

DEFINABILITY OF DERIVATIONS IN THE REDUCTS OF DIFFERENTIALLY CLOSED FIELDS

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Abstract. Let $\mathcal{F} = (F; +, \cdot, 0, 1, D)$ be a differentially closed field. We consider the question of definability of the derivation D in reducts of \mathcal{F} of the form $\mathcal{F}_R = (F; +, \cdot, 0, 1, P)_{P \in R}$ where R is some collection of definable sets in \mathcal{F} . We give examples and nonexamples and establish some criteria for definability of D . Finally, using the tools developed in the article, we prove that under the assumption of inductiveness of $\text{Th}(\mathcal{F}_R)$ model completeness is a necessary condition for definability of D . This can be seen as part of a broader project where one is interested in finding Ax-Schanuel type inequalities (or predimension inequalities) for differential equations.

§1. Introduction. For a differentially closed field $\mathcal{F} = (F; +, \cdot, 0, 1, D)$, we consider its reducts of the form $\mathcal{F}_R = (F; +, \cdot, 0, 1, P)_{P \in R}$ where R is some collection of definable sets in \mathcal{F} . Our main problem is to understand when the derivation D is definable in \mathcal{F}_R . Ideally, we would like to find a dividing line for definability of D like local modularity in the problem of recovering the field structure in the reducts of algebraically closed fields (see the discussion below).

QUESTION 1.1. *When is D definable in the reduct \mathcal{F}_R ?*

As we will see when D is definable, it is definable with using just one parameter, namely an element $t \in F$ with $Dt = 1$. So it is more convenient to add t to our language as a constant symbol and work in the reducts of $\mathcal{F} = (F; +, \cdot, 0, 1, t, D)$ (we do this starting from Section 5). So, we will assume for simplicity that the sets from R are 0-definable in this language and also we will be interested in 0-definability of D .

Note that one could be tempted to ask a more general question of whether there is a derivation definable in the reduct. But in that case such a derivation will also be definable in the differentially closed field \mathcal{F} . Since it is known that any such derivation is of the form $a \cdot D$ for some $a \in F$, it is no loss of generality if we restrict our attention to definability of D only. Another point is that, we can assume that R is finite since any possible definition of D can contain only finitely many relations from R .

This is by nature a classification problem. We do not have a comprehensive solution yet, but we give some partial answers to our question, and draw some

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conclusions based on our analysis. We will not pose any explicit conjectures, but one may nevertheless expect intuitively that definability of D is very rare, i.e., in most cases it is not definable. In other words, our general expectation is that for “generic” reducts D is not definable.

The motivation to consider this kind of problem comes from two independent sources. Firstly, the analogous problem for pure fields, that is, recovering the field structure from reducts of algebraically closed fields or from nonlocally modular strongly minimal sets in general, is a well studied question in model theory of fields and Zariski geometries. It was initiated by Zilber’s famous “Trichotomy conjecture” and is still not entirely resolved. It has been (and still is) a topic of active research during the past few decades and proved to be very useful. Zariski geometries, introduced by B. Zilber and E. Hrushovski, are structures where that theory works ideally. For more details on this, we refer the reader to [3, 8, 13, 20].

Secondly, this problem turns out to be related to the existence of an “Ax-Schanuel type theorem” for a given differential equation $E(x, y)$ (in this case, we will work in the reduct $\mathcal{F}_E = (F; +, \cdot, 0, 1, E)$ with $R = \{E\}$). Let us briefly explain what we mean by this.

James Ax has proved the following analogue of Schanuel’s conjecture in differential setting ([2]). Let K be a differential field and C be its field of constants. Let also $(x_1, y_1), \dots, (x_n, y_n)$ be nonconstant solutions to the exponential differential equation $Dx = \frac{Dy}{y}$ in K . Then

$$\delta(x_1, \dots, x_n) := \text{td}_C C(x_1, \dots, x_n, y_1, \dots, y_n) - \text{l. dim}_{\mathbb{Q}}(x_1, \dots, x_n/C) \geq 1,$$

where td stands for the transcendence degree and l. dim stands for the linear dimension (modulo C) as a vector space. This inequality is now known as the Ax-Schanuel inequality. The function δ here is a *predimension* function in the sense of Hrushovski ([4]). Thus the Ax-Schanuel inequality is a *predimension inequality*. This property gives a good understanding of the exponential differential equation. In particular, one can consider the corresponding reduct. Then the first order theory of the reduct is axiomatised by axioms of algebraically closed fields, functional equation(s), an axiom scheme for the Ax-Schanuel inequality and the strong existential closedness axiom scheme (see [6, 18]). This is exactly the axiomatisation that one obtains after carrying out a Hrushovski construction with the above predimension function. Thus, the reduct here is reconstructed by a Hrushovski construction. Zilber calls such predimension inequalities *adequate*.

After realising this one can ask whether it is possible to do something similar for other differential equations. One therefore poses a problem whether for a given differential equation there is an “Ax-Schanuel type” inequality (or a predimension inequality). It is useful to classify differential equations with respect to this property, i.e., whether there is an “adequate” predimension inequality or not. If there is one, then one will know the complete theory of the equation. One of the recent developments in this direction is the establishment of an Ax-Schanuel type inequality for the j -function by Jonathan Pila and Jacob Tsimerman ([12]). For details on Schanuel’s conjecture and the Ax-Schanuel inequality (and its generalised versions) see [6, 17–19, 21]. For Hrushovski constructions and predimensions, we refer the reader to [4, 16].

We are not going to consider these questions in this article, but let us see how this problem is related to definability of D in the corresponding reduct. The idea is that definability of a derivation would imply that there is no “nontrivial” adequate predimension inequality for the given differential equation. Roughly speaking, if D is definable then the problem is reduced to finding an Ax-Schanuel type inequality for the equation $y = Dx$. But one can argue that there is no such nontrivial inequality for the latter equation. We will support this viewpoint by a result in the last section. Indeed, as we will see if D is definable and the theory of the reduct is inductive then it must actually be model complete. But Hrushovski constructions yield inductive theories in nice examples, so it is a reasonable condition. Thus, assuming the reduct is inductive, definability of D implies model completeness which can be used to show that a possible adequate predimension must be trivial in some sense. Of course, these statements are pretty vague, and we presented them here just to give a basic idea about the connection of those two questions. We limit ourselves to these explanations and refer the reader to [1] for more details.

Let us briefly outline the article. After giving the necessary preliminaries in Section 2, we show in Section 3 that definable derivations in models of DCF_0 are the trivial ones. Then, we study the reducts of differentially closed fields from a general model theoretic point of view and establish some of their properties in Section 4. In Section 5, we will see that if E is a differential curve containing the graph of D then D is a quantifier-free definable in \mathcal{F}_E .

Furthermore, we will show in Section 6 that the behaviour of D at generic points is enough to understand whether it is definable. Indeed, we will prove that if for a generic element a the Morley rank (in the reduct) of $D a$ over a is finite, then D is definable (Theorem 6.4). Using the results on generic points, we will give further examples of differential equations that define D (Section 7). Theorem 7.10 will sum up most of our results obtained up to that point giving a list of conditions equivalent to definability of D in the reducts.

The last section will be devoted to the question of model completeness of reducts that define D . Namely, we will prove that if D is definable in \mathcal{F}_R and $\text{Th}(\mathcal{F}_R)$ is inductive, then this theory must in fact be model complete (Theorem 8.2). This will immediately imply that one cannot define D from the exponential differential equation $Dy = yDx$.

§2. Preliminaries. In this section, we present basic definitions and facts about differential fields. For more details and proofs of the results stated here, we refer the reader to [5, 8–10].

We assume all rings that we deal with are commutative rings with identity and have characteristic zero.

The language of differential rings is $\mathcal{L}_D = \{+, \cdot, 0, 1, D\}$. In this language, we can axiomatise the theory of differential (rings) fields with the axioms of (rings) fields with two extra axioms stating that D is additive and satisfies Leibniz’s rule, i.e., $\forall x, y \ D(x + y) = Dx + Dy$ and $\forall x, y \ D(xy) = xDy + yDx$. The theory of differential fields of characteristic zero is denoted by DF_0 .

The *field of constants* of a differential field $(F; +, \cdot, 0, 1, D)$ is defined as the kernel of the derivation, i.e., $C_F = \{x \in F : Dx = 0\}$. This is always a relatively algebraically closed subfield of F .

If F is a differential field then the ring of *differential polynomials* over F is a differential ring extension defined as $F\{X\} = F[X, D(X), D^2(X), \dots]$ with $D(D^n(X)) = D^{n+1}(X)$. Thus, differential polynomials are of the form $p(X, D X, \dots, D^n X)$ where $p(X_0, \dots, X_n) \in F[X_0, \dots, X_n]$ is an algebraic polynomial over F . A *differential rational function* over F is the quotient of two differential polynomials over F . The field of all differential rational functions of X over F will be denoted by $F\langle X \rangle$. We can also consider differential polynomials in several variables, which are defined analogously. If $f(X_1, X_2, \dots, X_n)$ is such a polynomial, then the equation $f = 0$ is a *differential equation* over F .

Further, for F a differential field and $A \subseteq F$ a subset, we denote by $\langle A \rangle$ or $\mathbb{Q}\langle A \rangle$ the differential subfield generated by A . If $K \subseteq F$ are differential fields, and $A \subseteq F$ then $K\langle A \rangle$ is the differential subfield generated by K and A . The algebraic subfield generated by K and A is denoted by $K(A)$. One can easily verify that $K\langle A \rangle = K(\{D^n a : a \in A, n \in \mathbb{N}\})$.

The *order* of f , denoted $\text{ord}(f)$, is the biggest n for which $D^n(X)$ occurs in f . In this case the highest power of $D^n(X)$ in f is the *degree* of f , written $\text{deg}(f)$. In the case of polynomials of several variables, we will write $\text{ord}_{X_i}(f)$ for the order of f with respect to X_i .

The theory DF_0 has a model completion. It is called the theory of *differentially closed fields* of characteristic zero. To axiomatise this theory, we add the existential closedness axiom scheme: a differential field $(F; +, \cdot, 0, 1, D)$ is *differentially closed* if for any nonconstant differential polynomials $f(X)$ and $g(X)$ over F with $\text{ord}(g) < \text{ord}(f)$ there exists $x \in F$ such that $f(x) = 0$ and $g(x) \neq 0$. We let DCF_0 denote the theory of differentially closed fields of characteristic 0. It immediately follows from the definition that differentially closed fields are algebraically closed (in the field theoretic sense). Hence, the field of constants is algebraically closed as well.

