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NONLINEAR FUNCTIONS AND CONVERGENCE TO BROWNIAN MOTION: BEYOND THE CONTINUOUS MAPPING THEOREM

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Weak convergence results for sample averages of nonlinear functions of (discrete-time) stochastic processes satisfying a functional central limit theorem (e.g., integrated processes) are given. These results substantially extend recent work by Park and Phillips (1999, *Econometric Theory* 15, 269–298) and de Jong (2002, working paper), in that a much wider class of functions is covered. For example, some of the results hold for the class of all locally integrable functions, thus avoiding any of the various regularity conditions imposed on the functions in Park and Phillips (1999) or de Jong (2002).

1. INTRODUCTION

A standard tool in the asymptotic theory of integrated processes and elsewhere is a functional central limit theorem. Typically, a real-valued stochastic process $(x_t)_{t\in\mathbb{N}}$ is considered such that $n^{-1/2}x_{[rn]}$, $0 \le r \le 1$, converges weakly to $\sigma W(r)$ (in the space $D[0,1]$ of cadlag functions), where $W(.)$ represents Brownian motion and $[x]$ denotes the integer part of x. Frequently, then the asymptotic behavior of a functional of the form $n^{-1}\sum_{t=1}^{n} T(n^{-1/2}x_t)$ is of interest. Such functionals arise in the construction of test statistics or in the theory of nonlinear estimation with integrated processes (Park and Phillips, 2001). For continuous real-valued functions T on $\mathbb{R}$ an argument based on the continuous mapping theorem shows that

$$n^{-1}\sum_{t=1}^{n} T(n^{-1/2}x_t) \xrightarrow{d} \int_0^1 T(\sigma W(r))\,dr, \tag{1.1}$$

where $\xrightarrow{d}$ signifies convergence in distribution; for convenience we include a formal statement and a proof in Appendix A; cf. Lemma A.1 and its proof. For an important subclass (cf. Park and Phillips, 1999, Assumption 2.1) of the class

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of all processes satisfying a functional central limit theorem, Theorem 3.2 in Park and Phillips (1999) shows that property (1.1) actually holds for a class of functions T wider than the class of continuous functions. Functions in that class are dubbed "regular" in Park and Phillips (1999). Apart from continuous functions, this class contains, e.g., locally bounded monotone functions and piecewise continuous functions.[1] However, it does not contain, e.g., every bounded (measurable) function. Furthermore, it also does not include (locally) unbounded functions such as $T(x) = \log|x|$ or $T(x) = |x|^{\alpha}$ for $-1 < \alpha < 0$ (cf. Park and Phillips, 1999, Remark 3.3(c)). [For functions of this latter kind, Theorem 3.4 of Park and Phillips (1999) presents a modified version of (1.1) with a suitably "clipped" approximation T_n replacing T on the l.h.s. of (1.1) under stronger conditions on the process x_t; cf. Assumption 2.2 in Park and Phillips, 1999. However, this theorem does not establish property (1.1) itself. We also note that the proof of this theorem, if not the theorem itself, seems to be in error.[2]] In a recent paper, de Jong (2002) establishes property (1.1) for a class of functions different from the class of "regular" functions appearing in Theorem 3.2 of Park and Phillips (1999) but covering functions such as $T(x) = \log|x|$ and $T(x) = |x|^{\alpha}$ for $-1 < \alpha < 0$. Roughly speaking, de Jong (2002) allows T to have finitely many "poles" and requires T to be continuous and monotone between poles; furthermore he requires T to be locally integrable.[3] De Jong's class neither contains all bounded (measurable) functions nor all "regular" functions in the sense of Park and Phillips. He establishes his result for processes satisfying (the stronger) Assumption 2.2 of Park and Phillips (1999).

In the present paper we establish the result (1.1) under the minimal condition that T is locally integrable (in the Lebesgue sense). In contrast to Park and Phillips (1999) and de Jong (2002) we thus are able to avoid *any* regularity condition on T save the unavoidable local integrability condition. Note that any "regular" function in the sense of Park and Phillips (1999) is locally bounded and thus is a fortiori locally integrable (cf. also Park and Phillips, 1999, Remark 3.3(a)). Thus, apart from covering a wider class of functions T than in Park and Phillips (1999) or de Jong (2002), our results also have the advantage of relieving one from the nontrivial burden of verifying regularity conditions as is necessary when using the results of Park and Phillips or de Jong. We first prove result (1.1) under high-level assumptions on the process x_t in Section 2. In Section 3 we provide sufficient conditions on the process x_t that imply these high-level assumptions, and we obtain corresponding corollaries. It turns out that one of these corollaries (Corollary 3.3) contains the result in de Jong (2002) as a special case in that it covers a much wider class of functions T (e.g., functions with infinitely many "poles," or functions that are neither piecewise monotone nor piecewise continuous) and at the same time imposes weaker conditions on x_t. Corollary 3.3, in fact, applies to any locally integrable function T that satisfies a certain growth condition at the origin. Corollary 3.2 moreover shows that this growth condition can be dispensed with if a rather mild condition on x_t, namely, that the innovations driving x_t have a bounded density, is added.

Corollary 3.2 thus only imposes the minimal condition of local integrability on T. Both of these corollaries cover classes of functions much wider than the class of "regular" functions considered in Theorem 3.2 of Park and Phillips (1999). Although the conditions on x_t maintained by these corollaries are somewhat stronger than the corresponding conditions in Theorem 3.2 of Park and Phillips (1999), we believe that the extra conditions on x_t in these corollaries are a small price to pay for the ability to cover much larger function classes. It should furthermore be noted that the assumptions on x_t in Corollary 3.3 are strictly weaker than the assumptions on x_t employed in Theorem 3.4 of Park and Phillips (1999), thus showing that the "clipping" device of that theorem can be avoided altogether. Section 4 concludes the main body of the paper and discusses some generalizations of the results in Sections 2 and 3. All proofs are relegated to the Appendixes.

After this paper had been written, the book by Borodin and Ibragimov (1995) came to my attention. In this important work results of the form (1.1) and also many other related results are established for the case when the process x_t is a random walk with increments that are independent and identically distributed (i.i.d.) and belong to the domain of attraction of a stable law. Their results always assume more in terms of the function T than we do in the present paper. (For example, one of the results in Borodin and Ibragimov, 1995, is for locally Riemann integrable functions, which constitute a much smaller class than the class of locally Lebesgue integrable functions. In particular, locally Riemann integrable functions are necessarily locally bounded, thus ruling out functions with poles.) Contrary to Park and Phillips (1999), de Jong (2002), and the present paper, Borodin and Ibragimov (1995) do not provide results for the case where the increments of x_t are dependent (e.g., follow a linear process). However, it should be noted that the results in Borodin and Ibragimov (1995) are more general along another dimension, namely, that the limiting behavior of x_t need not be given by Brownian motion but may be given by some stable process. A recent paper by Jeganathan (2002) takes up this issue and extends it to the case of dependent increments.

2. WEAK CONVERGENCE OF NONLINEAR FUNCTIONALS

Let $(x_t)_{t\in\mathbb{N}}$ be a stochastic process with values in $\mathbb{R}$. We shall make use of the following assumptions.

Assumption 2.1. The process $n^{-1/2}x_{[rn]}$, $0 \le r \le 1$, converges weakly to $\sigma W(r)$, where $W(.)$ is Brownian motion on $[0,1]$ and $\sigma \neq 0$, $\sigma \in \mathbb{R}$, holds. (As a convention, $x_{[rn]}$ is set equal to zero for $r < n^{-1}$.)[4]

As usual, it is understood that $W(0) = 0$ a.s. and that $W(.)$ has continuous sample paths a.s. Furthermore, weak convergence in the preceding assumption is understood w.r.t. the Skorohod topology on the space $D[0,1]$.

ASSUMPTION 2.2. For every $t \in \mathbb{N}$ the distribution of $t^{-1/2}x_t$ possesses a density, h_t, say, w.r.t. Lebesgue measure on $\mathbb{R}$. The densities h_t are uniformly bounded, i.e., $\sup_{t\in\mathbb{N}}\|h_t\|_\infty < \infty$ holds, where $\|.\|_\infty$ denotes the sup-norm.

In light of the fact that the distribution of $t^{-1/2}x_t$ converges to a normal distribution under Assumption 2.1, the conditions imposed by Assumption 2.2 have some intuitive appeal; in particular, if a local central limit theorem holds (cf. Ibragimov and Linnik, 1971, Theorem 4.3.1), then $\|h_t\|_\infty$ is automatically uniformly bounded (at least from some index onward). Sufficient conditions for Assumptions 2.1 and 2.2 will be discussed in the next section.

