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BOUNDED-HOP PERCOLATION AND WIRELESS COMMUNICATION

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Abstract

Motivated by an application in wireless telecommunication networks, we consider a two-type continuum-percolation problem involving a homogeneous Poisson point process of users and a stationary and ergodic point process of base stations. Starting from a randomly chosen point of the Poisson point process, we investigate the distribution of the minimum number of hops that are needed to reach some point of the base station process. In the supercritical regime of continuum percolation, we use the close relationship between Euclidean and chemical distance to identify the distributional limit of the rescaled minimum number of hops that are needed to connect a typical Poisson point to a point of the base station process as its intensity tends to 0. In particular, we obtain an explicit expression for the asymptotic probability that a typical Poisson point connects to a point of the base station process in a given number of hops.

Keywords: Ad hoc network; chemical distance; connection probability; continuum percolation

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1. Introduction and main results

We consider a model for a wireless telecommunication network where users are scattered at random in the entire Euclidean plane. In order to meet the users' communication demands, the operator sustains a network of base stations. In classical cellular networks, the base stations subdivide the plane into serving zones and all users inside a serving zone communicate directly with the associated base station. Although such networks exhibit a simple hierarchical topology, installation and upkeep are costly. Indeed, to guarantee good quality of service to all users, the operator either needs to install (and maintain) a relatively dense network of base stations, or the base stations' transmission powers must be sufficiently high so that distant users can also be served.

In the early 2010s network operators started to deploy wireless networks of the fourth generation, called the *long-term evolution* (LTE). Since the advent of LTE technology, operators have the possibility to reduce the number of required base stations substantially by using relays. As of today, this means installing fixed relays at locations that have been chosen in advance. For future generation networks it is desirable to extend this concept through the intelligent use of *ad hoc technology*. To be more precise, we assume that each user has a (comparatively small) transmission radius. A direct communication between users is possible if they are within each others communication radii. Additionally, by forwarding messages via chains of directly

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connected users, base stations can communicate with distant users, even if transmission radii are comparatively small.

Despite these virtues, having users act as relays entails a major drawback regarding the quality of service for delay-sensitive applications. Indeed, the forwarding of messages via several hops induces substantial delay in message transmission. Hence, in network planning, it is crucial to have detailed knowledge of distributional properties of the minimum number of hops to a base station.

In the random-graphs community, the minimum number of hops that are needed to connect two vertices of a graph is known as the *chemical distance*. In supercritical Bernoulli percolation on the lattice, chemical distance has been investigated in [1] and [2]. Loosely speaking, for distant points in the infinite connected component, the chemical distance is approximately proportional to the Euclidean distance, where the proportionality factor is called the *time constant*. The extension of this result to the setting of continuum percolation [15] will be the major tool for establishing the distributional limit of the rescaled minimum number of hops needed to connect a user to a base station.

Next, we provide a precise definition of the wireless spatial telecommunication network under consideration. It consists of two types of network components. The first component is formed by network users. They are modeled by a homogeneous Poisson point process X in $\mathbb{R}^d$, $d \geq 2$, with some intensity $\lambda \in (0, \infty)$. The base stations constitute the second component. We assume that they are of the form $Y = rY^{(1)}$, where $Y^{(1)}$ is assumed to be a stationary (with respect to translations in $\mathbb{R}^d$) and ergodic point process that is independent of X and has a finite and positive intensity. For ease of notation, we assume this intensity to be equal to 1. Here, $r \geq 0$ is some scaling parameter controlling the intensity of base stations. Since we only assume stationarity and ergodicity, our results are valid under rather weak conditions on the spatial distribution of base stations. For instance, they can be applied to stationary point processes that are obtained from $\mathbb{Z}^d$ through translation by a random vector uniformly distributed in $[0, 1]^d$ as well as to (nonshifted) homogeneous Poisson point processes. In other words, our results do not depend on the question as to whether the base stations are scattered at random in the Euclidean plane or are aligned according to a grid that is viewed from a random reference point.

The random network under consideration can be thought of as a model for a wireless telecommunication network, where users can connect to base stations indirectly via at most $k \geq 1$ hops of Euclidean distance at most 1 to other network users. To be more precise, we say that $x, y \in \mathbb{R}^d$ are *k-connectable* if there exist (not necessarily distinct) $X_{i_1}, X_{i_2}, \dots, X_{i_{k-1}} \in X$ such that $|X_{i_j} - X_{i_{j+1}}| \leq 1$ for all $j \in \{0, \dots, k-1\}$, where $X_{i_0} = x$ and $X_{i_k} = y$. Here, $|\cdot|$ denotes the standard Euclidean norm in $\mathbb{R}^d$. We say that x, y are *connectable* if they are k-connectable for *some* $k \geq 1$. In Figure 1, we show a realization of the network model, where the points of X and Y are represented by dots and squares, respectively. Points of X that are 1-connectable to some point of Y are shown as solid lines and open circles, while points of X that are 2-connectable but not 1-connectable to some point in Y appear as dotted lines and open dotted circles.

In the following, we write $H_r(x)$ for the smallest number $k \geq 1$ such that $x \in \mathbb{R}^d$ is k-connectable to some point of $Y = rY^{(1)}$. The main object of investigation in this paper is the quantity

$$\Theta(k, r) = \lambda^{-1}\mathbb{E}\#\big\{X_i \in X \cap \big[-\tfrac{1}{2}, \tfrac{1}{2}\big]^d : H_r(X_i) \leq k\big\},$$

i.e. the normalized expected number of points in $X \cap [-\frac{1}{2}, \frac{1}{2}]^d$ that are k-connectable to some

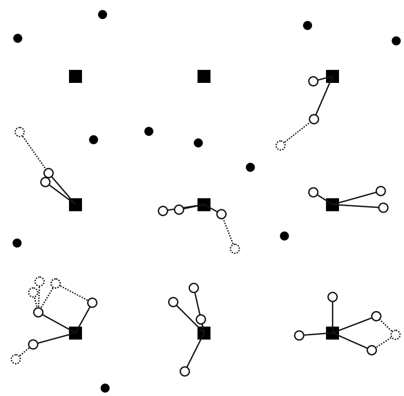

FIGURE 1: Realization of the network model.

base station. In fact, we show that $\Theta(k, r)$ admits a more natural representation as the limit of the average number of points in X inside a large box that are k-connectable to a point of Y.