Suppose $K \subseteq F$ are two models of DF_0 . For an element $a \in F$ one defines the *differential rank* (or *dimension* or *order*) of a over K , denoted $\text{DR}(a/K)$ (or $\text{dim}(a/K)$ or $\text{ord}(a/K)$), as the transcendence degree of $K\langle a \rangle$ over K . If it is finite, say n , then there is a differential polynomial $f(X) \in K\{X\}$ of order n with $f(a) = 0$. If f is the *simplest* among such polynomials, i.e., the pair $(\text{ord}(f), \text{deg}(f))$ is minimal with respect to the lexicographical order, then it is called *the minimal polynomial* of a over K . This polynomial must be irreducible. The elements $a, Da, \dots, D^{n-1}a$ are algebraically independent, while $a, Da, \dots, D^n a$ are algebraically dependent over K . In this case a is called *differentially algebraic* over K , otherwise it is called *differentially transcendental* over K . In the latter case $\text{DR}(a/K)$ is defined to be ω .

Suppose $K \models \text{DF}_0$ and $K \subseteq F$ is a differentially closed extension of K . Then for any element $a \in F$ the following inequality holds

$$U(a/K) \leq \text{MR}(a/K) \leq \text{DR}(a/K),$$

where $U(a/K)$ stands for the U-rank and $\text{MR}(a/K)$ stands for the Morley rank of a over K . Moreover, a is differentially transcendental over K if and only if $U(a/K) = \text{MR}(a/K) = \text{DR}(a/K) = \omega$. In this case a is called *generic* over K (if we omit K then it means a is generic over the empty set or, equivalently, over the prime differential subfield).

There is a unique complete type of a differentially transcendental element (over a subfield K) which is determined by formulas $\{f(x) \neq 0 : f(X) \in K\{X\}\}$.

The theory of differentially closed fields is model theoretically very nice. Namely, it admits elimination of quantifiers, elimination of imaginaries, it is complete and model complete. Further, DCF_0 is ω -stable with Morley rank ω . Every differential field K has a *differential closure* which is defined as the prime model of DCF_0 over K . The prime model always exists and is unique up to isomorphism (over K) in ω -stable theories. We will denote *the* differential closure of K by K^{dif} , while K^{alg} will denote the field theoretic algebraic closure.

Furthermore, it is easy to see that the Morley degree of DCF_0 is 1. This means that in a model $\mathcal{F} \models \text{DCF}_0$ any definable set $A \subseteq F$ is either of finite rank (it is *small*) or it has rank ω and its complement has finite rank (it is *big*).

Now, we define differential curves and make some easy observations about them that will be used later in the article.

DEFINITION 2.1. A *differential algebraic curve* E in a differential field K is a set in K^2 defined by a differential equation of two variables, i.e., $E = \{(x, y) \in K^2 : f(x, y) = 0\}$ for some $f(X, Y) \in K\{X, Y\}$. For brevity, we will sometimes say *differential curve* instead of differential algebraic curve.

Note also that by an algebraic curve, we mean a set defined by an algebraic equation of two variables. Let $\mathbb{D} := \{(x, Dx) : x \in K\}$ be the graph of D in K . This is an example of a differential curve.

DEFINITION 2.2. A *differential curve in general sense* in a differentially closed field \mathcal{F} is a definable subset of F^2 the generic fibres of which are of finite Morley rank.

Clearly any proper differential curve is a curve in general sense. On the other hand it is easy to notice that any curve E in general sense must be contained in a proper differential curve. This means, it must be defined by a formula of the form $\varphi(x, y) = [f(x, y) = 0 \wedge \psi(x, y)]$, where f is a differential polynomial and ψ is any formula. Indeed, otherwise E will contain a set of the form $f(x, y) \neq 0$ the generic fibres of which have rank ω .

We could alternatively define curves in general sense to be definable sets (in F^2) of Morley rank less than $\omega \cdot 2$. The above argument shows that this is equivalent to the above definition. Thus, if (a, b) is a pair of differentially independent elements and $\neg\varphi(a, b)$ holds in \mathcal{F} then $\varphi(x, y)$ defines a curve in general sense.¹

Finally let us fix some notations. We will use upper-case letters X, Y, \dots with possible subscripts for indeterminates of polynomials. We will use lower-case letters for elements of a set and for variables in formulas (it will be clear from the context which one we mean). In particular if $f(X) \in F\{X\}$ is a differential polynomial then $f(X) = 0$ means that f is identically zero, while $f(x) = 0$ means f vanishes at x (or it is a formula with a free variable x).

§3. Definable derivations. If D is a derivation on a field $(F; +, \cdot, 0, 1)$ then for any element $a \in F$ the map $a \cdot D$ will be a derivation as well. We show in this section that in a differentially closed field all definable derivations are of that form.

¹One can also require $\text{MR}(E)$ to be at least ω in order to avoid any degeneracies like $Dx = 0 \wedge Dy = 0$ (which correspond to finite sets in ACF_0), but it is not important for us.

THEOREM 3.1. *Let $\mathcal{F} = (F; +, \cdot, 0, 1, D)$ be a differentially closed field and \tilde{D} be a definable (possibly with parameters) derivation. Then there exists an element $a \in F$ such that $\tilde{D} = aD$.*

Though this fact is well known (a proof can be found for example in [14]), we nevertheless present our proof here as we are going to need it in Section 7.

The following well-known result is a characterisation of definable functions in a differentially closed field (see, for example, [9] or [15], Exercise 6.1.14).

LEMMA 3.2. *Let \mathcal{F} be a differentially closed field and $f : F^k \rightarrow F$ be a definable (possibly with parameters) function in \mathcal{F} . Then there is a partition of F^k into a finite number of definable subsets U_i such that f is given by a differential rational function on each of them (this means, in particular, that each of these rational functions is determined on the corresponding set).*

We establish one more result before proving the Theorem 3.1.

LEMMA 3.3. *Suppose D and D_1 are derivations on a field $(F; +, \cdot, 0, 1)$ such that there is $t \in F$ with $Dt = 1$. Let $P(X_0, \dots, X_n, Y)$ be a nonzero polynomial over F such that*

$$P(X, DX, \dots, D^n X, D_1 X) = 0. \tag{3.1}$$

Then $D_1 = a \cdot D$, where $a = D_1 t$.

PROOF. For an element $x \in F$ and an arbitrary rational number r one has $P(x + r, Dx, \dots, D^n x, D_1 x) = 0$, hence

$$P(X, Dx, \dots, D^n x, D_1 x) = 0$$

(as a polynomial of X). Therefore all coefficients of this polynomial are zeros. Since $P(X_0, \dots, X_n, Y)$ is nonzero, if we consider it as a polynomial of X_0 , it will have a nonzero coefficient that is a polynomial of X_1, \dots, X_n, Y . It must vanish at $(Dx, \dots, D^n x, D_1 x)$. This is true for all $x \in F$.

Thus for a nonzero polynomial P_1 , we have

$$P_1(DX, \dots, D^n X, D_1 X) = 0.$$

Again, fixing an element $x \in F$, we see that for any rational r one has $P_1(Dx + r, D^2 x, \dots, D^n x, D_1 x + ar) = 0$ (we substitute $X = x + rt$). This implies

$$P_1(X, D^2 x, \dots, D^n x, D_1 x - aDx + aX) = 0.$$

Replacing X by a fixed element $y \in F$ and taking $x + rt^2$ instead of x , we get

$$P_1(y, D^2 x + 2r, D^3 x, \dots, D^n x, D_1 x - aDx + ay) = 0.$$

Therefore

$$P_1(y, X, D^3 x, \dots, D^n x, D_1 x - aDx + ay) = 0.$$

Arguing as above, we show that for some nonzero polynomial P_2 , we have

$$P_2(y, D^3 x, \dots, D^n x, D_1 x - aDx + ay) = 0$$

for all $x, y \in F$. Proceeding this way one can prove that there is a nonzero polynomial $Q(Z_1, Z_2) \in F[Z_1, Z_2]$ such that

$$Q(Y, D_1 X - aDX + aY) = 0.$$

Now suppose for some $u \in F$, we have $D_1 u \neq a D u$. Then for any natural number n one has $D_1(nu) \neq a D(nu)$. This means that for any $y \in F$ the polynomial $Q(y, ay + Z)$ equals zero for infinitely many values of Z , hence, it is identically zero. This yields $Q(Y, Z) = 0$. We arrived at a contradiction, therefore $D = a D_1$. \dashv

PROOF OF THEOREM 3.1. From Lemma 3.2 it follows that there are definable sets $U_i \subseteq F$ such that \tilde{D} is given by a differential rational function on each U_i . Therefore there are differential polynomials $f_i(X), g_i(X) \in F\{X\}$ such that $f_i(x) \cdot \tilde{D}(x) = g_i(x)$ and $f_i(x) \neq 0$ for all $x \in U_i$. We know that $f_i(X) = P_i(X, D X, \dots, D^m X)$, $g_i(X) = Q_i(X, D X, \dots, D^m X)$ for some polynomials P_i and Q_i over F . Form the polynomial

$$P(X_0, \dots, X_m, Y) = \prod_i (P_i(X_0, \dots, X_m) \cdot Y - Q_i(X_0, \dots, X_m)).$$

This is a nonzero polynomial and

$$P(X, D X, \dots, D^m X, \tilde{D} X) = 0.$$

As \mathcal{F} is differentially closed, there exists $t \in F$ with $D t = 1$. Now Lemma 3.3 yields the desired result. \dashv

§4. Model theoretic properties of the reducts. From now on, we will work in a differentially closed field $\mathcal{F} = (F; +, \cdot, 0, 1, D)$, which we will assume to be sufficiently saturated. Thus, it will serve as a monster model for us.

For a collection R of definable sets in (Cartesian powers of) \mathcal{F} , we define the R -reduct \mathcal{F}_R of \mathcal{F} to be the structure $(F; +, \cdot, 0, 1, P)_{P \in R}$ in the language $\mathcal{L}_R = \{+, \cdot, 0, 1\} \cup R$ (the elements of R are relation symbols in the language \mathcal{L}_R). We will omit R and just say “reduct” whenever no confusion can arise. We will say that R (or the reduct \mathcal{F}_R) is *algebraic* if all relations of R can be defined in the pure field $(F; +, \cdot, 0, 1)$. If R consists of just one relation E then we will write \mathcal{F}_E for the corresponding E -reduct.

In this section, we examine basic model theoretic properties of the reducts \mathcal{F}_R . Though we will sometimes assume R is finite, most of our results will be valid for an arbitrary R . From the point of view of Question 1.1 the assumption of finiteness of R is no loss of generality as a possible definition of D would anyway contain only finitely many occurrences of relation symbols from R .

We start by introducing a piece of notation. In order to distinguish between the same concepts in the differentially closed field \mathcal{F} and in the reduct \mathcal{F}_R , we will add a subscript D or R , respectively, to their notations. Thus MR_D, MD_D, tp_D, dcl_D , and acl_D stand for Morley rank, Morley degree, type, definable closure and algebraic closure, respectively, in \mathcal{F} while MR_R, MD_R, tp_R, dcl_R , and acl_R stand for the same notions in \mathcal{F}_R .

Also, we will need to consider generic elements and types. *By generic, we will always mean generic in the differentially closed field \mathcal{F} (rather than in \mathcal{F}_R) unless explicitly stated otherwise.* If we do not specify over which set an element is generic then we mean over the empty set.