Let T be a real-valued Borel-measurable function on $\mathbb{R}$. We say that T is *locally integrable* if and only if[5]

$$\int_{-K}^{K} |T(x)|\, dx < \infty \quad \text{for any } 0 < K < \infty. \tag{2.2}$$

Condition (2.2) is certainly satisfied if T is locally bounded (i.e., $\sup_{|x|\le K}|T(x)| < \infty$ for any $0 < K < \infty$) but is much less restrictive because it allows also for many locally unbounded functions such as, e.g., $T(x) = \log|x|$ and $T(x) = |x|^\alpha$, $-1 < \alpha < 0$.[6] (For T in these latter two examples [or in similar cases] to be a proper real-valued function defined on *all* of $\mathbb{R}$, a real number has to be specified as the value of T at $x = 0$; if one desires to set $T(0) = -\infty$ or $T(0) = \infty$, respectively, T becomes a function with values in the extended real line; cf. Remarks 2.1 and 2.5, which follow.)

The following theorem establishes the main weak convergence result for locally integrable functions. It is remarkable in that it does not impose any regularity conditions on T beyond (2.2). Its generalization to the case of functions with values in the extended real line is given in Remark 2.1, which follows.

THEOREM 2.1. *Suppose Assumptions 2.1 and 2.2 hold and* $T{:}\mathbb{R} \to \mathbb{R}$ *is locally integrable. Then*

$$n^{-1}\sum_{t=1}^{n} T(n^{-1/2}x_t) \xrightarrow{d} \int_0^1 T(\sigma W(r))\, dr. \tag{2.3}$$

We note that the integral in (2.3) exists a.s. and is finite a.s. if and only if T is locally integrable (see Karatzas and Shreve, 1991, Ch. 3, Proposition 6.27 and Problem 6.29; cf. also Park and Phillips, 1999, Remark 3.3 (a)).[7] (In this sense the local integrability condition is an unavoidable requirement.)

We note for later use that the integral in (2.3) can equivalently be expressed in terms of local time. That is,

$$\int_0^1 T(\sigma W(r))\, dr = \int_{-\infty}^{\infty} T(\sigma x) L(1,x)\, dx \quad \text{a.s.,} \tag{2.4}$$

where $L(t,x)$ denotes Brownian local time (cf. Chung and Williams, 1990, Corollary 7.4).

Remark 2.1. (Extended Real Functions)

(a) Theorem 2.1 also holds if T is a Borel-measurable function from $\mathbb{R}$ to the extended real line $\mathbb{R} \cup \{-\infty,\infty\}$ that is locally integrable (e.g., if $T(x) = \log|x|$ for $x \neq 0$ and $=-\infty$ for $x = 0$ or if $T(x) = |x|^{\alpha}$ for $x \neq 0$ and $=\infty$ for $x = 0$, $-1 < \alpha < 0$). To see this, first note that we may change T into a locally integrable *real-valued* function T^* by modifying T only on a set of Lebesgue measure zero. Because the distribution of x_t is absolutely continuous by Assumption 2.2, $n^{-1}\sum_{t=1}^{n} T(n^{-1/2}x_t)$ coincides with $n^{-1}\sum_{t=1}^{n} T^*(n^{-1/2}x_t)$ a.s. and, in particular, is a.s. well defined. It hence suffices to show that the integral $\int_0^1 T(\sigma W(r))\,dr$ is well defined a.s. and coincides a.s. with $\int_0^1 T^*(\sigma W(r))\,dr$. For this it is enough to show that for almost every path of Brownian motion the set $D = \{r \in [0,1]\colon T(\sigma W(r)) \neq T^*(\sigma W(r))\}$ is a Lebesgue null set: Let A denote the Lebesgue null set $\{x \in \mathbb{R}\colon T(x) \neq T^*(x)\}$. Then $1_D(r) = 1_A(\sigma W(r))$ and hence $\int_0^1 1_D(r)\,dr = \int_0^1 1_A(\sigma W(r))\,dr$. Corollary 7.4 in Chung and Williams (1990) gives $\int_0^1 1_D(r)\,dr = \int_0^1 1_A(\sigma W(r))\,dr = \int_{-\infty}^{\infty} 1_A(\sigma x)L(1,x)\,dx = 0$ where $L(t,x)$ denotes local time and where the last equality follows because A is a Lebesgue null set and $\sigma \neq 0$. This establishes the claim.

(b) Similar reasoning as in (a) shows that equation (2.4) also holds for locally integrable functions $T\colon\mathbb{R} \to \mathbb{R} \cup \{-\infty,\infty\}$.

Remark 2.2. If T is a function from $\mathbb{R}$ to $\mathbb{R}^p$ (or to $(\mathbb{R} \cup \{-\infty,\infty\})^p$) with each component being locally integrable, then Theorem 2.1 continues to hold (where the r.h.s. of (2.3) is defined componentwise). This follows from Theorem 2.1 (and Remark 2.1) combined with the Cramér–Wold device.

Remark 2.3.

(a) If x_t satisfies the convergence condition in Assumption 2.1 and if Assumption 2.2 holds, then necessarily $\sigma \neq 0$ holds as is easily seen.

(b) If x_t satisfies the convergence condition in Assumption 2.1, but with $\sigma = 0$, and if T is continuous, then (2.3) continues to hold by Lemma A.1. However, this is not necessarily true for arbitrary locally integrable (even "regular") T as the following example shows.[8] Let $x_t = \sum_{j=1}^{t}(\varepsilon_j - \varepsilon_{j-1}) = \varepsilon_t - \varepsilon_0$ where the random variables ε_t are i.i.d. standard normal. Then the convergence condition in Assumption 2.1 is satisfied with $\sigma = 0$. Let $T(x) = 1_{(-\infty,0)}(x)$ and note that T is locally integrable and is even "regular" in the sense of Park and Phillips (1999). The l.h.s. of (2.3) is now equal to $n^{-1}\sum_{t=1}^{n} 1_{(-\infty,0)}(\varepsilon_t - \varepsilon_0)$, which converges a.s. to $E(1_{(-\infty,0)}(\varepsilon_t - \varepsilon_0)|\,\varepsilon_0) = \Phi(\varepsilon_0)$, which is positive ($\Phi$ denoting the standard normal cumulative distribution function [c.d.f.]). The r.h.s. of (2.3), however, is equal to $T(0) = 0$, because $\sigma = 0$.

Theorem 2.1 is in fact a special case of a more general result that makes use of a weaker version of Assumption 2.2.

Assumption 2.2*. There exists $a \in \mathbb{N}$ such that for every $t \geq a$ the distribution of $t^{-1/2}x_t$ possesses a density, h_t, say, w.r.t. Lebesgue measure on $\mathbb{R}$. Furthermore, $\sup_{t \geq a} \|h_t\|_\infty < \infty$ holds.

Assumption 2.2* does not restrict the distribution of $x_t, 1 \leq t < a$, at all. Of course, under any assumption implying existence and boundedness of h_t for $1 \leq t < a$, Assumption 2.2* becomes equivalent to Assumption 2.2. As with Theorem 2.1, Theorem 2.2 is formulated for real-valued functions. Its generalization to the case of extended real functions is given in Remark 2.5, which follows.

THEOREM 2.2. *Suppose Assumptions 2.1 and 2.2* hold and* $T:\mathbb{R} \to \mathbb{R}$ *is locally integrable. Then*

$$n^{-1} \sum_{t=a}^{n} T(n^{-1/2}x_t) \xrightarrow{d} \int_0^1 T(\sigma W(r))\, dr. \tag{2.5}$$

If, additionally,

$$n^{-1} \sum_{t=1}^{a-1} T(n^{-1/2}x_t) \to 0 \quad \text{in probability,} \tag{2.6}$$

then (2.3) also holds. (In case $a = 1$ *we use the convention that the sum in (2.6) is zero.)*

Of course, Theorem 2.1 is a special case of Theorem 2.2 (with $a = 1$). It is not difficult to see that existence and boundedness of h_t for $1 \leq t < a$ is a sufficient condition for (2.6) when T is locally integrable.[9] However, under this condition on h_t Assumptions 2.2 and 2.2* coincide. Theorem 2.2 is therefore useful as an alternative to Theorem 2.1 in situations where existence and uniform boundedness of the densities h_t can only be established from a certain index onward (cf. Section 3) and where (2.6) can be verified from some source other than boundedness of h_t for $1 \leq t < a$. For example, if T is bounded in a neighborhood of $x = 0$ (a fortiori if T is continuous at $x = 0$), condition (2.6) immediately follows regardless of the distribution of x_t, $1 \leq t < a$.[10] More general sufficient conditions are given in the following proposition.

PROPOSITION 2.3. *For Borel-measurable* $T:\mathbb{R} \to \mathbb{R}$ *consider the following conditions:*

(i) $T(x) = o(|x|^{-2})$ *for* $x \to 0$, $x \neq 0$.
(ii) T *is Lebesgue-integrable in a neighborhood of* $x = 0$, $|T(x)|$ *is increasing on* $(-\varepsilon, 0)$ *and decreasing on* $(0, \varepsilon)$ *for some* $\varepsilon > 0$.
(iii) T *is Lebesgue-integrable in a neighborhood of* $x = 0$, $|T(x)|$ *is increasing on* $(-\varepsilon, 0)$ *and bounded on* $(0, \varepsilon)$ *for some* $\varepsilon > 0$.

(iv) T is Lebesgue-integrable in a neighborhood of $x = 0$, $|T(x)|$ *is bounded on* $(-\varepsilon, 0)$ *and decreasing on* $(0, \varepsilon)$ *for some* $\varepsilon > 0$.