Proposition 1. *Let $k \geq 1$ and $r > 0$. Then, almost surely,*

$$\Theta(k, r) = \lim_{n \to \infty} \lambda^{-1} n^{-d} \# \left\{ X_i \in X \cap \left[-\frac{n}{2}, \frac{n}{2} \right]^d : H_r(X_i) \leq k \right\}.$$

Let $B_s(x) = \{y \in \mathbb{R}^d : |x - y| \leq s\}$ denote the Euclidean ball of radius $s > 0$ centered at $x \in \mathbb{R}^d$. Provided that k and r are of the same order, the asymptotic behavior of $\Theta(k, r)$ depends sensitively on whether the intensity λ is below or above the critical intensity λ_c in continuum percolation. To be more precise, λ_c is the infimum over all intensities $\lambda > 0$ for which the union $\bigcup_{i=1}^{\infty} B_{1/2}(X_i)$ almost surely has an unbounded connected component.

Concerning the subcritical regime, our first main result shows that $\Theta(k, r)$ decays polynomially in r as $r \to \infty$. In the following, we write o for the origin in $\mathbb{R}^d$ and $C(o)$ denotes the set of all $X_i \in X$ that are connectable to o.

Theorem 1. *Let $\lambda < \lambda_c$ and $r > 0$. Then,*

$$\sup_{k \geq 1} \Theta(k, r) \leq \lambda^{-1} r^{-d} \mathbb{E}\#C(o).$$

Note that for $\lambda < \lambda_c$, we have $\mathbb{E}\#C(o) < \infty$; see, e.g. [6, Theorem 12.35]. However, as $\lambda \to \lambda_c$, the bound in Theorem 1 becomes less useful as $\mathbb{E}\#C(o) \to \infty$. In dimension 2 the speed of divergence is conjectured to be polynomial in $(\lambda - \lambda_c)^{-1}$ of order $\frac{43}{18}$; see [6, Chapter 9].

Next, consider the supercritical case, i.e. let $\lambda > \lambda_c$. By a central result in continuum percolation [10, Theorem 2.1], the set $\bigcup_{i=1}^{\infty} B_{1/2}(X_i)$ contains a unique unbounded connected component. In the following, $\mathcal{C}_\infty \subset X$ denotes the subset of all elements of X that are contained in this unbounded connected component. We write θ for the probability that there exists $X_i \in \mathcal{C}_\infty$, with $|X_i| \leq 1$.

In order to describe the asymptotic behavior of $\Theta(k, r)$ for large k and r, it is important to understand that the chemical distance between two points of $\mathcal{C}_\infty$, i.e. the minimum number of hops needed to establish a connection, grows linearly in the Euclidean distance of the two points. This can be formalized in different ways.

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First, fixing any point $X_i \in \mathcal{C}_\infty$, there should exist an almost sure finite random variable ρ_i such that for every $X_j \in \mathcal{C}_\infty$ the chemical distance between X_i and X_j is at most $\rho_i |X_i - X_j|$. As observed in [4, Lemma 5.2], when considering Bernoulli site percolation on the lattice, the corresponding result can be derived by adapting the bond percolation argument established in [1, Lemma 2.4].

Additionally, when disregarding points in a small neighborhood of X_i, the random variable ρ_i can be replaced by a deterministic quantity $\mu \in (0, \infty)$ that does not depend on i. To be more precise, we put $q(x) = X_j$ if X_j is the element of $\mathcal{C}_\infty$ minimizing the distance to $x \in \mathbb{R}^d$. Then, D_n denotes the minimum integer $k \geq 1$ such that $q(o)$ and $q(n\boldsymbol{e}_1)$ are k-connectable, where $\boldsymbol{e}_1 = (1, 0, \dots, 0)$ is the first standard unit vector in $\mathbb{R}^d$. Using Kingman's subadditive ergodic theorem, it was shown in [15] that there exists a real number $\mu \in (0, \infty)$ such that almost surely, $\lim_{n\to\infty} n^{-1} D_n = \mu$; see also [2] for the corresponding statement on the lattice.

With this background, we can now provide a heuristic explanation for the asymptotic behavior of H_r as r tends to ∞, where we put $H_r = H_r(o)$. As a corollary, we deduce the limiting value of $\Theta(k, r)$ if the speed at which k and r tend to ∞ is chosen so that their quotient tends to some constant. The reader who is interested in the precise mathematical result may jump to Theorem 2 below. Regarding the heuristic, the Slivnyak–Mecke theorem [13, Corollary 3.2.3] implies that

$$\Theta(k, r) = \mathbb{P}(r^{-1} H_r \leq r^{-1} k).$$

Hence, it suffices to understand the asymptotic distribution of $r^{-1} H_r$ as $r \to \infty$. We stress that the following heuristic considerations are only valid in the regime $r \to \infty$. First, points of X can only connect to points of Y that are contained in the unbounded connected component of continuum percolation and the probability that a given point of Y is contained in the unbounded connected component is given by θ. Hence, instead of $rY^{(1)}$, we consider the process of relevant points $rY^{(\theta)}$, where $Y^{(\theta)}$ is obtained from $Y^{(1)}$ by independent thinning with survival probability θ. Then, for a given point of X to be connectable to some point of Y, the former must also belong to the unbounded connected component, which occurs with probability θ. Moreover, the closest point of $rY^{(\theta)}$ is at Euclidean distance $r \min\{|y| \colon y \in Y^{(\theta)}\}$ and it can be reached in at most $\mu r \min\{|y| \colon y \in Y^{(\theta)}\}$ hops. This heuristic is made precise in our second main result, where we use the convention $0 \cdot \infty = 0$.

Theorem 2. *Let $\lambda > \lambda_c$. Then, $r^{-1} H_r$ converges in distribution to the random variable*

$$(1 - Z) \cdot \infty + Z\mu \min\{|y| \colon y \in Y^{(\theta)}\},$$

where Z is a Bernoulli random variable that is independent of $Y^{(\theta)}$ and which assumes the value 1 with probability θ.

In other words, the asymptotic distribution of $r^{-1} H_r$ is a mixture between a Dirac measure at ∞ and the contact distribution of the point process $\mu Y^{(\theta)}$. In particular, Theorem 2 can be used to compute $\lim_{r\to\infty} \mathbb{P}(H_r \leq cr)$.

Corollary 1. *Let $\lambda > \lambda_c$ and assume that $\lim_{r\to\infty} r^{-1} k(r) = c$ for some $c \in (0, \infty)$. Then,*

$$\lim_{r\to\infty} \Theta(k, r) = \theta \mathbb{P}\bigg(o \in \bigcup_{Y_j \in Y^{(\theta)}} B_{c/\mu}(Y_j)\bigg).$$

If $Y^{(1)}$ is a homogeneous Poisson point process with intensity 1, then $Y^{(\theta)}$ is a homogeneous Poisson point process with intensity θ. Hence, we have the following result.