Finally, we turn to model theoretic properties of the reducts. Clearly \mathcal{F}_R is an ω -stable structure. We now find its Morley rank.

PROPOSITION 4.1. \mathcal{F}_R has Morley rank ω unless R is algebraic.

PROOF. First of all, since \mathcal{F}_R is a reduct of \mathcal{F} , and the latter has Morley rank ω , we have $\text{MR}(\mathcal{F}_R) \leq \omega$. So we need to prove $\text{MR}(\mathcal{F}_R) \geq \omega$.

It suffices to prove this for $R = \{P\}$ where P is a nonalgebraic unary relation which has finite Morley rank in the differentially closed field \mathcal{F} . The case $P = C$ (the field of constants) is a well-known example. In this case the reduct is just an algebraically closed field with a unary predicate for an algebraically closed subfield. Our proof below is an adaptation of a known proof for this special case (see, for example, [7], Exercise 6.6.17, d).

As P is nonalgebraic, it must be infinite and hence $\text{MR}_R(P) \geq 1$. Also P has finite Morley rank in \mathcal{F} , so $(\mathbb{Q}(P))^{\text{alg}} \neq F$. Now for an element $x \in F \setminus (\mathbb{Q}(P))^{\text{alg}}$ define

$$X_n = \left\{ y \in F : \exists a_0, \dots, a_{n-1} \in P \left(y = \sum a_i x^i \right) \right\}.$$

The map $\pi : P^{n+1} \rightarrow X_{n+1}$ given by $(a_0, \dots, a_n) \mapsto a_0 + a_1 x + \dots + a_n x^n$ is a definable bijection. Hence $\text{MR}_R(X_n) = \text{MR}_R(P^n) \geq n$. Therefore $\text{MR}_R(F) = \omega$. ⊔

We will assume throughout the article that R is not algebraic and so \mathcal{F}_R has Morley rank ω .

REMARK 4.2. As we saw in the proof, if $a \in F$ is a differentially transcendental element then for each $n < \omega$ there is a definable (in \mathcal{F}_R) set $X_n \subseteq F$, defined over a , such that $n \leq \text{MR}_R(X_n) < \omega$.

Further, observe that \mathcal{F}_R has Morley degree 1. If $\varphi(x)$ is a formula (of one variable) in the language $\mathcal{L}_R = \{+, \cdot, 0, 1\} \cup R$ then in the language \mathcal{L}_D it is equivalent to a quantifier-free formula. If it is an equation in conjunction with something else, then $\text{MR}_R(\varphi) < \omega$ otherwise $\text{MR}_R(\varphi) = \omega$. Also, $\text{MR}_R(\varphi) \leq \text{MR}_D(\varphi)$ and these ranks are finite or infinite simultaneously. Indeed, if $\text{MR}_D(\varphi) = \omega$ then $\text{MR}_D(\neg\varphi) < \omega$, and so $\text{MR}_R(\neg\varphi) < \omega$. Therefore, $\text{MR}_R(\varphi) = \omega$ since $\text{MR}_R(x = x) = \omega$ as proven above.

There is a unique generic 1-type in \mathcal{F}_R given by

$$\{\varphi(x) : \varphi \in \mathcal{L}_R, \text{MR}_R(\varphi) = \omega\} = \{\neg\varphi(x) : \varphi \in \mathcal{L}_R, \text{MR}_R(\varphi) < \omega\}.$$

Similarly, the unique generic n -type is given by formulas of Morley rank $\omega \cdot n$.

Now let us discuss the issue of quantifier elimination for \mathcal{F}_R . First notice that, even when $R = \{\mathbb{D}\}$, \mathcal{F}_R does not admit quantifier elimination for $y = \mathbb{D}^2 x$ is existentially definable but not quantifier-free definable. It turns out that this is a general phenomenon.

COROLLARY 4.3. *If R is nonalgebraic and finite then the reduct \mathcal{F}_R does not admit elimination of quantifiers.*

PROOF. Suppose R is not algebraic but \mathcal{F}_R has quantifier elimination. Then any formula with one free variable must be equivalent to a Boolean combination of algebraic polynomial equations (in the language of rings) and formulas of the form

$$Q(p_1(x), \dots, p_n(x)),$$

where $Q \in R$ is an n -ary predicate and p_i 's are algebraic polynomials. But clearly if such a formula has finite Morley rank then the latter is uniformly bounded, i.e.,

there is a bound which is the same for all formulas of finite Morley rank (remember that R is finite). This contradicts Proposition 4.1. \dashv

One sees that although in the case $R = \{\mathbb{D}\}$ the reduct does not have quantifier elimination, it is nevertheless model complete. In general it is true if \mathbb{D} is existentially definable. We show this below.

LEMMA 4.4. *Let \mathcal{M} be a structure. If a function $f : M^n \rightarrow M$ is existentially definable in \mathcal{M} then it is also universally definable.*

PROOF. If $\phi(\bar{x}, y)$ defines f then so does $\forall z(z = y \vee \neg\phi(\bar{x}, z))$. \dashv

PROPOSITION 4.5. *If \mathbb{D} is existentially definable in \mathcal{F}_R then $T_R := \text{Th}(\mathcal{F}_R)$ is model complete.*

PROOF. Suppose that \mathbb{D} is existentially definable. Take an arbitrary formula $\varphi \in \mathcal{L}_R$. In the language of differential rings it is equivalent to a quantifier-free formula, i.e., to a Boolean combination of differential equations. Each differential equation is existentially definable in the reduct and, by Lemma 4.4, it is also universally definable. Substituting existential definitions in positive parts (i.e., equations) and universal definitions in negative parts (inequations), we get an existential formula in the language \mathcal{L}_R . Thus any formula in the language of the reduct is equivalent to an existential formula. This is equivalent to model completeness. \dashv

Thus, model completeness is the deepest possible level of quantifier elimination that we can have for T_R . As we will see in the last section, under a natural assumption, definability of \mathbb{D} will imply that T_R is model complete.

§5. An example. In this section, we show that in a certain class of reducts \mathbb{D} is definable. It will be used later to establish some criteria for definability of \mathbb{D} .

Choose an element $t \in F$ with $\mathbb{D}t = 1$ (it exists because our field is differentially closed) and add it as a constant symbol to our language. Thus from now on, we work in the language $\{+, \cdot, \mathbb{D}, 0, 1, t\}$ for differential fields, which by abuse of notation, we will again denote by $\mathcal{L}_{\mathbb{D}}$. Correspondingly all reducts will be considered in the language $\mathcal{L}_R = \{+, \cdot, 0, 1, t\} \cup R$. Again abusing the nomenclatures, we will call $\mathcal{L}_{\mathbb{D}}$ the language of differential rings and \mathcal{L}_R the language of the reducts. This means that, we do not count t as a parameter in our formulas, i.e., we are free to use t in formulas and declare that something is definable without parameters. Note that this does not affect any of the results proved in the previous section. Let us also mention that after adding t to our language (and requiring that a derivation takes the value 1 at t), the only candidate for a definable derivation can be \mathbb{D} (see Theorem 3.1).

For a formula $\varphi(\bar{x})$ in the language $\mathcal{L}_{R, \mathbb{D}} = \mathcal{L}_R \cup \mathcal{L}_{\mathbb{D}}$ and a tuple $\bar{a} \in F$, we will sometimes write $\mathcal{F} \models \varphi(\bar{a})$. This is an abuse since in general φ is not in the language of differential rings, but clearly \mathcal{F} can be canonically made into an $\mathcal{L}_{R, \mathbb{D}}$ -structure.

In general, if the relations in R are defined with parameters and \mathbb{D} is definable then it will be definable with parameters as well. But in many cases, we do not use any extra parameters to define \mathbb{D} . So for simplicity, we will assume that R consists of 0-definable relations in \mathcal{F} , i.e., relations defined over $k_0 = \mathbb{Q}(t) = \text{dcl}(\emptyset)$. Thus from now on by definable, we will mean definable without parameters unless explicitly stated otherwise.

We denote the theory of the reduct by $T_R := \text{Th}(\mathcal{F}_R)$. We will say that there is a derivation D_K on a model $\mathcal{K}_R \models T_R$ which is *compatible* with R . This means

that $(K; +, \cdot, D_K, 0, 1, t, P)_{P \in R} \equiv (F; +, \cdot, D, 0, 1, t, P)_{P \in R}$, i.e., the differential field $\mathcal{K} = (K; +, \cdot, D_K, 0, 1, t)$ is differentially closed with $D_K t = 1$ and the sets from R are defined by the same formulas as in \mathcal{F} .

Throughout the article, we let E be a differential curve (possibly in general sense); as we noted above the corresponding reduct will be denoted \mathcal{F}_E . Recall also that $\mathbb{D} = \{(x, Dx) : x \in F\}$ is the graph of D .

Now, we prove an auxiliary result which will be used several times throughout the article. It states that $(\mathbb{Q}(t))^n$ is Kolchin-dense in F^n for each n .

LEMMA 5.1 (cf. [8], Lemma A.4). *For any nonzero differential polynomial $f(X_1, \dots, X_n)$ over $\mathbb{Q}(t)$ there are elements $t_1, \dots, t_n \in \mathbb{Q}[t]$ such that $f(t_1, \dots, t_n) \neq 0$.*

PROOF. First assume f is a polynomial of one variable X . Let $\text{ord}(f) = n$. Since \mathcal{F} is differentially closed, we can find an element $u \in F$ with $D^{n+1}u = 0 \wedge f(u) \neq 0$. Then clearly

$$u = c_n t^n + \dots + c_1 t + c_0$$

for some constants $c_0, \dots, c_n \in C$.

Now for constants $\lambda_0, \dots, \lambda_n$ denote

$$p(t, \vec{\lambda}) = \lambda_n t^n + \dots + \lambda_1 t + \lambda_0.$$

Since t is transcendental over C , there are algebraic polynomials $q_i(X_0, \dots, X_n) \in \mathbb{Q}[X_0, \dots, X_n]$, $i = 1, \dots, m$, such that for all $\vec{\lambda} \in C^{n+1}$

$$f(p(t, \vec{\lambda})) = 0 \text{ iff } \bigwedge_{i=1}^m q_i(\vec{\lambda}) = 0.$$

Let $V \subseteq C^{n+1}$ be the algebraic variety over \mathbb{Q} defined by $\bigwedge_{i=1}^m q_i(\vec{\lambda}) = 0$. Then as we saw above $V(C) \neq C^{n+1}$, and hence $V(\mathbb{Q}) \subsetneq \mathbb{Q}^{n+1}$. So there is a tuple $\vec{r} \in \mathbb{Q}^{n+1}$ with $\vec{r} \notin V(\mathbb{Q})$. Therefore $f(p(t, \vec{r})) \neq 0$ and $p(t, \vec{r}) \in \mathbb{Q}[t]$.