(v) T is bounded on $(-\varepsilon, 0)$ *and also on* $(0, \varepsilon)$ *for some* $\varepsilon > 0$.

Then each of conditions (ii)–(v) implies (i), which in turn implies (2.6).[11]

Simple corollaries to Theorem 2.2 that immediately follow from the preceding discussion are given next. In the important case where x_t has an absolutely continuous distribution for all t, the conditions in these corollaries can be weakened somewhat; see Remark 2.4, which follows.

COROLLARY 2.4. *Suppose Assumptions 2.1 and 2.2* hold and* $T: \mathbb{R} \to \mathbb{R}$ *is locally integrable. If T satisfies* $T(x) = o(|x|^{-2})$ *for* $x \to 0$, $x \neq 0$, *then the weak convergence result (2.3) holds.*

COROLLARY 2.5. *Suppose Assumptions 2.1 and 2.2* hold and* $T: \mathbb{R} \to \mathbb{R}$ *is locally bounded (and Borel-measurable). Then the weak convergence result (2.3) holds.*

As already noted, the "regularity" conditions on T in Park and Phillips (1999) and de Jong (2002), respectively, imply local integrability. Furthermore, their respective "regularity" conditions imply condition (i) in Proposition 2.3.[12] Hence, these "regularity" conditions also imply (2.6). Thus the "regularity" conditions in Park and Phillips (1999) and also in de Jong (2002) are stronger than the conditions imposed on T in Theorems 2.1 and 2.2 and Corollary 2.4; cf. also Example 3.2 in Section 3.

Remark 2.4. Suppose $T: \mathbb{R} \to \mathbb{R}$ is Borel-measurable and suppose that each x_t, $1 \leq t < a$, has a (possibly unbounded) density. Then (2.6) already follows if condition (i) in Proposition 2.3 holds only outside of a set of Lebesgue measure zero.[13] Consequently, given the above assumption on x_t, Corollary 2.4 already holds under this weaker form of condition (i). We also note that this weaker form of condition (i) is satisfied if, e.g., T is essentially bounded in $(-\varepsilon, 0) \cup (0, \varepsilon)$ for a suitable $\varepsilon > 0$.[14] In particular, given the above assumption on x_t, Corollary 2.5 then holds even for essentially locally bounded T.

Remark 2.5. (Extended Real Functions).

(a) Similarly as in Remark 2.1, the first claim in Theorem 2.2 also holds for locally integrable functions T from $\mathbb{R}$ to $\mathbb{R} \cup \{-\infty, \infty\}$; the second claim also holds provided that the expression in (2.6) is well defined (at least on a sequence of sets Ω_n with $P(\Omega_n) \to 1$ as $n \to \infty$).[15] This is, e.g., the case if each x_t, $1 \leq t < a$, has a (possibly unbounded) density. As another example, the expression in (2.6) is also always well defined, regardless of the distribution of x_t, $1 \leq t < a$, if T takes its values only in $\mathbb{R} \cup \{\infty\}$ (or in $\mathbb{R} \cup \{-\infty\}$).

(b) Suppose the function T in Proposition 2.3 takes now values in $\mathbb{R} \cup \{-\infty,\infty\}$.[16] Then again each of the conditions (ii)–(v) implies (i). (Note that under any of the conditions (i)–(v) the function T is in fact real-valued on $(-\varepsilon,\varepsilon)\backslash\{0\}$ for a suitable $\varepsilon > 0$.) Furthermore, condition (i) continues to imply (2.6) if $T(0)$ is finite or if none of the distributions of x_t, $1 \leq t < a$, has positive point mass at $x = 0$. In particular, Corollary 2.4 continues to hold for $T: \mathbb{R} \to \mathbb{R} \cup \{-\infty,\infty\}$ if additionally $T(0)$ is finite or if none of the distributions of x_t, $1 \leq t < a$, has positive point mass at $x = 0$.

(c) Remark 2.4 continues to hold for functions $T: \mathbb{R} \to \mathbb{R} \cup \{-\infty,\infty\}$. In particular, Corollary 2.4 also holds for $T: \mathbb{R} \to \mathbb{R} \cup \{-\infty,\infty\}$ already under the weaker form of condition (i), provided each x_t, $1 \leq t < a$, has a (possibly unbounded) density. Similarly, Corollary 2.5 already holds for essentially locally bounded $T: \mathbb{R} \to \mathbb{R} \cup \{-\infty,\infty\}$ under the same provision for x_t, $1 \leq t < a$. (If $T: \mathbb{R} \to \mathbb{R} \cup \{-\infty,\infty\}$ is locally bounded, we are back to the case of real-valued T, and hence Corollary 2.5 directly applies without any further provision on x_t, $1 \leq t < a$.)

Remark 2.6. Suppose $T: \mathbb{R} \to \mathbb{R}$ or $T: \mathbb{R} \to \mathbb{R} \cup \{-\infty,\infty\}$ is essentially locally bounded and suppose that Assumptions 2.1 and 2.2* hold. Then certainly (2.5) holds (even T locally integrable would suffice). We stress, however, that (2.3) need not follow in general without further assumptions. Remarks 2.4 and 2.5(c) provide such additional conditions. Alternatively, it follows from the preceding discussion that (2.3) also holds if we additionally assume that condition (i) in Proposition 2.3 holds, and that $T(0)$ is finite or none of the distributions of x_t, $1 \leq t < a$, has positive point mass at $x = 0$.

Remark 2.7. Similarly as in Remark 2.2, Theorem 2.2 continues to hold for functions T with values in $\mathbb{R}^p$. For functions T with values in $(\mathbb{R} \cup \{-\infty,\infty\})^p$ the same is also true for the first claim in Theorem 2.2, and it is true for the second claim provided (2.6) is well defined for any linear combination $\alpha' T$. A corresponding remark applies to Corollaries 2.4 and 2.5, Remark 2.4, and their extensions discussed in Remark 2.5.

3. SUFFICIENT CONDITIONS AND COROLLARIES

In this section we discuss the important special case when x_t is an integrated process, which is the case exclusively considered in Park and Phillips (1999) and de Jong (2002). Assume that for $n \geq 1$ the process x_n takes the form

$$x_n = x_0 + \sum_{t=1}^{n} w_t \tag{3.1}$$

with x_0 being independent of the process $(w_t)_{t\geq 1}$ and with w_t given by

$$w_t = \sum_{j=0}^{\infty} \phi_j \varepsilon_{t-j}. \tag{3.2}$$

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Here (ε_j) are i.i.d., $E(\varepsilon_j) = 0$, $E\varepsilon_j^2 < \infty$, $\sum_{j=0}^{\infty}|\phi_j| < \infty$, and $\sum_{j=0}^{\infty} \phi_j \neq 0$. Without loss of generality we shall set the variance of ε_j equal to one. Furthermore, it is assumed that ε_j has a density, say, q. The preceding assumptions will be kept throughout Section 3 and will be referred to as the maintained assumptions of Section 3.

3.1. Sufficient Conditions for Assumptions 2.2 and 2.2*

To begin with, note that x_n can be represented as

$$x_n = x_0 + \left[\sum_{j=1}^{n} c_{n-j}\,\varepsilon_j + \sum_{j=0}^{\infty} \gamma_{n,j}\,\varepsilon_{-j} \right], \tag{3.3}$$

where $c_{n-j} = \sum_{i=0}^{n-j} \phi_i$ and $\gamma_{n,j} = \sum_{i=j+1}^{n+j} \phi_i$. It immediately follows that $n^{-1/2}x_n$ has a density for every $n \geq 1$ (cf. Lukacs, 1970, Theorem 3.3.2, and observe that the term in brackets in (3.3) cannot be identically zero because $\sum_{j=0}^{\infty} \phi_j \neq 0$).

To motivate the sufficient conditions for Assumptions 2.2 and 2.2* given in Lemma 3.1, which follows, we start with a preparatory and informal discussion. It is easy to see that any distribution given by a convolution has a bounded density if at least one factor in the convolution has a bounded density. Consequently, the density h_n of $n^{-1/2}x_n$ is guaranteed to be bounded (for every fixed $n \geq 1$) if the (common) density q of ε_j is bounded. We note that a sufficient condition for boundedness of q is that ψ, the characteristic function of ε_j, is absolutely integrable (Lukacs, 1970, Theorem 3.2.2). However, the density h_n can be bounded even if the density q is unbounded. To see how this can happen, consider for the moment the special case where x_n is a random walk, i.e., where $w_t = \varepsilon_t$ and where $x_0 = 0$ (for simplicity). Because $n^{-1/2}x_n$ is then the sum of n i.i.d. random variables, its density h_n is the (scaled) n-fold convolution of q itself. Now, for example, if q has a pole, it can happen that this pole is "smoothed" out by the convolution operation, resulting in a bounded density h_n. Related to this observation is the fact that in cases where the characteristic function $\psi(s)$ of ε_j is not absolutely integrable, the characteristic function of $n^{-1/2}x_n$ can be integrable (implying that h_n is bounded) from some n onward, because it is the nth power of ψ (evaluated at $n^{-1/2}s$) and because $|\psi| \leq 1$ holds. It follows that absolute integrability of a power of ψ will imply (individual) boundedness of h_n, at least from a certain n onward, and thus will be a central condition in the following. As it turns out, this central condition implies not only *individual* boundedness of h_n (from a certain n onward) but also *uniform* boundedness (from a certain n onward). Returning to the case of general x_n as in (3.1), we note that (depending on the behavior of the coefficients ϕ_j) often h_n is in fact a convolution of much more than n factors of the form q (sometimes even of infinitely many factors). Not too surprisingly, in this case the previously mentioned central condition on ψ will automatically deliver individual boundedness of h_n for *every* $n \geq 1$.

