Sigma_0} \right)|_G = \mathrm{Ind}_P^G(\sigma^{\Sigma_0}|_M)$$

unless G is special even orthogonal and $M^{\Sigma_0} = M$, in which case

$$\left(\mathrm{Ind}_{P^{\Sigma_0}}^{G^{\Sigma_0}} \sigma^{\Sigma_0} \right)|_G = \mathrm{Ind}_P^G(\sigma^{\Sigma_0}|_M) \oplus \left(\mathrm{Ind}_P^G(\sigma^{\Sigma_0}|_M) \right)^{\theta_0}.$$

We can also define Jac_x on $\mathrm{Rep}(G^{\Sigma_0})$ as in the introduction.

6.1 Construction of a Class of Representations

The construction of Mœglin is by induction on the rank of the groups, and it also depends on certain so-called *basic properties*, which have to be established at the same time again by induction. So let us assume for $G = G(n')$ with $n' < n$ and elementary $\psi \in \overline{\Psi}(G)$, the irreducible representation $\pi^{\Sigma_0}(\psi, \varepsilon)$ of G^{Σ_0} is well defined and distinct for $\varepsilon \in \widehat{\mathcal{S}_\psi^{\Sigma_0}}$.

Let $b_{\rho,\psi,\varepsilon} \in \mathrm{Jord}_\rho(\psi_d)$ be the biggest integer such that ε is "ρ-cuspidal" for

$$\mathrm{Jord}_{\rho,cusp}(\psi) := \{(\rho, \alpha, \delta_\alpha) \in \mathrm{Jord}_\rho(\psi) : \alpha \leqslant b_{\rho,\psi,\varepsilon}\};$$

i.e.,

(a) if $(\rho, \alpha, \delta_\alpha) \in \mathrm{Jord}_{\rho, cusp}(\psi)$, then $(\rho, \alpha - 2, \delta_{\alpha-2}) \in \mathrm{Jord}_{\rho, cusp}(\psi)$ as long as $\alpha - 2 > 0$;

(b) if $(\rho, \alpha, \delta_\alpha), (\rho, \alpha - 2, \delta_{\alpha-2}) \in \mathrm{Jord}_{\rho, cusp}(\psi)$, then $\varepsilon(\rho, \alpha, \delta_\alpha)\varepsilon(\rho, \alpha - 2, \delta_{\alpha-2}) = -1$;

(c) if $(\rho, 2, \delta_2) \in \mathrm{Jord}_{\rho, cusp}(\psi)$, then $\varepsilon(\rho, 2, \delta_2) = -1$.

We allow $b_{\rho, \psi, \varepsilon}$ to be zero. Let $a_{\rho, \psi, \varepsilon} \in \mathrm{Jord}_\rho(\psi_d)$ be the smallest integer such that $a_{\rho, \psi, \varepsilon} > b_{\rho, \psi, \varepsilon}$, and let $\delta_{\rho, \psi, \varepsilon}$ be the associated sign. If such $a_{\rho, \psi, \varepsilon}$ does not exist, we say $a_{\rho, \psi, \varepsilon} = \infty$.

Along with our assumption on the existence of $\pi^{\Sigma_0}(\psi, \varepsilon)$, we also assume they satisfy the following basic properties.

Basic Properties [Mœg06b, Section 2.3]

(1) (Jacquet module): If $\mathrm{Jac}_{\rho\|^x} \pi^{\Sigma_0}(\psi, \varepsilon) \neq 0$, then there exists $b_{\rho, \psi, \varepsilon} < \alpha \in \mathrm{Jord}_\rho(\psi_d)$ such that $x = \delta_\alpha \alpha$.

(2) (Non-unitary irreducibility) : For $x \geq 1/2$, if $2x - 1 \notin \mathrm{Jord}_\rho(\psi_d) \cup \{0\}$ or $0 < x \leq (b_{\rho, \psi, \varepsilon} - 1)/2$, then $\rho\|^x \rtimes \pi^{\Sigma_0}(\psi, \varepsilon)$ is irreducible.

(3) (Unitary reducibility) : Suppose $\mathrm{Jord}_\rho(\psi_d)$ contains odd integers. Then $\rho \rtimes \pi^{\Sigma_0}(\psi, \varepsilon)$ is irreducible if $1 \in \mathrm{Jord}_\rho(\psi_d)$, and is semisimple of length 2 without multiplicities otherwise. Moreover, let σ^{Σ_0} be an irreducible subrepresentation of $\rho \rtimes \pi^{\Sigma_0}(\psi, \varepsilon)$ in both cases; then $\rho \times \cdots \times \rho \rtimes \sigma^{\Sigma_0}$ is irreducible.

Remark 6.2 Property (1) is proved in [Mœg06b, Section 2.5]; Property (2) is proved in [Mœg06b, Section 2.7]. In the tempered case, Property (1) can be deduced easily from [Xu15, Lemma 9.2]. But, the general proof of Property (1) depends on Property (2). Property (2) is not obvious even in the tempered case, and its proof in the tempered case is more or less the same as in the general case. A fundamental case of Property (2) is when $\pi^{\Sigma_0}(\psi, \varepsilon)$ is supercuspidal, and that follows from [Xu15, Corollary 9.1] (*cf.* Proposition 1.6). Property (3) is proved in [Mœg06b, Section 2.8] without assuming any unitarity results of Arthur, and in the tempered case it follows easily from Arthur's theory.

Based on our assumptions, we can now give the construction for $\pi^{\Sigma_0}(\psi, \varepsilon)$.

Definition 6.3 Suppose $\psi \in \overline{\Psi}(G(n))$ is an elementary parameter and $\varepsilon \in \widehat{\mathcal{S}_\psi^{\Sigma_0}}$.

(i) If $a_{\rho, \psi, \varepsilon} = \infty$ for all ρ, then let $(\phi_{\mathrm{cusp}}, \varepsilon_{\mathrm{cusp}}) := (\psi_d, \varepsilon)$, and we define $\pi^{\Sigma_0}(\psi, \varepsilon)$ to be $\pi_W^{\Sigma_0}(\phi_{\mathrm{cusp}}, \varepsilon_{\mathrm{cusp}})$ in Theorem 4.3, which is supercuspidal by [Xu15, Theorem 3.3] (*cf.* Theorem 1.4).

(ii) If $a_{\rho, \psi, \varepsilon} > b_{\rho, \psi, \varepsilon} + 2$ or $b_{\rho, \psi, \varepsilon} = 0$, we define

$$\pi^{\Sigma_0}(\psi, \varepsilon) \longrightarrow \rho\|^{\delta_{\rho, \psi, \varepsilon}(a_{\rho, \psi, \varepsilon} - 1)/2} \rtimes \pi^{\Sigma_0}(\psi', \varepsilon')$$

to be the unique irreducible subrepresentation, where (ψ', ε') is obtained from (ψ, ε) by changing $(\rho, a_{\rho, \psi, \varepsilon}, \delta_{\rho, \psi, \varepsilon})$ to $(\rho, a_{\rho, \psi, \varepsilon} - 2, \delta_{\rho, \psi, \varepsilon})$.

(iii) If $a_{\rho, \psi, \varepsilon} = b_{\rho, \psi, \varepsilon} + 2$, we need to divide into three cases.

(a) If $\mathrm{Jord}_\rho(\psi_d)$ contains even integers and $b_{\rho, \psi, \varepsilon} \neq 0$, then we define

$$\pi^{\Sigma_0}(\psi, \varepsilon) \longrightarrow \left\langle \delta_{\rho, \psi, \varepsilon}(a_{\rho, \psi, \varepsilon} - 1)/2, \dots, \delta_{\rho, \psi, \varepsilon}1/2 \right\rangle \rtimes \pi^{\Sigma_0}(\psi_-, \varepsilon_-)$$

to be the unique irreducible subrepresentation, where (ψ_-, ε_-) is obtained from (ψ, ε) by removing $(\rho, a_{\rho,\psi,\varepsilon}, \delta_{\rho,\psi,\varepsilon})$, and changing $(\rho, \alpha, \delta_\alpha)$ to $(\rho, \alpha, -\delta_{\rho,\psi,\varepsilon})$ with

$$\varepsilon_-(\rho, \alpha, -\delta_{\rho,\psi,\varepsilon}) = -\varepsilon(\rho, \alpha, \delta_\alpha)$$

for all $\alpha \leqslant b_{\rho,\psi,\varepsilon}$. Moreover,

$$\pi^{\Sigma_0}(\psi, \varepsilon) \hookrightarrow \left\langle \delta_{\rho,\psi,\varepsilon}(a_{\rho,\psi,\varepsilon} - 1)/2, \ldots, -\delta_{\rho,\psi,\varepsilon}(b_{\rho,\psi,\varepsilon} - 1)/2 \right\rangle \rtimes \pi^{\Sigma_0}(\psi', \varepsilon').$$

where (ψ', ε') is obtained from (ψ, ε) by removing $a_{\rho,\psi,\varepsilon}$ and $b_{\rho,\psi,\varepsilon}$ from $\mathrm{Jord}_\rho(\psi_d)$.

(b) If $\mathrm{Jord}_\rho(\psi_d)$ contains odd integers and $b_{\rho,\psi,\varepsilon} \neq 1$, then we define $\pi^{\Sigma_0}(\psi, \varepsilon)$ to be the unique common irreducible subrepresentation of

$$\left\langle \delta_{\rho,\psi,\varepsilon}(a_{\rho,\psi,\varepsilon} - 1)/2, \ldots, 0 \right\rangle \rtimes \pi^{\Sigma_0}(\psi_-, \varepsilon_-)$$

and

$$\left\langle \delta_{\rho,\psi,\varepsilon}(a_{\rho,\psi,\varepsilon} - 1)/2, \ldots, -\delta_{\rho,\psi,\varepsilon}(b_{\rho,\psi,\varepsilon} - 1)/2 \right\rangle \rtimes \pi^{\Sigma_0}(\psi', \varepsilon').$$

Here (ψ', ε') is obtained from (ψ, ε) by removing $a_{\rho,\psi,\varepsilon}$ and $b_{\rho,\psi,\varepsilon}$ from $\mathrm{Jord}_\rho(\psi_d)$; (ψ_-, ε_-) is obtained from (ψ, ε) by removing $(\rho, a_{\rho,\psi,\varepsilon}, \delta_{\rho,\psi,\varepsilon})$ and $(\rho, 1, \delta_1)$, and changing $(\rho, \alpha, \delta_\alpha)$ to $(\rho, \alpha, -\delta_{\rho,\psi,\varepsilon})$ with

$$\varepsilon_-(\rho, \alpha, -\delta_{\rho,\psi,\varepsilon}) = -\varepsilon(\rho, \alpha, \delta_\alpha)$$

for $1 < \alpha \leqslant b_{\rho,\psi,\varepsilon}$.

(c) If $a_{\rho,\psi,\varepsilon} = 3, b_{\rho,\psi,\varepsilon} = 1$, we have $(\psi_-, \varepsilon_-) = (\psi', \varepsilon')$ in the notation of (b). By Property (3), $\sigma^{\Sigma_0} = \rho \rtimes \pi^{\Sigma_0}(\psi', \varepsilon')$ is semisimple of length 2, and hence we can write $\sigma^{\Sigma_0} = \pi_+^{\Sigma_0} \oplus \pi_-^{\Sigma_0}$ according to the following two cases.

(1) When $\mathrm{Jord}_\rho(\psi_d)$ only contains 2 elements, we fix arbitrary parametrization in σ^{Σ_0}, and we define $\pi^{\Sigma_0}(\psi, \varepsilon)$ to be the unique irreducible subrepresentation of $\rho||^{\delta_3} \rtimes \pi_\zeta^{\Sigma_0}$, with $\zeta = \varepsilon(3)\delta_3$.

(2) When $|\mathrm{Jord}_\rho(\psi_d)| > 2$, i.e., $a_{\rho,\psi',\varepsilon'} \neq \infty$, we can specify the parametrization in σ^{Σ_0} as follows. Assume (ψ'', ε'') is obtained from (ψ', ε') by changing $(\rho, a_{\rho,\psi',\varepsilon'}, \delta_{\rho,\psi',\varepsilon'})$ to $(\rho, 1, \delta_{\rho,\psi',\varepsilon'})$. Let
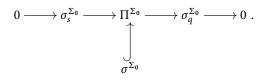
$$\Pi^{\Sigma_0} = \rho \times \left\langle \delta_{\rho,\psi',\varepsilon'}(a_{\rho,\psi',\varepsilon'} - 1)/2, \ldots, \delta_{\rho,\psi',\varepsilon'} \right\rangle \rtimes \pi^{\Sigma_0}(\psi'', \varepsilon''),$$

$$\sigma_q^{\Sigma_0} = \left\langle \delta_{\rho,\psi',\varepsilon'}(a_{\rho,\psi',\varepsilon'} - 1)/2, \ldots, 0 \right\rangle \rtimes \pi^{\Sigma_0}(\psi'', \varepsilon''),$$

and

$$\sigma_s^{\Sigma_0} = \left\langle \rho \times \langle \delta_{\rho,\psi',\varepsilon'}(a_{\rho,\psi',\varepsilon'} - 1)/2, \ldots, \delta_{\rho,\psi',\varepsilon'} \rangle \right\rangle \rtimes \pi^{\Sigma_0}(\psi'', \varepsilon'').$$

There is an exact sequence

$$0 \longrightarrow \sigma_s^{\Sigma_0} \longrightarrow \Pi^{\Sigma_0} \longrightarrow \sigma_q^{\Sigma_0} \longrightarrow 0 .$$

$$\sigma^{\Sigma_0}$$

We set $\pi_+^{\Sigma_0} = \sigma^{\Sigma_0} \cap (s.s.\sigma_q^{\Sigma_0})$ and $\pi_-^{\Sigma_0} = \sigma^{\Sigma_0} \cap (s.s.\sigma_s^{\Sigma_0})$. Then we define $\pi^{\Sigma_0}(\psi, \varepsilon)$ to be the unique irreducible subrepresentation of $\rho||^{\delta_3} \rtimes \pi_\zeta^{\Sigma_0}$, with

$\zeta = \varepsilon(a_{\rho,\psi',\varepsilon'})\delta_{\rho,\psi',\varepsilon'}\varepsilon(3)\delta_3$. Under such a choice this parametrization is compatible with Arthur's parametrization of discrete series representations in the case $\psi = \psi_d$ (*cf.* Proposition 1.5 and also [Xu15, Proposition 9.3]), and it also satisfies Theorem 6.10.

Remark 6.4 The uniqueness properties in the construction should follow from the property about Jacquet modules, *i.e.,* Property (1). The parametrization of representations of G^{Σ_0} in this construction is not uniquely determined by the choices we make in Step (c)(1). To fix this one can use the (twisted) endoscopy theory. In the tempered case, there are unique choices to be made here so that this parametrization is the same as Arthur's (*cf.* Theorem 1.3 and also [Xu15, Theorem 2.2]). In the nontempered case, we can fix the parametrization by that in the tempered case through the generalized Aubert involution, and we will denote such parametrization by $\pi_M^{\Sigma_0}(\psi, \varepsilon)$ later on.

In the next few sections, we would like to show that $\overline{\Pi}_\psi$ consists of $\overline{\mathcal{H}}(G)$-modules obtained from restriction of $\pi^{\Sigma_0}(\psi, \varepsilon)$ for $\varepsilon \in \widehat{\mathcal{S}_\psi^{\Sigma_0}}$. To do so, we will introduce two kinds of generalized Aubert involution operators, one on the Grothendieck group of representations of G^{Σ_0} (similarly also for representations of G viewed as $\overline{\mathcal{H}}(G)$-modules), and the other on that of $GL(N) \rtimes \langle \theta_N \rangle$. We will start with G^{Σ_0} following [Mœg06b, Section 4].

6.2 Aubert Involution for G^{Σ_0}

Let us fix a positive integer X_0 and write $x_0 = (X_0 - 1)/2$. We also fix a self-dual irreducible unitary supercuspidal representation ρ of $GL(d_\rho)$. We denote by $\mathcal{P}_{d_\rho}^{\Sigma_0}$ the set of Σ_0-conjugacy classes of standard parabolic subgroups P of G whose Levi component M is isomorphic to

$$(6.1) \qquad GL(a_1 d_\rho) \times \cdots \times GL(a_l d_\rho) \times G\Big(n - \sum_{i \in [1,l]} a_i d_\rho\Big).$$

Here we also require $G\big(n - \sum_{i \in [1,l]} a_i d_\rho\big) \neq SO(2)$ when $d_\rho \neq 1$. Let A_M be the maximal split central torus of M. For $P \in \mathcal{P}_{d_\rho}^{\Sigma_0}$ and $\sigma^{\Sigma_0} \in \mathrm{Rep}(M^{\Sigma_0})$, we denote by $\sigma_{<x_0}^{\Sigma_0}$ the direct sum of irreducible constitutes of σ whose cuspidal support on the general linear factors consist only of $\rho||^x$ with $|x| < x_0$. In particular, when $G\big(n - \sum_{i \in [1,l]} a_i d_\rho\big) = SO(2) \cong GL(1)$, we also impose this condition on $G\big(n - \sum_{i \in [1,l]} a_i d_\rho\big)$.

We define the generalized Aubert involution for G^{Σ_0} with respect to (ρ, X_0) as follows. For any $\pi^{\Sigma_0} \in \mathrm{Rep}(G^{\Sigma_0})$,

$$\mathrm{inv}_{<X_0}(\pi^{\Sigma_0}) := \sum_{P \in \mathcal{P}_{d_\rho}^{\Sigma_0}} (-1)^{\dim A_M} \mathrm{Ind}_{P^{\Sigma_0}}^{G^{\Sigma_0}} \Big(\widetilde{\mathrm{Jac}}_{P^{\Sigma_0}} (\pi^{\Sigma_0})_{<x_0} \Big),$$

where

$$\widetilde{\mathrm{Jac}}_{P^{\Sigma_0}}(\pi^{\Sigma_0}) = \begin{cases} \mathrm{Jac}_{P^{\Sigma_0}}(\pi^{\Sigma_0}) \otimes \omega_0, & \text{if } G\big(n - \sum_{i \in [1,l]} a_i d_\rho\big) = SO(2), \\ \mathrm{Jac}_{P^{\Sigma_0}}(\pi^{\Sigma_0}), & \text{otherwise.} \end{cases}$$

Analogously, we can define $\mathrm{inv}_{\leqslant X_0}$ if we change all strict inequalities to inequalities here. Just as for the usual Aubert involution, we have the following result.

Proposition 6.5 ([Mœg06b], Proposition 4)   *As defined above,* $\mathrm{inv}_{<X_0}$ *is an involution on the Grothendieck group of finite-length smooth representations of* G^{Σ_0}.

However, unlike the usual Aubert involution, it is by no means clear that $\mathrm{inv}_{<X_0}$ preserves irreducibility. Because of this we would like to show it preserves irreducibly at least for the class of representations that we have constructed in Section 6.1. The key ingredient of showing this is the following proposition.

Proposition 6.6 ([Mœg06b, Proposition 3])   *Let* $\pi^{\Sigma_0}(\psi, \varepsilon)$ *be a representation defined as in Section 6.1, and let* $\mathcal{E}$ *be an ordered multi-set of half-integers such that for all* $x \in \mathcal{E}, |x| < (a_{\rho,\psi,\varepsilon} - 1)/2$. *If* π^{Σ_0} *is an irreducible subquotient of* $\times_{x \in \mathcal{E}} \rho||^x \rtimes \pi^{\Sigma_0}(\psi, \varepsilon)$, *then there exists an ordered multi-set* $\mathcal{E}'$ *satisfying*

$$\{\mathcal{E}'\} \cup \{-\mathcal{E}'\} = \{\mathcal{E}\} \cup \{-\mathcal{E}\},$$

such that

$$\pi^{\Sigma_0} \hookrightarrow \times_{x \in \mathcal{E}'} \rho||^x \rtimes \pi^{\Sigma_0}(\psi, \varepsilon).$$

Combining Propositions 6.5 and 6.6, one can show the following theorem.

Theorem 6.7 ([Mœg06b, Theorem 4.1])   *We have that* $\mathrm{inv}_{<X_0} \pi^{\Sigma_0}(\psi, \varepsilon)$ *is irreducible with a sign in the Grothendieck group of representations of* G^{Σ_0}. *Moreover, the corresponding irreducible representation* $|\mathrm{inv}_{<X_0} \pi^{\Sigma_0}(\psi, \varepsilon)|$ *also belongs to the class of representations constructed in Section 6.1.*

One can also determine the sign in this theorem. Let

$$\mathrm{Jord}(\psi, \rho, < X_0) = \{\alpha \in \mathrm{Jord}_\rho(\psi_d) : \alpha < X_0\},$$

and we define

$\beta(\psi, \rho, < X_0) :=$
$$\begin{cases} (-1)^{|\mathrm{Jord}(\psi,\rho,<X_0)|(|\mathrm{Jord}(\psi,\rho,<X_0)|-1)/2} \cdot \prod_{\alpha \in \mathrm{Jord}(\psi,\rho,<X_0)} (-1)^{(\alpha-1)/2}, \\ \qquad\qquad\qquad\qquad\qquad \text{if } \mathrm{Jord}_\rho(\psi_d) \text{ contains odd integers;} \\ \prod_{\alpha \in \mathrm{Jord}(\psi,\rho,<X_0)} (-1)^{\alpha/2}, \qquad\qquad \text{if } \mathrm{Jord}_\rho(\psi_d) \text{ contains even integers.} \end{cases}$$

Proposition 6.8 ([Mœg06b, Proposition 4,2])

$\beta(\psi, \rho, < X_0) \, \mathrm{inv}_{<X_0} \pi^{\Sigma_0}(\psi, \varepsilon) =$
$$\begin{cases} \prod_{\alpha \in \mathrm{Jord}(\psi,\rho,<X_0)} \varepsilon(\rho, \alpha, \delta_\alpha) |\mathrm{inv}_{<X_0} \pi^{\Sigma_0}(\psi, \varepsilon)|, \\ \qquad\qquad\qquad\qquad\qquad \text{if } \mathrm{Jord}_\rho(\psi_d) \text{ contains even integers;} \\ |\mathrm{inv}_{<X_0} \pi^{\Sigma_0}(\psi, \varepsilon)|, \qquad\qquad \text{if } \mathrm{Jord}_\rho(\psi_d) \text{ contains odd integers.} \end{cases}$$

Next we want to illustrate the second part of Theorem 6.7. This makes use of a compatible relation between this Aubert involution and the Jacquet module. To describe this relation, let $P = MN$ be in $\mathcal{P}^{\Sigma_0}_{d_\rho}$ and let w_P be a Weyl group element in $W^{\Sigma_0}(M) := \mathrm{Norm}(A_M, G^{\Sigma_0})/M$ sending all positive roots outside M to negative roots. We can

also define $\mathrm{inv}_{<X_0}^{M^{\Sigma_0}}$ by taking the usual Aubert involution on the general linear factors of (6.1). For any representation π^{Σ_0} of G^{Σ_0}, let $\mathrm{Jac}_{P^{\Sigma_0},<x}(\pi^{\Sigma_0}) = (\mathrm{Jac}_{P^{\Sigma_0}}(\pi^{\Sigma_0}))_{<x}$. Then we have

$$\mathrm{Jac}_{P^{\Sigma_0},<x}\,|\,\mathrm{inv}_{<X_0}(\pi^{\Sigma_0})| = \mathrm{Ad}(w_P)|\,\mathrm{inv}_{<X_0}^{M^{\Sigma_0}}\,\mathrm{Jac}_{P^{\Sigma_0},<x}(\pi^{\Sigma_0})|$$

for all $x \leqslant x_0$ and $\pi^{\Sigma_0} \in \mathrm{Rep}(G^{\Sigma_0})$ (*cf.* [Mœg06b, Section 4.2]). From this equality, one can easily conclude the following corollary.

Corollary 6.9 ([Mœg06b, Corollary 4.2]) *Let* $\alpha \in \mathrm{Jord}_\rho(\psi)$ *with* $a_{\rho,\psi,\varepsilon} < \alpha$.

(i)   *If* $a_{\rho,\psi,\varepsilon} > b_{\rho,\psi,\varepsilon} + 2$, *then*

$$|\,\mathrm{inv}_{<\alpha}(\pi^{\Sigma_0}(\psi,\varepsilon))| \longrightarrow \rho||^{-\delta_{\rho,\psi,\varepsilon}(a_{\rho,\psi,\varepsilon}-1)/2} \rtimes |\,\mathrm{inv}_{<\alpha}(\pi^{\Sigma_0}(\psi',\varepsilon'))|,$$

where (ψ',ε') *is obtained by changing* $(\rho, a_{\rho,\psi,\varepsilon}, \delta_{\rho,\psi,\varepsilon})$ *to* $(\rho, a_{\rho,\psi,\varepsilon} - 2, \delta_{\rho,\psi,\varepsilon})$.