Corollary 2. *Let $Y^{(1)}$ be a homogeneous Poisson point process with intensity* 1. *Then, under the assumptions of Corollary 1,*

$$\lim_{r\to\infty}\Theta(k,r)=\theta(1-\exp(-\theta\kappa_d c^d\mu^{-d})),$$

where κ_d denotes the volume of the unit ball in $\mathbb{R}^d$.

The limiting distribution provided in Theorem 2 depends on λ implicitly via θ and μ. In order to develop an intuition on the order of λ that is needed to achieve a given (high) connectivity probability, it is useful to have some information on the behavior of the percolation probability θ and the time constant μ as a function of λ. Concerning θ, it was shown in [11, Corollary of Theorem 3] that $\theta=\theta(\lambda)$ converges exponentially fast to 1 as λ tends to ∞. In our third main result, we show that asymptotically $\mu-1=\mu(\lambda)-1$ tends to 0 as $\lambda\to\infty$ and that the convergence occurs at least at a polynomial speed.

Theorem 3. *We have $\mu(\lambda)-1\in O(\lambda^{-1/d}(\log\lambda)^{1/d})$ as $\lambda\to\infty$.*

This paper is organized as follows. In Section 2 we establish the ergodic representation of $\Theta(k,r)$ announced in Proposition 1 and investigate the asymptotic behavior of $\Theta(k,r)$ in the subcritical regime. That is, we prove Theorem 1. Section 3 is devoted to the proof of Theorem 2 which describes the distributional limit of the rescaled minimum number of hops $r^{-1}H_r$ in the supercritical regime. Finally, in Section 4 we prove Theorem 3, i.e. we show that the time constant μ tends to 1 as the intensity tends to ∞. Additionally, we provide a lower bound for the speed of this convergence.

2. Proofs of Proposition 1 and Theorem 1

The proof of Proposition 1 is based on the multidimensional ergodic theorem. To apply this result, it is important to note that the homogeneous Poisson point process is mixing [13, Theorem 9.3.5], so that the pair of independent stationary point processes (X,Y) are again ergodic; see [8, Theorem 3.6].

Proof of Proposition 1. For $z\in\mathbb{R}^d$ let

$$W_z=\#\big\{X_i\in X\cap\big(\big[-\tfrac12,\tfrac12\big]^d+z\big)\colon H_r(X_i)\le k\big\}$$

denote the number of points in $X\cap([-\frac12,\frac12]^d+z)$ that are at most k hops away from some point of Y. From the ergodic theorem for spatial processes (see, e.g. [8, Theorem 2.13]), we conclude that the random variable

$$\Xi_m=m^{-d}\int_{[-m/2,m/2]^d}W_z\,\mathrm{d}z$$

converges almost surely to

$$\mathbb{E}\int_{[-1/2,1/2]^d}W_z\,\mathrm{d}z=\mathbb{E}\#\big\{X_i\in X\cap\big[-\tfrac12,\tfrac12\big]^d\colon H_r(X_i)\le k\big\}.$$

Moreover, we assert, that for sufficiently large $n\ge1$, the expression

$$n^{-d}\#\big\{X_i\in X\cap\big[-\tfrac12 n,\tfrac12 n\big]^d\colon H_r(X_i)\le k\big\}$$

is bounded below and above by $n^{-d}(n-1)^d\,\Xi_{n-1}$ and $n^{-d}(n+1)^d\,\Xi_{n+1}$, respectively. Once this assertion is shown sending $n\to\infty$ completes the proof. In order to achieve this goal, we observe that

$$\begin{aligned}(n+1)^d\,\Xi_{n+1} &= \sum_{\{X_i\in X:\ H_r(X_i)\le k\}} \nu_d\Big(\big(-X_i+\big[-\tfrac12,\tfrac12\big]^d\big)\cap\big[-t\tfrac12(n+1),\tfrac12(n+1)\big]^d\Big)\\ &\ge \#\big\{X_i\in X\cap\big[-\tfrac12 n,\tfrac12 n\big]^d:\ H_r(X_i)\le k\big\},\end{aligned}$$

where ν_d denotes the Lebesgue measure in $\mathbb{R}^d$. Since the lower bound can be shown using a similar argument, the assertion follows. □

To prepare the proof of Theorem 1, we note that it is possible to express $\Theta(k,r)$ as the expected value of the suitably weighted size of the cluster at a typical point of Y. To be more precise, for $y\in\mathbb{R}^d$, let $C_k(y)$ denote the set of all $X_i\in X$ such that X_i is k-connectable to y. Additionally, put $\kappa(X_i)=\#\{Y_j\in Y:\ X_i\in C_k(Y_j)\}$.

Lemma 1. *Let $k\ge 1$ and $r>0$. Then,*

$$\Theta(k,r)=\lambda^{-1}\mathbb{E}\sum_{Y_j\in Y\cap[-1/2,1/2]^d}\ \sum_{X_i\in C_k(Y_j)}\kappa(X_i)^{-1}.$$

Proof. The claimed identity is a consequence of the mass-transport principle [3]. Indeed, define a function $\Phi\colon\mathbb{Z}^d\times\mathbb{Z}^d\to[0,\infty)$ by mapping a pair of sites $(z,z')\in\mathbb{Z}^d\times\mathbb{Z}^d$ to

$$\Phi(z,z')=\sum_{Y_j\in Y\cap([-1/2,1/2]^d+z)}\ \sum_{X_i\in C_k(Y_j)\cap([-1/2,1/2]^d+z')}\kappa(X_i)^{-1}.$$

Then, $\sum_{z'\in\mathbb{Z}^d}\Phi(o,z')=\sum_{Y_j\in Y\cap[-1/2,1/2]^d}\sum_{X_i\in C_k(Y_j)}\kappa(X_i)^{-1}$. On the other hand,

$$\begin{aligned}\sum_{z\in\mathbb{Z}^d}\Phi(z,o) &= \sum_{Y_j\in Y}\ \sum_{X_i\in C_k(Y_j)\cap[-1/2,1/2]^d}\kappa(X_i)^{-1}\\ &= \sum_{X_i\in X\cap[-1/2,1/2]^d}\ \sum_{\{Y_j\in Y:\ X_i\in C_k(Y_j)\}}\kappa(X_i)^{-1}\\ &= \#\big\{X_i\in X\cap\big[-\tfrac12,\tfrac12\big]^d:\ X_i \text{ is } k\text{-connectable to some point of } Y\big\}.\end{aligned}$$