With ψ denoting the characteristic function of ε_j, we shall therefore consider the following integrability condition:

$$\int_{-\infty}^{\infty} |\psi(s)|^{\nu} \, ds < \infty \quad \text{for some } \nu \in \mathbb{R} \quad \text{with } 1 \leq \nu < \infty. \tag{3.4}$$

Recall that (3.4) with $\nu = 1$ implies boundedness of q and that (3.4) becomes less stringent as ν increases. In particular, characteristic functions corresponding to unbounded densities can satisfy (3.4) with $\nu > 1$; cf. Remark 3.1(b), which follows. We mention here that a simple sufficient condition for (3.4) is $|\psi(s)| = O(s^{-\eta})$ as $s \to \infty$ for some $\eta > \nu^{-1}$. In particular, if $|\psi(s)| = O(s^{-\eta})$ for some $\eta > 0$, then (3.4) holds for some $\nu \geq 1$. The latter condition with "O" strengthened to "o" is used in Park and Phillips (1999) and de Jong (2002); see Section 3.3 for more discussion.

The following lemma provides sufficient conditions for Assumptions 2.2 and 2.2* and is inspired by Section 4.3 of Ibragimov and Linnik (1971). Part (i) of the lemma improves upon Lemma 1 in de Jong (2002). Recall that h_n denotes the density of $n^{-1/2}x_n$.

LEMMA 3.1. *Suppose condition (3.4) holds. Then the following statements are true.*

(i) There exist $n_0 \in \mathbb{N}$ and a real number C such that for $n \geq n_0$

$$\|h_n\|_\infty \leq C \tag{3.5}$$

holds; i.e., Assumption 2.2 is satisfied.*

(ii) If, for every $n \geq 1$, at least ν coefficients of the innovations ε_j, $-\infty < j \leq n$, in (3.3) are nonzero, then (3.5) holds for $n \geq n_0 = 1$.[17] *That is, Assumption 2.2 is satisfied.*

(iii) If $\nu = 1$, then (3.5) holds for $n \geq n_0 = 1$. That is, Assumption 2.2 is satisfied.

(The constants C in (i)–(iii) and also the index n_0 depend only on ψ and the coefficients ϕ_j.)

The more difficult part in the proof of the preceding lemma is to establish Assumption 2.2*, i.e., the uniform boundedness of the densities h_n from a certain index n_0 onward. Once Assumption 2.2* is known to hold, Assumption 2.2 then follows under *any* condition that implies (individual) boundedness of h_n for every n (in fact for every n, $1 \leq n < a$, suffices). Parts (ii) and (iii) provide such conditions. The basic observation here is that whenever (3.4) holds and the distribution of x_n is the convolution of not less than ν terms of the form q (not counting the factor corresponding to x_0), then $\|h_n\|_\infty$ is finite (cf. Lemma B.2 in Appendix B). The additional assumptions in parts (ii) and (iii) precisely imply this for the distribution of x_n. As already mentioned, boundedness of q, i.e.,

$$\|q\|_\infty < \infty, \tag{3.6}$$

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implies that $\|h_n\|_\infty$ is finite for every $n \geq 1$. Thus (3.6) is an alternative condition under which Assumptions 2.2 and 2.2* are equivalent. (We note that (3.4) with $\nu = 1$ implies (3.6); cf. Lukacs, 1970, Theorem 3.2.2.)[18]

Note that the conditions in Lemma 3.1(i) and (ii) allow the density q of ε_j to be unbounded, whereas the conditions for part (iii) imply boundedness of q.

Remark 3.1.

(a) The assumption in part (ii) is certainly satisfied if the coefficients ϕ_j are all positive (negative).

(b) The additional assumption in part (ii) cannot be removed. Consider the example where $w_t = \varepsilon_t - \varepsilon_{t-1} + \varepsilon_{t-2}$ and $\alpha^{1/2}\varepsilon_t + \alpha$ is gamma-distributed with shape parameter α satisfying $\frac{1}{3} < \alpha < \frac{1}{2}$ and scale parameter 1 and where $x_0 = 0$ (for simplicity). Then $x_2 = \varepsilon_2 + \varepsilon_{-1}$ whereas x_n for $n \neq 2$ is always the sum of at least three ε_j's. Consequently, the density of x_2 (being a shifted and scaled version of a gamma$(2\alpha,1)$-distribution) has a pole, whereas the density of x_n, $n \neq 2$ (being a shifted and scaled version of a gamma$(\beta,1)$-distribution with $\beta \geq 3\alpha > 1$) is bounded. Note that the characteristic function $\psi(s)$ of ε_j satisfies $|\psi(s)| = (1 + \alpha^{-1}s^2)^{-\alpha/2}$ and thus ψ satisfies (3.4) for $\nu > 1/\alpha > 2$ but not for $\nu \leq 1/\alpha$.

3.2. Sufficient Conditions for Assumption 2.1

Sufficient conditions for Assumption 2.1 abound in the literature. For the sake of comparability with Park and Phillips (1999) and de Jong (2002) we shall use the condition

$$\sum_{j=0}^{\infty} j^{1/2}|\phi_j| < \infty, \tag{3.7}$$

which is also used in Park and Phillips (1999) (cf. their Assumption 2.1). A stronger summability condition is used in Assumption 2.2 of Park and Phillips (1999) and also in de Jong (2002). It is well known that—under the maintained assumptions of Section 3—condition (3.7) implies our Assumption 2.1 with $\sigma = \sum_{j=0}^{\infty} \phi_j$ (cf., e.g., Phillips and Solo, 1992, Theorem 3.4 and Remarks 2.2(ii) and 3.5(i)).

3.3. Corollaries and Comparison with Park and Phillips (1999) and de Jong (2002)

The following corollary collects some of the results that can be obtained by combining Theorem 2.1 with the sufficient conditions discussed in Sections 3.1 and 3.2.

COROLLARY 3.2. *Suppose the process x_t satisfies the maintained assumptions of Section 3 and (3.4) and (3.7) hold. Let $T:\mathbb{R} \to \mathbb{R}$ or $T:\mathbb{R} \to \mathbb{R} \cup \{-\infty,\infty\}$ be locally integrable. Then*

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$$n^{-1}\sum_{t=1}^{n} T(n^{-1/2}x_t) \xrightarrow{d} \int_0^1 T(\sigma W(r))\,dr$$

holds with $\sigma = \sum_{j=0}^{\infty}\phi_j$*, provided the densities* h_t *are (individually) bounded for every* $t \geq 1$.[19] *This latter condition is satisfied if any of the following conditions holds.*

(i) The density of q *of* ε_j *is bounded; i.e., (3.6) holds.*
(ii) The characteristic function ψ *of* ε_j *is integrable; i.e., (3.4) holds with* $\nu = 1$.
(iii) $\phi_j > 0$ *for all* $j \geq 0$ *or* $\phi_j < 0$ *for all* $j \geq 0$.

The preceding corollary gives conditions that imply the desired convergence result for *all* locally integrable functions. The next corollary operates under weaker conditions on the process x_t at the expense of imposing a mild growth condition on the function T.

COROLLARY 3.3. *Suppose the process* x_t *satisfies the maintained assumptions of Section 3 and (3.4) and (3.7) hold. Let* $T:\mathbb{R}\to\mathbb{R}$ *or* $T:\mathbb{R}\to\mathbb{R}\cup\{-\infty,\infty\}$ *be locally integrable. Then* $n^{-1}\sum_{t=a}^{n} T(n^{-1/2}x_t) \xrightarrow{d} \int_0^1 T(\sigma W(r))\,dr$ *with* $\sigma = \sum_{j=0}^{\infty}\phi_j$ *holds for some* $a \geq 1$.[20] *Furthermore,*

$$n^{-1}\sum_{t=1}^{n} T(n^{-1/2}x_t) \xrightarrow{d} \int_0^1 T(\sigma W(r))\,dr$$

holds, provided $T(x) = o(|x|^{-2})$ *for* $x \to 0$, $x \neq 0$*, except possibly on a set of Lebesgue measure zero. This latter condition is satisfied if any of the conditions (ii)–(v) of Proposition 2.3 hold.*[21]

A simple special case of Corollary 3.3 is the following result.