(ii) *If* $a_{\rho,\psi,\varepsilon} = b_{\rho,\psi,\varepsilon} + 2$, *then*

$$|\,\mathrm{inv}_{<\alpha}(\pi^{\Sigma_0}(\psi,\varepsilon))| \longrightarrow \left\langle -\delta_{\rho,\psi,\varepsilon}(a_{\rho,\psi,\varepsilon}-1)/2, \ldots, \delta_{\rho,\psi,\varepsilon}(b_{\rho,\psi,\varepsilon}-1)/2\right\rangle$$
$$\rtimes \left|\,\mathrm{inv}_{<\alpha}(\pi^{\Sigma_0}(\psi',\varepsilon'))\right|,$$

where (ψ',ε') *is obtained by removing* $a_{\rho,\psi,\varepsilon}$ *and* $b_{\rho,\psi,\varepsilon}$ *from* $\mathrm{Jord}_\rho(\psi_d)$.

It is easy to see from this corollary that $|\,\mathrm{inv}_{<X_0}\,\pi^{\Sigma_0}(\psi,\varepsilon)|$ is in the class of Section 6.1. In fact, from here one can even describe the pair $(\psi^\sharp, \varepsilon^\sharp)$ that parametrizes $|\,\mathrm{inv}_{<X_0}\,\pi^{\Sigma_0}(\psi,\varepsilon)|$.

Theorem 6.10 ([Mœg06b, Theorem 5]) *For* $\pi^{\Sigma_0}(\psi,\varepsilon)$, *let* $\psi^\sharp$ *be obtained from* ψ *by changing* δ_α *to* $-\delta_\alpha$ *for all* $\alpha \in \mathrm{Jord}_\rho(\psi_d)$ *such that* $\alpha < X_0$, *and let* $\varepsilon^\sharp = \varepsilon$ *under this correspondence. Then one can make suitable choices in the construction of representation corresponding to this new pair* $(\psi^\sharp, \varepsilon^\sharp)$ *(see Section* 6.1, *(c)(1)) such that* $\pi^{\Sigma_0}(\psi^\sharp, \varepsilon^\sharp) = |\,\mathrm{inv}_{<X_0}\,\pi^{\Sigma_0}(\psi,\varepsilon)|$.

Let $\overline{\mathrm{Rep}}(G)$ be the category of finite-length smooth representations of G viewed as $\overline{\mathcal{H}}(G)$-modules. We denote the elements in $\overline{\mathrm{Rep}}(G)$ by $[\pi]$ for $\pi \in \mathrm{Rep}(G)$, and we call $[\pi]$ irreducible if π is irreducible. Let

$$\overline{\mathrm{Jac}}_P = \begin{cases} \mathrm{Jac}_P + \mathrm{Jac}_P \circ \theta_0, & \text{if } G = SO(2n) \text{ and } M^{\theta_0} \neq M, \\ \mathrm{Jac}_P, & \text{otherwise.} \end{cases}$$

We can define parabolic induction and the Jacquet module on $\overline{\mathrm{Rep}}(G)$ as follows:

$$\mathrm{Ind}_P^G[\sigma] := [\mathrm{Ind}_P^G\,\sigma] \quad \text{and} \quad \overline{\mathrm{Jac}}_P[\pi] := [\overline{\mathrm{Jac}}_P\pi].$$

Then the generalized Aubert involution $\mathrm{inv}_{<X_0}$ can also be defined for $\overline{\mathrm{Rep}}(G)$ in an analogous way, *i.e.*,

$$\overline{\mathrm{inv}}_{<X_0}([\pi]) := \sum_{P \in \mathcal{P}_{d_\rho}^{\Sigma_0}} (-1)^{\dim A_M} \mathrm{Ind}_P^G\left(\overline{\mathrm{Jac}}_P([\pi])_{<x_0}\right).$$

For $\pi^{\Sigma_0} \in \mathrm{Rep}(G^{\Sigma_0})$, we have

$$\left[\left(\mathrm{Ind}_{P^{\Sigma_0}}^{G^{\Sigma_0}}\,\mathrm{Jac}_{P^{\Sigma_0}}\,\pi^{\Sigma_0}\right)|_G\right] = \mathrm{Ind}_P^G\,\overline{\mathrm{Jac}}_P[\pi^{\Sigma_0}|_G],$$

so
$$\left[(\mathrm{inv}_{<X_0}\, \pi^{\Sigma_0})|_G\right] = \overline{\mathrm{inv}}_{<X_0}\left([\pi^{\Sigma_0}|_G]\right).$$

6.3 Twisted Aubert Involution for $GL(N)$

As in the previous section, we again fix X_0, x_0, and ρ. We denote by $\mathcal{P}^{\theta_N}_{d_\rho}$ the set of θ_N-invariant standard parabolic subgroups P of $GL(N)$ whose Levi component M is isomorphic to

$$GL(a_1 d_\rho) \times \cdots \times GL(a_l d_\rho) \times GL\Big(N - 2\sum_{i\in[1,l]} a_i d_\rho\Big) \times GL(a_l d_\rho) \times \cdots \times GL(a_1 d_\rho).$$

Let A_M be the maximal split central torus of M, and $(A_M)_{\theta_N}$ be the group of its θ_N-coinvariants. For $P \in \mathcal{P}^{\theta_N}_{d_\rho}$ and $\tau \in \mathrm{Rep}(M)$, we denote by $\tau_{<x_0}$ the direct sum of irreducible constitutes of τ whose cuspidal support on $\times_{i\in[1,l]} GL(a_i d_\rho)$ consists only of $\rho||^x$ with $|x| < x_0$. Then we define the generalized θ_N-twisted Aubert involution for $GL(N)$ with respect to (ρ, X_0) as follows. For any self-dual representation π of $GL(N)$, let π^+ be an extension of π to $GL(N) \rtimes \langle\theta_N\rangle$,

$$\mathrm{inv}^{\theta_N}_{<X_0}(\pi^+) := \sum_{P\in\mathcal{P}^{\theta_N}_{d_\rho}} (-1)^{\dim(A_M)_{\theta_N}} \mathrm{Ind}^{GL(N)}_P\left(\mathrm{Jac}_P(\pi^+)_{<x_0}\right).$$

We should point out that $\mathrm{inv}^{\theta_N}_{<X_0}$ is defined differently from that in [MW06, Section 3.1]). Here $\mathrm{inv}^{\theta_N}_{<X_0}(\pi^+)$ is only an element in the Grothendieck group of representations of $GL(N) \rtimes \langle\theta_N\rangle$ (see [MW06, Section 3.2],), even when we take $\pi = \pi(\psi)$. However, if we only consider the θ_N-twisted characters of $GL(N)$, we can still get a theorem parallel with Theorem 6.10.

Theorem 6.11 ([MW06, Proposition 3.1]) *Let $\psi^\sharp$ be defined as in Theorem 6.10,*

$$f_N\big(\mathrm{inv}^{\theta_N}_{<X_0}(\pi^+(\psi))\big) = f_N(\pi^+(\psi^\sharp)), \quad f \in C^\infty_c(GL(N) \rtimes \theta_N)$$

for certain normalization of $\pi^+(\psi^\sharp)$ with respect to that of $\pi^+(\psi)$.

To determine the normalization of $\pi^+(\psi^\sharp)$ in this theorem, we need the following proposition.

Proposition 6.12 ([MW06, Lemma 3.2.2]) *Suppose that $\pi^+(\psi)$ in Theorem 6.11 is normalized according to Mœglin and Waldspurger (cf. Section 5); then the corresponding normalization of θ_N on $\pi^+(\psi^\sharp)$ differs from $\theta_{MW}(\psi^\sharp)$ by $\beta(\psi, \rho, < X_0)$.*

The careful reader may notice that this proposition is slightly different from the original result of Mœglin and Waldspurger, and that is due to a sign error in the statement of [MW06, Lemma 3.2.2]. As a consequence of this proposition, we can rewrite Theorem 6.11 as follows.

Corollary 6.13
(6.2)
$$f_N\big(\mathrm{inv}^{\theta_N}_{<X_0}(\pi^+_{MW}(\psi))\big) = \beta(\psi, \rho, < X_0) f_N(\pi^+_{MW}(\psi^\sharp)), \ f \in C^\infty_c(GL(N) \rtimes \theta_N),$$

where $\pi^+_{MW}(\psi)$ and $\pi^+_{MW}(\psi^\sharp)$ are normalized extensions of $\pi(\psi)$ and $\pi(\psi^\sharp)$ according to Mœglin–Waldspurger.

6.4 Construction of Elementary Arthur Packets by Aubert Involution

In the tempered case, we already know that $\pi(\psi, \overline{\varepsilon})$ is a Σ_0-orbit of discrete series representations (*cf.* Proposition 1.5 and also [Xu15, Proposition 9.3]), and moreover its parametrization by $(\psi, \overline{\varepsilon})$ is the same as Arthur's if we make certain choices in our definition of $\pi(\psi, \overline{\varepsilon})$ (*cf.* Section 6.1, (c)(1)). To obtain the nontempered packet, we need to use (6.2) and show that the following diagram commutes when restricted to distributions associated with elementary parameters.

$$(6.3) \qquad \begin{array}{ccc} \widehat{SI}(G) & \longrightarrow & \widehat{I}(N^\theta) \\ \overline{\mathrm{inv}}_{<X_0} \downarrow & & \downarrow \mathrm{inv}^{\theta_N}_{<X_0} \\ \widehat{SI}(G) & \longrightarrow & \widehat{I}(N^\theta). \end{array}$$

Here, $\widehat{SI}(G)$ is the space of stable invariant distributions on G, $\widehat{I}(N^\theta)$ is the space of twisted invariant distributions on $GL(N)$, and the horizontal arrows denote the twisted spectral endoscopic transfers. The commutativity of this diagram (under our restriction) essentially follows from the compatibility of twisted endoscopic transfer with both Jacquet module and parabolic induction, and we will give its proof in Appendix A. If we apply this diagram to $\overline{\Pi}_{MW}(\psi)$ (see (5.8)) and expand using (5.3) and (6.2), we get

$$f^G \Big(\sum_{\overline{\varepsilon} \in \widehat{\mathcal{S}_\psi}} \overline{\varepsilon}(s_\psi) \overline{\mathrm{inv}}_{<X_0} \pi_{MW}(\psi, \overline{\varepsilon}) \Big) = \beta(\psi, \rho, <X_0) f_{N^\theta, MW}(\pi(\psi^\sharp))$$

$$= \beta(\psi, \rho, <X_0) f^G \Big(\sum_{\overline{\varepsilon} \in \widehat{\mathcal{S}_{\psi^\sharp}}} \overline{\varepsilon}(s_{\psi^\sharp}) \pi_{MW}(\psi^\sharp, \overline{\varepsilon}) \Big),$$

where $f \in C_c^\infty(GL(N))$, and $f^G \in C_c^\infty(G)$ is its twisted endoscopic transfer. Hence,

$$(6.4) \qquad \sum_{\overline{\varepsilon} \in \widehat{\mathcal{S}_\psi}} \overline{\varepsilon}(s_\psi) f_G \big(\overline{\mathrm{inv}}_{<X_0} \pi_{MW}(\psi, \overline{\varepsilon}) \big) =$$

$$\beta(\psi, \rho, <X_0) \sum_{\overline{\varepsilon} \in \widehat{\mathcal{S}_{\psi^\sharp}}} \overline{\varepsilon}(s_{\psi^\sharp}) f_G \big(\pi_{MW}(\psi^\sharp, \overline{\varepsilon}) \big),$$

for any $f \in \overline{\mathcal{H}}(G)$.

Lemma 6.14

$$\varepsilon(s_\psi)/\varepsilon(s_{\psi^\sharp}) = \begin{cases} \prod_{\alpha \in \mathrm{Jord}(\psi, \rho, <X_0)} \varepsilon(\rho, \alpha, \delta_\alpha), & \text{if } \mathrm{Jord}_\rho(\psi_d) \text{ contains even integers,} \\ 1, & \text{if } \mathrm{Jord}_\rho(\psi_d) \text{ contains odd integers.} \end{cases}$$

Proof It suffices to note that

$$s_\psi s_{\psi^\sharp}(\rho, \alpha, \delta_\alpha) = \begin{cases} -1 & \text{if } \alpha < X_0 \text{ and } \alpha \text{ is even,} \\ 1 & \text{otherwise.} \end{cases} \qquad \blacksquare$$

Equality (6.4) suggests we can construct the nontempered Arthur packet by applying the generalized Aubert involution consecutively to tempered packet. So we have the following definition.

Definition 6.15   Suppose $\psi \in \overline{\Psi}(G)$ is elementary; for $\varepsilon \in \widehat{S_{\psi}^{\Sigma_0}}$ we define

$$\pi_M^{\Sigma_0}(\psi, \varepsilon) := \circ_{(\rho, a, \delta_a) \in \text{Jord}(\psi): \delta_a = -1} \left(|\operatorname{inv}_{<a}| \circ |\operatorname{inv}_{\leqslant a}| \right) \pi_W^{\Sigma_0}(\psi_d, \varepsilon)$$

and

$$\pi_M(\psi, \overline{\varepsilon}) := \circ_{(\rho, a, \delta_a) \in \text{Jord}(\psi): \delta_a = -1} \left(|\overline{\operatorname{inv}}_{<a}| \circ |\overline{\operatorname{inv}}_{\leqslant a}| \right) \pi_W(\psi_d, \overline{\varepsilon}),$$

where we have $\widehat{S_{\psi}^{\Sigma_0}} \cong \widehat{S_{\psi_d}^{\Sigma_0}}$ (resp. $\widehat{S_{\psi}} \cong \widehat{S_{\psi_d}}$) by identifying $\text{Jord}(\psi)$ with $\text{Jord}(\psi_d)$.

From Theorem 6.10, it is clear that $\pi_M^{\Sigma_0}(\psi, \varepsilon) = \pi^{\Sigma_0}(\psi, \varepsilon)$ constructed in Section 6.1, but with fixed parametrization determined by that of tempered representations (*cf.* Remark 6.4). It follows from Theorem 4.3 that

$$(6.5) \qquad\qquad \pi_M^{\Sigma_0}(\psi, \varepsilon\varepsilon_0) \cong \pi_M^{\Sigma_0}(\psi, \varepsilon) \otimes \omega_0.$$

Moreover, $\left[\pi_M^{\Sigma_0}(\psi, \varepsilon)|_G \right] = 2\pi_M(\psi, \overline{\varepsilon})$ if G is special even orthogonal and $S_{\psi}^{\Sigma_0} = S_{\psi}$, or $\pi_M(\psi, \overline{\varepsilon})$ otherwise. In particular, $\pi_M(\psi, \overline{\varepsilon})$ is irreducible.

Theorem 6.16   Suppose $\psi \in \overline{\Psi}(G)$ is elementary; then

$$\overline{\Pi}_{MW}(\psi) = \sum_{\overline{\varepsilon} \in \widehat{S_{\psi}}} \overline{\varepsilon}(s_{\psi}) \pi_M(\psi, \overline{\varepsilon}).$$

Proof   Note that in the tempered case $\pi_M(\psi, \overline{\varepsilon}) = \pi_W(\psi, \overline{\varepsilon}) = \pi_{MW}(\psi, \overline{\varepsilon})$, so this is already known. Then from the tempered packet, one can apply the generalized Aubert involution and use equality (6.4) step by step. Finally, note

$$(6.6) \qquad \overline{\varepsilon}(s_{\psi}) \beta(\psi, \rho, <X_0) \overline{\operatorname{inv}}_{<X_0} \pi_M(\psi, \overline{\varepsilon}) = \overline{\varepsilon}(s_{\psi^{\sharp}}) \pi_M(\psi^{\sharp}, \overline{\varepsilon}),$$

which follows from Proposition 6.8 and Lemma 6.14.   ∎

At this point, we have shown the elementary Arthur packets of G do contain irreducible representations of G viewed as $\overline{\mathcal{H}}(G)$-modules obtained by restriction from the class of representations of G^{Σ_0} constructed in Section 6.1. However, to prove Theorem 6.1 we still need to find the relation between $\pi_W(\psi, \overline{\varepsilon})$ and $\pi_M(\psi, \overline{\varepsilon})$. One can think of this as a problem of parametrization, but in fact it is much more subtle than that, for we do not know a priori that $\pi_W(\psi, \overline{\varepsilon})$ is irreducible or not. Nonetheless, we will show they are irreducible, and at same time compute the difference of parametrization between $\pi_W(\psi, \overline{\varepsilon})$ and $\pi_M(\psi, \overline{\varepsilon})$.

To describe this difference, we have to introduce a special element $\varepsilon_{\psi}^{M/MW} \in \widehat{S_{\psi}^{\Sigma_0}}$. It is defined in the following way.

Definition 6.17   Suppose $\psi \in \overline{\Psi}(G)$ is elementary and $\alpha \in \text{Jord}_{\rho}(\psi_d)$.

(i)   If α is even, $\varepsilon_{\psi}^{M/MW}(\rho, \alpha, \delta_\alpha) = 1$.

(ii)   If α is odd, let

$$m = \sharp\{\alpha' \in \mathrm{Jord}_\rho(\psi_d) : \alpha' > \alpha, \delta_{\alpha'} = -1\} \quad \text{and} \quad n = \sharp\{\alpha' \in \mathrm{Jord}_\rho(\psi_d) : \alpha' < \alpha\}.$$

Then

$$\varepsilon_\psi^{M/MW}(\rho, \alpha, \delta_\alpha) = \begin{cases} (-1)^m & \text{if } \delta_\alpha = +1, \\ (-1)^{m+n} & \text{if } \delta_\alpha = -1. \end{cases}$$

Theorem 6.18   *Suppose $\psi \in \overline{\Psi}(G)$ is elementary; then*

$$\pi_M(\psi, \overline{\varepsilon}) = \pi_{MW}(\psi, \overline{\varepsilon\varepsilon}_\psi^{M/MW}).$$

Proof   The idea is similar to the proof of Theorem 6.16 in that we have to apply the generalized Aubert involution step by step. First note that in the tempered case, we have by definition $\pi_M(\psi, \overline{\varepsilon}) = \pi_{MW}(\psi, \overline{\varepsilon})$, and it is easy to check that $\varepsilon_\psi^{M/MW} = 1$ in this case. Next, let us assume ψ is some elementary parameter satisfying the theorem, and we would like to prove the theorem for $\psi^\sharp$. In fact this is the critical step in our proof. To be more precise, we now have

$$\pi_M(\psi, \overline{\varepsilon}) = \pi_{MW}(\psi, \overline{\varepsilon\varepsilon}_\psi^{M/MW})$$

under our assumption, and we want to show

$$\pi_M(\psi^\sharp, \overline{\varepsilon}) := |\overline{\mathrm{inv}}_{<X_0}\pi_M(\psi, \overline{\varepsilon})| = \pi_{MW}(\psi^\sharp, \overline{\varepsilon\varepsilon}_{\psi^\sharp}^{M/MW}).$$

The main ingredient of the proof is a commutative diagram analogous to the diagram (6.3). Note that we can identify $\mathcal{S}_\psi$ with $\mathcal{S}_{\psi^\sharp}$, and for any $s \in \mathcal{S}_\psi \cong \mathcal{S}_{\psi^\sharp}$, let $(H, \psi_H) \to (\psi, s)$ and $(H, \psi_H^\sharp) \to (\psi^\sharp, s)$, where $H = G_I \times G_{II}$ and $\psi_H = \psi_I \times \psi_{II}$. Then the following diagram commutes when restricting to distributions associated with elementary parameters, and this again follows from the compatibility of endoscopic transfer with Jacquet module and parabolic induction (see [Hir04] and Appendix A).

$$(6.7) \qquad \begin{array}{ccc} \widehat{SI}(H) & \longrightarrow & \widehat{I}(G) \\ {\scriptstyle \overline{\mathrm{inv}}_{<X_0}^H} \downarrow & & \downarrow {\scriptstyle \overline{\mathrm{inv}}_{<X_0}} \\ \widehat{SI}(H) & \longrightarrow & \widehat{I}(G). \end{array}$$

Here $\widehat{I}(G)$ is the space of invariant distributions on G, $\widehat{SI}(H)$ is the space of stable invariant distributions on H, and the horizontal arrows denote the spectral endoscopic transfers. We define

$$\overline{\mathrm{inv}}_{<X_0}^H := \overline{\mathrm{inv}}_{<X_0}^{G_I} \otimes \overline{\mathrm{inv}}_{<X_0}^{G_{II}}$$

with $\overline{\mathrm{inv}}_{<X_0}^{G_I}$ respecting $\rho \otimes \eta'$, where $\eta' = \eta_I$ in (Example 3.1(i), (ii)), and $\eta' = 1$ in (Example 3.1(iii)). Applying this diagram to $\overline{\Pi}_{MW}(\psi_H) := \overline{\Pi}_{MW}(\psi_I) \otimes \overline{\Pi}_{MW}(\psi_{II})$, we get

$$\beta(\psi_H, \rho, <X_0)f_{MW}^H(\psi_H^\sharp) = \sum_{\overline{\varepsilon} \in \widehat{\mathcal{S}_\psi}} \overline{\varepsilon}(ss_\psi)f_G\big(\mathrm{inv}_{<X_0}\pi_{MW}(\psi, \overline{\varepsilon})\big), \quad f \in \overline{\mathcal{H}}(G),$$

where $\beta(\psi_H, \rho, <X_0) = \beta(\psi_I, \rho \otimes \eta', <X_0)\beta(\psi_{II}, \rho, <X_0)$. By our assumption, the right-hand side can be written as

$$\sum_{\overline{\varepsilon} \in \widehat{\mathcal{S}_\psi}} \overline{\varepsilon}(ss_\psi) f_G\left(\mathrm{inv}_{<X_0} \pi_M(\psi, \overline{\varepsilon \varepsilon}_\psi^{M/MW})\right) = \sum_{\overline{\varepsilon} \in \widehat{\mathcal{S}_\psi}} \overline{\varepsilon \varepsilon}_\psi^{M/MW}(ss_\psi) f_G\left(\mathrm{inv}_{<X_0} \pi_M(\psi, \overline{\varepsilon})\right).$$

Combining (6.6), we have

$$f_{MW}^H(\psi_H^\natural) = \beta(\psi_H, \rho, <X_0) \sum_{\overline{\varepsilon} \in \widehat{\mathcal{S}_\psi}} \overline{\varepsilon \varepsilon}_\psi^{MW}(ss_\psi) f_G\left(\mathrm{inv}_{<X_0} \pi_M(\psi, \overline{\varepsilon})\right)$$

$$= \beta(\psi_H, \rho, <X_0) \sum_{\overline{\varepsilon} \in \widehat{\mathcal{S}_\psi}} \overline{\varepsilon \varepsilon}_\psi^{M/MW}(ss_\psi) \beta(\psi, \rho, <X_0)\overline{\varepsilon}(s_\psi s_{\psi^\natural}) f_G\left(\pi_M(\psi^\natural, \overline{\varepsilon})\right)$$

$$= \beta(\psi_H, \rho, <X_0)\beta(\psi, \rho, <X_0)\overline{\varepsilon}_\psi^{M/MW}(ss_\psi) \sum_{\overline{\varepsilon} \in \widehat{\mathcal{S}_\psi}} \overline{\varepsilon}(ss_{\psi^\natural}) f_G\left(\pi_M(\psi^\natural, \overline{\varepsilon})\right).$$

Finally, it is a simple fact that $\varepsilon_\psi^{M/MW}(s_\psi) = 1$. So

(6.8) $f_{MW}^H(\psi_H^\natural) =$

$$\beta(\psi_H, \rho, <X_0)\beta(\psi, \rho, <X_0)\overline{\varepsilon}_\psi^{M/MW}(s) \sum_{\overline{\varepsilon} \in \widehat{\mathcal{S}_\psi}} \overline{\varepsilon}(ss_{\psi^\natural}) f_G\left(\pi_M(\psi^\natural, \overline{\varepsilon})\right).$$

On the other hand, we have from the character relation that

$$f_{MW}^H(\psi_H^\natural) = \sum_{\overline{\varepsilon} \in \widehat{\mathcal{S}_{\psi^\natural}}} \overline{\varepsilon}(ss_{\psi^\natural}) f_G\left(\pi_{MW}(\psi^\natural, \overline{\varepsilon})\right).$$

Since we know from linear algebra that $\pi_{MW}(\psi^\natural, \overline{\varepsilon})$ are completely determined by these identities for all $s \in \mathcal{S}_{\psi^\natural}$, it remains for us to show

$$\beta(\psi_H, \rho, <X_0)\beta(\psi, \rho, <X_0) = \varepsilon_\psi^{M/MW}\varepsilon_{\psi^\natural}^{M/MW}(s).$$

If $\mathrm{Jord}_\rho(\psi_d)$ contains even integers, then it is easy to show from the definitions that both sides are equal to 1. So now let us assume that $\mathrm{Jord}_\rho(\psi_d)$ contains odd integers. Note that $\mathrm{Jord}(\psi) = \mathrm{Jord}(\psi_I \otimes \eta') \sqcup \mathrm{Jord}(\psi_{II})$. Let $u = |\mathrm{Jord}(\psi_I, \rho \otimes \eta', <X_0)|$ and $v = |\mathrm{Jord}(\psi_{II}, \rho, <X_0)|$; then

$$\beta(\psi_H, \rho, <X_0)\beta(\psi, \rho, <X_0) = (-1)^{u(u-1)/2 + v(v-1)/2 - (u+v)(u+v-1)/2} = (-1)^{uv}$$

On the other hand, we can index $\mathrm{Jord}_\rho(\psi_d)$ according to the natural order of integers and assume that $\mathrm{Jord}(\psi_I, \rho \otimes \eta', <X_0) = \{\alpha_{t_j}\}_{j=1}^u$. Then

$$\varepsilon_\psi^{M/MW}\varepsilon_{\psi^\natural}^{M/MW}(s) = \prod_{j=1}^u (-1)^{(u+v-t_j)+(t_j-1)} = (-1)^{u(u+v-1)} = (-1)^{uv}.$$

This finishes the proof.   ∎

Corollary 6.19   *Suppose $\psi \in \overline{\Psi}(G)$ is elementary; let $\varepsilon_\psi^{M/W} := \varepsilon_\psi^{M/MW}\varepsilon_\psi^{MW/W}$. Then*

$$\pi_W(\psi, \overline{\varepsilon \varepsilon}_\psi^{M/W}) = \pi_M(\psi, \overline{\varepsilon}).$$

Proof   The proof is clear from Proposition 5.6.   ∎

In particular, this proves Theorem 6.1.