Now, we prove the general case (when f has more than one variables) by induction on n . If $f = f(X_1, \dots, X_n)$ with $n > 1$ then consider it as a differential polynomial $g(X_1, \dots, X_{n-1})$ of $n - 1$ variables over the differential ring $\mathbb{Q}(t)\{X_n\}$. Choose a nonzero coefficient of g which will be a nonzero differential polynomial $h(X_n) \in \mathbb{Q}(t)\{X_n\}$. As we proved above there is $t_n \in \mathbb{Q}[t]$ such that $h(t_n) \neq 0$. Now the polynomial $f(X_1, \dots, X_{n-1}, t_n)$ is a nonzero polynomial of $n - 1$ variables over $\mathbb{Q}(t)$ and we are done by the induction hypothesis. \dashv

REMARK 5.2. The proof shows that, we can choose t_1, \dots, t_n from $\mathbb{Z}[t]$ (and even from $\mathbb{N}[t]$).

DEFINITION 5.3. Introduce the reverse lexicographical order on $(n + 1)$ -tuples of integers, that is, $(\alpha_0, \dots, \alpha_n) < (\beta_0, \dots, \beta_n)$ if and only if for some j , $\alpha_i = \beta_i$ for $i > j$ and $\alpha_j < \beta_j$. The *multidegree* of an algebraic polynomial $Q(X_0, \dots, X_n)$ is the greatest (with respect to this order) $(n + 1)$ -tuple $(\alpha_0, \dots, \alpha_n)$ for which $X_0^{\alpha_0} \dots X_n^{\alpha_n}$ appears in Q with a nonzero coefficient. The multidegree of a differential polynomial $f(X) = P(X, DX, \dots, D^n X)$ is defined as that of P .

THEOREM 5.4. *If E (a differential algebraic curve) contains the graph of D then D is quantifier-free definable in \mathcal{F}_E .*

Before proving the theorem, we give an example which helps to understand how the proof works.

EXAMPLE 5.5. Suppose E is given by $(Dy - D^2x) \cdot Dx = 0$. Then $E(x+t, y+1) = [(Dy - D^2x) \cdot (Dx + 1) = 0]$. The conjunction $E(x, y) \wedge E(x+t, y+1)$ implies $Dy - D^2x = 0$. Now, we substitute $x \mapsto tx, y \mapsto x + ty$ and get $t(Dy - D^2x) + y - Dx = 0$. Subtracting the previous equation multiplied by t , we get $y - Dx = 0$. Thus the formula $E(x, y) \wedge E(x+t, y+1) \wedge E(tx, x+ty) \wedge E(tx+t, x+ty+1)$ defines D .

PROOF. ²Let E be given by a differential equation $f(x, y) = 0$. We know that $f(X, DX)$ identically vanishes. Denote $U := Y - DX$ and consider the differential polynomial $g(X, U) := f(X, U + DX)$. Clearly $g(X, 0) = 0$.

First, we intersect additive translates to “eliminate” x and define a differential equation $h(u) = 0$ for some differential polynomial $h(U)$. If $g(X, U)$ depends on X (i.e., $g(X, U) \in k_0\{X, U\} \setminus k_0\{U\}$) then we can find (see Lemma 5.1) $p(t) \in \mathbb{Q}[t]$ such that $g(X + p(t), U) \neq g(X, U)$. Clearly, U is invariant under the transformation $X \mapsto X + p(t), Y \mapsto Y + p'(t)$ where $p'(Z) = \frac{\partial p}{\partial Z}$. So consider the formula $E(x, y) \wedge E(x + p(t), y + p'(t))$. It is equivalent to $g(x, u) = 0 \wedge g(x + p(t), u) = 0$ which implies $g_1(x, u) := g(x, u) - g(x + p(t), u) = 0$. The leading terms of the differential polynomials $g(X, U)$ and $g(X + p(t), U)$ in variable X (i.e., the sums of monomials in these polynomials that have highest multidegree in X) are the same and hence they cancel out in the difference $g_1(X, U) := g(X, U) - g(X + p(t), U)$. On the other hand $g_1(X, U) \neq 0$ by our choice of p and the multidegree of g_1 in X is strictly less than that of g . In other words, if the multidegree of g in X is bigger than $(0, \dots, 0)$, then we can reduce it. Now if $g_1(X, U)$ depends on X then, we do the same for g_1 . We keep repeating this process and reduce the multidegree of our differential polynomial step by step until it becomes $(0, \dots, 0)$. This means, we get a curve $h(u) = 0$ for a nonzero differential polynomial h , which contains a quantifier-free definable set in our reduct. It is also clear that the latter contains the curve $u = 0$ (the graph of D).

Now, we use multiplicative translates to define the curve $u = 0$ (which is actually $y = Dx$). Let $p(t) \in \mathbb{Q}[t]$. When we substitute $X \mapsto p(t)X, Y \mapsto p'(t)X + p(t)Y$ then U is replaced by $p(t)U$. Then $h(u) = 0 \wedge h(p(t)u) = 0$ is implied by a quantifier-free formula in the language of the reduct and implies $h_{\alpha,1}(u) := p(t)^\alpha h(u) - h(p(t)u) = 0$ for any positive integer α . If $(\alpha_0, \dots, \alpha_n)$ is the multidegree of h , then taking, $\alpha := \alpha_0 + \dots + \alpha_n$, the leading terms of the differential polynomials $p(t)^\alpha h(U)$ and $h(p(t)U)$ will coincide and will cancel out in the difference $h_{\alpha,1}(U) := p(t)^\alpha h(U) - h(p(t)U)$. By an appropriate choice of p , we can also guarantee that $h_{\alpha,1}(U)$ is nonzero unless $h(U) = h(1) \cdot U^\alpha$. Indeed, if $h(U) \neq h(1) \cdot U^\alpha$ then the polynomial $h(V \cdot U) - V^\alpha \cdot h(U)$ is nonzero and hence there is $p(t) \in \mathbb{Q}[t]$ such that $h(p(t) \cdot U) \neq p(t)^\alpha \cdot h(U)$, therefore $h_{\alpha,1}(U) \neq 0$. Thus, if $h(U)$ is not a homogeneous algebraic polynomial then $h_{\alpha,1}$ is nonzero and its multidegree is strictly less than that of h . Now if $h_{\alpha,1}(U)$ is not algebraic homogeneous, then we repeat the above procedure for $h_{\alpha,1}$. Iterating this process,

²I am grateful to Ehud Hrushovski for detecting a gap in the initial version of the proof and helping me to fix it.

we will eventually obtain an equation $u^\alpha = 0$ for some positive integer α which is equivalent to $u = 0$. Taking into account that all the sets defined this way contain $u = 0$, we see that at the last step we have defined $u = 0$ which, in terms of x and y , is the curve $y = D x$.

Finally note that, we only take conjunctions of atomic formulas here, hence the definition is quantifier-free. ⊢

REMARK 5.6. Strictly speaking, for the “quantifier-free” part of the theorem to be true, we need to pick $p(t) \in \mathbb{N}[t]$ each time. Alternatively, we could add unary functions for multiplicative and additive inverses to our language.

COROLLARY 5.7. *If E is a curve in general sense that contains \mathbb{D} then D is quantifier-free definable.*

PROOF. Being a curve in general sense, E is defined by a formula of the form $f(x, y) = 0 \wedge \psi(x, y)$ for ψ a quantifier free formula in the language of differential fields. Now for the curve E' given by the equation $f(x, y) = 0$, we have a definition of D . Suppose it is given by the formula $\varphi(x, y)$ in the reduct $\mathcal{F}_{E'}$. We claim that the same formula defines D in \mathcal{F}_E . Indeed, as we take only conjunctions to define D from E' , the set defined by $\varphi(x, y)$ in \mathcal{F}_E will be contained in \mathbb{D} . On the other hand it clearly contains \mathbb{D} . Therefore it defines \mathbb{D} . ⊢

We will give further examples and nonexamples (of differential equations defining D) in Section 7, but first, we need to establish some facts on generic points which we do in the next section.

§6. Generic points. Recall that we work in a saturated differentially closed field \mathcal{F} . From now on, we fix a generic (in the sense of DCF_0 , that is, differentially transcendental) point $a \in \mathcal{F}$. We first prove that if $D a$ can be defined from a then we can recover the whole of D .

PROPOSITION 6.1. *Suppose a formula $\varphi(x, y) \in \mathcal{L}_R$ defines $D a$ from a , that is,*

$$\mathcal{F} \models \forall y(\varphi(a, y) \leftrightarrow y = D a).$$

Then D is definable (without parameters). Moreover, if φ is existential then D is existentially definable.

FIRST PROOF. First of all observe that since the generic type is unique, for any differentially transcendental element $b \in \mathcal{F}$, we have

$$\mathcal{F} \models \forall y(\varphi(b, y) \leftrightarrow y = D b).$$

Let A be the set defined by $\varphi(x, y)$ and define

$$B := \{(b, D b) : b \text{ generic in } \mathcal{F}\} \subseteq A.$$

At generic points b , the formula φ defines $D b$, but we do not have any information about nongeneric points. So, we need to shrink the set A to a subset of \mathbb{D} in order to avoid any possible problems at nongeneric points. The set A , being a curve in general sense (its fibre over any generic point $x = b$ consists of one element and hence is small), must be defined by a formula $f(x, y) = 0 \wedge \psi(x, y)$ (in the language of differential rings). Then $f(a, D a) = 0$ and hence $f(X, D X) = 0$. Therefore D can be defined from the differential curve $f(x, y) = 0$ by Theorem 5.4. Taking into

account that for a generic element b the elements $b + p(t)$ and $p(t)b$ are generic as well for any $p(t) \in \mathbb{Q}[t] \setminus \{0\}$, we see that the sets $\varphi(x, y) \wedge \varphi(x + p(t), y + p'(t))$ and $\varphi(x, y) \wedge \varphi(p(t)x, p(t)y + p'(t)x)$ contain B . Arguing as in the proofs of Theorem 5.4 and Corollary 5.7, after taking sufficiently many conjunctions of such formulas, we will eventually define a set B' such that it contains B and is contained in the graph \mathbb{D} of D . Note that B' is 0-definable.

Treating \mathbb{D} as an additive group, we prove the following.

CLAIM. $\mathbb{D} = B' + B'$.

Clearly $B' + B' \subseteq \mathbb{D}$. Let us show that the converse inclusion holds. Any element $d \in F$ has a representation $d = b_1 + b_2$ with b_1 and b_2 generic. Indeed, take b_1 to be generic over d and choose $b_2 = d - b_1$. Hence $(d, Dd) = (b_1, Db_1) + (b_2, Db_2) \in B + B \subseteq B' + B'$.

This gives a definition of D without parameters. Moreover, if φ is existential then we get an existential definition. ⊢

REMARK 6.2. The group \mathbb{D} is in fact a connected ω -stable group (its Morley degree is one). Therefore the equality $\mathbb{D} = G + G$ holds for any definable subset G of \mathbb{D} with $\text{MR}(G) = \text{MR}(\mathbb{D})$ (see, for example, [7], Chapter 7, Corollary 7.2.7). We could use this to show that $\mathbb{D} = B' + B'$ since $\text{MR}(B') = \text{MR}(\mathbb{D}) = \omega$. In fact, the idea is the same as in the above claim; one just passes to a saturated extension and uses the above argument there.