COROLLARY 3.4. *Suppose the process* x_t *satisfies the maintained assumptions of Section 3 and (3.4) and (3.7) hold. Let* $T:\mathbb{R}\to\mathbb{R}$ *or* $T:\mathbb{R}\to\mathbb{R}\cup\{-\infty,\infty\}$ *be essentially locally bounded (and Borel-measurable). Then*

$$n^{-1}\sum_{t=1}^{n} T(n^{-1/2}x_t) \xrightarrow{d} \int_0^1 T(\sigma W(r))\,dr$$

holds with $\sigma = \sum_{j=0}^{\infty}\phi_j$.

Corollary 3.2 is based on Theorem 2.1 and hence on Assumption 2.2, whereas Corollary 3.3 derives from Theorem 2.2 and Assumption 2.2*. As already noted, Assumption 2.2 differs from Assumption 2.2* only in that it additionally requires the first few densities h_t to be (individually) bounded. As a consequence, the requirements on the process x_t in Corollary 3.2 are only marginally stronger than in Corollary 3.3; e.g., adding condition (3.6), i.e., that the density q of ε_j is bounded, suffices. The advantage of Corollary 3.2 thus is that it does not impose any regularity condition on the function T but delivers the desired convergence result for *any* locally integrable function T (at a small cost in terms of additional conditions on x_t).

It is easy to see that Corollary 3.3 contains the convergence result in de Jong (2002) as a special case: First, de Jong uses stronger assumptions on the process x_t (namely, the stronger summability condition $\sum_{j=0}^{\infty} j|\phi_j| < \infty$, existence of moments of ε_j of order higher than 2, and the stronger condition $|\psi(s)| = o(s^{-\eta})$ for some $\eta > 0$ on the characteristic function ψ). Second, the class of functions considered in Corollary 3.3 is much wider than the class considered in de Jong (2002) as the discussion subsequent to Corollary 2.5 has shown.

Comparing Corollary 3.3 with Theorem 3.2 in Park and Phillips (1999), we observe that Corollary 3.3 (and a fortiori Corollary 3.2) allows for a much wider class of functions than Theorem 3.2 in Park and Phillips (1999). In particular, Corollary 3.3 not only covers any (essentially) locally bounded function (cf. Corollary 3.4) but also allows for locally unbounded functions and extended real-valued functions. (Recall that any function that is “regular” in the sense of Park and Phillips, 1999, is locally bounded.) With respect to the conditions imposed on the process x_t, note that Corollary 3.3 makes use of the same assumptions as used in Theorem 3.2 in Park and Phillips (1999) plus the additional condition (3.4) and the assumption that the innovations ε_j possess an absolutely continuous distribution. (Comparing Corollary 3.2 with Theorem 3.2 in Park and Phillips (1999) we see that a further mild condition such as, e.g., boundedness of the density of ε_j has been added.) This seems to be a modest price to pay for the ability to cover much larger classes of functions. Finally, we also point out that the conditions on x_t in Corollary 3.3 are strictly weaker than the assumptions underlying Theorem 3.4 in Park and Phillips (1999) (cf. Park and Phillips, 1999, Assumption 2.2), which provides a weak convergence result for “clipped” versions of certain locally unbounded functions T. This shows that the “clipping” device of that theorem can be avoided altogether. (Recall from Section 1 that the proof of this theorem seems to be in error; cf. also note 2.)

We illustrate the corollaries with some examples.

Example 3.1

Suppose the process x_t satisfies the assumptions of Corollary 3.3. Let $T_1(x) = \log|x|$ and $T_2(x) = |x|^{\alpha}$ with $-1 < \alpha < 0$, where $T_1(0)$ and $T_2(0)$ are set to an arbitrary element of $\mathbb{R} \cup \{-\infty, \infty\}$. It is easy to see that both functions are locally integrable and satisfy $T_1(x) = o(|x|^{-2})$ and $T_2(x) = o(|x|^{-2})$ for $x \to 0$, $x \neq 0$. Corollary 3.3 then implies $n^{-1}\sum_{t=1}^{n} T_i(n^{-1/2}x_t) \xrightarrow{d} \int_0^1 T_i(\sigma W(r))\,dr$ for $i = 1,2$ with $\sigma = \sum_{j=0}^{\infty} \phi_j$.

The functions in the preceding example do not satisfy the “regularity” conditions for Theorem 3.2 in Park and Phillips (1999) but do satisfy the “regularity” conditions of de Jong (2002). The following example is covered neither by the results in Park and Phillips (1999) (because the functions are not locally bounded) nor by the results in de Jong (2002) (because the functions are not piecewise monotone).

Example 3.2

Suppose the process x_t satisfies the assumptions of Corollary 3.3. Let $T_3(x) = (\log|x|)\sin(x^{-1})$ and $T_4(x) = |x|^{\alpha}\sin(x^{-1})$ with $-1 < \alpha < 0$, where $T_3(0)$ and $T_4(0)$ are set to an arbitrary element of $\mathbb{R} \cup \{-\infty,\infty\}$. Again both functions are locally integrable and satisfy $T_3(x) = o(|x|^{-2})$ and $T_4(x) = o(|x|^{-2})$ for $x \to 0$, $x \neq 0$. Corollary 3.3 then implies $n^{-1}\sum_{t=1}^{n} T_i(n^{-1/2}x_t) \xrightarrow{d} \int_0^1 T_i(\sigma W(r))\,dr$ for $i = 3,4$ with $\sigma = \sum_{j=0}^{\infty}\phi_j$. In fact, Corollary 3.3 applies as well to the functions $T_5(x) = (\log|x|)S(x)$ and $T_6(x) = |x|^{\alpha}S(x)$ with $-1 < \alpha < 0$, where $T_5(0)$ and $T_6(0)$ are set to an arbitrary element of $\mathbb{R} \cup \{-\infty,\infty\}$ and where S is an arbitrary (essentially) local bounded Borel-measurable function.

4. EXTENSIONS

The results in Section 3 allow for dependence in the increments of the process x_t as they are modeled as a linear process. It is quite natural to ask to what extent the results in Section 3 can be generalized to other dependence structures such as mixing, near epoch dependence, and so on. Observe that the results in Section 2 are of a generic nature in that they rely only on Assumptions 2.1 and 2.2 (or 2.2*), which do not specify a particular dependence structure. Because functional central limit theorems as expressed in Assumption 2.1 are widely available for various dependence structures, including those mentioned previously, the question reduces to whether or not Assumption 2.2 (or 2.2*) holds for such dependence structures. In particular, the validity of a local central limit theorem would imply Assumption 2.2*. Not much seems to be available in the literature in that regard.

A key feature of the results in this paper is that the random variables $t^{-1/2}x_t$ have to have uniformly bounded densities (at least from some index onward). In view of local central limit theorems this appears to be a quite natural condition. Whether or not this condition can be relaxed while retaining the validity of the convergence result for all locally integrable functions, I do not know. Of course, relaxation is certainly possible if the convergence result is to be established only for a smaller class of functions T (e.g., Assumption 2.2 or 2.2* can be completely dropped for continuous T).

Suppose the convergence result (2.3) holds for a function H and suppose the function T satisfies $T(\lambda x) = g(\lambda)H(x)$ for all $\lambda > 0$ and all $x \in \mathbb{R}$ with a suitable function g (e.g., T is homogenous of degree α and $H = T$). Then (2.3) applied to H can be rewritten as

$$(ng(n^{1/2}))^{-1}\sum_{t=1}^{n} T(x_t) \xrightarrow{d} \int_0^1 H(\sigma W(r))\,dr. \tag{4.1}$$

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Now, if T does not satisfy a decomposition as before but does so approximately in a suitable sense, relation (4.1) can still be established. This then provides convergence results for nonlinear functions of unnormalized integrated processes. Section 5 of Park and Phillips (1999) carries through this program under the assumption that the function H appearing in the approximation is "regular" in their sense. De Jong and Whang (2002) obtain analogous results when H satisfies the "regularity" conditions of de Jong (2002). Based on the results of the present paper, both of these results can be extended to the situation where the function H in the approximation is locally integrable but does not satisfy the regularity conditions in Park and Phillips (1999) or de Jong (2002).

NOTES

1. The last claim is true if one adopts a definition of piecewise continuity such that the l.h.s. and r.h.s. limits exist and are finite at each point of discontinuity.

2. A function T can be constructed that satisfies all the conditions of Theorem 3.4 in Park and Phillips (1999) but does not satisfy $c_n T(c_n) \to 0$ as claimed in the proof of that theorem. It seems that to salvage that theorem a condition such as $T(x) = o(|x|^{-1})$ for $x \to 0$, $x \neq 0$, needs to be added.

3. For T to be defined as a real-valued function on all of $\mathbb{R}$, de Jong (2002) assigns the value zero to T at the pole locations. The results in the present paper do not rely on this (arbitrary) assignment and also work for functions that assume the values ∞ or $-\infty$; cf. Remarks 2.1 and 2.5. (The arguments in these remarks also show that an assignment such as the one in de Jong, 2002, is in fact inconsequential under the assumptions on the process made in that paper.)

4. If, instead, $x_{[rn]}$ is set equal to an arbitrary random variable x_* for $r < n^{-1}$, which is defined on the probability space supporting (x_t), an equivalent assumption is obtained. More generally, Assumption 2.1 is unaffected by any modification made to finitely many elements of (x_t).