Corollary 6.20 *Suppose G is special even orthogonal and $\psi \in \overline{\Psi}(G)$ is elementary. For $\overline{\varepsilon} \in \widehat{\mathcal{S}_\psi}$, let $\pi_W(\psi, \overline{\varepsilon}) = [\pi]$. Then $\pi^{\theta_0} \cong \pi$ if and only if $\mathcal{S}_\psi^{\Sigma_0} \neq \mathcal{S}_\psi$.*

Proof This follows from (6.5). ∎

If $\psi \in \overline{\Psi}(G)$ is elementary, we can define $\Pi_\psi^{\Sigma_0}$ to be the set of irreducible representations of G^{Σ_0}, whose restriction to G belongs to $\overline{\Pi}_\psi$. Then it follows from Corollary 6.20 and Theorem 4.2 that there is a canonical bijection between

$$\mathcal{S}_\psi^{\Sigma_0} \longrightarrow \Pi_\psi^{\Sigma_0}$$
$$\varepsilon \longmapsto \pi_W^{\Sigma_0}(\psi, \varepsilon),$$

such that:

- $[\pi_W^{\Sigma_0}(\psi, \varepsilon)|_G] = 2\pi_W(\psi, \overline{\varepsilon})$ if G is special even orthogonal and $\mathcal{S}_\psi^{\Sigma_0} = \mathcal{S}_\psi$, or $\pi_W(\psi, \overline{\varepsilon})$ otherwise.
- For any $s \in \mathcal{S}_\psi^{\Sigma_0}$ but not in $\mathcal{S}_\psi$ and $(H, \psi_H) \to (\psi, s)$, the following identity holds:

$$f_W^H(\psi_H) = \sum_{\overline{\varepsilon} \in \widehat{\mathcal{S}_\psi}} \varepsilon(s s_\psi) f_G\left(\pi_W^{\Sigma_0}(\psi, \varepsilon)\right) \quad f \in C_c^\infty(G \rtimes \theta_0).$$

Let us define $\pi_{MW}^{\Sigma_0}(\psi, \varepsilon) := \pi_W^{\Sigma_0}(\psi, \varepsilon \varepsilon_\psi^{MW/W})$ for $\varepsilon \in \widehat{\mathcal{S}_\psi^{\Sigma_0}}$; then we can show in the same way as Proposition 5.6 that for any $s \in \mathcal{S}_\psi^{\Sigma_0}$ but not in $\mathcal{S}_\psi$ and $(H, \psi_H) \to (\psi, s)$,

$$f_{MW}^H(\psi_H) = \sum_{\overline{\varepsilon} \in \widehat{\mathcal{S}_\psi}} \varepsilon(s s_\psi) f_G\left(\pi_{MW}^{\Sigma_0}(\psi, \varepsilon)\right) \quad f \in C_c^\infty(G \rtimes \theta_0).$$

At last, we can extend Theorem 6.18 to G^{Σ_0}.

Theorem 6.21 *Suppose $\psi \in \overline{\Psi}(G)$ is elementary; then*

$$\pi_M^{\Sigma_0}(\psi, \varepsilon) = \pi_{MW}^{\Sigma_0}(\psi, \varepsilon \varepsilon_\psi^{M/MW}).$$

Proof We can assume that G is special even orthogonal and $\mathcal{S}_\psi^{\Sigma_0} \neq \mathcal{S}_\psi$. Since $\pi_M(\psi, \overline{\varepsilon}) = \pi_{MW}(\psi, \overline{\varepsilon \varepsilon_\psi^{M/MW}})$,

$$\pi_{MW}^{\Sigma_0}(\psi, \varepsilon \varepsilon_\psi^{M/MW}) = \pi_M^{\Sigma_0}(\psi, \varepsilon) \quad \text{or} \quad \pi_M^{\Sigma_0}(\psi, \varepsilon) \otimes \omega_0.$$

Note that when ψ is tempered,

$$\varepsilon_\psi^{M/MW} = \varepsilon_\psi^{MW/W} = 1 \quad \text{and} \quad \pi_M^{\Sigma_0}(\psi, \varepsilon) = \pi_W^{\Sigma_0}(\psi, \varepsilon) = \pi_{MW}^{\Sigma_0}(\psi, \varepsilon).$$

So as in the proof of Theorem 6.18, we can assume

$$\pi_M^{\Sigma_0}(\psi, \varepsilon) = \pi_{MW}^{\Sigma_0}(\psi, \varepsilon \varepsilon_\psi^{M/MW})$$

for some parameter ψ by induction, and the critical step is to show that

$$\pi_M^{\Sigma_0}(\psi^\sharp, \varepsilon) := |\mathrm{inv}_{<X_0} \pi_M^{\Sigma_0}(\psi, \varepsilon)| = \pi_{MW}^{\Sigma_0}(\psi^\sharp, \varepsilon \varepsilon_{\psi^\sharp}^{M/MW}).$$

We identity $\mathcal{S}_\psi^{\Sigma_0} \cong \mathcal{S}_{\psi^\sharp}^{\Sigma_0}$, and choose $s^* \in \mathcal{S}_\psi^{\Sigma_0}$ but not in $\mathcal{S}_\psi$. Let $(H, \psi_H) \to (\psi, s^*)$ and $(H, \psi_H^\sharp) \to (\psi^\sharp, s)$, where $H = G_I \times G_{II}$ and $\psi_H = \psi_I \times \psi_{II}$. Then we can have the following commutative diagram analogous to (6.7) (see Appendix A):

(6.9)
$$
\begin{array}{ccc}
\widehat{SI}(H) & \longrightarrow & \widehat{I}(G^{\theta_0}) \\
{\scriptstyle \mathrm{inv}_{<X_0}^H} \downarrow & & \downarrow {\scriptstyle \mathrm{inv}_{<X_0}} \\
\widehat{SI}(H) & \longrightarrow & \widehat{I}(G^{\theta_0}).
\end{array}
$$

Here, $\widehat{I}(G^{\theta_0})$ is the space of θ_0-twisted invariant distributions on G, and the horizontal arrows denote the twisted spectral endoscopic transfers. We define

$$
\mathrm{inv}_{<X_0}^H := \mathrm{inv}_{<X_0}^{G_I} \otimes \mathrm{inv}_{<X_0}^{G_{II}}
$$

with $\mathrm{inv}_{<X_0}^{G_I}$ (resp. $\mathrm{inv}_{<X_0}^{G_{II}}$) respecting $\rho \otimes \eta_I$ (resp. $\rho \otimes \eta_{II}$). Applying this diagram to $\overline{\Pi}_{MW}(\psi_H) := \overline{\Pi}_{MW}(\psi_I) \otimes \overline{\Pi}_{MW}(\psi_{II})$, one can show

$$
f_{MW}^H(\psi_H^\sharp) = \beta(\psi_H, \rho, <X_0) \beta(\psi, \rho, <X_0) \varepsilon_\psi^{M/MW}(s^*) \sum_{\overline{\varepsilon} \in \widehat{\mathcal{S}_\psi}} \varepsilon(s^* s_{\psi^\sharp}) f_G\big(\pi_M^{\Sigma_0}(\psi^\sharp, \varepsilon)\big)
$$

for $f \in C_c^\infty(G \rtimes \theta_0)$ (cf. (6.8)). As in the proof of Theorem 6.18, we also have

$$
\beta(\psi_H, \rho, <X_0) \beta(\psi, \rho, <X_0) = \varepsilon_\psi^{M/MW} \varepsilon_{\psi^\sharp}^{M/MW}(s^*).
$$

Since

$$
f_{MW}^H(\psi_H^\sharp) = \sum_{\overline{\varepsilon} \in \widehat{\mathcal{S}_{\psi^\sharp}}} \varepsilon(s^* s_{\psi^\sharp}) f_G\big(\pi_{MW}^{\Sigma_0}(\psi^\sharp, \varepsilon)\big),
$$

$$
\sum_{\overline{\varepsilon} \in \widehat{\mathcal{S}_{\psi^\sharp}}} \varepsilon(s^* s_{\psi^\sharp}) f_G\big(\pi_{MW}^{\Sigma_0}(\psi^\sharp, \varepsilon)\big) = \sum_{\overline{\varepsilon} \in \widehat{\mathcal{S}_\psi}} \varepsilon \varepsilon_{\psi^\sharp}^{M/MW}(s^* s_{\psi^\sharp}) f_G\big(\pi_M^{\Sigma_0}(\psi^\sharp, \varepsilon)\big)
$$

$$
= \sum_{\overline{\varepsilon} \in \widehat{\mathcal{S}_{\psi^\sharp}}} \varepsilon(s^* s_{\psi^\sharp}) f_G\big(\pi_M^{\Sigma_0}(\psi^\sharp, \varepsilon \varepsilon_{\psi^\sharp}^{M/MW})\big).
$$

By the linear independence of twisted characters, we have for any $\overline{\varepsilon} \in \widehat{\mathcal{S}_{\psi^\sharp}}$,

$$
\varepsilon(s^* s_{\psi^\sharp}) f_G\big(\pi_{MW}^{\Sigma_0}(\psi^\sharp, \varepsilon)\big) = \varepsilon(s^* s_{\psi^\sharp}) f_G\big(\pi_M^{\Sigma_0}(\psi^\sharp, \varepsilon \varepsilon_{\psi^\sharp}^{M/MW})\big),
$$

and hence $f_G(\pi_{MW}^{\Sigma_0}(\psi^\sharp, \varepsilon)) = f_G(\pi_M^{\Sigma_0}(\psi^\sharp, \varepsilon \varepsilon_{\psi^\sharp}^{M/MW}))$, i.e.,

$$
\pi_M^{\Sigma_0}(\psi^\sharp, \varepsilon) = \pi_{MW}^{\Sigma_0}(\psi^\sharp, \varepsilon \varepsilon_{\psi^\sharp}^{M/MW}). \qquad \blacksquare
$$

Remark 6.22   Later on we will see that Mœglin defines $\pi_M^{\Sigma_0}(\psi, \varepsilon)$ in the general case, and if one also extends the definition of $\varepsilon_\psi^{M/MW}$ to the general case, then Theorem 6.21 is still valid (see Theorem 8.9).

7 The Case of Discrete Diagonal Restriction

In this section, we would like to look into the Arthur packets associated with parameters having discrete diagonal restrictions. To be more precise, we want to give a parametrization of irreducible constituents of $\pi_W(\psi, \overline{\varepsilon})$ (or equivalently $\pi_{MW}(\psi, \overline{\varepsilon})$) in this case. This parametrization is given by Mœglin, and we will follow her paper [Mœg09] closely.

As in the elementary case, we start by constructing certain elements in the Grothendieck group of representations of G^{Σ_0}. These elements are parametrized by $\psi \in \overline{\Psi}(G)$ with discrete diagonal restriction and $\varepsilon \in \widehat{S_\psi^{\Sigma_0}}$.

Definition 7.1 Suppose $\psi \in \overline{\Psi}(G)$ has discrete diagonal restriction, and there exists $(\rho, A, B, \zeta) \in \mathrm{Jord}(\psi)$ such that $A > B$. Let $\varepsilon \in \widehat{S_\psi^{\Sigma_0}}$ and $\eta_0 := \varepsilon(\rho, A, B, \zeta)$. Then we define

$$\pi_M^{\Sigma_0}(\psi, \varepsilon) :=$$
$$\bigoplus_{C \in]B,A]} (-1)^{A-C} \langle \zeta B, \ldots, -\zeta C \rangle \rtimes \mathrm{Jac}_{\zeta(B+2),\ldots,\zeta C}\, \pi_M^{\Sigma_0}\big(\psi', \varepsilon', (\rho, A, B+2, \zeta; \eta_0)\big) \oplus$$
$$\bigoplus_{\eta=\pm 1} (-1)^{[(A-B+1)/2]} \eta^{A-B+1} \eta_0^{A-B} \pi_M^{\Sigma_0}\big(\psi', \varepsilon', (\rho, A, B+1, \zeta; \eta), (\rho, B, B, \zeta; \eta\eta_0)\big),$$

where ψ' is obtained from ψ by removing (ρ, A, B, ζ), and $\varepsilon'(\cdot)$ is the restriction of $\varepsilon(\cdot)$.

Remark 7.2 (i) When $A = B+1$ and $\eta_0 = -1$, the term involving $(\rho, A, B+2, \zeta, \eta_0)$ does not appear because $\varepsilon'(\cdot)$ does not define a character of $S_{\psi'}^{\Sigma_0}$ in this case.

(ii) It is clear by induction that
$$\pi_M^{\Sigma_0}(\psi, \varepsilon\varepsilon_0) \cong \pi_M^{\Sigma_0}(\psi, \varepsilon) \otimes \omega_0.$$

(iii) We could also define $\pi_M(\psi, \overline{\varepsilon})$ in a similar way. Let
$$\mathrm{Jord}(\psi^1) = \mathrm{Jord}(\psi') \cup \{(\rho, A, B+2, \zeta)\},$$
and
$$\mathrm{Jord}(\psi^2) = \mathrm{Jord}(\psi') \cup \{(\rho, A, B+1, \zeta), (\rho, B, B, \zeta)\}.$$
We can identify $S_\psi \cong S_{\psi^1}$ by sending (ρ, A, B, ζ) to $(\rho, A, B+2, \zeta)$, and map $s \in S_\psi$ into S_{ψ^2} by letting
$$s(\rho, A, B+1, \zeta) = s(\rho, B, B, \zeta) := s(\rho, A, B, \zeta).$$
Then $S_\psi \hookrightarrow S_{\psi^2}$ is of index 1 or 2. We denote the image of $\overline{\varepsilon}$ in $\widehat{S_{\psi^1}}$ by $\overline{\varepsilon}_1$. Let us define

$$\pi_M(\psi, \overline{\varepsilon}) :=$$
$$\bigoplus_{C \in]B,A]} (-1)^{A-C} \langle \zeta B, \ldots, -\zeta C \rangle \rtimes \overline{\mathrm{Jac}}_{\zeta(B+2),\ldots,\zeta C}\, \pi_M(\psi^1, \overline{\varepsilon}_1) \oplus$$
$$\bigoplus_{\overline{\varepsilon} \leftarrow \overline{\varepsilon}_2 \in \widehat{S_{\psi^2}}} (-1)^{[(A-B+1)/2]} \varepsilon_2(\rho, A, B+1, \zeta)^{A-B+1} \varepsilon(\rho, A, B, \zeta)^{A-B} \pi_M(\psi^2, \overline{\varepsilon}_2).$$

Again by induction, one observes the restriction of $\pi_M^{\Sigma_0}(\psi, \varepsilon)$ to G viewed as $\overline{\mathcal{H}}(G)$-modules is $2\pi_M(\psi, \overline{\varepsilon})$ if G is special even orthogonal and $\mathcal{S}_\psi^{\Sigma_0} = \mathcal{S}_\psi$, or $\pi_M(\psi, \overline{\varepsilon})$ otherwise. Later we will show that $\pi_M^{\Sigma_0}(\psi, \varepsilon)$ is a representation of G^{Σ_0}, and $\pi_M(\psi, \overline{\varepsilon})$ consists of irreducible representations of G viewed as $\overline{\mathcal{H}}(G)$-modules in the restriction of $\pi_M^{\Sigma_0}(\psi, \varepsilon)$ to G without multiplicities.

Next we want to show that $\overline{\Pi}_\psi$ consists of $\pi_M(\psi, \overline{\varepsilon})$, and furthermore we would like to compute the difference between the parametrizations of $\pi_M(\psi, \overline{\varepsilon})$ and $\pi_{MW}(\psi, \overline{\varepsilon})$. To do so, we need to extend the definition of $\varepsilon_\psi^{M/MW} \in \widehat{\mathcal{S}_\psi^{\Sigma_0}}$ in the previous section.

Definition 7.3 Suppose $\psi \in \overline{\Psi}(G)$ has discrete diagonal restriction, and $(\rho, a, b) \in$ Jord(ψ).

(i) If $a + b$ is odd, $\varepsilon_\psi^{M/MW}(\rho, a, b) = 1$.
(ii) If $a + b$ is even, let

$$m = \sharp\{(\rho, a', b') \in \mathrm{Jord}(\psi) : a', b' \text{ odd}, \zeta_{a',b'} = -1, |a' - b'| > |a - b|\},$$

and

$$n = \sharp\{(\rho, a', b') \in \mathrm{Jord}(\psi) : a', b' \text{ odd}, |a' - b'| < |a - b|\}.$$

Then

$$\varepsilon_\psi^{M/MW}(\rho, a, b) = \begin{cases} 1 & \text{if } a, b \text{ even,} \\ (-1)^m & \text{if } a, b \text{ odd, } \zeta_{a,b} = +1, \\ (-1)^{m+n} & \text{if } a, b \text{ odd, } \zeta_{a,b} = -1. \end{cases}$$

There is a simple fact about this character $\varepsilon_\psi^{M/MW}$.

Lemma 7.4 Suppose $\psi \in \overline{\Psi}(G)$ has discrete diagonal restriction; then $\varepsilon_\psi^{M/MW}(s_\psi) = 1$.

Proof From the definition, we see $\varepsilon_\psi^{M/MW}(\rho, a, b) = 1$ if b is even. Then

$$\varepsilon_\psi^{M/MW}(s_\psi) = \prod_{\substack{(\rho, a, b) \in \mathrm{Jord}(\psi) \\ b \text{ even}}} \varepsilon_\psi^{M/MW}(\rho, a, b) = 1. \qquad \blacksquare$$

Theorem 7.5 Suppose $\psi \in \overline{\Psi}(G)$ has discrete diagonal restriction; then

$$\pi_M(\psi, \overline{\varepsilon}) = \pi_{MW}(\psi, \overline{\varepsilon\varepsilon}_\psi^{M/MW}).$$

Before we prove the theorem, for any $s \in \mathcal{S}_\psi$ let

$$\overline{\Pi}_{MW,s}(\psi) := \sum_{\overline{\varepsilon} \in \widehat{\mathcal{S}_\psi}} \overline{\varepsilon}(ss_\psi)\pi_{MW}(\psi, \overline{\varepsilon}),$$

$$\overline{\Pi}_{M,s}(\psi) := \sum_{\overline{\varepsilon} \in \widehat{\mathcal{S}_\psi}} \overline{\varepsilon}(ss_\psi)\pi_M(\psi, \overline{\varepsilon}).$$

In particular, $\overline{\Pi}_{MW}(\psi) = \overline{\Pi}_{MW,1}(\psi)$ and we denote $\overline{\Pi}_M(\psi) = \overline{\Pi}_{M,1}(\psi)$. For $\overline{\Pi}_{M,s}(\psi)$, we have the following recursive formula.

Lemma 7.6 *Suppose $\psi \in \overline{\Psi}(G)$ has discrete diagonal restriction and $s \in \mathcal{S}_\psi$. Let $(\rho, A, B, \zeta) \in \mathrm{Jord}(\psi)$ such that $A > B$; then*

$$\overline{\Pi}_{M,s}(\psi) =$$
$$\bigoplus_{C \in]B,A]} (-1)^{A-C} \langle \zeta B, \ldots, -\zeta C \rangle \rtimes \overline{\mathrm{Jac}}_{\zeta(B+2),\ldots,\zeta C} \overline{\Pi}_{M,s}\big(\psi', (\rho, A, B+2, \zeta)\big)$$
$$\oplus (-1)^{[(A-B+1)/2]} \overline{\Pi}_{M,s}\big(\psi', (\rho, A, B+1, \zeta), (\rho, B, B, \zeta)\big),$$

where we let $s(\rho, A, B, \zeta) = s(\rho, A, B+2, \zeta) = s(\rho, A, B+1, \zeta) = s(\rho, B, B, \zeta)$.

Proof By definition we have for any $\overline{\varepsilon} \in \widehat{\mathcal{S}_\psi}$,

$$\overline{\varepsilon}(ss_\psi)\,\pi_M(\psi, \overline{\varepsilon}) =$$
$$\bigoplus_{C \in]B,A]} (-1)^{A-C} \langle \zeta B, \ldots, -\zeta C \rangle \rtimes \overline{\mathrm{Jac}}_{\zeta(B+2),\ldots,\zeta C}\,\overline{\varepsilon}(ss_\psi)\,\pi_M(\psi^1, \overline{\varepsilon}_1)$$
$$\bigoplus_{\overline{\varepsilon} \leftarrow \overline{\varepsilon}_2 \in \widehat{\mathcal{S}_{\psi^2}}} (-1)^{[(A-B+1)/2]} \varepsilon_2(\rho, A, B+1, \zeta)^{A-B+1} \varepsilon(\rho, A, B, \zeta)^{A-B}$$
$$\cdot\,\overline{\varepsilon}(ss_\psi)\,\pi_M(\psi^2, \overline{\varepsilon}_2).$$

So it suffices to show $\overline{\varepsilon}_1(ss_{\psi^1}) = \overline{\varepsilon}(ss_\psi)$ and

$$\overline{\varepsilon}_2(ss_{\psi^2}) = \varepsilon_2(\rho, A, B+1, \zeta)^{A-B+1} \varepsilon(\rho, A, B, \zeta)^{A-B}\overline{\varepsilon}(ss_\psi).$$

The first one is easy, because $s_{\psi^1} = s_\psi$ under our identification. For the second one, note that $\overline{\varepsilon}_2(s) = \overline{\varepsilon}(s)$ and

$$\overline{\varepsilon}(s_\psi) = \prod_{(\rho,a,b) \in \mathrm{Jord}(\psi)} \varepsilon(\rho, a, b)^{b-1} = \prod_{(\rho,A,B,\zeta) \in \mathrm{Jord}(\psi)} \varepsilon(\rho, A, B, \zeta)^{A-\zeta B}.$$

Then

$$\overline{\varepsilon}_2(s_{\psi^2})/\overline{\varepsilon}(s_\psi) = \varepsilon_2(\rho, A, B+1, \zeta)^{A-\zeta(B+1)} \varepsilon_2(\rho, B, B, \zeta)^{B-\zeta B}/\varepsilon(\rho, A, B, \zeta)^{A-\zeta B}.$$

Using the fact that $\varepsilon_2(\rho, A, B+1, \zeta)\varepsilon_2(\rho, B, B, \zeta) = \varepsilon(\rho, A, B, \zeta)$, we have

$$\overline{\varepsilon}_2(s_{\psi^2})/\overline{\varepsilon}(s_\psi)$$
$$= \varepsilon_2(\rho, A, B+1, \zeta)^{A-\zeta(B+1)}$$
$$\cdot\,\varepsilon(\rho, A, B, \zeta)^{B-\zeta B} \varepsilon_2(\rho, A, B+1, \zeta)^{-B+\zeta B}/\varepsilon(\rho, A, B, \zeta)^{A-\zeta B}$$
$$= \varepsilon_2(\rho, A, B+1, \zeta)^{A-B-1} \varepsilon(\rho, A, B, \zeta)^{B-A}$$
$$= \varepsilon_2(\rho, A, B+1, \zeta)^{A-B+1} \varepsilon(\rho, A, B, \zeta)^{A-B}.$$

This finishes the proof. ∎

Lemma 7.7 *Suppose $\psi \in \overline{\Psi}(G)$ has discrete diagonal restriction; then $\overline{\Pi}_{MW}(\psi) = \overline{\Pi}_M(\psi)$.*

Proof Lemma 7.6 and Proposition 5.9 allow us to reduce this lemma to the case of elementary Arthur packets, where the statement is already known. ∎

Now we can give the poof of Theorem 7.5.