We will shortly give another proof to Proposition 6.1. For this, we first observe that if D is definable with independent parameters then it is also definable without parameters.

LEMMA 6.3. *Suppose $\psi(x, y, u_1, \dots, u_n) \in \mathfrak{L}_R$ and b_1, \dots, b_n are differentially independent elements in \mathcal{F} . If the formula $\psi(x, y, \bar{b})$ defines $y = Dx$ then there are 0-definable elements $t_1, \dots, t_n \in k_0 = \mathbb{Q}(t)$ such that $\psi(x, y, \bar{t})$ defines D (and so D is 0-definable).*

PROOF. We have

$$\mathcal{F} \models \psi(x, y, \bar{b}) \iff y = Dx.$$

Therefore

$$q(\bar{z}) := \text{tp}_D(\bar{b}) \models \psi(x, y, \bar{z}) \iff y = Dx.$$

Since $q(\bar{z})$ is the generic m -type in DCF_0 , it consists only of differential inequations. Applying compactness and taking into account that conjunction of finitely many inequations is an inequation as well, we conclude that there is a differential polynomial $f(Z_1, \dots, Z_m)$ over k_0 such that

$$\mathcal{F} \models \forall \bar{z} (f(\bar{z}) \neq 0 \implies \forall x, y (\psi(x, y, \bar{z}) \iff y = Dx)).$$

By Lemma 5.1, we can find elements $t_1, \dots, t_m \in k_0$ such that $f(t_1, \dots, t_m)$ is nonzero. Now, we see that

$$\mathcal{F} \models \psi(x, y, \bar{t}) \iff y = Dx,$$

and we are done. ⊢

SECOND PROOF OF PROPOSITION 6.1. Let $(b_1, b_2) \in F^2$ be a differentially independent tuple. Then for every $d \in F$ the differential transcendence degree of $d, d + b_1, d + b_2$ is at least 2. It is easy to deduce from this that the following formula defines D :

$$\exists u_1, u_2(\varphi(b_1, u_1) \wedge \varphi(b_2, u_2) \wedge [(\varphi(x, y) \wedge \varphi(x + b_1, y + u_1)) \vee (\varphi(x, y) \wedge \varphi(x + b_2, y + u_2)) \vee (\varphi(x + b_2, y + u_2) \wedge \varphi(x + b_1, y + u_1))])$$

Now Lemma 6.3 concludes the proof. ⊢

The idea that the behaviour of D at generic (differentially transcendental) points determines its global behaviour as a function can be developed further. We proceed towards this goal in the rest of this section.

Next, we show that if $D a$ is not generic over a (in the reduct), then it is in fact definable and hence D is definable. Let $p(y) := \text{tp}_R(D a/a)$ be the type of $D a$ over a in \mathcal{F}_R .

THEOREM 6.4. *The derivation D is definable in \mathcal{F}_R if and only if p has finite Morley rank (in \mathcal{F}_R).*

PROOF. Obviously, if D is definable then p is algebraic and hence has Morley rank 0. Let us prove the other direction.

Let $\varphi(a, y) \in p$ be a formula of finite Morley rank. Trivially $\mathcal{F} \models \varphi(a, D a)$ and $\varphi(x, y)$ defines a curve in general sense. As in the proof of Proposition 6.1, we can define a big subset $\psi(x, y)$ of \mathbb{D} , that is, a subset of Morley rank ω . This set certainly contains the point $(a, D a)$ and $\psi(a, y)$ defines $D a$. Thus $D a$ is definable over a and Proposition 6.1 finishes the proof. ⊢

REMARK 6.5. The proof shows that if $\varphi(x, y)$ is an existential formula of rank $< \omega \cdot 2$ which is true of $(a, D a)$ then D is existentially definable.

COROLLARY 6.6. *In the reduct, $D a$ is either generic or algebraic (in fact, definable) over a .*

LEMMA 6.7. *If p is isolated then it has finite Morley rank (in the reduct).*

PROOF. The argument here is an adaptation of the proof of the fact that in differentially closed fields the generic type is not isolated.

Suppose p is isolated but has rank ω , i.e., it is the generic type over a (in the reduct). Then

$$p(y) = \{\neg\varphi(a, y) : \varphi \in \mathcal{L}_R, \mathcal{F} \models \varphi(a, D a) \text{ and } \text{MR}_R(\varphi(a, y)) < \omega\}.$$

Suppose $\neg\psi(a, y)$ isolates p . By Remark 4.2 there is a formula $\varphi(a, y)$ for which $\text{MR}_R(\psi(a, y)) < \text{MR}_R(\varphi(a, y)) < \omega$. Then $\varphi(a, y) \wedge \neg\psi(a, y)$ is consistent. A realisation of this formula cannot be generic, for φ has finite Morley rank. This is a contradiction. ⊢

As an immediate consequence one gets the following result.

COROLLARY 6.8. *The derivation D is definable in \mathcal{F}_R if and only if p is isolated.*

REMARK 6.9. We can consider the quantifier-free type $q(y) := \text{qftp}(D a/a)$. Then D is quantifier-free definable if and only if this type is isolated, if and only if it has finite Morley rank.

Notice that in stability-theoretic language, we have proved that D is definable if and only if $\text{tp}_R(Da/a)$ forks over the empty set. Indeed, $\text{MR}_R(Da) = \omega$ (since it is generic in the differentially closed field) and forking in ω -stable theories means that Morley rank decreases, hence $\text{tp}_R(Da/a)$ forks over \emptyset if and only if $\text{MR}_R(Da/a) < \omega$. In terms of forking independence, we have the following formulation: D is definable if and only if $a \not\perp Da$ in \mathcal{F}_R . This will be generalised in the next section. Note also that all the above results will remain true if we replace Morley rank everywhere with U-rank.

Now add a differentially transcendental element a to our language and consider the reducts in this new language. Denote the theory of \mathcal{F}_R in this language by T_R^+ . Assume that each model of T_R^+ comes from a differentially closed field, that is, each model \mathcal{K}_R is the reduct of a differentially closed field $\mathcal{K} = (K; +, \cdot, D_K, 0, 1, t, a)$ in which a is generic (differentially transcendental) and relations from R are interpreted canonically (i.e., they are defined in \mathcal{K} by the same formulas as in \mathcal{F}). Then the type $p(y)$ will be realised by $D_K a$ in \mathcal{K}_R . The omitting types theorem now yields that p must be isolated. Thus, we have established the following result.

THEOREM 6.10. *If each model of T_R^+ is the R -reduct (with canonical interpretation) of a model of DCF_0 , then D is definable.*

In other words, this means that if each model of T_R^+ is equipped with a derivation which is compatible with R then D is definable. The converse of this holds as well trivially.

This is similar to Beth’s definability theorem in spirit (see [11]). Beth’s theorem in this setting means that if each model of T_R^+ has at most one derivation compatible with R then D is definable. We showed that if each model has at least one derivation then D is definable. Also it is worth mentioning that unlike Beth’s definability theorem, this statement is not true in general for arbitrary theories.

§7. Further examples. In this section, we will give more examples of differential equations defining D . Those examples will be used to characterise definable and algebraic closures of generic elements in the reducts. At the end of the section, we will give two nonexamples. Note that the results of this section will not be used later.

We will show first that differential rational functions define the derivation.

PROPOSITION 7.1. *If $E(x, y)$ is given by $g(x) \cdot y = f(x)$ where $\frac{f(x)}{g(x)}$ is a differential rational function which is not an algebraic rational function, then D is definable in \mathcal{F}_E .*

LEMMA 7.2. *Let D_1 and D_2 be derivations on a field K and $t \in K$ be such that $D_1 t = D_2 t = 1$. If there is a nonzero algebraic polynomial $P(X_0, \dots, X_n, Y_1, \dots, Y_m)$ over K such that*

$$P(X, D_1 X, \dots, D_1^n X, D_2 X, \dots, D_2^m X) = 0$$

then $D_1 = D_2$.

PROOF. We can assume without loss of generality that $n = m$. As in the proof of Lemma 3.3, we can show there is a nonzero polynomial $P_1(\bar{X}, \bar{Y})$ such that

$$P_1(X_1, \dots, X_n, D_2 Y - D_1 Y + X_1, \dots, D_2^n Y - D_1^n Y + X_n) = 0.$$

Clearly $D := D_2 - D_1$ is a derivation of K . The above identity implies that for some nonzero polynomial Q , we have

$$Q(DX, D^2X, \dots, D^nX) = 0.$$

If $D_1 \neq D_2$ then $D \neq 0$ and there is an element $b \in K$ with $Db \neq 0$. Dividing D by Db , we can assume that $Db = 1$. But then substituting $X \mapsto X + rb^j$ for $r \in \mathbb{Q}$ and $j = 1, \dots, n$, we see that $Q = 0$, which is a contradiction. \dashv

PROOF OF PROPOSITION 7.1. Suppose

$$f(X) = P(X, DX, \dots, D^nX), \quad g(X) = Q(X, DX, \dots, D^mX).$$

We will use Beth’s definability theorem to show that D is definable in $T_E := \text{Th}(\mathcal{F}_E)$. Indeed, if we have two derivations D_1 and D_2 on a model $\mathcal{K}_E \models T_E$ that are compatible with E (and K is differentially closed with either of these derivations and $D_1t = D_2t = 1$), then

$$\begin{aligned} P(X, D_1X, \dots, D_1^nX) \cdot Q(X, D_2X, \dots, D_2^mX) \\ = P(X, D_2X, \dots, D_2^nX) \cdot Q(X, D_1X, \dots, D_1^mX). \end{aligned}$$

Since $f(X)/g(X)$ is not an algebraic rational function, the above identity shows that the conditions of Lemma 7.2 are satisfied. Therefore $D_1 = D_2$. \dashv

REMARK 7.3. Note that even in the simple cases $y = D^2x$ and $y = (Dx)^2$ the differentiation is not definable without using t since we can not distinguish between D and $-D$.

Now, we prove that if $E(x, y)$ defines an algebraic function of $x, Dx, \dots, D^n x$, i.e., E is given by an equation $f(x, y) = 0$ with $\text{ord}_Y(f) = 0$, then one can define Dx . But first, we need to exclude some trivial counterexamples like $y \cdot Dx = 0$ (see Example 7.11).

DEFINITION 7.4. A differential polynomial $f(X, Y)$ is said to be *non-degenerate* if it cannot be decomposed into a product $g(X)h(X, Y)$ where g is a differential polynomial and h is an algebraic polynomial. An irreducible nonalgebraic polynomial which depends on both variables is obviously nondegenerate.

PROPOSITION 7.5. Suppose $E(x, y)$ is defined by a nondegenerate equation $f(x, y) = 0$ where $\text{ord}_X(f) > 0$ and $\text{ord}_Y(f) = 0$. Then D is definable in \mathcal{F}_E .