5. The integral in expression (2.2) is to be understood in the sense of Lebesgue.

6. Condition (2.2) is of course also satisfied if T is only essentially locally bounded (i.e., if $\text{ess-sup}_{|x|\leq K}|T(x)| < \infty$ for any $0 < K < \infty$, where ess-sup denotes the essential supremum w.r.t. Lebesgue measure).

7. The integral over the positive part $T^+(\sigma W(r))$ and also the integral over the negative part $T^-(\sigma W(r))$ exist a.s. for every Borel-measurable T, because almost every sample path of $W(.)$ is continuous. The argument in the proof of Theorem 2.1 then also establishes a.s. finiteness of both these integrals under local integrability.

8. Of course, it is trivially true for any real-valued T if, e.g., $x_t = 0$ with probability one for all $t \in \mathbb{N}$.

9. To see this note that for every t, $1 \leq t < a$, (and $M > 0$) we have $P(n^{-1}|T(n^{-1/2}x_t)| > \delta) \leq P(n^{-1}|T(n^{-1/2}x_t)| > \delta, |n^{-1/2}x_t| \leq M) + P(|n^{-1/2}x_t| > M) \leq (n\delta)^{-1}\int_{-\infty}^{\infty}|T((t/n)^{1/2}x)| \times 1_{[-M,M]}((t/n)^{1/2}x)h_t(x)\,dx + o(1) = (n\delta)^{-1}(n/t)^{1/2}\int_{-M}^{M}|T(z)|h_t((n/t)^{1/2}z)\,dz + o(1) \leq \delta^{-1} \times n^{-1/2}t^{-1/2}\|h_t\|_\infty \int_{-M}^{M}|T(z)|\,dz + o(1) = o(1)$ by local integrability of T.

10. Although (2.6) is true for such functions T, we stress that (2.6) is in general *not* true without further conditions even for locally integrable T.

11. In fact, each one of (ii)–(v) even implies $T(x) = o(|x|^{-1})$ for $x \to 0$, $x \neq 0$.

12. Observe that any function "regular" in the sense of Park and Phillips (1999) is locally bounded and thus satisfies (v) of Proposition 2.3. Furthermore, any function "regular" in the sense of de Jong (2002) satisfies at least one of (ii)–(v) in Proposition 2.3 as is easily seen.

13. To see this, note that after suitably modifying T on a set of Lebesgue measure zero, condition (i) in Proposition 2.3 is satisfied for the modified function and that this modification changes the sum in (2.6) at most on a set of probability zero as a result of the assumption on x_t, $1 \leq t < a$.

14. More generally, if modifying T on a set of Lebesgue measure zero results in a function that satisfies one of conditions (ii)–(v) of Proposition 2.3, then condition (i) holds outside a set of Lebesgue measure zero.

15. That is, there exists a sequence of sets Ω_n with $P(\Omega_n) \to 1$ as $n \to \infty$ such that $T(n^{-1/2}x_t(\omega)) = \infty$ and $T(n^{-1/2}x_s(\omega)) = -\infty$ do not hold simultaneously for $\omega \in \Omega_n$ and some $1 \leq s < a$, $1 \leq t < a$.

16. As a point of interest we note that for real-valued T condition (v) is in fact equivalent to boundedness of T on $(-\varepsilon, \varepsilon)$, but this is not necessarily so if T takes its values in $\mathbb{R} \cup \{-\infty,\infty\}$.

17. Lemma B.2 in fact shows that if at least ν coefficients of ε_j, $-\infty < j \leq n$, in (3.3) are nonzero for a *given* n, then $\|h_n\|_\infty$ is finite for this n.

18. Together with Lemma 3.1(i) this provides an alternative proof of part (iii) of Lemma 3.1.

19. In fact, boundedness of h_t for $1 \leq t < n_0$ suffices, where n_0 is as in Lemma 3.1(i).

20. Namely, for $a \geq n_0$; cf. Lemma 3.1(i).

21. For a minor generalization of this implication see Remark 2.4 and note 14.

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APPENDIX A: PROOFS FOR SECTION 2

LEMMA A.1. *Suppose $T:\mathbb{R} \to \mathbb{R}$ is continuous and Assumption 2.1 holds with the requirement $\sigma \neq 0$ omitted. Then (1.1) holds.*

Proof. Define $\mathcal{I}_T(f)$ as $\int_0^1 T(f(r))\,dr$ for every $f \in D[0,1]$. Because each $f \in D[0,1]$ is bounded and measurable (Billingsley, 1968, p. 110) and because T is continuous, $\mathcal{I}_T(f)$ is well defined and finite. Observe that (1.1) can be rewritten as

$$\mathcal{I}_T(n^{-1/2}x_{[.n]}) + n^{-1}[T(n^{-1/2}x_n) - T(0)] \xrightarrow{d} \mathcal{I}_T(\sigma W(.)),$$

where the second term on the l.h.s. is $o_p(1)$ because T is continuous and $n^{-1/2}x_n$ converges in distribution. It hence suffices to establish that $\mathcal{I}_T(n^{-1/2}x_{[.n]})$ converges to $\mathcal{I}_T(\sigma W(.))$ in distribution. Suppose now that $f_k \in D[0,1]$ converges to $f \in C[0,1]$ (the subset of all continuous functions on $[0,1]$) w.r.t. the Skorohod topology. Then this convergence is in fact uniform (Billingsley, 1968, p. 112). In particular, it follows that f_k and f are uniformly bounded (w.r.t. $r \in [0,1]$ and $k \geq 1$) by a finite positive constant, say, M. Because T restricted to $[-M, M]$ is uniformly continuous, it follows that $T(f_k(r))$ converges to $T(f(r))$ uniformly on $[0,1]$. Thus, $\mathcal{I}_T(f_k)$ converges to $\mathcal{I}_T(f)$. It follows that the set of continuity points of $\mathcal{I}_T$ contains $C[0,1]$. Because almost every sample path of Brownian motion is an element of $C[0,1]$, it follows that the set of continuity points of $\mathcal{I}_T$ is a set of measure one under the measure induced by $\sigma W(.)$. Applying the continuous mapping theorem in its extended form (e.g., Billingsley, 1968, Theorem 5.1) then establishes (1.1). ∎

LEMMA A.2. *Let $T: \mathbb{R} \to \mathbb{R}$ be a locally integrable function. For every $\varepsilon > 0$ there exists a continuous function $\tilde{T}: \mathbb{R} \to \mathbb{R}$ such that $\|T - \tilde{T}\|_1 < \varepsilon$, where $\|T - \tilde{T}\|_1$ denotes $\int_{-\infty}^{\infty} |T(x) - \tilde{T}(x)|\, dx$.*

Proof. For any $m \in \mathbb{Z}$ define $T_m(x) = T(x)1_{[m,m+1)}(x)$. Because T is locally integrable, the function T_m is certainly Lebesgue-integrable over $[m, m+1]$. Hence, there exists a continuous function $\bar{T}_m: [m, m+1] \to \mathbb{R}$ such that $\int_m^{m+1} |T_m(x) - \bar{T}_m(x)|\, dx < (\varepsilon/3)2^{-|m|-1}$ (cf. Bauer, 1978, (43.6) and (44.2)). Extend $\bar{T}_m$ to a function on all of $\mathbb{R}$ by setting $\bar{T}_m(x) = 0$ for $x \notin [m, m+1]$. Obviously then $\|T_m - \bar{T}_m\|_1 < (\varepsilon/3)2^{-|m|-1}$ holds. Note that $\bar{T}_m$ is continuous on $\mathbb{R}$ except possibly at $x = m$ and $x = m+1$. For $0 < \eta < \frac{1}{2}$ let $g_{m,\eta}$ denote the "trapezoidal" function given by $g_{m,\eta}(x) = 1$ for $m + \eta \leq x \leq m + 1 - \eta$, $g_{m,\eta}(x) = 0$ for $x \leq m$ and for $x \geq m+1$ and that linearly interpolates between $x = m$ and $x = m + \eta$ and also between $x = m + 1 - \eta$ and $x = m + 1$. Then the function $\bar{T}_m g_{m,\eta}$ is continuous on all of $\mathbb{R}$ and vanishes outside of $(m, m+1)$. By choosing $\eta(m)$ small enough (depending on T and ε) we obtain $\|\bar{T}_m - \bar{T}_m g_{m,\eta(m)}\|_1 < (\varepsilon/3)2^{-|m|-1}$. Define $\tilde{T} = \sum_{m\in\mathbb{Z}} \bar{T}_m g_{m,\eta(m)}$ and note that $\tilde{T}$ is continuous on $\mathbb{R}$. Since clearly $T = \sum_{m\in\mathbb{Z}} T_m$ holds, we arrive at $\|T - \tilde{T}\|_1 \leq \sum_{m\in\mathbb{Z}} \|T_m - \bar{T}_m g_{m,\eta(m)}\|_1 \leq \sum_{m\in\mathbb{Z}} (\|T_m - \bar{T}_m\|_1 + \|\bar{T}_m - \bar{T}_m g_{m,\eta(m)}\|_1) < \sum_{m\in\mathbb{Z}} (\varepsilon/3)2^{-|m|} = \varepsilon$. ∎

Proof of Theorem 2.1. The idea of the proof is to use Lemma A.2 to reduce the case of locally integrable T to the case of continuous T and then to appeal to Lemma A.1, which in turn rests on the continuous mapping theorem.