By stationarity, we obtain

$$\mathbb{E}\sum_{z\in\mathbb{Z}^d}\Phi(z,o)=\sum_{z\in\mathbb{Z}^d}\mathbb{E}\Phi(z,o)=\sum_{z\in\mathbb{Z}^d}\mathbb{E}\Phi(o,-z)=\mathbb{E}\sum_{z\in\mathbb{Z}^d}\Phi(o,z),$$

which concludes the proof. □

Since $\kappa(X_i)\ge 1$ for all $X_i\in C_k(Y_j)$, the independence of X and Y implies that

$$\begin{aligned}\mathbb{E}\sum_{Y_j\in Y\cap[-1/2,1/2]^d}\ \sum_{X_i\in C_k(Y_j)}\kappa(X_i)^{-1} &\le \mathbb{E}\sum_{Y_j\in Y\cap[-1/2,1/2]^d}\#C_k(Y_j)\\ &\le \mathbb{E}\sum_{Y_j\in Y\cap[-1/2,1/2]^d}\mathbb{E}\#C_k(o).\end{aligned}$$

In particular, Lemma 1 gives rise to a simple upper bound for $\Theta(k,r)$.

Proposition 2. *Let $k \geq 1$ and $r > 0$. Then,* $\Theta(k, r) \leq \lambda^{-1} r^{-d} \mathbb{E}\#C_k(o)$.

We note two corollaries of Proposition 2. First, k must grow at least linearly in r for $\Theta(k, r)$ to have a nonzero limit.

Corollary 3. *If* $k = k(r) \in o(r)$ *then* $\lim_{r\to\infty} \Theta(k, r) = 0$.

Proof. Since $C_k(o)$ is contained in $B_k(o)$, we deduce that $\mathbb{E}\#C_k(o) \leq k^d \mathbb{E}\#(X \cap B_1(o))$. In particular, applying the upper bound from Proposition 2 proves the claim. □

Moreover, Proposition 2 is also useful for proving Theorem 1.

Proof of Theorem 1. Combining the trivial inequality $\#C_k(o) \leq \#C(o)$ with Proposition 2 yields the desired bound. □

3. Proof of Theorem 2

In this section we prove Theorem 2. To this end, we fix $\lambda > \lambda_c$ throughout the entire section. Using the notation of Theorem 2, let $W = (1 - Z) \cdot \infty + Z\mu \min\{|y| \colon y \in Y^{(\theta)}\}$. In order to show that $r^{-1} H_r$ converges to W in distribution, we fix an arbitrary $a \geq 0$. Then, we proceed in three steps.

Lemma 2. *It holds that* $\lim_{r\to\infty} \mathbb{P}(H_r = \infty) = 1 - \theta$.

Lemma 3. *It holds that* $\liminf_{r\to\infty} \mathbb{P}(H_r \leq ra) \geq \mathbb{P}(W \leq a)$.

Lemma 4. *It holds that* $\limsup_{r\to\infty} \mathbb{P}(H_r \leq ra) \leq \mathbb{P}(W \leq a)$.

As a first auxiliary result, we note that asymptotically the events that points in $\mathbb{R}^d$ belong to the unbounded connected component become independent.

Lemma 5. *Let $\lambda > \lambda_c$ and $z_1, \dots, z_m$ be distinct points in $\mathbb{R}^d \setminus \{o\}$. Furthermore, let $E_r = E_r(\{z_1, \dots, z_m\})$ denote the event that $\#C(o) = \infty$ and $\#C(rz_i) = \infty$ for some $i \in \{1, \dots, m\}$. Then,* $\lim_{r\to\infty} \mathbb{P}(E_r) = \theta(1 - (1 - \theta)^m)$.

Proof. Choose $\delta > 0$ such that the cubes $[-\delta, \delta]^d, z_1 + [-\delta, \delta]^d, \dots, z_m + [-\delta, \delta]^d$ are disjoint. Furthermore, let $G(y, r)$ denote the event that the connected component of $B_{1/2}(y) \cup \bigcup_{j\geq 1} B_{1/2}(X_j)$ at $y \in \mathbb{R}^d$ is not contained in $y + [-r\delta + 1, r\delta - 1]^d$. Since the events $G(o, r), G(rz_1, r), \dots, G(rz_m, r)$ are independent, we can conclude that

$$\lim_{r\to\infty} \mathbb{P}(E_r) = \lim_{r\to\infty} \mathbb{P}(G(o, r))\biggl(1 - \prod_{i=1}^{m} (1 - \mathbb{P}(G(rz_i, r)))\biggr) = \theta(1 - (1 - \theta)^m),$$

if we can show that $\lim_{r\to\infty} \mathbb{P}(G'(y, r)) = 0$ holds for every $y \in \mathbb{R}^d$, where $G'(y, r)$ denotes the event that the connected component of $B_{1/2}(y) \cup \bigcup_{j\geq 1} B_{1/2}(X_j)$ at y is finite, but not contained in $y + [-r\delta + 1, r\delta - 1]^d$. To achieve this goal, note that under the event $\bigcap_{r>0} G'(y, r)$ the connected component of $B_{1/2}(y) \cup \bigcup_{j\geq 1} B_{1/2}(X_j)$ at y is infinite. Hence,

$$\lim_{r\to\infty} \mathbb{P}(G'(y, r)) = \mathbb{P}\biggl(\bigcap_{r>0} G'(y, r)\biggr) = 0,$$

as required. □

Lemma 5 allows us to compute $\lim_{r\to\infty} \mathbb{P}(H_r = \infty)$.

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Proof of Lemma 2. First, we note that $\limsup_{r\to\infty} \mathbb{P}(H_r < \infty) \le \theta$. Indeed, applying Fatou's lemma to the nonpositive random variable $\mathbb{P}(H_r < \infty \mid Y^{(1)}) - 1$, we obtain

$$\begin{aligned}\limsup_{r\to\infty} \mathbb{P}(H_r < \infty) &= 1 + \limsup_{r\to\infty} \mathbb{P}(H_r < \infty) - 1\\ &\le 1 + \mathbb{E}\Big(\limsup_{r\to\infty} \mathbb{P}(H_r < \infty \mid Y^{(1)}) - 1\Big),\end{aligned}$$

which is equal to θ, i.e. the probability that $C(o)$ is unbounded. For the reverse inequality, let $n \ge 1$ be arbitrary. Uniqueness of the infinite connected component [10, Theorem 2.1] shows that $H_r < \infty$ almost surely under the event that $\#C(o) = \infty$ and $\#C(rY_j) = \infty$ for some $Y_j \in Y^{(1)} \cap [-n/2, n/2]^d$. Hence, by Fatou's lemma and Lemma 5,

$$\begin{aligned}&\liminf_{r\to\infty} \mathbb{P}(H_r < \infty)\\ &\quad\ge \mathbb{E}\Big(\liminf_{r\to\infty} \mathbb{P}(\#C(o) = \infty \text{ and } \sup_{Y_j \in Y^{(1)}\cap[-n/2,n/2]^d} \#C(rY_j) = \infty \mid Y^{(1)})\Big)\\ &\quad= \theta\mathbb{E}(1 - (1-\theta)^{\#(Y^{(1)}\cap[-n/2,n/2]^d)}).\end{aligned}$$

Letting $n \to \infty$ completes the proof of the lower bound. □

For the proofs of Lemmas 3 and 4, we need two further auxiliary results which are immediate corollaries to the shape theorem [15, Theorem 2.2]. Loosely speaking, the following auxiliary results encode the intuition that asymptotically two users in the unbounded connected component of continuum percolation are k-connectable if and only if their Euclidean distance is of order at most k/μ.