PROOF. Pick a differentially transcendental element $a \in F$ and let

$$f(a, Y) = \prod_{i=1}^k f_i(a, Y)^{e_i}$$

be the irreducible factorisation of $f(a, Y)$ over $k_0\langle a \rangle$. Denote

$$g(a, Y) := \prod_{i=1}^k f_i(a, Y) = \sum_{i=0}^m g_i(a) \cdot Y^i,$$

where $g_i(X) \in k_0\langle X \rangle$ and $g_m \neq 0$.

Consider the formula

$$\psi(x, z_0, \dots, z_m) = \exists y_1, \dots, y_m \left(\bigwedge_{i \neq j} y_i \neq y_j \wedge \bigwedge_{i=1}^m E(x, y_i) \wedge \bigwedge_{i=1}^m \sum_{j=0}^m z_j \cdot y_i^j = 0 \right).$$

Clearly, $E(a, y)$ holds if and only if $g(a, y) = 0$. The polynomial $g(a, Y)$ has m different roots. Therefore $\varphi(a, z_0, \dots, z_m)$ holds if and only if the roots of $\sum_{i=0}^m z_i \cdot Y^i$ are exactly the same as those of $g(a, Y)$ (as these two polynomials have the same degree in Y). This can happen if and only if $\sum_{i=0}^m z_i \cdot Y^i$ is equal to $g(a, Y)$ up to a constant which depends on a . This means that

$$\frac{z_i}{z_m} = \frac{g_i(a)}{g_m(a)},$$

for all i . At least one of $\frac{g_i(X)}{g_m(X)}$ is not an algebraic rational function since otherwise f would be degenerate. But then, we can define $D a$ from that differential rational function by Proposition 7.1 and we are done. \dashv

Next, we will apply Proposition 7.5 to work out definable and algebraic closures of generic points in the reducts. As before, let $a \in F$ be a generic point. We will show that the definable closure of a in \mathcal{F}_R coincides either with the definable closure in the differentially closed field or with that in the pure algebraically closed field.

It is well known what the definable and algebraic closures of arbitrary sets in differentially closed fields look like. Taking into account the fact that we have added t as a constant symbol to the language, we see that for a set $A \subseteq F$ the definable and algebraic closures in \mathcal{F} are given by $\text{dcl}_D(A) = k_0\langle A \rangle$ and $\text{acl}_D(A) = (k_0\langle A \rangle)^{\text{alg}}$, where $k_0 = \mathbb{Q}(t)$ and $k_0\langle A \rangle$ is the differential subfield generated by k_0 and A . This immediately implies that in the reduct we have $k_0(A) \subseteq \text{dcl}_R(A) \subseteq k_0\langle A \rangle$ and $(k_0(A))^{\text{alg}} \subseteq \text{acl}_R(A) \subseteq (k_0\langle A \rangle)^{\text{alg}}$.

We show that for generic elements one of these two extremal cases must happen.

THEOREM 7.6. *For $a \in F$ a generic point exactly one of the following statements holds:*

- $\text{dcl}_R(a) = k_0(a)$; this holds if and only if $\text{acl}_R(a) = (k_0(a))^{\text{alg}}$ if and only if D is not definable;
- $\text{dcl}_R(a) = k_0\langle a \rangle$; this holds if and only if $\text{acl}_R(a) = (k_0\langle a \rangle)^{\text{alg}}$ if and only if D is definable.

PROOF. It will be enough to show that if $\text{acl}_R(a) \supsetneq (k_0(a))^{\text{alg}}$ then D is definable. Thus, let $\text{acl}_R(a) \supsetneq (k_0(a))^{\text{alg}}$. Choose $b \in (k_0\langle a \rangle)^{\text{alg}} \setminus (k_0(a))^{\text{alg}}$ which is algebraic (in the model theoretic sense) over a in \mathcal{F}_R . There is a formula $\varphi(x, y) \in \mathcal{L}_R$ such that $\varphi(a, b)$ holds and $\varphi(a, y)$ has finitely many realisations. Because $\varphi(a, y)$ defines a finite set in the differentially closed field \mathcal{F} , it is equivalent to an algebraic polynomial equation over $k_0\langle a \rangle$. The latter is clearly nondegenerate and is not defined over $k_0(a)$ since b is its root. Applying Proposition 7.5, we define $D a$ (over a). Hence D is definable. \dashv

Now using Proposition 7.1, we generalise Theorem 6.4.

PROPOSITION 7.7. *Let $a \in F$ be a differentially transcendental element. If*

$$\text{MR}_R(a, D a, \dots, D^n a) < \omega \cdot (n + 1)$$

for some n then D is definable.

PROOF. We proceed to the proof by induction on n . The case $n = 1$ is done in Theorem 5.4. Assuming the theorem is true for all numbers less than n , we prove it for n .

There is a formula $\varphi(x_0, x_1, \dots, x_n) \in \mathcal{L}_R$ with $\text{MR}_R(\varphi) < \omega \cdot (n + 1)$ and

$$\mathcal{F}_R \models \varphi(a, \text{D} a, \dots, \text{D}^n a).$$

Since φ does not have “full” rank, we can assume without loss of generality it is given by a differential equation $f(x_0, x_1, \dots, x_n) = 0$ in the language of differential rings. Since $f(a, \text{D} a, \dots, \text{D}^n a) = 0$ and a is generic, f must be equal to $g(X, U_1, \dots, U_n)$ for some differential polynomial g with $g(X, 0, \dots, 0) = 0$ where $X := X_0, U_i := X_i - \text{D}^i X$. Further, applying the method of additive translates as in the proof of Theorem 5.4, we can assume that g does not depend on the first variable, so we write $g(U_1, \dots, U_n)$. However, we cannot proceed as in Theorem 5.4 and use multiplicative translates as there exist nonalgebraic “homogeneous” differential polynomials of several variables.

CLAIM. $\text{D}^n a \in \text{dcl}_R(a, \text{D} a, \dots, \text{D}^{n-1} a)$.

PROOF. The set defined by $\varphi(a, \text{D} a, \dots, \text{D}^{n-1} a, y)$ contains $\text{D}^n a$. Moreover, that formula is given by $h(y - \text{D}^n a) = 0$ where $h(U) = g(0, \dots, 0, U)$. Consider the formula

$$\varphi [p(t)a, \text{D}(p(t)a), \dots, \text{D}^{n-1}(p(t)a), \text{D}^n(p(t)a) - p(t)\text{D}^n a + p(t)y], \quad (7.1)$$

for a nonzero polynomial $p(t) \in \mathbb{Q}[t]$.

It is easy to see that this is a formula in the language of reducts with parameters $a, \text{D} a, \dots, \text{D}^{n-1} a$ and it is true of $y = \text{D}^n a$. The formula (7.1) is equivalent to $h(p(t)(y - \text{D}^n a)) = 0$. Taking the conjunction of $\varphi(a, \text{D} a, \dots, \text{D}^{n-1} a, y)$ and the formula (7.1), we get a formula in the language \mathcal{L}_R equivalent to³

$$h[y - \text{D}^n a] = 0 \wedge h[p(t)(y - \text{D}^n a)] = 0.$$

This contains the point $\text{D}^n a$ since $h(0) = 0$ and is contained in sets defined by $h[p(t)(y - \text{D}^n a)] - (p(t))^\alpha h[y - \text{D}^n a] = 0$ for α a positive integer. By an appropriate choice of α , we reduce the multidegree of h and by a choice of p , we make sure the difference is not identically zero (see the proof of Theorem 5.4). This can be done unless h is an algebraic homogeneous polynomial. Iterating this process, we will eventually reach a situation where h has been replaced by an algebraic homogeneous polynomial in which case our formula defines $\text{D}^n a$. \dashv

We proved that there is a formula $\psi(\bar{x}) \in \mathcal{L}_R$ such that $\psi(a, \text{D} a, \dots, \text{D}^{n-1} a, y)$ has a unique solution which is $\text{D}^n a$. We can assume $\text{MR}_R(a, \text{D} a, \dots, \text{D}^{n-1} a) = \omega \cdot n$ since otherwise D is definable by the induction hypothesis. Let b_1, \dots, b_{n-1} be differentially independent elements over a . Then

$$\text{tp}_R(a, b_1, \dots, b_{n-1}) = \text{tp}_R(a, \text{D} a, \dots, \text{D}^{n-1} a) =: p(x_0, \dots, x_{n-1}).$$

Evidently $\exists! y \psi(x_0, \dots, x_{n-1}, y) \in p$ where “ $\exists!$ ” stands for “there is a unique” (it is obviously first-order expressible). Therefore

$$\mathcal{F}_R \models \exists! y \psi(a, b_1, \dots, b_{n-1}, y),$$

³Here we use square brackets for ease of reading. They do not have any special meaning.

and the unique solution of $\psi(a, b_1, \dots, b_{n-1}, y)$ is a differential rational function of a, b_1, \dots, b_{n-1} . Denote it by $r(a, b_1, \dots, b_{n-1})$. If r is an algebraic rational function then

$$\xi(\bar{x}) := \forall y(\psi(x_0, \dots, x_{n-1}, y) \leftrightarrow y = r(x_0, \dots, x_{n-1}))$$

is a formula in the language of reducts and is true of (a, b_1, \dots, b_{n-1}) . Hence it must be true of $(a, D a, \dots, D^{n-1} a)$ too, which means $D^n a = r(a, D a, \dots, D^{n-1} a)$ which is impossible since a is differentially transcendental.

Thus, r is not algebraic. By a compactness argument (as in Lemma 6.3), we can choose $t_1, \dots, t_{n-1} \in \mathbb{Q}(t, a)$ such that $\mathcal{F} \models \xi(a, t_1, \dots, t_{n-1})$ and $r(a, t_1, \dots, t_{n-1}) \in \mathbb{Q}(t)\langle a \rangle \setminus \mathbb{Q}(t, a)$. This guarantees that the formula $\psi(a, t_1, \dots, t_{n-1}, y)$ (which is in the language of reducts) defines a nonalgebraic (in the field theoretic sense) element over a and so $\text{dcl}_R(a) \supsetneq k_0(a)$. So D is definable due to Theorem 7.6. \dashv

Recall that in a stable theory a set A (in the monster model) is called *independent* (over B) if for any $a \in A$, we have $a \perp_B A \setminus \{a\}$.

COROLLARY 7.8. D is definable in \mathcal{F}_R if and only if the sequence $a, D a, D^2 a, \dots$ is not independent (over the empty set) in \mathcal{F}_R .

PROOF. If the sequence $a, D a, D^2 a, \dots$ is not independent then for some n the set $\{a, D a, \dots, D^n a\}$ is not independent. Therefore $\text{MR}_R(a, D a, \dots, D^n a) < \omega \cdot (n + 1)$. \dashv

As a common generalisation of Theorem 5.4 and Proposition 7.5, we prove the following result.

PROPOSITION 7.9. Suppose E (a curve in general sense) contains a differential curve defined by a nondegenerate equation $f(x, y) = 0$ where $\text{ord}_X(f) > 0$ and $\text{ord}_Y(f) = 0$. Then D is definable in \mathcal{F}_E .