Step 1. Let $\varepsilon > 0$ and let $\tilde{T}$ be the continuous function guaranteed by Lemma A.2. Then for all $n \geq 1$ we have

$$
\begin{aligned}
E\Big|n^{-1} \sum_{t=1}^{n} T(n^{-1/2}x_t) - n^{-1} \sum_{t=1}^{n} \tilde{T}(n^{-1/2}x_t)\Big| &\leq n^{-1} \sum_{t=1}^{n} E|T(n^{-1/2}x_t) - \tilde{T}(n^{-1/2}x_t)| \\
&= n^{-1} \sum_{t=1}^{n} \int_{-\infty}^{\infty} |T((t/n)^{1/2}x) - \tilde{T}((t/n)^{1/2}x)|\, h_t(x)\, dx \\
&= n^{-1/2} \sum_{t=1}^{n} t^{-1/2} \int_{-\infty}^{\infty} |T(z) - \tilde{T}(z)|\, h_t((n/t)^{1/2}z)\, dz \\
&\leq n^{-1/2} \sum_{t=1}^{n} t^{-1/2} \|T - \tilde{T}\|_1 \|h_t\|_\infty \leq 2\varepsilon \sup_{t\geq 1} \|h_t\|_\infty. \qquad \textbf{(A.1)}
\end{aligned}
$$

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Step 2. Let $\varepsilon > 0$ and let $\tilde{T}$ be as in step 1. Observing that $|T(\sigma x) - \tilde{T}(\sigma x)|$ is locally integrable we may apply Corollary 7.4 in Chung and Williams (1990) to obtain

$$E\left|\int_0^1 T(\sigma W(r))\,dr - \int_0^1 \tilde{T}(\sigma W(r))\,dr\right| \le E\int_0^1 |T(\sigma W(r)) - \tilde{T}(\sigma W(r))|\,dr$$

$$= E\int_{-\infty}^{\infty} |T(\sigma x) - \tilde{T}(\sigma x)| L(1,x)\,dx = \int_{-\infty}^{\infty} |T(\sigma x) - \tilde{T}(\sigma x)| EL(1,x)\,dx. \qquad \textbf{(A.2)}$$

For the last equality in (A.2) we have used Fubini's theorem. This is justified because the functions involved are nonnegative and because $L(1,x)$ is a measurable stochastic process. (That is, the map $(\omega, x) \to L(1,x)(\omega)$ is measurable w.r.t. the product σ-field $\mathcal{A} \otimes \mathcal{B}(\mathbb{R})$ where $\mathcal{A}$ is the σ-field on the probability space supporting $W(.)$ and $\mathcal{B}(\mathbb{R})$ is the Borel-σ-field on $\mathbb{R}$. This is true because L has continuous sample paths; cf. Chung and Williams, 1990, p. 146; Karatzas and Shreve, 1991, Remark 1.14.) Now, for every $x \in \mathbb{R}$, the local time $L(1,x)$ has a distribution that has point mass $2\Phi(|x|) - 1$ at the origin and otherwise has a density given by $k(y) = (2/\pi)^{1/2}\exp[-0.5(y + |x|)^2]$ for $y > 0$ and $k(y) = 0$ else (cf. Borodin and Salminen, 1996, p. 127, eq. (1.3.4)). Consequently,

$$EL(1,x) = 2[\phi(|x|) - |x|(1 - \Phi(|x|))] \le 2\phi(|x|) \le (2/\pi)^{1/2}$$

for all $x \in \mathbb{R}$, and hence the r.h.s. of (A.2) is not less than

$$(2/\pi)^{1/2}\int_{-\infty}^{\infty} |T(\sigma x) - \tilde{T}(\sigma x)|\,dx = (2/\pi)^{1/2}|\sigma^{-1}|\,\|T - \tilde{T}\|_1 \le (2/\pi)^{1/2}|\sigma^{-1}|\varepsilon. \qquad \textbf{(A.3)}$$

Step 3. It follows from steps 1 and 2 that for every $\eta > 0$ we can find a continuous function $\tilde{T}_\eta : \mathbb{R} \to \mathbb{R}$ such that

$$\sup_{n\ge 1} E\left|n^{-1}\sum_{t=1}^{n} T(n^{-1/2}x_t) - n^{-1}\sum_{t=1}^{n} \tilde{T}_\eta(n^{-1/2}x_t)\right| < \eta \qquad \textbf{(A.4)}$$

and

$$E\left|\int_0^1 T(\sigma W(r))\,dr - \int_0^1 \tilde{T}_\eta(\sigma W(r))\,dr\right| < \eta \qquad \textbf{(A.5)}$$

hold. By Lemma A.1 we have

$$n^{-1}\sum_{t=1}^{n} \tilde{T}_\eta(n^{-1/2}x_t) \xrightarrow{d} \int_0^1 \tilde{T}_\eta(\sigma W(r))\,dr. \qquad \textbf{(A.6)}$$

Relations (A.4)–(A.6) establish the result (2.3) by a standard argument (cf. Anderson, 1971, Theorem 7.7.1). ■

Proof of Theorem 2.2. The proof of (2.5) is identical to the proof of Theorem 2.1 apart from mainly notational differences. (For step 3 observe that because of continuity of $\tilde{T}_\eta$ the first $a - 1$ terms in (A.6) are $o_p(1)$ and hence can be omitted.) The second claim then follows from (2.5) and (2.6). ■

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Proof of Proposition 2.3. That (ii) implies (i) is seen as follows. Because of the monotonicity property we have for $0 < x < \varepsilon$ the inequality

$$x|T(x)| \leq \int_0^x |T(\xi)|\, d\xi \leq \int_0^\varepsilon |T(\xi)|\, d\xi < \infty,$$

the final integral being finite because of integrability in a neighborhood of zero. Hence, $x|T(x)| \to 0$ for $x \to 0$, $x > 0$. A similar argument for $-\varepsilon < x < 0$ then shows that $T(x) = o(|x|^{-1})$ and hence is $o(|x|^{-2})$ for $x \to 0$ and $x \neq 0$. The implication (v) $\Rightarrow$ (i) is trivial. The implications (iii) $\Rightarrow$ (i) and (iv) $\Rightarrow$ (i) follow by combining the arguments for the proofs of (ii) $\Rightarrow$ (i) and (v) $\Rightarrow$ (i). It remains to prove (i) $\Rightarrow$ (2.6), and for this it suffices to show that $n^{-1}T(n^{-1/2}x_t) \to 0$ as $n \to \infty$ for any given t and any value of x_t. If $x_t = 0$, this follows trivially, because $T(0)$ is a real number. Otherwise, we obtain $n^{-1}T(n^{-1/2}x_t) = x_t^{-2}o(1) = o(1)$ as $n \to \infty$. ■

APPENDIX B: PROOFS FOR SECTION 3

LEMMA B.1. *Let ψ be the characteristic function of a distribution with mean zero and variance 1. Then there exists Δ, $0 < \Delta < 1$, such that $|\psi(s)| \leq \exp(-s^2/8)$ holds for $-\Delta \leq s \leq \Delta$.*

Proof. Theorem 2.3.3 in Lukacs (1970) implies that $\psi(s) = 1 - s^2/2 + \zeta(s)$ where $\zeta(s) = o(s^2)$ as $s \to 0$ and $\zeta(0) = 0$. Hence, there exists Δ', $0 < \Delta' < 1$, such that $|\zeta(s)| \leq s^2/4$ for $-\Delta' \leq s \leq \Delta'$. It follows that $|\psi(s)| \leq |1 - s^2/2| + |\zeta(s)| \leq 1 - s^2/2 + s^2/4 = 1 - s^2/4$ for $-\Delta' \leq s \leq \Delta'$. Since $\psi(0) = 1$ and ψ is continuous, it follows that there exists Δ'', $0 < \Delta'' \leq \Delta'$, such that $|\psi(s)| > 0$ holds for $-\Delta'' \leq s \leq \Delta''$. Hence, $\log|\psi(s)|$ is well defined on $-\Delta'' \leq s \leq \Delta''$ and satisfies $\log|\psi(s)| \leq \log(1 - s^2/4)$ on that interval. A Taylor series expansion of $\log(1 + x)$ around $x = 0$ then shows that for $-\Delta'' \leq s \leq \Delta''$

$$\log|\psi(s)| \leq \log(1 - s^2/4) = -s^2/4 + \xi(s),$$

where $\xi(s) = o(s^2)$ for $s \to 0$ and $\xi(0) = 0$. Choosing Δ, $0 < \Delta \leq \Delta'' < 1$, small enough we obtain $|\xi(s)| \leq s^2/8$ for $-\Delta \leq s \leq \Delta$. This implies $\log|\psi(s)| \leq -s^2/8$ for $-\Delta \leq s \leq \Delta$. ■

We note that a more careful choice of constant in the preceding proof establishes that for any $0 < \delta < \frac{1}{2}$ there exists a $\Delta = \Delta(\delta)$ as in the lemma such that $|\psi(s)| \leq \exp(-\delta s^2)$ holds for $-\Delta \leq s \leq \Delta$.