Lemma 6. *Let $a > 0$ and $\lambda > \lambda_c$ be arbitrary. Then, for every $\varepsilon \in (0, 1)$,*

$$\lim_{r\to\infty} \mathbb{P}(E(r, \varepsilon)) = 0,$$

where $E(r, \varepsilon)$ denotes the event that there exists $Y_j \in Y^{(1)} \cap B_{a(1-\varepsilon)/\mu}(o)$ such that $\#C(o) = \#C(rY_j) = \infty$, but o is not $\lfloor ra \rfloor$-connectable to rY_j.

Lemma 7. *Let $a > 0$ and $\lambda > \lambda_c$ be arbitrary. Then, for every $\varepsilon \in (0, 1)$,*

$$\lim_{r\to\infty} \mathbb{P}(F(r, \varepsilon)) = 0,$$

where $F(r, \varepsilon)$ is the event that the origin is $\lceil ra \rceil$-connectable to some point in $\mathbb{R}^d \setminus B_{ra(1+\varepsilon)/\mu}(o)$.

After these preliminary results, we now proceed with the proof of Lemma 3.

Proof of Lemma 3. Put $E^*(r, \varepsilon) = \{\#C(o) = \infty\} \cap E^{**}(r, \varepsilon)$, where $E^{**}(r, \varepsilon)$ denotes the event that there exists $Y_j \in Y^{(1)} \cap B_{a(1-\varepsilon)/\mu,}(o)$ with $\#C(rY_j) = \infty$. Then,

$$\mathbb{P}(H_r \le ra) \ge \mathbb{P}(E^*(r, \varepsilon)) - \mathbb{P}(E(r, \varepsilon)).$$

By Lemma 6, $\mathbb{P}(E(r, \varepsilon))$ is negligible as $r \to \infty$. Hence, by Lemma 5,

$$\liminf_{r\to\infty} \mathbb{P}(H_r \le ra) \ge \theta\mathbb{E}(1 - (1-\theta)^{\#(Y^{(1)}\cap B_{a(1-\varepsilon)/\mu}(o))}).$$

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Finally, as $Y^{(\theta)}$ is an independent thinning of $Y^{(1)}$, we obtain

$$\begin{aligned}\mathbb{P}(\mu \min\{|Y_j|\colon Y_j \in Y^{(\theta)}\} \leq a(1-\varepsilon)) &= \mathbb{E}(\mathbb{P}(Y^{(\theta)} \cap B_{a(1-\varepsilon)/\mu}(o) \neq \varnothing \mid Y^{(1)})) \\ &= \mathbb{E}(1-(1-\theta)^{\#(Y^{(1)} \cap B_{a(1-\varepsilon)/\mu}(o))}).\end{aligned}$$

Letting $\varepsilon \to 0$ completes the proof. □

In order to complete the proof of Theorem 2, it remains to prove Lemma 4. First, we note that, asymptotically, distinct points that are connectable must be contained in the unbounded connected component of continuum percolation.

Lemma 8. *Let $\lambda > \lambda_c$ and $z_1, \dots, z_m$ be distinct points in $\mathbb{R}^d \setminus \{o\}$. Let $F_r = F_r(\{z_1, \dots, z_m\})$ denote the event that $\#C(o) < \infty$ and o is connectable to some rz_i. Then, $\lim_{r\to\infty} \mathbb{P}(F_r) = 0$.*

Proof. Denote by δ the minimum of the pairwise distances between elements of the set $\{o, z_1, \dots, z_m\}$. Let $F_r'(z)$ denote the event that $\#C(z) < \infty$, but $z \in \mathbb{R}^d$ is connectable to some point with distance at least $r\delta$. Then, by stationarity,

$$\mathbb{P}(F_r) \leq \mathbb{P}(F_r'(o)) + \sum_{i=1}^{m} \mathbb{P}(F_r'(z_i)) = (m+1)\mathbb{P}(F_r'(o)).$$

Moreover, if o is connectable to some point with distance at least $r\delta$ for every $r > 0$, then $\#C(o) = \infty$. In particular, $\lim_{r\to\infty} \mathbb{P}(F_r'(o)) = \mathbb{P}(\bigcap_{r>0} F_r'(o)) = 0$. □

Now, we can complete the proof of Theorem 2.

Proof of Lemma 4. First, we observe that

$$\begin{aligned}\mathbb{P}(H_r \leq ra) \leq \mathbb{P}(F(r,\varepsilon)) &+ \mathbb{E}(\mathbb{P}(F_r(Y^{(1)} \cap B_{a(1+\varepsilon)/\mu}(o)) \mid Y^{(1)})) \\ &+ \mathbb{E}(\mathbb{P}(E_r(Y^{(1)} \cap B_{a(1+\varepsilon)/\mu}(o)) \mid Y^{(1)})).\end{aligned}$$

Hence, we conclude from Lemmas 7 and 8 that it suffices to investigate the third summand. Now, applying Lemma 5, we have

$$\begin{aligned}\limsup_{r\to\infty} \mathbb{E}(\mathbb{P}(E_r(Y^{(1)} \cap B_{a(1+\varepsilon)/\mu}(o)) \mid Y^{(1)})) &\leq \mathbb{E}(\lim_{r\to\infty} \mathbb{P}(E_r(Y^{(1)} \cap B_{a(1+\varepsilon)/\mu}(o)) \mid Y^{(1)})) \\ &= \theta\mathbb{E}(1-(1-\theta)^{\#(Y^{(1)} \cap B_{a(1+\varepsilon)/\mu}(o))}).\end{aligned}$$

Repeating the final steps used in the derivation of Lemma 3 completes the proof. □

4. Proof of Theorem 3

Loosely speaking, in order to prove Theorem 3, we can proceed similarly as in [15, Lemma 3.4] and modify the arguments used in the lattice setting [2]. The general construction presented in these papers is useful for the proof of Theorem 3, but the identification of the behavior of $\mu = \mu(\lambda)$ as $\lambda \to \infty$ requires a more refined analysis.