PROOF. Let $g(X, Y) = p(X, D X, \dots, D^n X, Y)$ be an irreducible nondegenerate factor of $f(X, Y)$. Furthermore, as $\text{ord}_X(f) > 0$, we can assume that $\text{ord}_X(g) > 0$. Consider the formula

$$\varphi(x, y_1, \dots, y_n) := \exists z(E(x, z) \wedge p(x, y_1, \dots, y_n, z) = 0).$$

Clearly $\mathcal{F}_E \models \varphi(a, D a, \dots, D^n a)$. Further, if $\varphi(a, b_1, \dots, b_n)$ holds then for some c , we have

$$p(a, \bar{b}, c) = 0 \wedge E(a, c).$$

Since p is irreducible, a, b_1, \dots, b_n are algebraically dependent over c . Moreover, $\text{ord}_X(g) > 0$ implies that b_1, \dots, b_n are algebraically dependent over $\{a, c\}$. On the other hand, c is differentially algebraic over a . Therefore a, \bar{b} are differentially dependent and hence $\text{MR}_D(\varphi) < \omega \cdot (n + 1)$. Now Proposition 7.7 finishes the proof. \dashv

One will certainly notice at this point that we found a number of conditions on \mathcal{F}_R which are all equivalent to definability of D . We sum up all these conditions in the following theorem.

THEOREM 7.10. For a differentially transcendental element $a \in F$ the following are equivalent:

1. D is definable in the reduct \mathcal{F}_R without parameters,
2. $\text{MR}_R(D a/a) < \omega$,
3. $\text{MR}_R(D a/a) = 0$,

4. $\text{tp}_R(\mathbb{D} a/a)$ forks over the empty set,
5. The sequence $(\mathbb{D}^n a)_{n \geq 0}$ is not (forking) independent,
6. $\text{dcl}_R(a) \supsetneq k_0(a)$,
7. $\text{acl}_R(a) \supsetneq (k_0(a))^{\text{alg}}$,
8. Every model of T_R^+ is the \mathcal{L}_R -reduct (with canonical interpretation) of a differentially closed field,
9. Every automorphism of \mathcal{F}_R fixes \mathbb{D} setwise.

PROOF. We need only show $9 \Rightarrow 1$. Take any automorphism σ of \mathcal{F}_R which fixes a . It fixes \mathbb{D} setwise, hence $(\sigma(a), \sigma(\mathbb{D} a)) \in \mathbb{D}$. This means $\sigma(\mathbb{D} a) = \mathbb{D}(\sigma a) = \mathbb{D} a$. Thus any automorphism of \mathcal{F}_R fixing a fixes $\mathbb{D} a$. Since \mathcal{F}_R is saturated, $\mathbb{D} a$ is definable over a . Therefore \mathbb{D} is definable. \dashv

We conclude this section by giving examples of differential equations that do not define \mathbb{D} .

EXAMPLE 7.11. We will show that unary relations cannot define \mathbb{D} .

Let R consist of unary relations, i.e., definable subsets of F (by quantifier elimination of DCF_0 , we may assume R consists of sets of solutions of one-variable equations). Then \mathbb{D} is not definable in \mathcal{F}_R .

Consider the differential closure K of k_0 inside \mathcal{F} , that is,

$$K = \{d \in F : \text{DR}(d) < \omega\}.$$

This is by definition a differentially closed field. Take a generic element $a \in F$, i.e., an element outside K . Let $L \supseteq K$ be the differential closure of $K\langle a \rangle$ inside \mathcal{F} . Further, denote $a_i = \mathbb{D}^i a$, $i \geq 0$, and let A be a transcendence basis of L over K containing these elements (not differential transcendence basis, which would consist only of a).

Define a new derivation \mathbb{D}_1 on L as follows. Set $\mathbb{D}_1 = \mathbb{D}$ on $K \cup A \setminus \{a_0, a_1\}$ and $\mathbb{D}_1 a_0 = a_2$, $\mathbb{D}_1 a_1 = a_0$. This can be uniquely extended to a derivation of L . The field automorphism $\sigma \in \text{Aut}(L/K)$ which fixes $A \setminus \{a_0, a_1\}$ and swaps a_0 and a_1 is in fact an isomorphism of differential fields $\mathcal{L} = (L; +, \cdot, \mathbb{D})$ and $\mathcal{L}_1 = (L; +, \cdot, \mathbb{D}_1)$. Therefore the latter is differentially closed.

Thus, we have a field L equipped with two different derivations \mathbb{D} and \mathbb{D}_1 and L is a differentially closed field with respect to each of them. Further, $K \subset L$ consists of all differentially algebraic elements in \mathcal{L} . Since \mathcal{L} and \mathcal{L}_1 are isomorphic over K , the differential closure of k_0 in \mathcal{L}_1 is equal to K as well. Therefore the interpretations of relation symbols for one-variable differential equations in \mathcal{L} and \mathcal{L}_1 are contained in K . But \mathbb{D} and \mathbb{D}_1 agree on K and therefore those interpretations agree in \mathcal{L} and \mathcal{L}_1 . This shows that \mathbb{D} is not definable in the structure \mathcal{F}_R .

EXAMPLE 7.12. Now, we give a more interesting example.

PROPOSITION 7.13. The exponential differential equation $\mathbb{D} y = y \mathbb{D} x$ does not define \mathbb{D} .

We show first that for a differential equation E if \mathbb{D} is definable in T_E then E is uniquely determined by T_E .

LEMMA 7.14. If \mathbb{D} is definable in T_E then for any differential equation $E'(x, y)$

$$T_E = T_{E'} \Rightarrow E = E'.$$

PROOF. Let E be given by the equation $f(x, y) = 0$. Since D is definable, the formula $\forall x, y(E(x, y) \leftrightarrow f(x, y) = 0)$ (more precisely, its translation into the language of the reducts) is in T_E . In other words, the fact that E is defined by the equation $f(x, y) = 0$ is captured by T_E . Therefore if E' has the same theory as E it must be defined by the same equation $f(x, y) = 0$. \dashv

PROOF OF PROPOSITION 7.13. An axiomatisation of the complete theory of the exponential differential equation is given in [6]. One can deduce from the axioms that the equation $Dy = 2yDx$ is elementarily equivalent to the exponential equation. But clearly those two equations define different sets in differentially closed fields. Hence the previous lemma shows that D is not definable if E is given by $Dy = yDx$. \dashv

We will give another proof of Proposition 7.13 in Section 8.

§8. Model completeness. In Section 6, we showed that if a formula $\varphi(x, y)$ defines a small set which contains the point (a, Da) for a differentially transcendental element a then D is definable. Moreover, if φ is existential then D is existentially definable. Recall that smallness of a set can be verified as follows: if b is a generic (differentially transcendental) element over a , that is, (a, b) is a generic pair (differentially independent), then $\varphi(x, y)$ defines a small set if and only if $\neg\varphi(a, b)$. Thus, instead of working with formulas defining D , we can work with formulas $\varphi(x, y)$ with $\varphi(a, Da) \wedge \neg\varphi(a, b)$.

DEFINITION 8.1. A formula $\varphi(x, y) \in \mathcal{L}_R$ is a *D-formula* if $\mathcal{F} \models \varphi(a, Da) \wedge \neg\varphi(a, b)$, where (a, b) is a differentially independent pair.

Here, we worked over the empty set. In particular, a is differentially transcendental over the empty set and the definitions that we consider are again over the empty set, i.e., without parameters. However, it is clear that, we could in fact work over any set $A \subseteq F$. In this case, we should let a be differentially transcendental over A . If $\varphi(x, y)$ is a formula over A such that $\varphi(a, Da) \wedge \neg\varphi(a, b)$ holds where b is differentially transcendental over Aa (in this case, we will say φ is a *D-formula over A*), then certainly D is definable over A . Moreover, if $\varphi(x, y)$ is existential then D is existentially definable over A . In this section, we use this fact to prove that under a natural assumption, if D is definable then it is existentially definable.

As above $a \in F$ is a differentially transcendental element and $k_0 = \mathbb{Q}(t) = \text{dcl}_R(\emptyset)$ (recall that t is an element with $Dt = 1$).

THEOREM 8.2. *If T_R is inductive (i.e., $\forall\exists$ -axiomatisable) and defines D then it defines D existentially and, therefore, T_R is model complete.*

This is similar to Lindström's theorem in spirit stating that an inductive theory, which is categorical in some infinite cardinality, is model complete (see [15]). We can also consider another formulation of Theorem 8.2: if T_R is not model complete and is inductive, then it does not define D . In general, T_R is not expected to be model complete, so in inductive reducts, definability of a derivation is expected to be rare.

We now establish an auxiliary result which will be used in the proof of Theorem 8.2.

LEMMA 8.3. Let $\varphi(x, \bar{u}) \in \mathcal{L}_R$ be a quantifier-free formula and $p(X, Y, \bar{U}) \in k_0[X, Y, \bar{U}]$ be an algebraic polynomial which is monic in the Y variable. Denote

$$\chi(x, y) := \forall \bar{u}(\varphi(x, \bar{u}) \rightarrow p(x, y, \bar{u}) = 0).$$

If $\exists \bar{u}\varphi(a, \bar{u})$ and $\chi(a, D a)$ hold then D is existentially definable.

PROOF. Let the tuple $(b_1, \dots, b_m, e_1, \dots, e_s)$ be of maximal differential transcendence degree m over a such that $\mathcal{F}_R \models \varphi(a, b_1, \dots, b_m, e_1, \dots, e_s)$, and assume that b_1, \dots, b_m are differentially independent over a .

Consider the formula

$$\psi(x, y, \bar{z}) = \exists v_1, \dots, v_s(\varphi(x, \bar{z}, \bar{v}) \wedge p(x, y, \bar{z}, \bar{v}) = 0).$$

Clearly $\psi(a, D a, \bar{b})$ holds. Moreover, if $\psi(a, d, \bar{b})$ holds for some d , then for some d_1, \dots, d_s , we have

$$\mathcal{F}_R \models \varphi(a, \bar{b}, \bar{d}),$$

which implies that d_1, \dots, d_s have finite rank over $\{a, b_1, \dots, b_m\}$. Since p is monic as a polynomial of Y and $p(a, d, \bar{b}, \bar{d}) = 0$, we conclude that $d \in (k_0(a, \bar{b}, \bar{d}))^{\text{alg}}$ and hence d is not generic over $\{a, b_1, \dots, b_m\}$.

Thus working over the parameter set $B = \{b_1, \dots, b_m\}$, we see that a is generic over B and $\psi(x, y, \bar{b})$ is a D -formula over B . Hence, we can make it into a proper definition of D with parameters from B . Thus, we get an existential definition of D with differentially independent parameters b_1, \dots, b_m . By Lemma 6.3, we have an existential definition without parameters. \dashv

Now, we are ready to prove our main theorem.