Proof   Since $\varepsilon_\psi^{M/MW}(s_\psi) = 1$, it is enough to show that

$$\overline{\Pi}_{M,s}(\psi) = \varepsilon_\psi^{M/MW}(s)\overline{\Pi}_{MW,s}(\psi)$$

for all $s \in \mathcal{S}_\psi$. From the previous lemma, we know this is true for $s = 1$. So we can assume that $s \neq 1$ in the rest of the proof. By induction, we may assume the theorem is true for ψ^1 and ψ^2, *i.e.*,

$$\overline{\Pi}_{M,s}(\psi^1) = \varepsilon_{\psi^1}^{M/MW}(s)\overline{\Pi}_{MW,s}(\psi^1),$$

$$\overline{\Pi}_{M,s}(\psi^2) = \varepsilon_{\psi^2}^{M/MW}(s)\overline{\Pi}_{MW,s}(\psi^2).$$

Suppose $(H, \psi_H) \to (\psi, s)$ and $\psi_s := \psi_H = \psi_I \times \psi_{II}$. We can assume $(\rho, A, B, \zeta) \in$ Jord(ψ_{II}) because the other case is similar. Let $\psi_s^1 = \psi_I^1 \times \psi_{II}^1$ and $\psi_s^2 = \psi_I^2 \times \psi_{II}^2$. In particular, $\psi_I = \psi_I^1 = \psi_I^2$. Note that $\overline{\Pi}_{MW,s}(\psi^1)$ (resp. $\overline{\Pi}_{MW,s}(\psi^2)$) is the spectral endoscopic transfer of $\overline{\Pi}_{MW}(\psi_I^1) \otimes \overline{\Pi}_{MW}(\psi_{II}^1)$ (resp. $\overline{\Pi}_{MW}(\psi_I^2) \otimes \overline{\Pi}_{MW}(\psi_{II}^2)$). By the compatibility of endoscopic transfer with Jacquet module and parabolic induction, we can conclude that $\overline{\Pi}_{M,s}(\psi)$ is the spectral endoscopic transfer of

$$\bigoplus_{C \in]B,A]} (-1)^{A-C}\varepsilon_{\psi^1}^{M/MW}(s)\langle \zeta B, \ldots, -\zeta C\rangle \quad \rtimes$$
$$\overline{\mathrm{Jac}}_{\zeta(B+2),\ldots,\zeta C}(\overline{\Pi}_{MW}(\psi_I^1) \otimes \overline{\Pi}_{MW}(\psi_{II}^1))$$
$$\oplus (-1)^{[(A-B+1)/2]}\varepsilon_{\psi^2}^{M/MW}(s)\overline{\Pi}_{MW}(\psi_I^2) \otimes \overline{\Pi}_{MW}(\psi_{II}^2).$$

Note that $\overline{\mathrm{Jac}}_{\zeta D}\overline{\Pi}_{MW}(\psi_I^1) = 0$ for any $B + 2 \leqslant D \leqslant A$, which follows from the corresponding vanishing fact for Jacquet modules of $\pi(\psi_I^1)$. Then we can rewrite it as

$$\bigoplus_{C \in]B,A]} (-1)^{A-C}\varepsilon_{\psi^1}^{M/MW}(s)\overline{\Pi}_{MW}(\psi_I^1)$$
$$\otimes \left(\langle \zeta B, \ldots, -\zeta C\rangle \rtimes \overline{\mathrm{Jac}}_{\zeta(B+2),\ldots,\zeta C}\overline{\Pi}_{MW}(\psi_{II}^1)\right)$$
$$\oplus (-1)^{[(A-B+1)/2]}\varepsilon_{\psi^2}^{M/MW}(s)\overline{\Pi}_{MW}(\psi_I^2) \otimes \overline{\Pi}_{MW}(\psi_{II}^2).$$

If we can show

$$(7.1) \qquad\qquad \varepsilon_\psi^{M/MW}(s) = \varepsilon_{\psi^1}^{M/MW}(s) = \varepsilon_{\psi^2}^{M/MW}(s),$$

then that means $\overline{\Pi}_{M,s}(\psi)$ is the spectral endoscopic transfer of $\varepsilon_\psi^{M/MW}(s)\overline{\Pi}_{MW}(\psi_I) \otimes \overline{\Pi}_{MW}(\psi_{II})$. Hence,

$$\overline{\Pi}_{M,s}(\psi) = \varepsilon_\psi^{M/MW}(s)\overline{\Pi}_{MW,s}(\psi).$$

Finally, it is an easy exercise to verify (7.1). In fact, one can assume that $s(\rho, A, B, \zeta) = 1$; then the set of Jordan blocks (ρ, a', b') such that $s(\rho, a', b') = -1$ is the same for ψ, ψ^1, and ψ^2, and it is enough to show

$$\varepsilon_\psi^{M/MW}(\rho, a', b') = \varepsilon_{\psi^1}^{M/MW}(\rho, a', b') = \varepsilon_{\psi^2}^{M/MW}(\rho, a', b')$$

for any (ρ, a', b') in this set. Recall that

$$(\rho, A, B + 2, \zeta) = (\rho, a + 2\zeta, b - 2\zeta),$$
$$(\rho, A, B + 1, \zeta) = (\rho, a + \zeta, b - \zeta),$$
$$(\rho, B, B, \zeta) = (\rho, \sup(0, a - b) + 1, \sup(0, b - a) + 1).$$

One checks easily that the contribution of (ρ, A, B, ζ) to the numbers m, n in Definition 7.3 for ψ is the same as $(\rho, A, B+2, \zeta)$ for ψ^1, and $(\rho, A, B+1, \zeta)$, (ρ, B, B, ζ) for ψ^2 modulo 2. Then the rest is clear. ∎

One consequence of Theorem 7.5 is that $\pi_M(\psi, \overline{\varepsilon})$ is an $\overline{\mathcal{H}}(G)$-module, which is by no means clear from our definition. In fact, the main goal of [Mœg09] is to show that $\pi_M^{\Sigma_0}(\psi, \varepsilon)$ is a representation of G^{Σ_0} and characterize its irreducible constituents, which also implies that $\pi_M(\psi, \overline{\varepsilon})$ is an $\overline{\mathcal{H}}(G)$-module independently of Arthur's theory.

Theorem 7.8 ([Mœg09, Theorem 4.2]) *Suppose $\psi \in \overline{\Psi}(G)$ has discrete diagonal restriction, and there exists $(\rho, A, B, \zeta) \in \mathrm{Jord}(\psi)$ such that $A > B$. Let $\varepsilon \in \mathcal{S}_\psi^{\Sigma_0}$ and $\eta_0 := \varepsilon(\rho, A, B, \zeta)$. Then we have*

$$\pi_M^{\Sigma_0}(\psi, \varepsilon) = \bigoplus_{l \in [0, [(A-B+1)/2]]} \ \bigoplus_{\eta = \pm 1 : \eta_0 = \eta^{A-B+1} \prod_{C \in [B+l, A-l]} (-1)^{[C]}}$$
$$\Big\langle \langle \zeta B, \ldots, -\zeta A \rangle \times \cdots \times \langle \zeta(B+l-1), \ldots, -\zeta(A-l+1) \rangle$$
$$\rtimes \pi_M^{\Sigma_0}(\psi', \varepsilon', \cup_{C \in [B+l, A-l]} (\rho, C, C, \zeta; \eta(-1)^{[C]})) \Big\rangle,$$

where ψ' is obtained from ψ by removing (ρ, A, B, ζ), and $\varepsilon'(\cdot)$ is the restriction of $\varepsilon(\cdot)$. In particular, when $l = (A - B + 1)/2$ and $\eta_0 = 1$, we will just take one value for η, since both values give the same term.

Remark 7.9 The complicated condition on η comes from the fact that $\eta(-1)^{[C]}$ with $\varepsilon'(\cdot)$ needs to define a character ε_- of $\mathcal{S}_{\psi_-}^{\Sigma_0}$, where $\mathrm{Jord}(\psi_-)$ is obtained from $\mathrm{Jord}(\psi')$ by adding $\cup_{C \in [B+l, A-l]} (\rho, C, C, \zeta)$.

This theorem shows that $\pi_M^{\Sigma_0}(\psi, \varepsilon)$ is a representation G^{Σ_0} and allows us to decompose it according to two parameters $\underline{l}, \underline{\eta}$, where $\underline{l}$ is an integer-valued function on $\mathrm{Jord}(\psi)$ and $\underline{\eta}$ is a $\mathbb{Z}_2$-valued function on $\mathrm{Jord}(\psi)$. In the notation of this theorem, we let $\underline{l}(\rho, A, B, \zeta) = l$ and $\underline{\eta}(\rho, A, B, \zeta) = \eta(-1)^{[B+l]}$. Then

$$\underline{l}(\rho, A, B, \zeta) \in \big[0, [(A-B+1)/2]\big],$$

and

$$\varepsilon(\rho, A, B, \zeta) = \underline{\eta}(\rho, A, B, \zeta)^{A-B+1} (-1)^{[(A-B+1)/2] + \underline{l}(\rho, A, B, \zeta)}.$$

Let us denote by $\varepsilon_{\underline{l}, \underline{\eta}}$ the character of $\mathcal{S}_\psi^{\Sigma_0}$ defined by $(\underline{l}, \underline{\eta})$ through this formula. Then we define for any pair $(\underline{l}, \underline{\eta})$ such that $\varepsilon_{\underline{l}, \underline{\eta}} \in \widehat{\mathcal{S}_\psi^{\Sigma_0}}$,

$$\pi_M^{\Sigma_0}(\psi, \underline{l}, \underline{\eta}) := \Big\langle \langle \zeta B, \ldots, -\zeta A \rangle \times \cdots \times$$
$$\langle \zeta(B + \underline{l}(\rho, A, B, \zeta) - 1), \ldots, -\zeta(A - \underline{l}(\rho, A, B, \zeta) + 1) \rangle$$
$$\rtimes \pi_M^{\Sigma_0}(\psi_-, \underline{l}_-, \underline{\eta}_-) \Big\rangle,$$

where ψ_- is defined as in the remark, and $\underline{l}_-$ and $\underline{\eta}_-$ are extended from $\underline{l}$ and $\underline{\eta}$ by letting $\underline{l}_-(\rho, C, C, \zeta) = 0$ and $\underline{\eta}_-(\rho, C, C, \zeta) = \eta(-1)^{[C]}$. In the theorem, Mœglin

shows that $\pi_M^{\Sigma_0}(\psi, \underline{l}, \underline{\eta})$ is irreducible. In fact, one can also show that

$$\pi_M^{\Sigma_0}(\psi, \underline{l}, \underline{\eta}) \hookrightarrow$$

$$\underset{(\rho,A,B,\zeta)\in\mathrm{Jord}(\psi)}{\times} \begin{pmatrix} \zeta B & \cdots & -\zeta A \\ \vdots & & \vdots \\ \zeta(B+\underline{l}(\rho,A,B,\zeta)-1) & \cdots & -\zeta(A-\underline{l}(\rho,A,B,\zeta)+1) \end{pmatrix}$$

$$\rtimes \pi_M^{\Sigma_0}\Big(\underset{(\rho,A,B,\zeta)\in\mathrm{Jord}(\psi)}{\bigcup} \underset{C\in[B+\underline{l}(\rho,A,B,\zeta),A-\underline{l}(\rho,A,B,\zeta)]}{\bigcup}$$

$$\big(\rho,C,C,\zeta;\underline{\eta}(\rho,A,B,\zeta)(-1)^{C-B-\underline{l}(\rho,A,B,\zeta)}\big)\Big)$$

as the unique irreducible subrepresentation. We define $\pi_M(\psi, \underline{l}, \underline{\eta})$ to be the irreducible representation of G viewed as $\overline{\mathcal{H}}(G)$-module in the restriction of $\pi_M^{\Sigma_0}(\psi, \underline{l}, \underline{\eta})$ to G. Then

$$\pi_M(\psi, \underline{l}, \underline{\eta}) \hookrightarrow$$

$$\underset{(\rho,A,B,\zeta)\in\mathrm{Jord}(\psi)}{\times} \begin{pmatrix} \zeta B & \cdots & -\zeta A \\ \vdots & & \vdots \\ \zeta(B+\underline{l}(\rho,A,B,\zeta)-1) & \cdots & -\zeta(A-\underline{l}(\rho,A,B,\zeta)+1) \end{pmatrix}$$

$$\rtimes \pi_M\Big(\underset{(\rho,A,B,\zeta)\in\mathrm{Jord}(\psi)}{\bigcup} \underset{C\in[B+\underline{l}(\rho,A,B,\zeta),A-\underline{l}(\rho,A,B,\zeta)]}{\bigcup}$$

$$\big(\rho,C,C,\zeta;\underline{\eta}(\rho,A,B,\zeta)(-1)^{C-B-\underline{l}(\rho,A,B,\zeta)}\big)\Big)$$

as the unique irreducible element in $\overline{\mathrm{Rep}}(G)$ forming an $\overline{\mathcal{H}}(G)$-submodule.

We define an equivalence relation on pairs $(\underline{l}, \underline{\eta})$ such that $(\underline{l}, \underline{\eta}) \sim_{\Sigma_0} (\underline{l}', \underline{\eta}')$ if and only if $\underline{l} = \underline{l}'$ and $(\underline{\eta}/\underline{\eta}')(\rho, A, B, \zeta) = 1$ unless $\underline{l}(\rho, A, B, \zeta) = (A - B + 1)/2$. It is clear that $\pi_M^{\Sigma_0}(\psi, \underline{l}, \underline{\eta}) \cong \pi_M^{\Sigma_0}(\psi, \underline{l}', \underline{\eta}')$ if $(\underline{l}, \underline{\eta}) \sim_{\Sigma_0} (\underline{l}', \underline{\eta}')$. In fact, the converse is also true.

Proposition 7.10 *Suppose $\psi \in \overline{\Psi}(G)$ has discrete diagonal restriction and $\varepsilon \in \widehat{\mathcal{S}_\psi^{\Sigma_0}}$; then*

$$\pi_M^{\Sigma_0}(\psi, \varepsilon) = \bigoplus_{\{(\underline{l},\underline{\eta}): \varepsilon = \varepsilon_{\underline{l},\underline{\eta}}\}/\sim_{\Sigma_0}} \pi_M^{\Sigma_0}(\psi, \underline{l}, \underline{\eta}).$$

Moreover, $\pi_M^{\Sigma_0}(\psi, \underline{l}, \underline{\eta}) \cong \pi_M^{\Sigma_0}(\psi, \underline{l}', \underline{\eta}')$ if and only if $(\underline{l}, \underline{\eta}) \sim_{\Sigma_0} (\underline{l}', \underline{\eta}')$.

Proof The only thing that may not be obvious from Theorem 7.8 is the fact that $\pi_M^{\Sigma_0}(\psi, \underline{l}, \underline{\eta}) \not\cong \pi_M^{\Sigma_0}(\psi, \underline{l}', \underline{\eta}')$ if $(\underline{l}, \underline{\eta}) \not\sim_{\Sigma_0} (\underline{l}', \underline{\eta}')$. But this can be shown by comparing the Jacquet modules of these representations.   ∎

Remark 7.11 If $\mathrm{Jord}(\psi)$ contains (ρ, a, b) with $a = b$, then our definition of $\pi_M^{\Sigma_0}(\psi, \underline{l}, \underline{\eta})$ will depend on the choice of sign $\zeta_{a,b}$. However, it is not hard to show that the representation $\pi_M^{\Sigma_0}(\psi, \underline{l}, \underline{\eta})$ is independent of $\zeta_{a,b}$.

If G is special even orthogonal, and $\psi \in \overline{\Psi}(G)$ has discrete diagonal restriction, we define a $\mathbb{Z}_2$-valued function on $\mathrm{Jord}(\psi)$ by

$$\underline{\eta}_0(\rho, A, B, \zeta) = \begin{cases} -1, & \text{if } d_\rho \text{ is odd and } A \in \mathbb{Z}, \\ 1, & \text{otherwise.} \end{cases}$$

Then $\varepsilon_0(\rho, A, B, \zeta) = \underline{\eta}_0(\rho, A, B, \zeta)^{A-B+1}$, and hence $\varepsilon_{\underline{l}, \underline{\eta}\, \underline{\eta}_0} = \varepsilon_{\underline{l}, \underline{\eta}}\, \varepsilon_0$. In general, we let $\underline{\eta}_0 = 1$ if G is not special even orthogonal.

Corollary 7.12 *Suppose $\psi \in \overline{\Psi}(G)$ has discrete diagonal restriction; then*

$$(7.2) \qquad \pi_M^{\Sigma_0}(\psi, \underline{l}, \underline{\eta}\, \underline{\eta}_0) \cong \pi_M^{\Sigma_0}(\psi, \underline{l}, \underline{\eta}) \otimes \omega_0.$$

Proof This follows from the formula of $\pi_M^{\Sigma_0}(\psi, \underline{l}, \underline{\eta})$ and (6.5) in the elementary case. ∎

We define another equivalence relation on pairs $(\underline{l}, \underline{\eta})$ such that $(\underline{l}, \underline{\eta}) \sim (\underline{l}', \underline{\eta}')$ if and only if $(\underline{l}, \underline{\eta}) \sim_{\Sigma_0} (\underline{l}', \underline{\eta}')$ or $(\underline{l}, \underline{\eta}) \sim_{\Sigma_0} (\underline{l}', \underline{\eta}'\, \underline{\eta}_0)$. It follows from this corollary that $\pi_M(\psi, \underline{l}, \underline{\eta}) = \pi_M(\psi, \underline{l}', \underline{\eta}')$ if and only if $(\underline{l}, \underline{\eta}) \sim (\underline{l}', \underline{\eta}')$.

Corollary 7.13 *Suppose $\psi \in \overline{\Psi}(G)$ has discrete diagonal restriction and $\overline{\varepsilon} \in \widehat{\mathcal{S}_\psi}$; then*

$$\pi_M(\psi, \overline{\varepsilon}) = \bigoplus_{\{(\underline{l}, \underline{\eta}) : \overline{\varepsilon} = \overline{\varepsilon}_{\underline{l}, \underline{\eta}}\}/\sim} \pi_M(\psi, \underline{l}, \underline{\eta}).$$

Moreover,

$$\bigoplus_{\overline{\varepsilon} \leftarrow \varepsilon \in \widehat{\mathcal{S}_\psi^{\Sigma_0}}} \pi_M^{\Sigma_0}(\psi, \varepsilon)$$

consists of all irreducible representations of G^{Σ_0}, whose restriction to G belong to $\pi_M(\psi, \overline{\varepsilon})$.

Proof We can assume G is special even orthogonal. It follows from Proposition 7.10 that

$$m \cdot \pi_M(\psi, \overline{\varepsilon}) = \pi_M^{\Sigma_0}(\psi, \varepsilon)|_G = \bigoplus_{\{(\underline{l}, \underline{\eta}) : \varepsilon = \varepsilon_{\underline{l}, \underline{\eta}}\}/\sim_{\Sigma_0}} \pi_M^{\Sigma_0}(\psi, \underline{l}, \underline{\eta})|_G,$$

where $m = 2$ if $\mathcal{S}_\psi^{\Sigma_0} = \mathcal{S}_\psi$, and $m = 1$ otherwise. By (7.2), one can easily see the right hand side is

$$m \bigoplus_{\{(\underline{l}, \underline{\eta}) : \overline{\varepsilon} = \overline{\varepsilon}_{\underline{l}, \underline{\eta}}\}/\sim} \pi_M(\psi, \underline{l}, \underline{\eta}).$$

This proves the first part, and the second part should then be clear. ∎

Motivated by this corollary, we can define $\Pi_\psi^{\Sigma_0}$ to be the set of irreducible representations of G^{Σ_0}, whose restriction to G belong to $\overline{\Pi}_\psi$. In the case where G is special even orthogonal and $\psi \in \overline{\Psi}(G)$ has discrete diagonal restriction, suppose $\mathcal{S}_\psi^{\Sigma_0} \neq \mathcal{S}_\psi$; then for any $(\underline{l}, \underline{\eta})$,

$$(\underline{l}, \underline{\eta}\, \underline{\eta}_0) \nsim_{\Sigma_0} (\underline{l}, \underline{\eta}),$$

and hence $\pi^{\theta_0} \cong \pi$ for any irreducible constituent $[\pi]$ in $\pi_M(\psi, \bar{\varepsilon})$ by (7.2). Then it follows from Theorem 4.2 that there is a canonical disjoint decomposition

$$\Pi_\psi^{\Sigma_0} = \bigsqcup_{\varepsilon \in \widehat{\mathcal{S}_\psi^{\Sigma_0}}} \pi_W^{\Sigma_0}(\psi, \varepsilon)$$

such that

- $[\pi_W^{\Sigma_0}(\psi, \varepsilon)|_G] = 2\pi_W(\psi, \bar{\varepsilon})$ if G is special even orthogonal and $\mathcal{S}_\psi^{\Sigma_0} = \mathcal{S}_\psi$, or $\pi_W(\psi, \bar{\varepsilon})$ otherwise.
- For any $s \in \mathcal{S}_\psi^{\Sigma_0}$ but not in $\mathcal{S}_\psi$ and $(H, \psi_H) \to (\psi, s)$, the following identity holds

$$f_W^H(\psi_H) = \sum_{\bar{\varepsilon} \in \widehat{\mathcal{S}_\psi}} \varepsilon(s s_\psi) f_G(\pi_W^{\Sigma_0}(\psi, \varepsilon)) \quad f \in C_c^\infty(G \rtimes \theta_0).$$

Let us define $\pi_{MW}^{\Sigma_0}(\psi, \varepsilon) := \pi_W^{\Sigma_0}(\psi, \varepsilon \varepsilon_\psi^{MW/W})$ for $\varepsilon \in \widehat{\mathcal{S}_\psi^{\Sigma_0}}$; then we can show in the same way as Proposition 5.6 that for any $s \in \mathcal{S}_\psi^{\Sigma_0}$ but not in $\mathcal{S}_\psi$ and $(H, \psi_H) \to (\psi, s)$,

$$f_{MW}^H(\psi_H) = \sum_{\bar{\varepsilon} \in \widehat{\mathcal{S}_\psi}} \varepsilon(s s_\psi) f_G(\pi_{MW}^{\Sigma_0}(\psi, \varepsilon)) \quad f \in C_c^\infty(G \rtimes \theta_0).$$

At last, we can extend Theorem 7.5 to G^{Σ_0}.