It is convenient to introduce a specific family of site percolation processes. For this purpose, we describe certain useful configurations. Loosely speaking, it is only possible to connect $q(o)$ to $q(n\boldsymbol{e}_1)$ by a path of at most μn hops with μ very close to 1 if the vast majority of segments in this path are almost horizontal. We achieve this geometry by considering regularly placed positions on the segment $[o, n\boldsymbol{e}_1]$ and imposing that there is a point of the Poisson

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point process close to most of these positions. The precise level of closeness is controlled by a parameter $\varepsilon \in (0, 1/d)$. First, for $z \in \mathbb{Z}^d$, we need to ensure that any two points of $X \cap (z + [-(1-\varepsilon)/2, (1-\varepsilon)/2]^d)$ can be connected via hops of distance at most 1 to other points of $X \cap (z + [-(1-\varepsilon)/2, (1-\varepsilon)/2]^d)$. To be more precise, $E_{1,\varepsilon}(z)$ denotes the event that $(X - (1-\varepsilon)z) \cap Q_i \neq \varnothing$ for all $i \in \{1, \dots, (2d)^d\}$, where $Q_1, \dots, Q_{(2d)^d}$ is a subdivision of $[-(1-\varepsilon)/2, (1-\varepsilon)/2]^d$ into congruent subcubes of side length $(1-\varepsilon)/(2d)$. In particular, if $Q_i \cap Q_j \neq \varnothing$ then

$$|x_i - x_j| \leq \frac{2\sqrt{d}(1-\varepsilon)}{2d} \leq 1$$

holds for all $x_i \in Q_i$, $x_j \in Q_j$.

Second, we demand that X has a point close to $z \in \mathbb{Z}^d$. This will allow us to pass through linear arrangements of adjacent cubes without deviating too much from the line segment connecting the centers of these cubes. More precisely, $E_{2,\varepsilon}(z)$ denotes the event that $(X - (1-\varepsilon)z) \cap [-\varepsilon/4, \varepsilon/4]^d \neq \varnothing$. Note that $|x - y| \leq 1$ for all $x \in [-\varepsilon/4, \varepsilon/4]^d$ and $y \in ((1-\varepsilon)\boldsymbol{e}_1 + [-\varepsilon/4, \varepsilon/4]^d)$. Finally, for $\varepsilon \in (0, 1)$, we say that a site $z \in \mathbb{Z}^d$ is *ε-good* if $E_{1,\varepsilon}(z) \cap E_{2,\varepsilon}(z)$ occurs.

To begin with, we show that linear arrangements of good sites can be traversed quickly.

Lemma 9. *Let $j \geq 1$ and $\varepsilon \in (0, 1)$ be such that the site ie_1 is ε-good for all $i \in \{0, \dots, j\}$. Furthermore, let $X_{i_0}, X_{i_1} \in X$ be such that $X_{i_0} \in [-\varepsilon/4, \varepsilon/4]^d$ and $X_{i_1} \in (j(1-\varepsilon)\boldsymbol{e}_1 + [-\varepsilon/4, \varepsilon/4]^d)$. Then, X_{i_0} and X_{i_1} are j-connectable.*

Proof. Proceeding inductively, it suffices to consider the $j = 1$ case. But for $j = 1$, the claim follows from the previous observation that $|X_{i_0} - X_{i_1}| \leq 1$. □

Even for large values of the intensity λ, the probability that the site ie_1 is ε-good for all $i \in \{0, \dots, m\}$ decays exponentially fast in m. Therefore, we have to deal with the occasional occurrence of defects. In the following, we say that a set of sites $\Lambda \subset \mathbb{Z}^d$ is *$*$-connected* if it forms a connected set in the graph whose vertices are given by $\mathbb{Z}^d$ and where $z, z' \in \mathbb{Z}^d$ are connected by an edge if $|z - z'|_\infty \leq 1$. We need a crude upper bound for the number of steps required to traverse a set of cubes associated with a $*$-connected set of ε-good sites.

Lemma 10. *Let $\varepsilon > 0$ and $\Lambda \subset \mathbb{Z}^d$ be a finite $*$-connected set of ε-good sites. Furthermore, let $X_{i_0}, X_{i_1} \in X$ be such that $X_{i_0} \in (1-\varepsilon)(z + [-\frac{1}{2}, \frac{1}{2}]^d)$, $X_{i_1} \in (1-\varepsilon)(z' + [-\frac{1}{2}, \frac{1}{2}]^d)$ for some $z, z' \in \Lambda$. Then X_{i_0} and X_{i_1} are k-connectable for $k = (3 + (2d)^d)\#\Lambda$.*

Proof. If $z = z'$, then the definition of ε-goodness implies that X_{i_0} and X_{i_1} are k'-connectable for $k' = 2 + (2d)^d$. Next, if z, z' are such that $|z - z'|_\infty \leq 1$, then, again by the definition of ε-goodness, there exist $X_{j_0}, X_{j_1} \in X$ with $X_{j_0} \in (1-\varepsilon)(z + [-\frac{1}{2}, \frac{1}{2}]^d)$, $X_{j_1} \in (1-\varepsilon)(z' + [-\frac{1}{2}, \frac{1}{2}]^d)$, and $|X_{j_0} - X_{j_1}| \leq 1$. Hence, the proof of Lemma 10 is completed by an elementary induction argument on the length of the path in Λ connecting z and z'. □

The next step is to combine Lemmas 9 and 10 into an upper bound that is useful in situations where the $*$-connected ε-bad components associated with the sites ie_1, $i \in \{0, \dots, m\}$ only cover a small proportion of these sites. More precisely, let U_m be the union of the $*$-connected ε-bad components associated with the sites ie_1, $i \in \{0, \dots, m\}$. If ie_1 is ε-good, then we define its $*$-connected ε-bad component to be empty. Note that U_m is almost surely finite provided that λ is sufficiently large.