Proof of Lemma 3.1. It follows from Theorem 3.2.2. of Lukacs (1970) that $\|h_n\|_\infty \leq (2\pi)^{-1}\|\Psi_n\|_1$ provided the latter is finite, where Ψ_n denotes the characteristic function of $n^{-1/2}x_n$ and $\|.\|_1$ denotes the L_1-norm w.r.t. Lebesgue measure on $\mathbb{R}$. It hence suffices to bound $\|\Psi_n\|_1$.

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(i) Note that x_0 is independent of the term in brackets in the representation (3.3) and that both sums in the brackets are independent of each other. Hence,

$$\Psi_n(s) = E\exp(isn^{-1/2}x_n)$$
$$= E\exp(isn^{-1/2}x_0)E\exp\left(isn^{-1/2}\sum_{j=1}^{n} c_{n-j}\varepsilon_j\right)E\exp\left(isn^{-1/2}\sum_{j=0}^{\infty}\gamma_{n,j}\varepsilon_{-j}\right).$$

Consequently,

$$|\Psi_n(s)| \le \left|E\exp\left(isn^{-1/2}\sum_{j=1}^{n} c_{n-j}\varepsilon_j\right)\right| = \left|\prod_{j=1}^{n} E\exp(isn^{-1/2}c_{n-j}\varepsilon_j)\right|$$
$$= \prod_{j=1}^{n}|\psi(sn^{-1/2}c_{n-j})| = \prod_{j=1}^{n}|\psi(sn^{-1/2}|c_{n-j}|)|, \quad \textbf{(B.1)}$$

the final equality following from $|\psi(-s)| = |\psi(s)|$.

Now,

$$\int_{-\infty}^{\infty}|\Psi_n(s)|\,ds = \int_{|s|\le An^{1/2}}|\Psi_n(s)|\,ds + \int_{|s|>An^{1/2}}|\Psi_n(s)|\,ds, \quad \textbf{(B.2)}$$

for every $A > 0$. Performing the substitution $s \to sn^{-1/2}$ and using (B.1), the first integral on the r.h.s. of (B.2) can be bounded by

$$n^{1/2}\int_{|s|\le A}\prod_{j=1}^{n}|\psi(s|c_{n-j}|)|\,ds. \quad \textbf{(B.3)}$$

Choose $A = (2|c|)^{-1}\Delta > 0$, where Δ is as in Lemma B.1 and where $c = \sum_{i=0}^{\infty}\phi_i$, which is nonzero by assumption. Note that the coefficients c_k converge to c. Hence there is a $K \in \mathbb{N}$ such that $|c|/2 \le |c_k| \le 2|c|$ whenever $k \ge K$. Because every characteristic function is bounded by one in absolute value, and because $-A \le s \le A$ implies $-\Delta \le s|c_{n-j}| \le \Delta$ for $n-j \ge K$, the expression in (B.3) for $n > K$ is in view of Lemma B.1 bounded by

$$n^{1/2}\int_{|s|\le A}\prod_{j=1}^{n-K}|\psi(s|c_{n-j}|)|\,ds \le n^{1/2}\int_{|s|\le A}\prod_{j=1}^{n-K}\exp(-s^2c_{n-j}^2/8)\,ds$$
$$= n^{1/2}\int_{|s|\le A}\exp\left(-s^2\sum_{j=1}^{n-K}c_{n-j}^2/8\right)ds$$
$$\le n^{1/2}\int_{|s|\le A}\exp(-s^2c^2(n-K)/32)\,ds$$
$$\le (32\pi)^{1/2}|c^{-1}|n^{1/2}(n-K)^{-1/2} \le (32\pi)^{1/2}|c^{-1}|(K+1)^{1/2}.$$

Because (B.3) for $1 \le n \le K$ is clearly bounded by $2An^{1/2} \le |c|^{-1}\Delta K^{1/2}$, the expression in (B.3) is bounded by $C_1 = \max((32\pi)^{1/2}(K+1)^{1/2}, \Delta K^{1/2})/|c| < \infty$ for all $n \ge 1$.

To deal with the second term on the r.h.s. of (B.2), perform the same substitution as before and use (B.1) to obtain

$$\int_{|s|>An^{1/2}} |\Psi_n(s)|\,ds \leq n^{1/2} \int_{|s|>A} \prod_{j=1}^{n} |\psi(s|c_{n-j}|)|\,ds. \tag{B.4}$$

With K as defined after (B.3), we can then for $n > K$ bound (B.4) by

$$n^{1/2} \int_{|s|>A} \prod_{k=K}^{n-1} |\psi(s|c_k|)|\,ds, \tag{B.5}$$

because $|\psi(.)| \leq 1$. Applying Hölder's inequality successively $n-K$ times, (B.5) can for $n \geq K+\nu$ be bounded by

$$\begin{aligned} & n^{1/2} \prod_{k=K}^{n-1} \left(\int_{|s|>A} |\psi(s|c_k|)|^{n-K}\,ds \right)^{1/(n-K)} \\ & \quad \leq n^{1/2} \prod_{k=K}^{n-1} \left(|c_k|^{-1} \int_{|r|>A|c_k|} |\psi(r)|^{n-K}\,dr \right)^{1/(n-K)} \\ & \quad \leq 2|c|^{-1} n^{1/2} \left(\int_{|r|>A|c|/2} |\psi(r)|^{n-K}\,dr \right) \\ & \quad \leq 2|c|^{-1} n^{1/2} (\sup\{|\psi(r)| : |r| > A|c|/2\})^{n-K-\nu} \left(\int_{-\infty}^{\infty} |\psi(r)|^{\nu}\,dr \right). \end{aligned} \tag{B.6}$$

Because $A|c|/2 = \Delta/4 > 0$ and ψ is the characteristic function of an absolutely continuous distribution, the supremum in (B.6) is less than one. In view of (3.4), the r.h.s. of (B.6) is therefore bounded by a finite constant for $n \geq K+\nu$. This completes the proof of part (i).

(ii) In view of part (i) it suffices to show that $\|\Psi_n\|_1 < \infty$ holds for $1 \leq n < K+\nu$. Note that $|\Psi_n(s)| \leq |E\exp(isn^{-1/2}[\sum_{j=1}^{n} c_{n-j}\varepsilon_j + \sum_{j=0}^{\infty} \gamma_{n,j}\varepsilon_{-j}])|$. The result then follows from Lemma B.2, which is given subsequently.

(iii) This follows from part (ii), observing that the maintained assumption $\sum_{j=0}^{\infty} \phi_j \neq 0$ implies that at least one coefficient in the representation (3.3) is nonzero for every $n \geq 1$. ■

LEMMA B.2. *Suppose* $Z = W + \sum_{j=1}^{m} \alpha_j \varepsilon_{t(j)}$ *with* $\alpha_j \neq 0$ *for* $1 \leq j \leq m$ *and* W *is independent of* $\sum_{j=1}^{m} \alpha_j \varepsilon_{t(j)}$. *Then*

$$\int_{-\infty}^{\infty} |E\exp(isZ)|\,ds \leq (\min|\alpha_j|)^{-1} \int_{-\infty}^{\infty} |\psi(s)|^{\nu}\,ds < \infty,$$

provided (3.4) with $\nu \leq m$ *holds.*

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Proof. Observe that $|E\exp(isZ)| \leq |E\exp(is\sum_{j=1}^{m}\alpha_j\varepsilon_{t(j)})|$. Hence

$$\int_{-\infty}^{\infty}|E\exp(isZ)|\,ds \leq \int_{-\infty}^{\infty}\prod_{j=1}^{m}|\psi(s\alpha_j)|\,ds = \int_{-\infty}^{\infty}\prod_{j=1}^{m}|\psi(s|\alpha_j|)|\,ds$$

$$\leq \prod_{j=1}^{m}\left(\int_{-\infty}^{\infty}|\psi(s|\alpha_j|)|^m\,ds\right)^{1/m} = \prod_{j=1}^{m}\left(|\alpha_j|^{-1}\int_{-\infty}^{\infty}|\psi(s)|^m\,ds\right)^{1/m}$$

$$\leq (\min|\alpha_j|)^{-1}\int_{-\infty}^{\infty}|\psi(s)|^m\,ds \leq (\min|\alpha_j|)^{-1}\int_{-\infty}^{\infty}|\psi(s)|^{\nu}\,ds < \infty,$$

where the second inequality follows from Hölder's inequality. ■

Proof of Corollary 3.2. This follows from Theorem 2.1, Remark 2.1, and the discussion in Sections 3.1 and 3.2, in particular, Lemma 3.1 and Remark 3.1. ■

Proof of Corollary 3.3. This follows from Theorem 2.2, Proposition 2.3, Remarks 2.4 and 2.5 (note that each x_t has an absolutely continuous distribution), and the discussion in Sections 3.1 and 3.2, in particular, Lemma 3.1. ■

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