Theorem 7.14 *Suppose $\psi \in \overline{\Psi}(G)$ has discrete diagonal restriction; then*

$$\pi_M^{\Sigma_0}(\psi, \varepsilon) = \pi_{MW}^{\Sigma_0}(\psi, \varepsilon \varepsilon_\psi^{M/MW}).$$

Proof We can assume that G is special even orthogonal and $\mathcal{S}_\psi^{\Sigma_0} \neq \mathcal{S}_\psi$. The proof goes in the same way as that of Theorem 7.5. First we choose $s^* \in \mathcal{S}_\psi^{\Sigma_0}$ but not in $\mathcal{S}_\psi$, and we define

$$\Pi_{MW,s^*}^{\Sigma_0}(\psi) := \sum_{\bar{\varepsilon} \in \widehat{\mathcal{S}_\psi}} \varepsilon(s^* s_\psi) \pi_{MW}^{\Sigma_0}(\psi, \varepsilon),$$

$$\Pi_{M,s^*}^{\Sigma_0}(\psi) := \sum_{\bar{\varepsilon} \in \widehat{\mathcal{S}_\psi}} \varepsilon(s^* s_\psi) \pi_M^{\Sigma_0}(\psi, \varepsilon).$$

Secondly, we can extend Lemma 7.6 to this case, *i.e.*, for $(\rho, A, B, \zeta) \in \mathrm{Jord}(\psi)$ such that $A > B$,

$$\Pi_{M,s^*}^{\Sigma_0}(\psi) = \bigoplus_{C \in]B,A]} (-1)^{A-C} \langle \zeta B, \ldots, -\zeta C \rangle \rtimes \mathrm{Jac}_{\zeta(B+2),\ldots,\zeta C} \Pi_{M,s^*}^{\Sigma_0}(\psi', (\rho, A, B+2, \zeta))$$

$$\oplus (-1)^{[(A-B+1)/2]} \Pi_{M,s^*}^{\Sigma_0}(\psi', (\rho, A, B+1, \zeta), (\rho, B, B, \zeta)),$$

where we let $s^*(\rho, A, B, \zeta) = s^*(\rho, A, B+2, \zeta) = s^*(\rho, A, B+1, \zeta) = s^*(\rho, B, B, \zeta)$. The proof is the same. Then we can show by induction that

$$\Pi_{M,s^*}^{\Sigma_0}(\psi) = \varepsilon_\psi^{M/MW}(s^*) \Pi_{MW,s^*}^{\Sigma_0}(\psi).$$

This is because of Theorem 6.21 and the fact that (7.1) still holds in this case. Finally, since $\pi_M(\psi, \overline{\varepsilon}) = \pi_{MW}(\psi, \overline{\varepsilon \varepsilon}_\psi^{M/MW})$, we have

$$\varepsilon(s^* s_\psi) \pi_M^{\Sigma_0}(\psi, \varepsilon) = \varepsilon_\psi^{M/MW}(s^*) \cdot \varepsilon \varepsilon_\psi^{M/MW}(s^* s_\psi) \pi_{MW}^{\Sigma_0}(\psi, \varepsilon \varepsilon_\psi^{M/MW})$$

$$= \varepsilon(s^* s_\psi) \pi_{MW}^{\Sigma_0}(\psi, \varepsilon \varepsilon_\psi^{M/MW})$$

by the linear independence of twisted characters. Hence,

$$\pi_M^{\Sigma_0}(\psi, \varepsilon) = \pi_{MW}^{\Sigma_0}(\psi, \varepsilon \varepsilon_\psi^{M/MW}). \qquad \blacksquare$$

8 The General Case

In this section, we consider Mœglin's parametrization of elements in $\overline{\Pi}_\psi$ for general $\psi \in \overline{\Psi}(G)$. The idea is similar to Section 5. We first assume that $\psi = \psi_p$, and fix an order $>_\psi$ on $\mathrm{Jord}(\psi)$ satisfying condition ($\mathcal{P}$). We also choose a parameter $\psi_\gg$ dominating ψ with discrete diagonal restriction and natural order, and we identify $\mathcal{S}_{\psi>}^{\Sigma_0} \cong \mathcal{S}_{\psi_\gg}^{\Sigma_0}$. Then we define for $\varepsilon \in \widehat{\mathcal{S}_{\psi>}^{\Sigma_0}}$,

$$\pi_M^{\Sigma_0}(\psi, \varepsilon) := \circ_{(\rho, A, B, \zeta) \in \mathrm{Jord}(\psi)} \mathrm{Jac}_{(\rho, A_\gg, B_\gg, \zeta) \mapsto (\rho, A, B, \zeta)} \pi_M^{\Sigma_0}(\psi_\gg, \varepsilon),$$

where the composition is taken in the decreasing order. Since

$$\pi_M^{\Sigma_0}(\psi_\gg, \varepsilon \varepsilon_0) \cong \pi_M^{\Sigma_0}(\psi_\gg, \varepsilon) \otimes \omega_0,$$

we have

$$\pi_M^{\Sigma_0}(\psi, \varepsilon \varepsilon_0) \cong \pi_M^{\Sigma_0}(\psi, \varepsilon) \otimes \omega_0.$$

We also define

$$(8.1) \qquad \pi_M(\psi, \overline{\varepsilon}) := \circ_{(\rho, A, B, \zeta) \in \mathrm{Jord}(\psi)} \overline{\mathrm{Jac}}_{(\rho, A_\gg, B_\gg, \zeta) \mapsto (\rho, A, B, \zeta)} \pi_M(\psi_\gg, \overline{\varepsilon}).$$

It follows from the case of discrete diagonal restriction that the restriction of $\pi_M^{\Sigma_0}(\psi, \varepsilon)$ to G viewed as $\overline{\mathcal{H}}(G)$-modules is $2\pi_M(\psi, \overline{\varepsilon})$ if G is special even orthogonal and $\mathcal{S}_\psi^{\Sigma_0} = \mathcal{S}_\psi$, or $\pi_M(\psi, \overline{\varepsilon})$ otherwise.

Next we extend the definition of $\varepsilon_\psi^{M/MW} \in \widehat{\mathcal{S}_{\psi>}^{\Sigma_0}}$ to this case.

Definition 8.1 Suppose $\psi = \psi_p \in \overline{\Psi}(G)$ and $(\rho, a, b) \in \mathrm{Jord}(\psi)$. We fix an order $>_\psi$ on $\mathrm{Jord}(\psi)$ satisfying condition ($\mathcal{P}$).

(i) If $a + b$ is odd, $\varepsilon_\psi^{M/MW}(\rho, a, b) = 1$.
(ii) If $a + b$ is even, let

$$m = \sharp\{(\rho, a', b') \in \mathrm{Jord}(\psi) : a', b' \text{ odd}, \zeta_{a', b'} = -1, (\rho, a', b') >_\psi (\rho, a, b)\},$$

and

$$n = \sharp\{(\rho, a', b') \in \mathrm{Jord}(\psi) : a', b' \text{ odd}, (\rho, a', b') <_\psi (\rho, a, b)\}.$$

Then

$$\varepsilon_\psi^{M/MW}(\rho, a, b) = \begin{cases} 1 & \text{if } a, b \text{ even}, \\ (-1)^m & \text{if } a, b \text{ odd}, \zeta_{a,b} = +1, \\ (-1)^{m+n} & \text{if } a, b \text{ odd}, \zeta_{a,b} = -1. \end{cases}$$

Proposition 8.2   *Suppose $\psi = \psi_p \in \overline{\Psi}(G)$ and $\overline{\varepsilon} \in \widehat{\mathcal{S}_{\psi^>}}$; then*

$$\pi_M(\psi, \overline{\varepsilon}) = \pi_{MW}(\psi, \overline{\varepsilon\varepsilon}_\psi^{M/MW}).$$

Proof   By the definition of (5.6) and (8.1), it suffices to show that

$$\pi_M(\psi_\gg, \overline{\varepsilon}) = \pi_{MW}(\psi_\gg, \overline{\varepsilon\varepsilon}_\psi^{M/MW}).$$

One checks easily $\varepsilon_\psi^{M/MW} = \varepsilon_{\psi_\gg}^{M/MW}$ by the definition. So now this proposition will follow from Theorem 7.5 directly.   ∎

As a consequence, we have the following result.

Proposition 8.3   *Suppose $\psi = \psi_p \in \overline{\Psi}(G)$ and $\overline{\varepsilon} \in \widehat{\mathcal{S}_{\psi^>}}$. Let ρ be a unitary irreducible supercuspidal representation of $GL(d_\rho)$.*

(i)   *For $\zeta \in \{\pm 1\}$ and segment $[x, y]$ with $0 \leqslant x \leqslant y$, $\overline{\mathrm{Jac}}_{\zeta x, \dots, \zeta y} \pi_M(\psi, \overline{\varepsilon}) = 0$ unless there exists a sequence of Jordan blocks $\{(\rho, A_i, B_i, \zeta)\}_{i=1}^n \subseteq \mathrm{Jord}(\psi)$ such that $B_1 = x, A_n \geqslant y$, and $B_i \leqslant B_{i+1} \leqslant A_i + 1$.*

(ii)   *For $x \in \mathbb{R}$, let $m = \sharp\{(\rho, A, B, \zeta) \in \mathrm{Jord}(\psi) : \zeta B = x\}$; then*

$$\underbrace{\overline{\mathrm{Jac}}_{x, \dots, x}}_{n} \pi_M(\psi, \overline{\varepsilon}) = 0$$

if $n > m$.

Proof   Note that $\pi_M(\psi, \overline{\varepsilon}) = \pi_{MW}(\psi, \overline{\varepsilon\varepsilon}^{M/MW})$ and

$$\pi_{MW}(\psi, \overline{\varepsilon\varepsilon}^{M/MW}) = \begin{cases} \pi_W(\psi, \overline{\varepsilon\varepsilon}^{M/MW}\overline{\varepsilon}^{MW/W}), & \text{if } \overline{\varepsilon\varepsilon}^{M/MW}\overline{\varepsilon}^{MW/W} \in \widehat{\mathcal{S}_\psi}, \\ 0, & \text{otherwise.} \end{cases}$$

So it suffices to show the proposition for $\pi_W(\psi, \overline{\varepsilon})$ and $\overline{\varepsilon} \in \widehat{\mathcal{S}_\psi}$. As we see from the proof of Proposition 5.7,

$$\pi_W(\psi, \overline{\varepsilon}) = \frac{\overline{\varepsilon}(s_\psi)}{|\mathcal{S}_\psi|} \sum_{s \in \mathcal{S}_\psi} \overline{\varepsilon}(s) \overline{\Pi}_{W,s}(\psi),$$

where $\overline{\Pi}_{W,s}(\psi)$ is transferred from $\overline{\Pi}_{\psi_H}$ for $(H, \psi_H) \to (\psi, s)$. By (4.1), it suffices to show the vanishing of the corresponding Jacquet modules for $\pi_{\psi_H} := \pi_{\psi_I} \otimes \pi_{\psi_{II}}$. In fact, it suffices to consider

$$\pi_\psi = \underset{(\rho, a, b) \in \mathrm{Jord}(\psi)}{\times} \mathrm{Sp}(\mathrm{St}(\rho, a), b).$$

Then one can easily check that $\mathrm{Jac}_{\zeta x, \dots, \zeta y}^\theta \pi_\psi = 0$ unless there exists a sequence of Jordan blocks

$$\{(\rho, A_i, B_i, \zeta)\}_{i=1}^n \subseteq \mathrm{Jord}(\psi)$$

such that $B_1 = x, A_n \geqslant y$, and $B_i \leqslant B_{i+1} \leqslant A_i + 1$. It is also easy to see that

$$\underbrace{\mathrm{Jac}_{x, \dots, x}^\theta}_{n} \pi_\psi = 0$$

if $n > m$.   ∎

Remark 8.4 This proposition implies that the same kind of statements are also true for $\pi_M^{\Sigma_0}(\psi, \varepsilon)$.

For functions $\underline{l}(\rho, A, B, \zeta) \in \left[0, \left[(A - B + 1)/2\right]\right]$ and $\underline{\eta}(\rho, A, B, \zeta) \in \mathbb{Z}_2$ on $\mathrm{Jord}(\psi)$ such that

$$\varepsilon_{\underline{l}, \underline{\eta}}(\rho, A, B, \zeta) := \underline{\eta}(\rho, A, B, \zeta)^{A - B + 1}(-1)^{\left[(A - B + 1)/2\right] + \underline{l}(\rho, A, B, \zeta)}$$

defines a character $\varepsilon_{\underline{l}, \underline{\eta}}$ of $\mathcal{S}_{\psi_\gg}^{\Sigma_0}$, we define

$$\pi_M^{\Sigma_0}(\psi, \underline{l}, \underline{\eta}) := \circ_{(\rho, A, B, \zeta) \in \mathrm{Jord}(\psi)} \mathrm{Jac}_{(\rho, A_\gg, B_\gg, \zeta) \mapsto (\rho, A, B, \zeta)} \pi_M^{\Sigma_0}(\psi_\gg, \underline{l}, \underline{\eta}),$$

where the composition is taken in the decreasing order,

$$\underline{l}(\rho, A, B, \zeta) = \underline{l}(\rho, A_\gg, B_\gg, \zeta) \quad \text{and} \quad \underline{\eta}(\rho, A, B, \zeta) = \underline{\eta}(\rho, A_\gg, B_\gg, \zeta).$$

Then we have the following result about this representation.

Proposition 8.5 ([Mœg10], Proposition 2.8.1) *For $\psi = \psi_p \in \overline{\Psi}(G)$, $\pi_M^{\Sigma_0}(\psi, \underline{l}, \underline{\eta})$ only depends on $>_\psi$, but not on $\psi_\gg$. Moreover, $\pi_M^{\Sigma_0}(\psi, \underline{l}, \underline{\eta})$ is either zero or irreducible. If $\pi_M^{\Sigma_0}(\psi, \underline{l}, \underline{\eta}) \neq 0$, then*

$$\pi_M^{\Sigma_0}(\psi_\gg, \underline{l}, \underline{\eta}) \hookrightarrow \Big(\underset{(\rho, A, B, \zeta) \in \mathrm{Jord}(\psi)}{\times} \langle X_{(\rho, A, B, \zeta)}^\gg \rangle \Big) \rtimes \pi_M^{\Sigma_0}(\psi, \underline{l}, \underline{\eta}),$$

where the product is taken in the increasing order.

Proof First, we would like to show $\pi_M^{\Sigma_0}(\psi, \underline{l}, \underline{\eta})$ only depends on $>_\psi$. Suppose there are two dominating parameters $\psi_\gg^1$ and $\psi_\gg^2$ with discrete diagonal restriction and natural order; we can always choose a third one $\psi_\gg^*$ that dominates both $\psi_\gg^1$ and $\psi_\gg^2$. It is clear that

$$\pi_M^{\Sigma_0}(\psi_\gg^i, \underline{l}, \underline{\eta}) = \circ_{(\rho, A, B, \zeta) \in \mathrm{Jord}(\psi)} \mathrm{Jac}_{(\rho, A_\gg^*, B_\gg^*, \zeta) \mapsto (\rho, A_\gg^i, B_\gg^i, \zeta)} \pi_M^{\Sigma_0}(\psi_\gg^*, \underline{l}, \underline{\eta})$$

for $i = 1, 2$, where the composition is taken in the decreasing order. For all $(\rho', A', B', \zeta') >_\psi (\rho, A, B, \zeta)$, it is easy to check that

$$\mathrm{Jac}_{(\rho, A_\gg^i, B_\gg^i, \zeta) \mapsto (\rho, A, B, \zeta)} \quad \text{and} \quad \mathrm{Jac}_{(\rho', A_\gg'^*, B_\gg'^*, \zeta') \mapsto (\rho', A'^i, B'^i, \zeta')}$$

commute (*cf.* [Xu15, Lemma 5.6]). Also note that

$$\mathrm{Jac}_{(\rho, A_\gg^i, B_\gg^i, \zeta) \mapsto (\rho, A, B, \zeta)} \circ \mathrm{Jac}_{(\rho, A_\gg^*, B_\gg^*, \zeta) \mapsto (\rho, A_\gg^i, B_\gg^i, \zeta)} = \mathrm{Jac}_{(\rho, A_\gg^*, B_\gg^*, \zeta) \mapsto (\rho, A, B, \zeta)}.$$

Then

$$\circ_{(\rho, A, B, \zeta) \in \mathrm{Jord}(\psi)} \mathrm{Jac}_{(\rho, A_\gg^i, B_\gg^i, \zeta) \mapsto (\rho, A, B, \zeta)} \pi_M^{\Sigma_0}(\psi_\gg^i, \underline{l}, \underline{\eta}) =$$

$$\circ_{(\rho, A, B, \zeta) \in \mathrm{Jord}(\psi)} \mathrm{Jac}_{(\rho, A_\gg^*, B_\gg^*, \zeta) \mapsto (\rho, A, B, \zeta)} \pi_M^{\Sigma_0}(\psi_\gg^*, \underline{l}, \underline{\eta}).$$

This finishes the first part of the proof.

Next we index $\mathrm{Jord}(\psi)$ according to $>_\psi$ such that

$$(\rho_i, A_i, B_i, \zeta_i) >_\psi (\rho_{i-1}, A_{i-1}, B_{i-1}, \zeta_{i-1}).$$

Let $\psi_\gg$ be obtained from ψ by shifting $(\rho_i, A_i, B_i, \zeta_i)$ to $(\rho_i, A_i + T_i, B_i + T_i, \zeta_i)$. We also define ψ^k from $\psi_\gg$ by shifting $(\rho_i, A_i + T_i, B_i + T_i, \zeta_i)$ back to $(\rho_i, A_i, B_i, \zeta_i)$ for

$i \leqslant k$. Suppose $\pi_M^{\Sigma_0}(\psi, \underline{l}, \underline{\eta}) \neq 0$; then $\pi_M^{\Sigma_0}(\psi^k, \underline{l}, \underline{\eta}) \neq 0$ by definition. We would like to show by induction that $\pi_M^{\Sigma_0}(\psi^k, \underline{l}, \underline{\eta})$ is irreducible and that

$$(8.2) \quad \pi_M^{\Sigma_0}(\psi^{k-1}, \underline{l}, \underline{\eta}) \longhookrightarrow \begin{pmatrix} \zeta_k(B_k + T_k) & \cdots & \zeta_k(B_k + 1) \\ \vdots & & \vdots \\ \zeta_k(A_k + T_k) & \cdots & \zeta_k(A_k + 1) \end{pmatrix} \rtimes \pi_M^{\Sigma_0}(\psi^k, \underline{l}, \underline{\eta})$$

is the unique irreducible subrepresentation. Note that $\psi^0 = \psi_{\gg}$ and $\psi^n = \psi$, where $n = |\mathrm{Jord}(\psi)|$. So let us assume $\pi_M^{\Sigma_0}(\psi^{k-1}, \underline{l}, \underline{\eta})$ is irreducible. For $0 \leqslant l \leqslant T_k - 1$, we denote

$$\tau_l := \begin{pmatrix} \zeta_k(B_k + T_k) & \cdots & \zeta_k(B_k + l + 1) \\ \vdots & & \vdots \\ \zeta_k(A_k + T_k) & \cdots & \zeta_k(A_k + l + 1) \end{pmatrix}.$$

Let $\psi^{k-1,l}$ be obtained from ψ^{k-1} by shifting $(\rho_k, A_k + T_k, B_k + T_k, \zeta_k)$ to $(\rho_k, A_k + l, B_k + l, \zeta_k)$. We claim that $\pi_M^{\Sigma_0}(\psi^{k-1,l}, \underline{l}, \underline{\eta})$ is irreducible and

$$\pi_M^{\Sigma_0}(\psi^{k-1}, \underline{l}, \underline{\eta}) \longhookrightarrow \tau_l \rtimes \pi_M^{\Sigma_0}(\psi^{k-1,l}, \underline{l}, \underline{\eta})$$

is the unique irreducible subrepresentation. In particular, $\psi^{k-1,0} = \psi^k$, so this is what we want.

To prove the claim, we assume it is true for $l + 1$, and we would like to establish it for l.

$$\pi_M^{\Sigma_0}(\psi^{k-1}, \underline{l}, \underline{\eta}) \longhookrightarrow \tau_{l+1} \rtimes \pi_M^{\Sigma_0}(\psi^{k-1,l+1}, \underline{l}, \underline{\eta}).$$

Since

$$\mathrm{Jac}_{\zeta_k(B_k+l+1),\dots,\zeta_k(A_k+l+1)} \pi_M^{\Sigma_0}(\psi^{k-1,l+1}, \underline{l}, \underline{\eta}) \neq 0,$$

there exists an irreducible representation $\sigma_l^{\Sigma_0}$ and $C \in [B_k + l + 1, A_k + l + 1]$ such that

$$\pi_M^{\Sigma_0}(\psi^{k-1,l+1}, \underline{l}, \underline{\eta}) \longhookrightarrow \langle \zeta_k C, \dots, \zeta_k(A_k + l + 1) \rangle \rtimes \sigma_l^{\Sigma_0}.$$

If $C > B_k + l + 1$, then by Proposition 8.3 there exists $(\rho_i, A_i, B_i, \zeta_i) \in \mathrm{Jord}(\psi)$ for $i < k$ such that

$$\rho_i = \rho_k, \zeta_i = \zeta_k, B_i > B_k + l + 1 \quad \text{and} \quad A_i \geqslant A_k + l + 1.$$

But this is impossible by the condition $(\mathcal{P})$ on $>_\psi$. Therefore, we must have $C = B_k + l + 1$. It follows that $\sigma_l^{\Sigma_0}$ is a constituent of $\pi_M^{\Sigma_0}(\psi^{k-1,l}, \underline{l}, \underline{\eta})$. Apply Proposition 8.3 to $\psi^{k-1,l}$, and we have

$$(8.3) \qquad\qquad\qquad \mathrm{Jac}_{\zeta_k C', \dots, \zeta_k C''} \sigma_l^{\Sigma_0} = 0$$

for $C' \in [B_k + l + 1, A_k + T_k]$, $C'' \in [A_k + l + 1, A_k + T_k]$. To sum up,

$$\pi_M^{\Sigma_0}(\psi^{k-1}, \underline{l}, \underline{\eta}) \longhookrightarrow \tau_{l+1} \times \begin{pmatrix} \zeta_k(B_k + l + 1) \\ \vdots \\ \zeta_k(A_k + l + 1) \end{pmatrix} \rtimes \sigma_l^{\Sigma_0}.$$

If we apply $\mathrm{Jac}_{(\rho_k, A_k + T_k, B_k + T_k, \zeta_k) \mapsto (\rho_k, A_k + l, B_k + l, \zeta_k)}$ to

$$(8.4) \qquad \tau_{l+1} \times \begin{pmatrix} \zeta_k(B_k + l + 1) \\ \vdots \\ \zeta_k(A_k + l + 1) \end{pmatrix} \rtimes \sigma_l^{\Sigma_0},$$

we should get $\sigma_l^{\Sigma_0}$ by (8.3). So

$$\pi_M^{\Sigma_0}(\psi^{k-1,l}, \underline{l}, \underline{\eta}) := \mathrm{Jac}_{(\rho_k, A_k + T_k, B_k + T_k, \zeta_k) \mapsto (\rho_k, A_k + l, B_k + l, \zeta_k)} \, \pi_M^{\Sigma_0}(\psi^{k-1}, \underline{l}, \underline{\eta}) = \sigma_l^{\Sigma_0},$$

and (8.4) has a unique irreducible subrepresentation. Hence,

$$\pi_M^{\Sigma_0}(\psi^{k-1}, \underline{l}, \underline{\eta}) \hookrightarrow \tau_l \rtimes \pi_M^{\Sigma_0}(\psi^{k-1,l}, \underline{l}, \underline{\eta})$$

is the unique irreducible subrepresentation. This completes the proof of our claim. ∎

Remark 8.6 It is an interesting problem to determine when $\pi_M^{\Sigma_0}(\psi, \underline{l}, \underline{\eta})$ is not zero, and a solution to such problem would have many applications (*e.g.*, [Mœg11b, Mœg11a]). In a sequel to this paper, we will give a procedure for finding explicit non-vanishing conditions on $(\underline{l}, \underline{\eta})$ for $\pi_M^{\Sigma_0}(\psi, \underline{l}, \underline{\eta})$.

Corollary 8.7 *For $\psi = \psi_p \in \overline{\Psi}(G)$, if $\pi_M^{\Sigma_0}(\psi, \underline{l}, \underline{\eta}) \cong \pi_M^{\Sigma_0}(\psi, \underline{l}', \underline{\eta}') \neq 0$, then $(\underline{l}, \underline{\eta}) \sim_{\Sigma_0} (\underline{l}', \underline{\eta}')$.*

Proof Suppose $\pi_M^{\Sigma_0}(\psi, \underline{l}, \underline{\eta}) \cong \pi_M^{\Sigma_0}(\psi, \underline{l}', \underline{\eta}') \neq 0$; then by applying (8.2) step by step, one can conclude that $\pi_M^{\Sigma_0}(\psi_\gg, \underline{l}, \underline{\eta}) \cong \pi_M^{\Sigma_0}(\psi_\gg, \underline{l}', \underline{\eta}')$. This implies $(\underline{l}, \underline{\eta}) \sim_{\Sigma_0} (\underline{l}', \underline{\eta}')$. ∎

Let $\pi_M(\psi, \underline{l}, \underline{\eta})$ be the irreducible representation of G viewed as $\overline{\mathcal{H}}(G)$-module in the restriction of $\pi_M^{\Sigma_0}(\psi, \underline{l}, \underline{\eta})$ to G if $\pi_M^{\Sigma_0}(\psi, \underline{l}, \underline{\eta}) \neq 0$, and zero otherwise. Then

$$\pi_M(\psi, \underline{l}, \underline{\eta}) = \circ_{(\rho, A, B, \zeta) \in \mathrm{Jord}(\psi)} \overline{\mathrm{Jac}}_{(\rho, A_\gg, B_\gg, \zeta) \mapsto (\rho, A, B, \zeta)} \pi_M(\psi_\gg, \underline{l}, \underline{\eta}),$$

where the composition is taken in the decreasing order. The following proposition follows easily from the definitions and similar statements in the case of discrete diagonal restriction (*cf.* Proposition 7.10 and Corollary 7.13).