Let $U_m^{(\infty)}$ denote the unbounded connected component of $\mathbb{Z}^d \setminus U_m$. In particular, $U_m^{(\infty)}$ is also $*$-connected. Then, $U'_m = \mathbb{Z}^d \setminus U_m^{(\infty)}$ consists of $m' \geq 1$ $*$-connected components

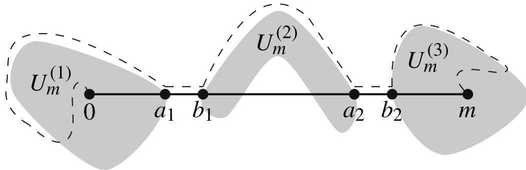

FIGURE 2: Construction of the sequences $\{a_i\}_{1\le i\le m''}$ and $\{b_i\}_{1\le i\le m''}$. The dashed line shows a possibility of a short path circumventing the obstacles.

$U_m^{(1)},\dots,U_m^{(m')}$. Let $\partial U_m^{(i)}$ denote the *outer boundary* of $U_m^{(i)}$, i.e. $\partial U_m^{(i)}$ consists of all $z\in\mathbb{Z}^d\setminus U_m^{(i)}$ such that $|z-z'|_\infty=1$ for some $z'\in U_m^{(i)}$. Note that $\partial U_m^{(i)}$ is $*$-connected, since the outer boundary of any $*$-connected set is again $*$-connected, see [7, Lemma 2.23] (related results can be found in [5] and [14]).

Next, we identify subsets of $\{o,\boldsymbol{e}_1,\dots,m e_1\}$ that form linear arrangements of ε-good sites. To be more precise, we construct two finite increasing subsequences $\{a_i\}_{1\le i\le m''}$ and $\{b_i\}_{1\le i\le m''}$ of $\{0,\dots,m\}$ inductively as follows. If $\{o,\boldsymbol{e}_1,\dots,me_1\}\subset U'_m$, then we put $m''=0$. Otherwise, choose $a_1=\min\{i\ge 0\colon ie_1\notin U'_m\}$ as the first site that is not contained in U'_m. If neither of $a_1\boldsymbol{e}_1,(a_1+1)\boldsymbol{e}_1,\dots,me_1$ is contained in U'_m, then we put $b_1=m$, $m''=1$, and terminate the construction. Otherwise, let $b_1=\inf\{i\in\{a_1,\dots,m\}\colon ie_1\in U'_m\}-1$ be the predecessor of the next site after a_1 that is contained in U'_m. In particular, there is some $i_1\in\{1,\dots,m'\}$ such that $(b_1+1)\boldsymbol{e}_1\in U_m^{(i_1)}$. Define $a'_2=\max\{i\in\{b_1,\dots,m\}\colon ie_1\in\partial U_m^{(i_1)}\}$. If $a'_2=b_1$ then put $m''=1$ and terminate the construction. Otherwise, define $a_2=a'_2$ and continue inductively. See Figure 2 for an illustration of this construction.

We make two crucial observations. First, the sites je_1 are ε-good for all $j\in\{a_i,\dots,b_i\}$ and $i\in\{1,\dots,m''\}$. Second, if $j<m'$ then the sites b_je_1, $a_{j+1}\boldsymbol{e}_1$ are contained in the $*$-connected set $\partial U_m^{(i_j)}$. This allows us to make use of Lemma 10.

To summarize, we have derived bounds on the number of hops for traversing linear arrangements of ε-good cubes and for making detours around defects. These bounds are sufficient for our purposes provided that neither o nor me_1 are contained in U'_m. In that situation, we need the following auxiliary result, where for $A\subset\mathbb{Z}^d$ we put $A\oplus[-\frac12,\frac12]^d=\bigcup_{z\in A}(z+[-\frac12,\frac12]^d)$.

Lemma 11. *Let $i\in\{1,\dots,m'\}$ and $X_{i_0}\in\mathcal{C}_\infty$ be such that $X_{i_0}\in(1-\varepsilon)(U_m^{(i)}\oplus[-\frac12,\frac12]^d)$. Then, there exists $X_{i_1}\in X\cap(1-\varepsilon)(\partial U_m^{(i)}\oplus[-\frac12,\frac12]^d)$ such that X_{i_0} and X_{i_1} are $(c_1\#U_m^{(i)})$-connectable, where $c_1=c_1(d)\ge 1$ is a constant depending only on the dimension d.*

Proof. Loosely speaking, we proceed as follows. Since X_{i_0} is contained in $\mathcal{C}_\infty$, it is k-connectable to the boundary of $(1-\varepsilon)(U_m^{(i)}\oplus[-\frac12,\frac12]^d)$ for some $k\ge 1$. Then, we make use of the observation in [15, Lemma 3.4] that the minimum such k cannot be too large in comparison to $\#U_m^{(i)}$. To be more precise, let $\gamma=\langle X_{i_0}=X_{j_1},\dots,X_{j_k}\rangle$ be some path in X consisting of hops of distance at most 1 such that $i_1=j_k$ is contained in $(1-\varepsilon)(\partial U_m^{(i)}\oplus[-\frac12,\frac12]^d)$. We note that there is a constant $c'_1=c'_1(d)\ge 1$ with the following property. There exists a finite subset S of $\mathbb{R}^d$ consisting of at most $c'_1\#U_m^{(i)}$ elements and such that for every $y\in(1-\varepsilon)((U_m^{(i)}\cup\partial U_m^{(i)})\oplus[-\frac12,\frac12]^d)$ there exists $y'\in S$ with $|y-y'|\le\frac12$. If there exist $y_1,\dots,y_k\in S$ with $|X_{j_\ell}-y_\ell|\le\frac12$ for every $\ell\in\{1,\dots,k\}$ and such that for every $\ell\in\{1,\dots,k\}$ there exists at most one $\ell'\in\{1,\dots,k\}\setminus\{\ell\}$ with $y_\ell=y_{\ell'}$, then the claim follows from the observation that $k\le 2\#S\le 2c'_1\#U_m^{(i)}$. Hence, it remains to transform γ into a path γ' with that property. This can be achieved by using Lawler's method of loop erasure [9].