Proposition 8.8 *For $\psi = \psi_p \in \overline{\Psi}(G)$ and $\varepsilon \in \widehat{\mathcal{S}_{\psi_>}^{\Sigma_0}}$,*

$$\pi_M^{\Sigma_0}(\psi, \varepsilon) = \bigoplus_{\{(\underline{l}, \underline{\eta}) : \varepsilon = \varepsilon_{\underline{l}, \underline{\eta}}\}/\sim_{\Sigma_0}} \pi_M^{\Sigma_0}(\psi, \underline{l}, \underline{\eta}),$$

$$\pi_M(\psi, \overline{\varepsilon}) = \bigoplus_{\{(\underline{l}, \underline{\eta}) : \overline{\varepsilon} = \overline{\varepsilon}_{\underline{l}, \underline{\eta}}\}/\sim} \pi_M(\psi, \underline{l}, \underline{\eta}).$$

Moreover,

$$\bigoplus_{\overline{\varepsilon} \leftarrow \varepsilon \in \widehat{\mathcal{S}_{\psi_>}^{\Sigma_0}}} \pi_M^{\Sigma_0}(\psi, \varepsilon)$$

consists of all irreducible representations of G^{Σ_0}, whose restriction to G belong to $\pi_M(\psi, \overline{\varepsilon})$.

As a consequence, for $\psi = \psi_p \in \overline{\Psi}(G)$ we can define $\Pi_\psi^{\Sigma_0}$ to be the set of irreducible representations of G^{Σ_0}, whose restriction to G belong to $\overline{\Pi}_\psi$. In the case where G is special even orthogonal, if $\mathcal{S}_\psi^{\Sigma_0} \neq \mathcal{S}_\psi$, then $\pi^{\theta_0} \cong \pi$ for any irreducible constituent $[\pi]$ in $\pi_M(\psi, \overline{\varepsilon})$. So it follows from Theorem 4.2 that there is a canonical disjoint decomposition

$$\Pi_\psi^{\Sigma_0} = \bigsqcup_{\varepsilon \in \widehat{\mathcal{S}_\psi^{\Sigma_0}}} \pi_W^{\Sigma_0}(\psi, \varepsilon)$$

such that:

- $[\pi_W^{\Sigma_0}(\psi, \varepsilon)|_G] = 2\pi_W(\psi, \overline{\varepsilon})$ if G is special even orthogonal and $\mathcal{S}_\psi^{\Sigma_0} = \mathcal{S}_\psi$, or $\pi_W(\psi, \overline{\varepsilon})$ otherwise.
- For any $s \in \mathcal{S}_\psi^{\Sigma_0}$ but not in $\mathcal{S}_\psi$ and $(H, \psi_H) \to (\psi, s)$, the following identity holds:

$$f_W^H(\psi_H) = \sum_{\overline{\varepsilon} \in \widehat{\mathcal{S}_\psi}} \varepsilon(ss_\psi) f_G\big(\pi_W^{\Sigma_0}(\psi, \varepsilon) \big) \quad f \in C_c^\infty(G \rtimes \theta_0).$$

Let us also define for $\varepsilon \in \widehat{\mathcal{S}_{\psi^>}^{\Sigma_0}}$,

$$\pi_{MW}^{\Sigma_0}(\psi, \varepsilon) := \circ_{(\rho,A,B,\zeta) \in \mathrm{Jord}(\psi)} \mathrm{Jac}_{(\rho,A_\gg,B_\gg,\zeta) \mapsto (\rho,A,B,\zeta)} \pi_{MW}^{\Sigma_0}(\psi_\gg, \varepsilon).$$

Then we have the following theorem.

Theorem 8.9 *Suppose $\psi = \psi_p \in \overline{\Psi}(G)$ and $\varepsilon \in \widehat{\mathcal{S}_{\psi^>}^{\Sigma_0}}$,*

$$\pi_{MW}^{\Sigma_0}(\psi, \varepsilon) = \begin{cases} \pi_W^{\Sigma_0}(\psi, \varepsilon\varepsilon_\psi^{MW/W}), & \text{if } \varepsilon\varepsilon_\psi^{MW/W} \in \widehat{\mathcal{S}_\psi^{\Sigma_0}}, \\ 0, & \text{otherwise,} \end{cases}$$

and

$$\pi_M^{\Sigma_0}(\psi, \varepsilon) = \pi_{MW}^{\Sigma_0}(\psi, \varepsilon\varepsilon_\psi^{M/MW}).$$

Proof   We can assume G is special even orthogonal and $\mathcal{S}_\psi^{\Sigma_0} \neq \mathcal{S}_\psi$. Since

$$\pi_{MW}(\psi, \overline{\varepsilon}) = \begin{cases} \pi_W(\psi, \overline{\varepsilon\varepsilon_\psi^{MW/W}}), & \text{if } \overline{\varepsilon\varepsilon_\psi^{MW/W}} \in \widehat{\mathcal{S}_\psi}, \\ 0, & \text{otherwise} \end{cases}$$

we have $\pi_{MW}^{\Sigma_0}(\psi, \varepsilon\varepsilon_\psi^{MW/W}) \neq 0$ only if $\varepsilon \in \widehat{\mathcal{S}_\psi^{\Sigma_0}}$.

Let us choose $s^* \in \mathcal{S}_{\psi^>}^{\Sigma_0}$ but not in $\mathcal{S}_{\psi^>}$, and we denote its image in $\mathcal{S}_\psi^{\Sigma_0}$ again by s^*. Then let us define

$$\Pi_{MW,s^*}^{\Sigma_0}(\psi) := \sum_{\overline{\varepsilon} \in \widehat{\mathcal{S}_{\psi^>}}} \varepsilon(s^* s_\psi^>) \pi_{MW}^{\Sigma_0}(\psi, \varepsilon),$$

$$\Pi_{W,s^*}^{\Sigma_0}(\psi) := \sum_{\overline{\varepsilon} \in \widehat{\mathcal{S}_\psi}} \varepsilon(s^* s_\psi) \pi_W^{\Sigma_0}(\psi, \varepsilon).$$

As in Proposition 5.7 one can show

$$\Pi_{MW,s^*}^{\Sigma_0}(\psi) = \varepsilon_\psi^{MW/W}(s_\psi^> s^*) \Pi_{W,s^*}^{\Sigma_0}(\psi)$$

(*cf.* (5.7)). By the linear independence of twisted characters, we have for $\varepsilon \in \widehat{\mathcal{S}_{\psi}^{\Sigma_0}}$,

$$\varepsilon \varepsilon_{\psi}^{MW/W}(s^* s_{\psi}^>) \pi_{MW}^{\Sigma_0}(\psi, \varepsilon \varepsilon_{\psi}^{MW/W}) = \varepsilon_{\psi}^{MW/W}(s_{\psi}^> s^*) \cdot \varepsilon(s^* s_{\psi}) \pi_W^{\Sigma_0}(\psi, \varepsilon).$$

And hence,

$$\pi_{MW}^{\Sigma_0}(\psi, \varepsilon \varepsilon_{\psi}^{MW/W}) = \pi_W^{\Sigma_0}(\psi, \varepsilon).$$

This proves the first part. The second part follows from the case of the discrete diagonal restriction and the fact that $\varepsilon_{\psi \gg}^{M/MW} = \varepsilon_{\psi}^{M/MW}$. ∎

The following corollary is a direct consequence of Theorem 8.9.

Corollary 8.10 *Suppose $\psi = \psi_p \in \overline{\Psi}(G)$ and $\varepsilon \in \widehat{\mathcal{S}_{\psi^>}^{\Sigma_0}}$, let $\varepsilon_{\psi}^{M/W} := \varepsilon_{\psi}^{M/MW} \varepsilon_{\psi}^{MW/W}$. Then*

$$\pi_M^{\Sigma_0}(\psi, \varepsilon) = \begin{cases} \pi_W^{\Sigma_0}(\psi, \varepsilon \varepsilon_{\psi}^{M/W}), & \text{if } \varepsilon \varepsilon_{\psi}^{M/W} \in \widehat{\mathcal{S}_{\psi}^{\Sigma_0}}, \\ 0, & \text{otherwise.} \end{cases}$$

Finally for $\psi \in \overline{\Psi}(G)$, $\overline{\Pi}_{\psi} = \pi_{\psi_{np}} \rtimes \overline{\Pi}_{\psi_p}$. We define

$$\Pi_{\psi}^{\Sigma_0} := \Big(\underset{(\rho,a,b) \in \mathrm{Jord}(\psi_{np})}{\times} \mathrm{Sp}(\mathrm{St}(\rho, a), b) \Big) \rtimes \Pi_{\psi_p}^{\Sigma_0},$$

$$\pi_W^{\Sigma_0}(\psi, \varepsilon) := \Big(\underset{(\rho,a,b) \in \mathrm{Jord}(\psi_{np})}{\times} \mathrm{Sp}(\mathrm{St}(\rho, a), b) \Big) \rtimes \pi_W^{\Sigma_0}(\psi_p, \varepsilon),$$

$$\pi_M^{\Sigma_0}(\psi, \varepsilon) := \Big(\underset{(\rho,a,b) \in \mathrm{Jord}(\psi_{np})}{\times} \mathrm{Sp}(\mathrm{St}(\rho, a), b) \Big) \rtimes \pi_M^{\Sigma_0}(\psi_p, \varepsilon)$$

for $\varepsilon \in \widehat{\mathcal{S}_{\psi}^{\Sigma_0}}$. Then we have

$$\pi_M^{\Sigma_0}(\psi, \varepsilon) = \begin{cases} \pi_W^{\Sigma_0}(\psi, \varepsilon \varepsilon_{\psi}^{M/W}), & \text{if } \varepsilon \varepsilon_{\psi}^{M/W} \in \widehat{\mathcal{S}_{\psi}^{\Sigma_0}}, \\ 0, & \text{otherwise.} \end{cases}$$

For $\underline{l}(\rho, A, B, \zeta) \in [0, [(A - B + 1)/2]]$ and $\underline{\eta}(\rho, A, B, \zeta) \in \mathbb{Z}_2$ on $\mathrm{Jord}(\psi_p)$ such that $\varepsilon_{\underline{l},\underline{\eta}} \in \widehat{\mathcal{S}_{\psi^>}^{\Sigma_0}}$, we also define

$$\pi_M^{\Sigma_0}(\psi, \underline{l}, \underline{\eta}) = \Big(\underset{(\rho,a,b) \in \mathrm{Jord}(\psi_{np})}{\times} \mathrm{Sp}(\mathrm{St}(\rho, a), b) \Big) \rtimes \pi_M^{\Sigma_0}(\psi_p, \underline{l}, \underline{\eta}),$$

$$\pi_M(\psi, \underline{l}, \underline{\eta}) = \Big(\underset{(\rho,a,b) \in \mathrm{Jord}(\psi_{np})}{\times} \mathrm{Sp}(\mathrm{St}(\rho, a), b) \Big) \rtimes \pi_M(\psi_p, \underline{l}, \underline{\eta}).$$

Proposition 8.11 ([Mœg06a, Theorem 6]) *For $\psi \in \overline{\Psi}(G)$, $\pi_M^{\Sigma_0}(\psi, \underline{l}, \underline{\eta})$ is irreducible or zero.*

As a consequence of this proposition, $\pi_M(\psi, \underline{l}, \underline{\eta})$ is the irreducible representation of G viewed as $\overline{\mathcal{H}}(G)$-module in the restriction of $\pi_M^{\Sigma_0}(\psi, \underline{l}, \underline{\eta})$ to G if $\pi_M^{\Sigma_0}(\psi, \underline{l}, \underline{\eta}) \neq 0$, and zero otherwise. To summarize, we obtain Mœglin's multiplicity free result for Arthur packets.

Theorem 8.12 (Mœglin)   *For $\psi \in \overline{\Psi}(G)$,*

$$\Pi^{\Sigma_0}(\psi) := \bigoplus_{\varepsilon \in \overline{\mathcal{S}_\psi^{\Sigma_0}}} \pi_W^{\Sigma_0}(\psi, \varepsilon) \quad \left(resp. \ \overline{\Pi}(\psi) := \bigoplus_{\overline{\varepsilon} \in \overline{\mathcal{S}_\psi}} \pi_W(\psi, \overline{\varepsilon}) \right)$$

is a multiplicity-free representation of G^{Σ_0} (resp. $\overline{\mathcal{H}}(G)$-module).

A Compatibility of Endoscopic Transfer with Aubert Involution

In this section, we want to establish the compatibility of (twisted) endoscopic transfer with generalized (twisted) Aubert involution (*cf.* (6.3), (6.7), and (6.9)). We will start by considering the usual (twisted) Aubert involution. Let F be a p-adic field and let G be a quasisplit connected reductive group over F. Let θ be an F-automorphism of G preserving an F-splitting. We denote the space of (resp. twisted) invariant distributions on G by $\widehat{I}(G)$ (resp. $\widehat{I}(G^\theta)$), and denote the space of stable invariant distributions on G by $\widehat{SI}(G)$. Let $\mathcal{P}^\theta$ be the set of θ-stable standard parabolic subgroups of G. Let $G^+ = G \rtimes \langle \theta \rangle$. For any $\pi^+ \in \mathrm{Rep}(G^+)$, we define the θ-twisted Aubert involution as follows:

$$\mathrm{inv}^\theta(\pi^+) = \sum_{P \in \mathcal{P}^\theta} (-1)^{\dim(A_P)_\theta} \mathrm{Ind}_P^G(\mathrm{Jac}_P \pi^+),$$

where A_P is the maximal split central torus of the Levi component M of P. Let H be a twisted endoscopic group of G, and we denote by inv^H the Aubert involution on Grothendieck group of $\mathrm{Rep}(H)$. Then we want to show that the following diagram commutes, where the horizontal maps correspond to the twisted spectral endoscopic transfer:

(A.1)
$$\begin{array}{ccc} \widehat{SI}(H) & \longrightarrow & \widehat{I}(G^\theta) \\ {\scriptstyle \mathrm{inv}^H} \downarrow & & \downarrow {\scriptstyle \mathrm{inv}^\theta} \\ \widehat{SI}(H) & \longrightarrow & \widehat{I}(G^\theta), \end{array}$$

To establish this diagram, we need to know the compatibility of twisted endoscopic transfer with Jacque modules, and we will recall its formulation here following [Xu15, Appendix C].

For simplicity, we will assume there is an embedding

$$\xi \colon {}^L H \longrightarrow {}^L G,$$

and $\xi({}^L H) \subseteq \mathrm{Cent}(s, {}^L G)$ and $\widehat{H} \cong \mathrm{Cent}(s, \widehat{G})^0$ for some semisimple $s \in \widehat{G} \rtimes \widehat{\theta}$. We fix ($\widehat{\theta}$-stable) Γ_F-splittings $(\mathcal{B}_H, \mathcal{T}_H, \{\mathcal{X}_{\alpha_H}\})$ and $(\mathcal{B}_G, \mathcal{T}_G, \{\mathcal{X}_\alpha\})$ for $\widehat{H}$ and $\widehat{G}$ respectively. By taking certain $\widehat{G}$-conjugate of ξ, we can assume $s \in \mathcal{T}_G \rtimes \widehat{\theta}$ and $\xi(\mathcal{T}_H) = (\mathcal{T}_G^{\widehat{\theta}})^0$ and $\xi(\mathcal{B}_H) \subseteq \mathcal{B}_G$. Let $W_H = W(\widehat{H}, \mathcal{T}_H)$ and $W_{G^\theta} = W(\widehat{G}, \mathcal{T}_G)^{\widehat{\theta}}$; then W_H can be viewed as a subgroup of W_{G^θ}. We also view ${}^L H$ as a subgroup of ${}^L G$ through ξ. For $P = MN \in \mathcal{P}^\theta$ with standard embedding ${}^L P \hookrightarrow {}^L G$, there exists a torus $S \subseteq (\mathcal{T}_G^{\widehat{\theta}})^0$ such that ${}^L M = \mathrm{Cent}(S, {}^L G)$. Let $W_{M^\theta} = W(\widehat{M}, \mathcal{T}_G)^{\widehat{\theta}}$. We define

$$W_{G^\theta}(H, M) := \{w \in W_{G^\theta} \mid \mathrm{Cent}(w(S), {}^L H) \to W_F \text{ surjective }\}.$$

For any $w \in W_{G^\theta}(H, M)$, let us take $g \in \widehat{G}$ such that $\mathrm{Int}(g)$ induces w. Since $\mathrm{Cent}(w(S), {}^L H) \to W_F$ is surjective, $g^L P g^{-1} \cap {}^L H$ defines a parabolic subgroup of ${}^L H$ with Levi component $g^L M g^{-1} \cap {}^L H$. So we can choose a standard parabolic subgroup $P'_w = M'_w N'_w$ of H with standard embedding ${}^L P'_w \hookrightarrow {}^L H$ such that ${}^L P'_w$ (resp. ${}^L M'_w$) is $\widehat{H}$-conjugate to $g^L P g^{-1} \cap {}^L H$ (resp. $g^L M g^{-1} \cap {}^L H$). In particular, M'_w can be viewed as a twisted endoscopic group of M, and the embedding $\xi_{M'_w} : {}^L M'_w \to {}^L M$ is given by the following diagram:

$$
\begin{array}{ccccccc}
{}^L P'_w & \longleftarrow & {}^L M'_w & \xrightarrow{\ \xi_{M'_w}\ } & {}^L M & \longrightarrow & {}^L P \\
\Big\uparrow & & \Big\downarrow & & & & \Big\uparrow \\
{}^L H & \xrightarrow{\ \mathrm{Int}(h)\ } & {}^L H & \xrightarrow{\ \xi\ } & {}^L G & \xleftarrow{\ \mathrm{Int}(g)\ } & {}^L G,
\end{array}
$$

where $h \in \widehat{H}$ induces an element in W_H. Note that the choice of h is unique up to $\widehat{M}'_w$-conjugation, and so is $\xi_{M'_w}$. If we change g to $h'gm$, where $h' \in \widehat{H}$ induces an element in W_H and $m \in \widehat{M}$ induces an element in W_{M^θ}, then we still get P'_w, but $\xi_{M'_w}$ changes to $\mathrm{Int}(m^{-1}) \circ \xi_{M'_w}$ up to $\widehat{M}'_w$-conjugation. To summarize, for any element w in

$$
W_H \backslash W_{G^\theta}(H, M) / W_{M^\theta}
$$

we can associate a standard parabolic subgroup $P'_w = M'_w N'_w$ of H and a $\widehat{M}$-conjugacy class of embeddings $\xi_{M'_w} : {}^L M'_w \to {}^L M$. Then the following diagram commutes, where the sum is over $W_H \backslash W_{G^\theta}(H, M) / W_{M^\theta}$ and the horizontal maps correspond to the twisted spectral endoscopic transfers with respect to ξ on the top and $\xi_{M'_w}$ on the bottom:

(A.2)
$$
\begin{array}{ccc}
\widehat{SI}(H) & \longrightarrow & \widehat{I}(G^\theta) \\
{\scriptstyle \oplus_w \mathrm{Jac}_{P'_w}} \Big\downarrow & & \Big\downarrow {\scriptstyle \mathrm{Jac}_P} \\
\oplus_w \widehat{SI}(M'_w) & \longrightarrow & \widehat{I}(M^\theta).
\end{array}
$$

Let us denote the twisted spectral endoscopic transfer from H to G by $\mathrm{Tran}_H^{G^\theta}$, and the twisted spectral endoscopic transfer from M'_w to M by $\mathrm{Tran}_{M'_w}^{M^\theta}$. Then we can translate the diagram (A.2) into the following identity. For $\Theta^H \in \widehat{SI}(H)$,

$$
\sum_w \mathrm{Tran}_{M'_w}^{M^\theta} \mathrm{Jac}_{P'_w} \Theta^H = \mathrm{Jac}_P \mathrm{Tran}_H^{G^\theta} \Theta^H.
$$

It follows that

$$
\sum_w \mathrm{Ind}_P^G \big(\mathrm{Tran}_{M'_w}^{M^\theta} \mathrm{Jac}_{P'_w} \Theta^H \big) = \mathrm{Ind}_P^G \big(\mathrm{Jac}_P \mathrm{Tran}_H^{G^\theta} \Theta^H \big).
$$

By the compatibility of twisted endoscopic transfer with parabolic induction,

$$
\mathrm{Ind}_P^G \mathrm{Tran}_{M'_w}^{M^\theta} \big(\mathrm{Jac}_{P'_w} \Theta^H \big) = \mathrm{Tran}_H^{G^\theta} \mathrm{Ind}_{P'_w}^H \big(\mathrm{Jac}_{P'_w} \Theta^H \big).
$$

So

$$
\mathrm{Tran}_H^{G^\theta} \Big(\sum_w \mathrm{Ind}_{P'_w}^H \mathrm{Jac}_{P'_w} \Theta^H \Big) = \mathrm{Ind}_P^G \mathrm{Jac}_P \big(\mathrm{Tran}_H^{G^\theta} \Theta^H \big).
$$

We can multiply both sides by $(-1)^{\dim(A_P)_\theta}$, and then sum over $P \in \mathcal{P}^\theta$,

$$\mathrm{Tran}_H^{G^\theta} \Big(\sum_{P \in \mathcal{P}^\theta} (-1)^{\dim(A_P)_\theta} \sum_w \mathrm{Ind}_{P'_w}^H \mathrm{Jac}_{P'_w} \Theta^H \Big) = \mathrm{inv}^\theta \Big(\mathrm{Tran}_H^{G^\theta} \Theta^H \Big).$$

To establish the diagram (A.1), it is enough to show

$$\sum_{P \in \mathcal{P}^\theta} (-1)^{\dim(A_P)_\theta} \sum_w \mathrm{Ind}_{P'_w}^H \mathrm{Jac}_{P'_w} \Theta^H = \mathrm{inv}^H \Theta^H.$$

By the definition,

$$\mathrm{inv}^H \Theta^H = \sum_{P' \in \mathcal{P}^H} (-1)^{\dim A_{P'}} \mathrm{Ind}_{P'}^H \mathrm{Jac}_{P'} \Theta^H,$$

where $\mathcal{P}^H$ denotes the set of standard parabolic subgroups of H. So it suffices to prove the following proposition.

Proposition A.1 *For any $P' = M'N' \in \mathcal{P}^H$,*

$$(A.3) \qquad\qquad \sum_{P \in \mathcal{P}^\theta} (-1)^{\dim(A_P)_\theta} a_{M',H,M} = (-1)^{\dim A_{P'}},$$

where

$$a_{M',H,M} := \sharp\{w \in W_H \backslash W_{G^\theta}(H,M)/W_{M^\theta} | P'_w = P'\}.$$

Hiraga proved this proposition in the non-twisted case (see [Hir04]), and we will extend his arguments to prove the twisted case here. First we need to introduce some more notation.

Let $A^{\widehat{G},\widehat{\theta}}$ be the identity component of Γ_F-invariant elements in $(\mathcal{T}_G^{\widehat{\theta}})^0$, and let $A^{\widehat{H}}$ be the identity component of Γ_F-invariant elements in $\mathcal{T}_H$. By the choice of $\widehat{G}$-conjugate of ξ, we can further assume that $\xi(A^{\widehat{H}}) \subseteq A^{\widehat{G},\widehat{\theta}}$ and there is a θ-stable standard Levi subgroup M^H of G such that ${}^L M^H = \mathrm{Cent}(A^{\widehat{H}}, {}^L G)$.

For any θ-stable standard Levi subgroup M of G, we denote by $R_{\mathrm{res}}(\widehat{M})$ the root system (not necessarily reduced) obtained by restriction from the root system $R(\widehat{M}, \mathcal{T}_G)$ to $(\mathcal{T}_G^{\widehat{\theta}})^0$, and we denote the set of simple roots in $R_{\mathrm{res}}(\widehat{M})$ by $\Delta_{\mathrm{res}}(\widehat{M})$. Let $R_{\mathrm{res}}^\pm(\widehat{M})$ be the set of positive (negative) roots. We write $r_{\mathrm{res}}(M)$ for the number of Γ_F-orbits in $\Delta_{\mathrm{res}}(\widehat{M})$. Note that $\mathcal{P}^\theta$ is in bijection with the Γ_F-stable subsets of $\Delta_{\mathrm{res}}(\widehat{G})$.