To be more precise, let $\ell \in \{1, \dots, k\}$ be the largest index such that $|X_{j_\ell} - y_1| \le \frac{1}{2}$. In particular, $|X_{j_1} - X_{j_\ell}| \le 1$ and $|X_{j_\ell} - X_{j_{\ell+1}}| \le 1$. Now the construction proceeds inductively by defining γ' as the path obtained by pasting the paths $\langle X_{j_1}, X_{j_\ell}, X_{j_{\ell+1}}\rangle$ and γ'', where γ'' is the loop erasure of the path $\langle X_{j_{\ell+1}}, \dots, X_{j_k}\rangle$. □

Let $m_\varepsilon(n)$ be the unique integer contained in the interval

$$\Big[\frac{n}{1-\varepsilon} - \frac{1}{2}, \frac{n}{1-\varepsilon} + \frac{1}{2}\Big).$$

Combining Lemmas 9–11, we can now construct a short path connecting $q(o)$ and $q(n\boldsymbol{e}_1)$. First, by Lemma 11, $q(o)$ and $q(n\boldsymbol{e}_1)$ can be connected to points in $X \cap (1-\varepsilon)(\partial U_m^{(1)} \oplus [-\frac{1}{2}, \frac{1}{2}]^d)$ and $X \cap (1-\varepsilon)(\partial U_m^{(m')} \oplus [-\frac{1}{2}, \frac{1}{2}]^d)$, respectively, by paths of at most $2c_1 \# U'_{m_\varepsilon(n)}$ hops in total. Next, by Lemma 9, for any $i \in \{1, \dots, m''\}$ every point in $X \cap (a_i(1-\varepsilon)\boldsymbol{e}_1 \oplus [-\varepsilon/4, \varepsilon/4]^d)$ is $(b_i - a_i)$-connectable to every point in $X \cap (b_i(1-\varepsilon)\boldsymbol{e}_1 \oplus [-\varepsilon/4, \varepsilon/4]^d)$. Aggregating over $i \in \{1, \dots, m''\}$ this gives paths of at most $m_\varepsilon(n)$ hops in total. Finally, using Lemma 10 to provide the missing links between these paths, we arrive at a path connecting $q(o)$ to $q(n\boldsymbol{e}_1)$ in at most

$$k = m_\varepsilon(n) + (3 + (2d)^d) \sum_{i=1}^{m'} \# \partial U^{(i)}_{m_\varepsilon(n)} + 2c_1 \# U'_{m_\varepsilon(n)} \tag{1}$$

hops. This construction is illustrated in Figure 2. In order to translate this observation into an upper bound for μ, it is important to have some control on the size of the random variables $\sum_{i=1}^{m'} \# \partial U^{(i)}_{m_\varepsilon(n)}$ and $\# U'_{m_\varepsilon(n)}$. In the following, we write $q_{\lambda,\varepsilon}$ for the probability that a fixed site is ε-bad. In particular,

$$q_{\lambda,\varepsilon} \le (2d)^d \exp(-\lambda(1-\varepsilon)^d (2d)^{-d}) + \exp(-\lambda 2^{-d} \varepsilon^d). \tag{2}$$

Lemma 12. *If* $q_{\lambda,\varepsilon} < 2^{-3^d - 1}$ *then* $\lim_{m\to\infty} \mathbb{P}(\sum_{i=1}^{m'} \# \partial U_m^{(i)} \ge 2^{3^d+2} 3^d q_{\lambda,\varepsilon} m) = 0$.

Proof. Note that any site in $\bigcup_{i=1}^{m'} \partial U_m^{(i)}$ is $*$-adjacent to an ε-bad $*$-connected component intersecting $\{o, \boldsymbol{e}_1, \dots, m\boldsymbol{e}_1\}$. It may happen that several sites in $\bigcup_{i=1}^{m'} \partial U_m^{(i)}$ share the same neighbor in U_m, but for each site in U_m the number of such sites in $\bigcup_{i=1}^{m'} \partial U_m^{(i)}$ does not exceed 3^d. Therefore,

$$\sum_{i=1}^{m'} \# \partial U_m^{(i)} \le 3^d \# U_m.$$

Furthermore, as shown in [5, Lemma 2.3], $\# U_m$ is stochastically dominated by $\sum_{i=0}^m R_i$, where $\{R_i\}_{0 \le i \le m}$ is a family of independent and identically distributed random variables such that R_i has the distribution of the size of the open $*$-connected component at the origin when considering Bernoulli site percolation with parameter $q_{\lambda,\varepsilon}$. The number of $*$-connected subsets of sites containing the origin and consisting of exactly $k \ge 1$ sites is bounded above by $2^{3^d k}$; see [12, Lemma 9.3]. Therefore,

$$\mathbb{E} R_0 \le \sum_{k=0}^{\infty} k 2^{3^d k} q_{\lambda,\varepsilon}^k = \frac{2^{3^d} q_{\lambda,\varepsilon}}{(1 - 2^{3^d} q_{\lambda,\varepsilon})^2} < 2^{3^d+2} q_{\lambda,\varepsilon}.$$

The claim now follows from the law of large numbers. □

Lemma 13. *If* $q_{\lambda,\varepsilon} < 2^{-3^d - 1}$ *then* $\lim_{m\to\infty} \mathbb{P}(\# U'_m \ge 2^{3^d+4} 9^d d^2 q_{\lambda,\varepsilon} m) = 0$.

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Proof. By the isoperimetric inequality [5, Equation (2.1)], $\#U_m^{(i)} \le d^2(\#\partial U_m^{(i)})^2$ holds for all $i \in \{1, \dots, m'\}$. Indeed, [5, Equation (2.1)] yields the desired inequality if the boundary ∂ is formed with respect to the standard adjacency relation on $\mathbb{Z}^d$ and the $*$-boundary is always at least as large as that. Moreover, using the same notation as in the proof of Lemma 12, the sum $\sum_{i=1}^{m'}(\#\partial U_m^{(i)})^2$ is stochastically dominated by $9^d \sum_{i=0}^{m} R_i^2$, where

$$\mathbb{E}R_0^2 \le \sum_{k=0}^{\infty} k^2 2^{3^d k} q_{\lambda,\varepsilon}^k = \frac{(2^{3^d} q_{\lambda,\varepsilon} + 1)2^{3^d} q_{\lambda,\varepsilon}}{(1 - 2^{3^d} q_{\lambda,\varepsilon})^3} < 2^{3^d+4} q_{\lambda,\varepsilon}.$$

As before, the law of large numbers now implies the claim. □

In order to prove Theorem 3, we need to decrease ε accordingly in the size of λ. By the upper bound on $q_{\lambda,\varepsilon}$ derived in (2), we conclude that if we choose

$$\varepsilon = \varepsilon(\lambda) = 2\lambda^{-1/d}(\log \lambda)^{1/d}, \tag{3}$$

then $\lim_{\lambda\to\infty} \varepsilon^{-1} q_{\lambda,\varepsilon} = 0$.

Proof of Theorem 3. Choose ε as in (3) and put $\mu^+ = 1 + 3\varepsilon$. Then, it suffices to show that $\mathbb{P}(D_n \ge n\mu^+) \to 0$ as $n \to \infty$. Combining (1) with Lemmas 12 and 13, we see that it suffices to show that $m_\varepsilon(n) \le n(1 + 2\varepsilon)$. But since $1/(1 - \varepsilon) < 1 + 2\varepsilon$, this is an immediate consequence of the definition of $m_\varepsilon(n)$. □

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