For any standard Levi subgroup M' of H, we denote by $R(\widehat{M'})$ the root system $R(\widehat{M'}, \mathcal{T}_H)$ and we denote the set of simple roots in $R(\widehat{M'})$ by $\Delta(\widehat{M'})$. Let $R^\pm(\widehat{M'})$ be the set of positive (negative) roots. We write $r(M')$ for the number of Γ_F-orbits in $\Delta(\widehat{M'})$. Note that $\mathcal{P}^H$ is in bijection with the Γ_F-stable subsets of $\Delta(\widehat{H})$. It is easy to see that $R^\pm(\widehat{H}) \subseteq R_{\mathrm{res}}^\pm(\widehat{G})$.

If we multiply both sides of (A.3) by $(-1)^{\dim A^{\widehat{G},\widehat{\theta}}}$, then we will get

$$(A.4) \qquad\qquad \sum_{P \in \mathcal{P}^\theta} (-1)^{r_{\mathrm{res}}(M)} a_{M',H,M} = (-1)^{r_{\mathrm{res}}(M^H)+r(M')}.$$

We will break the proof of this identity into four steps.

Step 1: We fix a θ-stable standard Levi subgroup M of G. Let

$$D_{M^\theta} = \{w \in W_{G^\theta} | w^{-1}(\Delta_{\mathrm{res}}(\widehat{M})) \subseteq R^+_{\mathrm{res}}(\widehat{G})\},$$

$$D_H = \{w \in W_{G^\theta} | w^{-1}(\Delta(\widehat{H})) \subseteq R^+_{\mathrm{res}}(\widehat{G})\}.$$

We would like to show that $D_{H,M^\theta} := D_{M^\theta}^{-1} \cap D_H$ is a set of representatives of $W_H \backslash W_{G^\theta} / W_{M^\theta}$.

Lemma A.2 *D_H (resp. D_{M^θ}) is a set of representatives of $W_H \backslash W_{G^\theta}$ (resp. $W_{M^\theta} \backslash W_{G^\theta}$).*

Proof For any $w \in W_{G^\theta}$, let $\widehat{B}_H := \widehat{H} \cap w(\mathcal{B}_G)$. Then $\widehat{B}_H$ is a Borel subgroup of $\widehat{H}$. So there exists a unique $w_H \in W_H$ such that $w_H(\widehat{B}_H) = \mathcal{B}_H$. It follows that $\mathcal{B}_H = w_H(\widehat{H} \cap w(\mathcal{B}_G)) = \widehat{H} \cap w_H w(\mathcal{B}_G)$, and hence $w_H w \in D_H$. By the uniqueness of w_H, we see D_H is a set of representatives of $W_H \backslash W_{G^\theta}$.

The proof for $W_{M^\theta} \backslash W_{G^\theta}$ is similar. One just needs to notice $W_{G^\theta} \cong W(\widehat{G}^1, (\mathcal{T}_G^{\widehat{\theta}})^0)$ and $W_{M^\theta} \cong W(\widehat{M}^1, (\mathcal{T}_G^{\widehat{\theta}})^0)$, where $\widehat{G}^1$ (resp. $\widehat{M}^1$) is the identity component of $\widehat{\theta}$-invariant elements in $\widehat{G}$ (resp. $\widehat{M}$). ∎

For $w \in W_{G^\theta}$, we define

$$l_{M^\theta}(w) = \sharp\{\alpha \in R^+_{\mathrm{res}}(\widehat{M}) | w\alpha \in R^-_{\mathrm{res}}(\widehat{G})\}$$

$$l_H(w) = \sharp\{\alpha \in R^+(\widehat{H}) | w\alpha \in R^-_{\mathrm{res}}(\widehat{G})\}.$$

Lemma A.3 *For any $w \in W_{G^\theta}$, $D_{H,M^\theta} \cap W_H w W_{M^\theta} \neq \varnothing$.*

Proof Since D_H is a set of representatives of $W_H \backslash W_{G^\theta}$, we can choose $w_0 \in W_H w W_{M^\theta}$ such that $w_0 \in D_H$. Note that $w_0^{-1} \in D_{M^\theta}$ if and only if $l_{M^\theta}(w_0) = 0$. So we can make an induction on $l_{M^\theta}(w_0)$. Suppose $l_{M^\theta}(w_0) > 0$; then there exists $\alpha \in \Delta_{\mathrm{res}}(\widehat{M})$ such that $w_0 \alpha \in R^-_{\mathrm{res}}(\widehat{G})$. We claim

$$l_{M^\theta}(w_0 s_\alpha) < l_{M^\theta}(w_0),$$

where s_α is corresponding the simple reflection. To see this, note that

$$s_\alpha(R^+_{\mathrm{res}}(\widehat{M}) - \mathbb{Z}_+\alpha) = R^+_{\mathrm{res}}(\widehat{M}) - \mathbb{Z}_+\alpha \quad \text{and} \quad w_0\alpha \in R^-_{\mathrm{res}}(\widehat{G}).$$

So

$$l_{M^\theta}(w_0 s_\alpha) = \sharp\{\alpha' \in R^+_{\mathrm{res}}(\widehat{M}) - \mathbb{Z}_+\alpha | w_0 s_\alpha \alpha' \in R^-_{\mathrm{res}}(\widehat{G})\}$$

$$= \sharp\{\alpha'' \in R^+_{\mathrm{res}}(\widehat{M}) - \mathbb{Z}_+\alpha | w_0 \alpha'' \in R^-_{\mathrm{res}}(\widehat{G})\}.$$

Then

$$l_{M^\theta}(w_0) = l_{M^\theta}(w_0 s_\alpha) + |\mathbb{Z}_+\alpha \cap R^+_{\mathrm{res}}(\widehat{M})| > l_{M^\theta}(w_0 s_\alpha).$$

We still need to show $w_0 s_\alpha \in D_H$. For that, let us consider $(w_0 s_\alpha)^{-1}(\Delta(\widehat{H})) = s_\alpha w_0^{-1}(\Delta(\widehat{H}))$. Since

$$s_\alpha(R^+_{\mathrm{res}}(\widehat{G}) - \mathbb{Z}_+\alpha) = R^+_{\mathrm{res}}(\widehat{G}) - \mathbb{Z}_+\alpha,$$

we only need to show $w_0^{-1}(\Delta(\widehat{H})) \cap \mathbb{Z}_+\alpha = \varnothing$. This is guaranteed by the fact that $w_0\alpha \in R^-_{\mathrm{res}}(\widehat{G})$. ∎

Now we have the following proposition.

Proposition A.4   *D_{H,M^θ} is a set of representatives of $W_H \backslash W_{G^\theta} / W_{M^\theta}$.*

Proof   In view of Lemma A.3, we just need to show $W_H w W_{M^\theta}$ contains a unique element in D_{H,M^θ} for any $w \in W_{G^\theta}$. Suppose $w_0, w_0' \in D_{H,M^\theta} \cap W_H w W_{M^\theta}$; then we can assume that $w_0' = w_H w_0 w_{M^\theta}$ for $w_H \in W_H$ and $w_{M^\theta} \in W_{M^\theta}$. First we want to show that w_H can be chosen to be trivial. Note that $w_H = 1$ if and only if $l_H(w_H^{-1}) = 0$. Suppose $l_H(w_H^{-1}) > 0$; then there exists $\alpha \in \Delta(\widehat{H})$ such that $w_H^{-1}(\alpha) \in R^-(\widehat{H})$. Since $w_0, w_0' \in D_H$, we have $\beta = w_0^{-1} w_H^{-1} \alpha \in R_{\mathrm{res}}^-(\widehat{G})$ and $w_{M^\theta}^{-1} \beta = (w_0')^{-1} \alpha \in R_{\mathrm{res}}^+(\widehat{G})$. So $\beta \in R_{\mathrm{res}}^-(\widehat{M})$. Hence

$$w_0' = w_H w_0 w_{M^\theta} = (s_\alpha \cdot s_\alpha) w_H w_0 w_{M^\theta} = s_\alpha w_H s_{w_H^{-1}\alpha} w_0 w_{M^\theta}$$
$$= (s_\alpha w_H) w_0 (s_{w_0^{-1} w_H^{-1}\alpha} w_{M^\theta}) = (s_\alpha w_H) w_0 (s_\beta w_{M^\theta}).$$

As in the proof of Lemma A.3, one can show $l_H(w_H^{-1} s_\alpha) < l_H(w_H^{-1})$. So by induction on $l_H(w_H^{-1})$, we can assume that $w_0' = w_0 w_{M^\theta}$. Since $w_0, w_0' \in D_{M^\theta}^{-1}$, we must have $w_{M^\theta} = 1$, and hence $w_0' = w_0$.   ∎

Next we would like to describe $D_{H,M^\theta} \cap W_{G^\theta}(H, M)$, which is a set of representatives of $W_H \backslash W_{G^\theta}(H, M) / W_{M^\theta}$. Since $^L M = \mathrm{Cent}((A_{\widehat{M}}^{\widehat{\theta}})^0, {}^L G)$, $w \in W_{G^\theta}(H, M)$ is characterized by the condition that

$$\mathrm{Cent}\big(w(A_{\widehat{M}}^{\widehat{\theta}})^0, {}^L H\big) \longrightarrow W_F$$

is surjective. For $w \in D_H$, the above condition is equivalent to requiring $w(A_{\widehat{M}}^{\widehat{\theta}})^0 \subseteq A^{\widehat{H}}$. So let us define

$$\widetilde{D}_{M^\theta} = \left\{ w \in D_{M^\theta} \mid w^{-1}(A_{\widehat{M}}^{\widehat{\theta}})^0 \subseteq A^{\widehat{H}} \right\}.$$

Then $\widetilde{D}_{H,M^\theta} := \widetilde{D}_{M^\theta}^{-1} \cap D_H$ is equal to $D_{H,M^\theta} \cap W_{G^\theta}(H, M)$.

For $w \in \widetilde{D}_{H,M^\theta}$, it is easy to see that $\widehat{M_w'} = w(\widehat{M}) \cap \widehat{H}$. So we would like to define $\widehat{M_w'} := w(\widehat{M}) \cap \widehat{H}$ for all $w \in D_{H,M^\theta}$, and note that M_w' is only a standard Levi subgroup of H over $\overline{F}$ in this case. For any standard Levi subgroup M' of H over $\overline{F}$, let us define

$$D_{M'} = \left\{ w \in W_{G^\theta} \mid w^{-1}(\Delta(\widehat{M'})) \subseteq R_{\mathrm{res}}^+(\widehat{G}) \right\}.$$

We also define

$$D_{M',H,M^\theta} := \{ w \in D_{H,M^\theta} \mid M_w' = M' \}, \quad \widetilde{D}_{M',H,M^\theta} := \{ w \in \widetilde{D}_{H,M^\theta} \mid M_w' = M' \}.$$

It is clear that $\widetilde{D}_{M',H,M^\theta} \neq \varnothing$ only when M' is defined over F.

Step 2: We again fix a θ-stable standard Levi subgroup M of G, and we will take M' to be standard Levi subgroups of H over $\overline{F}$ (if not specified). Let

$$\widetilde{\xi}_{M^\theta} = \sum_{w \in \widetilde{D}_{M^\theta}} w \quad \text{and} \quad \xi_{M'} = \sum_{w \in D_{M'}} w.$$

For any $\xi = \sum_{w \in W_{G^\theta}} a_w w$, let us write

$$[\xi]_H = \sum_{\substack{w \in W_{G^\theta} \\ w(A^{\widehat{H}}) = A^{\widehat{H}}}} a_w w.$$

Then we want to show

$$(A.5) \qquad [\xi_H \widetilde{\xi}_{M^\theta}]_H = \sum_{P' \in \mathcal{P}^H} a_{M',H,M^\theta} [\xi_{M'}]_H.$$

For any $x \in W_{G^\theta}$ satisfying $x(A^{\widehat{H}}) = A^{\widehat{H}}$, its coefficient in $[\xi_H \widetilde{\xi}_{M^\theta}]_H$ is given by the number of pairs $(d_H, d_{M^\theta}) \in D_H \times \widetilde{D}_{M^\theta}$ such that $x = d_H d_{M^\theta}$; in other words, we need to count $x\widetilde{D}_{M^\theta}^{-1} \cap D_H$.

By Proposition A.4, it is enough to count

$$(A.6) \qquad (x\widetilde{D}_{M^\theta}^{-1} \cap D_H) \cap W_H w W_{M^\theta}$$

for all $w \in D_{H,M^\theta}$. Let

$$(A.7) \qquad w^{-1}x = w_{M^\theta}(x,w) \cdot d_{M^\theta}(x,w)$$

for $w_{M^\theta}(x,w) \in W_{M^\theta}$ and $d_{M^\theta}(x,w) \in D_{M^\theta}$. Note that this decomposition makes sense for all $x \in W_{G^\theta}$.

Lemma A.5 *Suppose $x \in W_{G^\theta}$ satisfies $x(A^{\widehat{H}}) = A^{\widehat{H}}$ and $w \in D_{H,M^\theta}$; then $d_{M^\theta}(x,w) \in \widetilde{D}_{M^\theta}$ if and only if $w \in \widetilde{D}_{H,M^\theta}$.*

Proof Since $xd_{M^\theta}(x,w)^{-1}(A_{\widehat{M}}^{\widehat{\theta}})^0 = ww_{M^\theta}(x,w)(A_{\widehat{M}}^{\widehat{\theta}})^0 = w(A_{\widehat{M}}^{\widehat{\theta}})^0$, the lemma is clear. ∎

Before we give the result for (A.6), we would like to consider a slightly more general situation.

Proposition A.6 *For $x \in W_{G^\theta}$ and $w \in D_{H,M^\theta}$,*

$$(xD_{M^\theta}^{-1} \cap D_H) \cap W_H w W_{M^\theta} = \begin{cases} \{xd_{M^\theta}(x,w)^{-1}\}, & \text{if } xd_{M^\theta}(x,w)^{-1} \in D_H, \\ \varnothing, & \text{otherwise.} \end{cases}$$

To prove this proposition, we need the following lemma.

Lemma A.7 *Suppose $w \in D_{H,M^\theta}$, and every element in $W_H w W_{M^\theta}$ has a unique expression as $w_H w w_{M^\theta}$ for $w_{M^\theta} \in W_{M^\theta}$ and $w_H \in D_{M'}^{-1} \cap W_H$, where $M' = M'_w$. Moreover,*

$$l_H(w_{M^\theta}^{-1} w^{-1} w_H^{-1}) \geqslant l_H(w_H^{-1}).$$

Proof As in Lemma A.2, one can show that $D_{M'}^{-1} \cap W_H$ is a set of representatives of $W_H/W_{M'}$. Then

$$w_H w w_{M^\theta} = (d_{M'}^{-1} w_{M'}) w w_{M^\theta} = d_{M'}^{-1} w(w^{-1} w_{M'} w) w_{M^\theta}$$

for $d_{M'} \in D_{M'}$ and $w_{M'} \in W_{M'}$. Since $W_H \cap wW_{M^\theta}w^{-1} = W_{M'}$, we have $w^{-1}w_{M'}w \in W_{M^\theta}$. This proves the existence of the expression. To see the uniqueness, we can assume that $w_H w w_{M^\theta} = w'_H w w'_{M^\theta}$ in both the desired expressions. Then $w_H w w_{M^\theta}(w'_{M^\theta})^{-1} = w'_H w$. So we can assume instead that $w_H w w_{M^\theta} = w'_H w$. It follows $ww_{M^\theta} = w_H^{-1} w'_H w \in W_H w$. So $ww_{M^\theta}w^{-1} \in W_H$. Hence, $w_{M'} := ww_{M^\theta}w^{-1} \in W_{M'}$. Now we get $w_H w_{M'} = w'_H$. Since $w_H, w'_H \in D_{M'}^{-1} \cap W_H$, we must have $w_{M'} = 1$. Then $w_H = w'_H$ and $w_{M^\theta} = 1$. ∎

Next we want to show

$$l_H(w_{M^\theta}^{-1} w^{-1} w_H^{-1}) \geqslant l_H(w_H^{-1})$$

for $w_{M^\theta} \in W_{M^\theta}$ and $w_H \in D_{M'}^{-1} \cap W_H$. Note that

$$R^+(\widehat{H}) = \left(R^+(\widehat{H}) - w_H(R^+(\widehat{M'})) \right) \bigsqcup w_H(R^+(\widehat{M'})).$$

Then

$$w_H^{-1}(R^+(\widehat{H})) = \left(w_H^{-1}(R^+(\widehat{H})) - R^+(\widehat{M'}) \right) \bigsqcup R^+(\widehat{M'}).$$

We claim that $\alpha \in w_H^{-1}(R^+(\widehat{H})) - R^+(\widehat{M'})$ is positive if and only if $w_{M^\theta}^{-1} w^{-1}\alpha$ is positive. It is clear that for $\alpha \in R(\widehat{H})$, α is positive if and only if $w^{-1}\alpha$ is positive. So we only need to show $w^{-1}\alpha \notin R_{\mathrm{res}}(\widehat{M})$ for $\alpha \in w_H^{-1}(R^+(\widehat{H})) - R^+(\widehat{M'})$, or equivalently, $\alpha \notin w(R_{\mathrm{res}}(\widehat{M}))$. To see this, we consider

$$R^+(\widehat{H}) \cap w_H w(R_{\mathrm{res}}(\widehat{M})) = R^+(\widehat{H}) \cap w_H\left(R(\widehat{H}) \cap w(R_{\mathrm{res}}(\widehat{M})) \right)$$
$$= R^+(\widehat{H}) \cap w_H(R(\widehat{M'})).$$

Since $w_H \in D_{M'}^{-1} \cap W_H$, $w_H(R^\pm(\widehat{M'})) \subseteq R^\pm(\widehat{H})$, and we have

$$R^+(\widehat{H}) \cap w_H\left(R(\widehat{M'}) \right) = w_H\left(R^+(\widehat{M'}) \right).$$

Therefore,

$$R^+(\widehat{H}) \cap w_H w\left(R_{\mathrm{res}}(\widehat{M}) \right) = w_H\left(R^+(\widehat{M'}) \right).$$

Multiplying both sides by w_H^{-1}, and we get

$$w_H^{-1}\left(R^+(\widehat{H}) \right) \cap w\left(R_{\mathrm{res}}(\widehat{M}) \right) = R^+(\widehat{M'}).$$

From this identity, one can easily see $\alpha \notin w(R_{\mathrm{res}}(\widehat{M}))$ for $\alpha \in w_H^{-1}(R^+(\widehat{H})) - R^+(\widehat{M'})$. This shows our claim. Consequently, we have

$$l_H(w_{M^\theta}^{-1} w^{-1} w_H^{-1}) = l_H(w_H^{-1}) + \sharp\left\{ \alpha \in R^+(\widehat{M'}) \mid w_{M^\theta}^{-1} w^{-1}\alpha \in R_{\mathrm{res}}^-(\widehat{G}) \right\} \geq l_H(w_H^{-1}). \quad \blacksquare$$

Corollary A.8 *For $w \in D_{H,M^\theta}$, $D_H \cap W_H w W_{M^\theta} \subseteq w W_{M^\theta}$.*

Proof For $w_H w w_{M^\theta} \in D_H \cap W_H w W_{M^\theta}$, we can assume that $w_H \in D_{M'}^{-1} \cap W_H$ by Lemma A.7. Then $0 = l_H(w_{M^\theta}^{-1} w^{-1} w_H^{-1}) \geqslant l_H(w_H^{-1})$. So $l_H(w_H^{-1}) = 0$, and hence $w_H = 1$. $\blacksquare$

Now we will prove Proposition A.6.

Proof For $x \in W_{G^\theta}$ and

$$y \in (x D_{M^\theta}^{-1} \cap D_H) \cap W_H w W_{M^\theta},$$

we can assume $y = w w_{M^\theta}$ for $w_{M^\theta} \in W_{M^\theta}$ by Corollary A.8. There exists $d_{M^\theta} \in D_{M^\theta}$ such that

$$x d_{M^\theta}^{-1} = y = w w_{M^\theta}.$$

So $w^{-1} x = w_{M^\theta} d_{M^\theta}$. Compared with (A.7), we get $d_{M^\theta} = d_{M^\theta}(x, w)$ and $w_{M^\theta} = w_{M^\theta}(x, w)$. Then $y = x d_{M^\theta}(x, w)^{-1} \in D_H$. On the other hand, suppose $x d_{M^\theta}(x, w)^{-1} \in D_H$; it is clear that $x d_{M^\theta}(x, w)^{-1} \in x D_{M^\theta}^{-1} \cap D_H$. Moreover,

$$x d_{M^\theta}(x, w)^{-1} = w w_{M^\theta}(x, w) \in W_H w W_{M^\theta}.$$

So $xd_{M^\theta}(x,w)^{-1} \in (xD_{M^\theta}^{-1} \cap D_H) \cap W_H w W_{M^\theta}$. This finishes the proof. ∎

Since there is a decomposition

$$D_{H,M^\theta} = \bigsqcup_{P'} D_{M',H,M^\theta},$$

where the sum is over all standard parabolic subgroup P' of H over $\overline{F}$, we would like to refine Proposition A.6 by restricting to D_{M',H,M^θ}.

Proposition A.9 *For $x \in W_{G^\theta}$ and $w \in D_{M',H,M^\theta}$, $(xD_{M^\theta}^{-1} \cap D_H) \cap W_H w W_{M^\theta} \neq \varnothing$ if and only if $x \in D_{M'}$.*

Proof By Proposition A.6, it is enough to show that $xd_{M^\theta}(x,w)^{-1} \in D_H$ if and only if $x \in D_{M'}$. Since

$$R^+(\widehat{H}) \cap w(R_{\text{res}}(\widehat{M})) = R^+(\widehat{M'})$$

and $xd_{M^\theta}(x,w)^{-1} = ww_{M^\theta}(x,w)$, we have

$$d_{M^\theta}(x,w)x^{-1}(R^+(\widehat{M'})) = d_{M^\theta}(x,w)x^{-1}(R^+(\widehat{H})) \cap w_{M^\theta}(x,w)^{-1}w^{-1}w(R_{\text{res}}(\widehat{M}))$$

$$= d_{M^\theta}(x,w)x^{-1}(R^+(\widehat{H})) \cap R_{\text{res}}(\widehat{M}).$$

If $xd_{M^\theta}(x,w)^{-1} \in D_H$, then $d_{M^\theta}(x,w)x^{-1}(R^+(\widehat{H})) \subseteq R_{\text{res}}^+(\widehat{G})$. So

$$d_{M^\theta}(x,w)x^{-1}(R^+(\widehat{M'})) \subseteq R_{\text{res}}^+(\widehat{M}).$$

Then

$$x^{-1}(R^+(\widehat{M'})) \subseteq d_{M^\theta}(x,w)^{-1}(R_{\text{res}}^+(\widehat{M})) \subseteq R_{\text{res}}^+(\widehat{G}).$$

This means $x \in D_{M'}$.

Conversely, suppose $x \in D_{M'}$; then $x^{-1}(R^+(\widehat{M'})) \subseteq R_{\text{res}}^+(\widehat{G})$. We can rewrite it as

$$d_{M^\theta}(x,w)^{-1}(d_{M^\theta}(x,w)x^{-1})(R^+(\widehat{M'})) \subseteq R_{\text{res}}^+(\widehat{G}).$$

Since $d_{M^\theta}(x,w)x^{-1}(R^+(\widehat{M'})) = w_{M^\theta}(x,w)^{-1}w^{-1}(R^+(\widehat{M'})) \subseteq R_{\text{res}}(\widehat{M})$, we must have

$$d_{M^\theta}(x,w)x^{-1}(R^+(\widehat{M'})) \subseteq R_{\text{res}}^+(\widehat{M}).$$

So it is enough to consider

$$d_{M^\theta}(x,w)x^{-1}(R^+(\widehat{H}) - R^+(\widehat{M'}))$$

$$= w_{M^\theta}(x,w)^{-1}w^{-1}(R^+(\widehat{H}) - R^+(\widehat{M'}))$$

$$= w_{M^\theta}(x,w)^{-1}\big(w^{-1}(R^+(\widehat{H})) - w^{-1}(R^+(\widehat{M'}))\big)$$

$$= w_{M^\theta}(x,w)^{-1}\big(w^{-1}(R^+(\widehat{H})) - w^{-1}(R^+(\widehat{H}) \cap w(R_{\text{res}}(\widehat{M})))\big)$$

$$= w_{M^\theta}(x,w)^{-1}\big(w^{-1}(R^+(\widehat{H})) - w^{-1}(R^+(\widehat{H})) \cap R_{\text{res}}(\widehat{M})\big)$$

$$= w_{M^\theta}(x,w)^{-1}\big(w^{-1}(R^+(\widehat{H})) - R_{\text{res}}(\widehat{M})\big).$$

Since $\alpha \in w^{-1}(R^+(\widehat{H})) - R_{\text{res}}(\widehat{M})$ is positive and not in $R_{\text{res}}(\widehat{M})$, $w_{M^\theta}(x,w)^{-1}\alpha$ is also positive. Therefore,

$$d_{M^\theta}(x,w)x^{-1}(R^+(\widehat{H}) - R^+(\widehat{M'})) \subseteq R_{\text{res}}^+(\widehat{G}).$$

This implies $xd_{M^\theta}(x,w)^{-1} \in D_H$. ∎

Next, we will modify Propositions A.6 and A.9 for counting (A.6).

Proposition A.10 *For $x \in W_{G^\theta}$ satisfying $x(A^{\widehat{H}}) = A^{\widehat{H}}$ and $w \in D_{H,M^\theta}$,*

$$(x\widetilde{D}_{M^\theta}^{-1} \cap D_H) \cap W_H w W_{M^\theta} =$$
$$\begin{cases} \{xd_{M^\theta}(x,w)^{-1}\}, & \text{if } w \in \widetilde{D}_{H,M^\theta} \text{ and } xd_{M^\theta}(x,w)^{-1} \in D_H, \\ \varnothing, & \text{otherwise.} \end{cases}$$

Proof By Proposition A.6,

$$(xD_{M^\theta}^{-1} \cap D_H) \cap W_H w W_{M^\theta} = \begin{cases} \{xd_{M^\theta}(x,w)^{-1}\}, & \text{if } xd_{M^\theta}(x,w)^{-1} \in D_H, \\ \varnothing, & \text{otherwise.} \end{cases}$$

So $(x\widetilde{D}_{M^\theta}^{-1} \cap D_H) \cap W_H w W_{M^\theta} \neq \varnothing$ if and only if $xd_{M^\theta}(x,w)^{-1} \in D_H$ and $d_{M^\theta}(x,w) \in \widetilde{D}_{M^\theta}$. By Lemma A.5, this is equivalent to requiring $xd_{M^\theta}(x,w)^{-1} \in D_H$ and $w \in \widetilde{D}_{H,M^\theta}$. ∎

As a consequence, we can restrict ourselves to the set $\widetilde{D}_{H,M^\theta}$ when counting (A.6). Since

$$(A.8) \qquad\qquad \widetilde{D}_{H,M^\theta} = \bigsqcup_{P' \in \mathcal{P}^H} \widetilde{D}_{M',H,M^\theta},$$

we can further restrict to each $\widetilde{D}_{M',H,M^\theta}$.

Proposition A.11 *For $x \in W_{G^\theta}$ satisfying $x(A^{\widehat{H}}) = A^{\widehat{H}}$ and $w \in \widetilde{D}_{M',H,M^\theta}$,*

$$(x\widetilde{D}_{M^\theta}^{-1} \cap D_H) \cap W_H w W_{M^\theta} \neq \varnothing$$

if and only if $x \in D_{M'}$.

Proof By definition, $\widetilde{D}_{M',H,M^\theta} \subseteq D_{M',H,M^\theta}$. In view of Proposition A.9, it suffices to show that for $x \in D_{M'}$,

$$(x\widetilde{D}_{M^\theta}^{-1} \cap D_H) \cap W_H w W_{M^\theta} \neq \varnothing.$$

Since in this case

$$(xD_{M^\theta}^{-1} \cap D_H) \cap W_H w W_{M^\theta} \neq \varnothing,$$

we have $xd_{M^\theta}(x,w)^{-1} \in D_H$ by Proposition A.6. Then the result follows immediately from Proposition A.10. ∎

Corollary A.12

$$[\xi_H \widetilde{\xi}_{M^\theta}]_H = \sum_{P' \in \mathcal{P}^H} a_{M',H,M^\theta} [\xi_{M'}]_H.$$

Proof Since $a_{M',H,M^\theta} = |\widetilde{D}_{M',H,M^\theta}|$, this identity is an easy consequence of (A.8) and Proposition A.11. ∎

Step 3: In this step, we would like to establish the following two identities:

$$(A.9) \qquad \sum_{P \in \mathcal{P}^\theta} (-1)^{r_{\mathrm{res}}(M)} \widetilde{\xi}_{M^\theta} = (-1)^{r_{\mathrm{res}}(M^H)} w_-^G w_-^{M^H},$$

$$(A.10) \qquad \sum_{P' \in \mathcal{P}^H} (-1)^{r(M')} [\xi_{M'}]_H = [\xi_H w_-^G w_-^{M^H}]_H.$$

Here w_-^G (resp. $w_-^{M^H}$) is the longest element in W_G (resp. W_{M^H}). It is an easy exercise to show that $w_-^G \in W_{G^\theta}$ (resp. $w_-^{M^H} \in W_{(M^H)^\theta}$). Moreover, we have $w_-^G(A^{\widehat{G},\widehat{\theta}}) = A^{\widehat{G},\widehat{\theta}}$ (resp. $w_-^{M^H}(A^{\widehat{G},\widehat{\theta}}) = A^{\widehat{G},\widehat{\theta}}$), i.e., $w_-^G, w_-^{M^H} \in W_{G^\theta}^{\Gamma_F}$.

First let us consider (A.9). Recall that the left-hand side of (A.9) is equal to

$$\mathrm{LHS}(A.9) = \sum_{P \in \mathcal{P}^\theta} (-1)^{r_{\mathrm{res}}(M)} \sum_{w \in \widetilde{D}_{M^\theta}} w,$$

and we make the following observation.

Lemma A.13 *If $w \in \widetilde{D}_{M^\theta}$, then $w \in W_{G^\theta}^{\Gamma_F}$.*

Proof For $w \in \widetilde{D}_{M^\theta}$, we have

$$w^{-1}(\Delta_{\mathrm{res}}(\widehat{M})) \subseteq R_{\mathrm{res}}^+(\widehat{G}) \quad \text{and} \quad w^{-1}(A_{\widehat{M}}^{\widehat{\theta}})^0 \subseteq A^{\widehat{H}}$$

by the definition. We take any $\sigma \in \Gamma_F$. Since $A^{\widehat{H}} \subseteq A^{\widehat{G},\widehat{\theta}}$, it is easy to see that $\sigma(w) \in W_{M^\theta} w$. On the other hand,

$$\sigma(w)^{-1}(\Delta_{\mathrm{res}}(\widehat{M})) = \sigma(w^{-1}(\Delta_{\mathrm{res}}(\widehat{M}))) \subseteq \sigma(R_{\mathrm{res}}^+(\widehat{G})) = R_{\mathrm{res}}^+(\widehat{G}).$$

So $\sigma(w) \in D_{M^\theta}$. By Lemma A.2, $\sigma(w) = w$. Hence, $w \in W_{G^\theta}^{\Gamma_F}$. ∎

As a consequence, we can restrict the summation on the left-hand side of (A.9) to $W_{G^\theta}^{\Gamma_F}$. Moreover, for $w \in W_{G^\theta}^{\Gamma_F}$, the condition that $w \in \widetilde{D}_{M^\theta}$ is equivalent to

$$R_{\mathrm{res}}^+(\widehat{M^H}) \subseteq w^{-1}(R_{\mathrm{res}}^+(\widehat{M})) \subseteq R_{\mathrm{res}}^+(\widehat{G}).$$

So

$$\mathrm{LHS}(A.9) = \sum_{\substack{P \in \mathcal{P}^\theta \\ }} (-1)^{r_{\mathrm{res}}(M)} \sum_{\substack{w \in W_{G^\theta}^{\Gamma_F} \\ R_{\mathrm{res}}^+(\widehat{M^H}) \subseteq w^{-1}(R_{\mathrm{res}}^+(\widehat{M})) \subseteq R_{\mathrm{res}}^+(\widehat{G})}} w$$

$$= \sum_{w \in W_{G^\theta}^{\Gamma_F}} \Big(\sum_{\substack{P \in \mathcal{P}^\theta \\ w(R_{\mathrm{res}}^+(\widehat{M^H})) \subseteq R_{\mathrm{res}}^+(\widehat{M}) \subseteq w(R_{\mathrm{res}}^+(\widehat{G}))}} (-1)^{r_{\mathrm{res}}(M)} \Big) w$$

For $w \in W_{G^\theta}^{\Gamma_F}$, we define

$$I_w = \{ \alpha \in \Delta_{\mathrm{res}}(\widehat{G}) | n_{\beta\alpha} \neq 0 \text{ for some } \beta \in \Delta_{\mathrm{res}}(\widehat{M^H}) \},$$

where $w\beta = \sum_{\alpha \in \Delta_{\mathrm{res}}(\widehat{G})} n_{\beta\alpha} \alpha$. Then we have the following lemma.

Lemma A.14 *For $w \in W_{G^\theta}^{\Gamma_F}$, $I_w = w(\Delta_{\mathrm{res}}(\widehat{M^H}))$ if and only if*

$$w\left(R_{\mathrm{res}}^+(\widehat{M^H})\right) \subseteq R_{\mathrm{res}}^+(\widehat{M}) \subseteq w\left(R_{\mathrm{res}}^+(\widehat{G})\right)$$

for some $P \in \mathcal{P}^\theta$.

Proof If there exists $P \in \mathcal{P}^\theta$ such that

$$w\left(R_{\mathrm{res}}^+(\widehat{M^H})\right) \subseteq R_{\mathrm{res}}^+(\widehat{M}) \subseteq w\left(R_{\mathrm{res}}^+(\widehat{G})\right),$$

then $I_w \subseteq \Delta_{\mathrm{res}}(\widehat{M}) \subseteq w(R_{\mathrm{res}}^+(\widehat{G}))$. So $w^{-1}(I_w) \subseteq R_{\mathrm{res}}^+(\widehat{G})$. We claim $w(\Delta_{\mathrm{res}}(\widehat{M^H})) \subseteq \Delta_{\mathrm{res}}(\widehat{G})$. Suppose $\beta \in \Delta_{\mathrm{res}}(\widehat{M^H})$; since $w\beta \in R_{\mathrm{res}}^+(\widehat{M})$, we can assume that $w\beta = \sum_{\alpha \in I_w} n_{\beta\alpha} \alpha$ where $n_{\beta\alpha} \geqslant 0$. Hence,

$$\beta = w^{-1}(w\beta) = \sum_{\alpha \in I_w} n_{\beta\alpha}\,(w^{-1}\alpha).$$

Since $w^{-1}\alpha \in R_{\mathrm{res}}^+(\widehat{G})$ for $\alpha \in I_w$, this can only happen when $n_{\beta\alpha} = 0$ except for one simple root, *i.e.*, $w\beta \in \Delta_{\mathrm{res}}(\widehat{G})$. This proves our claim. As a consequence, $I_w = w(\Delta_{\mathrm{res}}(\widehat{M^H}))$.

Conversely, if $I_w = w(\Delta_{\mathrm{res}}(\widehat{M^H}))$, we can let M_{I_w} be the standard Levi subgroup of G associated with the subset of simple roots I_w. Then we have

$$w\left(R_{\mathrm{res}}^+(\widehat{M^H})\right) \subseteq R_{\mathrm{res}}^+(\widehat{M}_{I_w}) \subseteq w\left(R_{\mathrm{res}}^+(\widehat{G})\right). \qquad \blacksquare$$

In view of this lemma, we can assume that $I_w = w(\Delta_{\mathrm{res}}(\widehat{M^H}))$. Let $M(w)$ be the standard Levi subgroup of G associated with the subset of $\alpha \in \Delta_{\mathrm{res}}(\widehat{G})$ such that $w^{-1}\alpha \in R_{\mathrm{res}}^+(\widehat{G})$. It is clear that $M(w) \supseteq M_{I_w}$ under our assumption. Then

$$\mathrm{LHS(A.9)} = \sum_{\substack{w \in W_{G^\theta}^{\Gamma_F} \\ I_w = w(\Delta_{\mathrm{res}}(\widehat{M^H}))}} \left(\sum_{\substack{P \in \mathcal{P}^\theta \\ M_{I_w} \subseteq M \subseteq M(w)}} (-1)^{r_{\mathrm{res}}(M)} \right) w$$

$$= \sum_{\substack{w \in W_{G^\theta}^{\Gamma_F} \\ I_w = w(\Delta_{\mathrm{res}}(\widehat{M^H})),\, M_{I_w} = M(w)}} (-1)^{r_{\mathrm{res}}(M_{I_w})} w$$

Note that $r_{\mathrm{res}}(M_{I_w}) = r_{\mathrm{res}}(M^H)$, so

$$\mathrm{LHS(A.9)} = (-1)^{r_{\mathrm{res}}(M^H)} \sum_{\substack{w \in W_{G^\theta}^{\Gamma_F} \\ I_w = w(\Delta_{\mathrm{res}}(\widehat{M^H})),\, M_{I_w} = M(w)}} w.$$

Then (A.9) follows from the following lemma.

Lemma A.15 *Suppose $w \in W_{G^\theta}^{\Gamma_F}$ satisfies $I_w = w(\Delta_{\mathrm{res}}(\widehat{M^H}))$ and $M_{I_w} = M(w)$; then $w = w_-^G w_-^{M^H}$.*

Proof Since $(w_-^G)^2 = (w_-^{M^H})^2 = 1$, it is equivalent to show that $w_-^{M^H} w^{-1} = w_-^G$, *i.e.*,

$$w_-^{M^H} w^{-1}\left(\Delta_{\mathrm{res}}(\widehat{G})\right) \subseteq R_{\mathrm{res}}^-(\widehat{G}).$$

Since $w^{-1}(I_w) = \Delta_{\mathrm{res}}(\widehat{M^H})$, $w_-^{M^H} w^{-1}(I_w) \subseteq R_{\mathrm{res}}^-(\widehat{G})$. Since $M_{I_w} = M(w)$,

$$w^{-1}(\Delta_{\mathrm{res}}(\widehat{G}) - I_w) \subseteq R_{\mathrm{res}}^-(\widehat{G}).$$

By $w^{-1}(I_w) = \Delta_{\mathrm{res}}(\widehat{M^H})$ again, we have $w^{-1}(\Delta_{\mathrm{res}}(\widehat{G}) - I_w) \cap R_{\mathrm{res}}(\widehat{M^H}) = \varnothing$. Hence

$$w_-^{M^H} w^{-1}(\Delta_{\mathrm{res}}(\widehat{G}) - I_w) \subseteq R_{\mathrm{res}}^-(\widehat{G}).$$

This finishes the proof. ■

Next let us consider (A.10). Recall that the left-hand side of (A.10) is equal to

$$\mathrm{LHS(A.10)} = \sum_{P' \in \mathcal{P}^H} (-1)^{r(M')} \sum_{\substack{w \in D_{M'} \\ w(A^{\widehat{H}}) = A^{\widehat{H}}}} w.$$

For $w \in W_{G^\theta}$ satisfying $w(A^{\widehat{H}}) = A^{\widehat{H}}$, we have that for any $\sigma \in \Gamma_F$ and $\alpha \in R(\widehat{H})$, $w^{-1}(\alpha)$ and $w^{-1}(\sigma_H(\alpha))$ are both positive or negative, where σ_H is the Galois action in $^L H$. This is because

$$w^{-1}(\alpha)|_{A^{\widehat{H}}} = w^{-1}(\alpha|_{A^{\widehat{H}}}) = w^{-1}(\sigma_H(\alpha)|_{A^{\widehat{H}}}) = w^{-1}(\sigma_H(\alpha))|_{A^{\widehat{H}}} \neq 0.$$

So the subset of $\alpha \in \Delta(\widehat{H})$ satisfying $w^{-1}\alpha \in R_{\mathrm{res}}^+(\widehat{G})$ determines a standard Levi subgroup $M'(w)$ of H. Then

$$\mathrm{LHS(A.10)} = \sum_{\substack{w \in W_{G^\theta} \\ w(A^{\widehat{H}}) = A^{\widehat{H}}}} \left(\sum_{\substack{P' \in \mathcal{P}^H \\ M' \subseteq M'(w)}} (-1)^{r(M')} \right) w = \sum_{\substack{w \in W_{G^\theta} \\ w(A^{\widehat{H}}) = A^{\widehat{H}} \\ w^{-1}(\Delta(\widehat{H})) \subseteq R_{\mathrm{res}}^-(\widehat{G})}} w.$$

On the other hand, the right-hand side of (A.10) is equal to

$$\mathrm{RHS(A.10)} = \left[\xi_H w_-^G \right]_H \cdot w_-^{M^H}.$$

One can check easily that $D_H w_-^G$ consists of $w \in W_{G^\theta}$ such that $w^{-1}(\Delta(\widehat{H})) \subseteq R_{\mathrm{res}}^-(\widehat{G})$. So

$$\mathrm{RHS(A.10)} = \left(\sum_{\substack{w \in W_{G^\theta} \\ w(A^{\widehat{H}}) = A^{\widehat{H}} \\ w^{-1}(\Delta(\widehat{H})) \subseteq R_{\mathrm{res}}^-(\widehat{G})}} w \right) w_-^{M^H} = \sum_{\substack{w \in W_{G^\theta} \\ w(A^{\widehat{H}}) = A^{\widehat{H}} \\ w^{-1}(\Delta(\widehat{H})) \subseteq R_{\mathrm{res}}^-(\widehat{G})}} w.$$

The last equality is due to the fact that for $w \in W_{G^\theta}$ satisfying $w(A^{\widehat{H}}) = A^{\widehat{H}}$,

$$w^{-1}\big(\Delta(\widehat{H})\big) \subseteq R_{\mathrm{res}}^-(\widehat{G})$$

if and only if

$$(w w_-^{M^H})^{-1}\big(\Delta(\widehat{H})\big) \subseteq R_{\mathrm{res}}^-(\widehat{G}).$$

One can show this by restricting the roots to $A^{\widehat{H}}$. Then the proof is completed by comparing the last expressions of RHS (A.10) and LHS (A.10).

Step 4: We will establish (A.4) by using the identities (A.9) and (A.10). First, we multiply (A.9) by ξ_H, and compare it with (A.10):

$$(A.11) \quad \sum_{P \in \mathcal{P}^\theta} (-1)^{r_{\text{res}}(M)} \big[\xi_H \widetilde{\xi}_{M^\theta}\big]_H = (-1)^{r_{\text{res}}(M^H)} \big[\xi_H w_-^G w_-^{M^H}\big]_H$$

$$= (-1)^{r_{\text{res}}(M^H)} \sum_{P' \in \mathcal{P}^H} (-1)^{r(M')} \big[\xi_{M'}\big]_H.$$

Then we can use (A.5) to expand the left-hand side:

$$\text{LHS}(A.11) = \sum_{P \in \mathcal{P}^\theta} (-1)^{r_{\text{res}}(M)} \sum_{P' \in \mathcal{P}^H} a_{M',H,M^\theta} \big[\xi_{M'}\big]_H$$

$$= \sum_{P' \in \mathcal{P}^H} \Big(\sum_{P \in \mathcal{P}^\theta} (-1)^{r_{\text{res}}(M)} a_{M',H,M^\theta} \Big) \big[\xi_{M'}\big]_H.$$

By the linear independence of $\big[\xi_{M'}\big]_H$, we get

$$\sum_{P \in \mathcal{P}^\theta} (-1)^{r_{\text{res}}(M)} a_{M',H,M^\theta} = (-1)^{r_{\text{res}}(M^H)+r(M')}$$

for any $P' \in \mathcal{P}^H$.

A.1 Generalized Aubert Involution

We would like to generalize the diagram (A.1) to (6.3), (6.7), and (6.9). Note that we will only show the commutativity of (6.3) and (6.7) when restricting to distributions associated with elementary parameters. In fact, they are not commutative in general. Let G be a quasisplit symplectic or special orthogonal group. We fix a positive integer X_0 and write $x_0 = (X_0-1)/2$. We also fix a self-dual irreducible unitary supercuspidal representation ρ of $GL(d_\rho)$. Let $\mathcal{P}_{d_\rho}$ be the set of standard parabolic subgroups P of G whose Levi component M is isomorphic to

$$GL(a_1 d_\rho) \times \cdots \times GL(a_l d_\rho) \times G\Big(n - \sum_{i \in [1,l]} a_i d_\rho \Big).$$

Then we can define for $\pi \in \text{Rep}(G)$,

$$\text{inv}_{<X_0}(\pi) := \sum_{P \in \mathcal{P}_{d_\rho}} (-1)^{\dim A_M} \text{Ind}_P^G \big(\text{Jac}_P(\pi)_{<x_0}\big).$$

It is clear that

$$\big[\text{inv}_{<X_0}(\pi)\big] = \overline{\text{inv}}_{<X_0}([\pi]).$$

So (6.3) is equivalent to

$$
\begin{array}{ccc}
\widehat{SI}(G) & \longrightarrow & \widehat{I}(N^\theta) \\
\text{inv}_{<X_0} \downarrow & & \downarrow \text{inv}_{<X_0}^{\theta_N} \\
\widehat{SI}(G) & \longrightarrow & \widehat{I}(N^\theta).
\end{array}
$$

To prove this, we can follow the argument for (A.1). For $P \in \mathcal{P}_{d_\rho}^{\theta_N}$, we specialize the diagram (A.2) in our case:

(A.12)

$$
\begin{array}{ccc}
\widehat{SI}(G) & \longrightarrow & \widehat{I}(N^\theta) \\
{\scriptstyle \oplus_w (\mathrm{Jac}_{P'_w})_{<x_0}} \downarrow & & \downarrow {\scriptstyle (\mathrm{Jac}_P)_{<x_0}} \\
\oplus_w \widehat{SI}(M'_w) & \longrightarrow & \widehat{I}(M^\theta),
\end{array}
$$

where the sum is restricted to those w satisfying $P'_w \in \mathcal{P}_{d_\rho}$. Unlike (A.2), the above diagram may not commute in certain cases when we apply it to distributions not associated with elementary parameters. This is the reason that we want to restrict (6.3) (similarly (6.7)) to distributions associated with elementary parameters. By (A.12), it suffices to show that for any $P' \in \mathcal{P}_{d_\rho}$,

(A.13)
$$
\sum_{P \in \mathcal{P}_{d_\rho}^{\theta_N}} (-1)^{\dim(A_P)_\theta} a_{M',G,M} = (-1)^{\dim A_{P'}}.
$$

By Proposition A.1, we have

$$
\sum_{P \in \mathcal{P}^{\theta_N}} (-1)^{\dim(A_P)_\theta} a_{M',G,M} = (-1)^{\dim A_{P'}}.
$$

Therefore, (A.13) follows from the simple fact that $a_{M',G,M} = 0$ when $P \notin \mathcal{P}_{d_\rho}^{\theta_N}$.

The case of (6.7) is similar. For (6.9), let $\mathcal{P}_{d_\rho}^{\theta_0}$ be the set of θ_0-stable standard parabolic subgroups in $\mathcal{P}_{d_\rho}$. Then we can define for $\pi^{\Sigma_0} \in \mathrm{Rep}(G^{\Sigma_0})$,

$$
\mathrm{inv}_{<X_0}^{\theta_0}(\pi^{\Sigma_0}) := \sum_{P \in \mathcal{P}_{d_\rho}^{\theta_0}} (-1)^{\dim(A_M)_{\theta_0}} \mathrm{Ind}_{P^{\Sigma_0}}^{G^{\Sigma_0}} (\mathrm{Jac}_{P^{\Sigma_0}}(\pi^{\Sigma_0})_{<x_0}).
$$

For $P \in \mathcal{P}_{d_\rho}^{\theta_0}$ and $G(n - \sum_{i \in [1,l]} a_i d_\rho) \neq SO(2)$, it is clear that $(A_M)_{\theta_0} = A_M$ and $\mathrm{Jac}_{P^{\Sigma_0}} = \widetilde{\mathrm{Jac}}_{P^{\Sigma_0}}$. If $G(n - \sum_{i \in [1,l]} a_i d_\rho) = SO(2)$, then $\dim(A_M)_{\theta_0} = \dim(A_M) - 1$, but the effect of $\widetilde{\mathrm{Jac}}_{P^{\Sigma_0}}$ in taking the twisted character also differs from $\mathrm{Jac}_{P^{\Sigma_0}}$ by a negative sign. So we have

$$
f_G(\mathrm{inv}_{<X_0}^{\theta_0}(\pi^{\Sigma_0})) = f_G(\mathrm{inv}_{<X_0}(\pi^{\Sigma_0})), \quad f \in C_c^\infty(G \rtimes \theta_0).
$$

As a result, (6.9) is equivalent to

$$
\begin{array}{ccc}
\widehat{SI}(H) & \longrightarrow & \widehat{I}(G^{\theta_0}) \\
{\scriptstyle \mathrm{inv}_{<X_0}^H} \downarrow & & \downarrow {\scriptstyle \mathrm{inv}_{<X_0}^{\theta_0}} \\
\widehat{SI}(H) & \longrightarrow & \widehat{I}(G^{\theta_0}).
\end{array}
$$

The rest of the argument is similar to (6.3).

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