PROOF OF THEOREM 8.2. Let $\delta(x, y)$ be a formula defining D . We assume that D is not existentially definable, hence δ is not existential. The main idea of the proof is that unless one says explicitly that $\forall x \exists y \delta(x, y)$, one cannot guarantee that δ defines a function. In other words, we will prove that $\forall x \exists y \delta(x, y)$ (which is not an $\forall \exists$ -sentence) is not implied by the $\forall \exists$ -part of T_R , as otherwise we will be able to find an existential definition of D . This will contradict our assumption of inductiveness.

Let T be the $\forall \exists$ -part of T_R , i.e., the subset of T_R consisting of $\forall \exists$ -sentences. In other words

$$T = \{\forall \bar{x} \exists \bar{y} \varphi(\bar{x}, \bar{y}) : \varphi \text{ is a quantifier-free formula in } \mathcal{L}_R, \mathcal{F}_R \models \forall \bar{x} \exists \bar{y} \varphi(\bar{x}, \bar{y})\}.$$

Denote $\Phi := \{\varphi(\bar{x}, \bar{y}) : \forall \bar{x} \exists \bar{y} \varphi(\bar{x}, \bar{y}) \in T\}$.

By our assumption, T is an axiomatisation of T_R . However, we will get a contradiction to this by showing that T has a model in which $\forall x \exists y \delta(x, y)$ does not hold. The construction of that model will go as follows. We start with the field $k = \mathbb{Q}(t, a) = k_0(a)$ and add solutions of the formulas $\varphi \in \Phi$ step by step (for $\varphi(\bar{x}, \bar{y}) \in \Phi$, we think of \bar{x} as coefficients and of \bar{y} as solutions). We also make sure that, we do not add $D a$ in any step. If the latter is not possible then we show that D is existentially definable.

In order to implement this idea, we expand the language by adding constant⁴ symbols for solutions of all $\varphi \in \Phi$. First, take $C_0 = \{a\}$. We will inductively add new constant symbols to C_0 countably many times.

If C_l is constructed then C_{l+1} is the expansion of C_l by new constant symbols as follows. For each $\varphi(\bar{x}, \bar{y}) \in \Phi$ with $|\bar{x}| = m$, $|\bar{y}| = n$ say, and for all $\bar{c} \in C_l^m$, add new constant symbols $d_{\varphi, \bar{c}}^1, \dots, d_{\varphi, \bar{c}}^n$. After adding these new constants for all $\varphi \in \Phi$, we get C_{l+1} . Finally set $C = \bigcup_l C_l$. This is a countable set.

Now consider the following sets of sentences in the expanded language $\mathcal{L}_R \cup C$. First, denote

$$\Gamma(C) := \{\varphi(c_1, \dots, c_m, d_{\varphi, \bar{c}}^1, \dots, d_{\varphi, \bar{c}}^n) : \varphi(\bar{x}, \bar{y}) \in \Phi, |\bar{x}| = m, |\bar{y}| = n, \bar{c} \in C^m\}.$$

Furthermore, let

$$\Delta(C) := \{\neg\delta(a, c) : c \in C\}.$$

Finally, we set

$$\Sigma(C) := T_R \cup \text{tp}_R(a) \cup \Gamma(C) \cup \Delta(C).$$

CLAIM. $\Sigma := \Sigma(C)$ is satisfiable.

PROOF. If it is not satisfiable, then a finite subset $\Sigma_0 \subseteq \Sigma$ is not satisfiable. Denote the set of constants from C that occur in sentences from Σ_0 by $\{a, e_1, \dots, e_n\}$ (if necessary, we can assume a occurs in Σ_0 inessentially). We are going to give a its canonical interpretation in \mathcal{F} , and this is the reason that we separated it from the other constant symbols. Let $\psi(a, e_1, \dots, e_n) := \bigwedge(\Sigma_0 \cap \Gamma)$. The formula $\psi(x, u_1, \dots, u_n)$ is clearly quantifier-free and without parameters.

Thus

$$T_R \cup \text{tp}_R(a) \cup \{\psi(a, e_1, \dots, e_n)\} \cup \{\neg\delta(a, e_i) : i = 1, \dots, n\}$$

is inconsistent. This means that in particular, we cannot find interpretations for e_1, \dots, e_n in \mathcal{F}_R which will make the latter into a model of Σ_0 . As already mentioned above, a is interpreted canonically in \mathcal{F} , i.e., its interpretation is the element $a \in F$.

Therefore

$$\mathcal{F}_R \not\models \exists u_1, \dots, u_n \left[\psi(a, \bar{u}) \wedge \bigwedge_i \neg\delta(a, u_i) \right].$$

This means

$$\mathcal{F}_R \models \forall \bar{u} \left[\psi(a, \bar{u}) \longrightarrow \bigvee_i u_i = D a \right].$$

Note that evidently $\mathcal{F}_R \models \exists \bar{u} \psi(a, \bar{u})$, i.e., the implication above does not hold vacuously. So the formula

$$\chi(x, y) := \forall \bar{u} \left[\psi(x, \bar{u}) \longrightarrow \prod_{i=1}^n (y - u_i) = 0 \right]$$

satisfies the conditions of Lemma 8.3. Hence, D is existentially definable. This contradiction proves the claim. \dashv

⁴In this proof we use the word “constant” for constant symbols only and not for constants in the sense of differential algebra. In particular, the interpretations of those constant symbols may not be constants in the differential sense.

Thus Σ is satisfiable. Take a model \mathcal{M} of Σ and inside this model consider the subset K consisting of interpretations of the constant symbols from C . We claim that K is closed under addition and multiplication and contains $0, 1, t$. This is because the sentence $\forall x, y \exists z, w (x + y = z \wedge x \cdot y = w)$, being $\forall \exists$, belongs to T . So, by our construction of C , for each $c_1, c_2 \in C$, we have elements $d_1, d_2 \in C$ such that the sentences $c_1 + c_2 = d_1, c_1 \cdot c_2 = d_2$ are in Σ . Similarly $0, 1, t \in K$ since the sentences $\exists x(x = 0), \exists x(x = 1)$, and $\exists x(x = t)$ are in T . Therefore K is a structure in the language of rings. In fact it is an algebraically closed field (containing k) since ACF_0 is $\forall \exists$ -axiomatisable. Hence K is a structure $\mathcal{K}_R = (K; +, \cdot, 0, 1, t, P)_{P \in R}$ in the language of the reducts (with the induced structure from \mathcal{M}). By the choice of Σ , we know that \mathcal{K}_R is a model of T .

If we choose \mathcal{M} to be saturated of cardinality $|F|$ (such a model exists due to stability) then we can identify it with \mathcal{F}_R .⁵ In that case \mathcal{K}_R is a substructure obtained by starting with $k_0(a)$ and inductively adding solutions to formulas from Φ .

Suppose for a moment that δ is universal in order to illustrate what we are going to do next. Let

$$\delta(x, y) = \forall \bar{v} \rho(x, y, \bar{v})$$

with ρ quantifier-free. Since $\mathcal{F}_R \models \neg \delta(a, s)$ for any $s \in K$, there is a witness $\bar{l}_s \in F$ such that $\mathcal{F}_R \models \neg \rho(a, s, \bar{l}_s)$. However this witness may not be in K . So, we add all those witnesses to K and then repeat the above procedure to make it a model of T . We also make sure we never add $D a$, which is possible as above (otherwise D would be existentially definable). Iterating this process countably many times and taking the union of all the constructed substructures, we end up with a structure \mathcal{N}_R in the language of reducts which is a model of T and contains witnesses for each of the formulas $\exists \bar{v} \neg \rho(a, s, \bar{v})$ where $s \in N$. Thus, $\mathcal{N}_R \models \neg \exists y \delta(a, y)$ which means that T is not an axiomatisation of T_R . This contradiction proves the theorem.

Now, we consider the general case. Let δ be of the form

$$\delta(x, y) = \forall \bar{v}_1 \exists \bar{w}_1 \forall \bar{v}_2 \cdots \forall \bar{v}_n \exists \bar{w}_n \rho(x, y, \bar{v}_1, \dots, \bar{v}_n),$$

where ρ is quantifier-free (the tuples \bar{v}_i and \bar{w}_i can be empty). Then

$$\neg \delta(x, y) = \exists \bar{v}_1 \forall \bar{w}_1 \exists \bar{v}_2 \cdots \exists \bar{v}_n \forall \bar{w}_n \neg \rho(x, y, \bar{v}_1, \dots, \bar{v}_n).$$

We add new constant symbols as follows. Firstly, for each $s \in C$, we add a tuple of constants \bar{l}_s^1 of the same length as \bar{v}_1 . Then for each i and each tuple $\bar{c} \in C^{|\bar{w}_i|}$, we add new constants $\bar{l}_{\bar{c}}^{i+1}$ with $|\bar{l}_{\bar{c}}^{i+1}| = |\bar{v}_{i+1}|$. Denote this extension of C by C' . Then, we add new constant symbols to C' for solutions of all formulas $\varphi \in \Phi$ as above. We denote this set by C^1 . Then, we iterate this procedure by adding new constants to witness $\neg \delta(a, s)$ (for each s from the set of constants already constructed) and then adding new constants for solutions of $\varphi \in \Phi$. Thus, we get a chain $C \subseteq C^1 \subseteq C^2 \subseteq \dots$. Let \tilde{C} be their union.⁶

⁵Note that this is not essential, but it helps to understand how the proof works.

⁶In this construction for an element $s \in C$ (and for tuples \bar{c}_i), we add one \bar{l}_s^1 in each step. Though this does not cause any problems, at each step we can add the corresponding sets of constants only for new constant symbols. In particular, after adding one \bar{l}_s^1 we do not add any such tuple for the same s any more. Alternatively, we could just require all those different tuples for the same element s to be equal by adding the appropriate formulas stating their equality.

For $A \subseteq \tilde{C}$ denote

$$\Xi(A) := \{-\rho(a, s, \bar{l}_s^1, \bar{c}_1, \bar{l}_{\bar{c}_1}^2, \dots, \bar{l}_{\bar{c}_{n-1}}^n, \bar{c}_n) : s \in A, \bar{c}_i \in A^{|\bar{w}_i|}\}.$$

For any $s \in K$, we know that $\mathcal{F}_R \models \neg\delta(a, s)$, therefore $\Sigma(C) \cup \Xi(C)$ is satisfiable (note that this collection of sentences contains parameters from C'). The proof of the above claim shows that $\Sigma(C^1) \cup \Xi(C)$ is satisfiable and so $\Sigma(C^1) \cup \Xi(C^1)$ is satisfiable too. Proceeding inductively, we see that $\Sigma(C^i) \cup \Xi(C^i)$ is satisfiable for each $i < \omega$. Hence, by compactness, $\Sigma(\tilde{C}) \cup \Xi(\tilde{C})$ is satisfiable.

The interpretation of \tilde{C} in a model of $\Sigma(\tilde{C}) \cup \Xi(\tilde{C})$ gives a structure \mathcal{N}_R in the language of reducts which is a model of T and contains witnesses for each of the formulas $\exists \bar{v}_1 \forall \bar{w}_1 \exists \bar{v}_2 \dots \exists \bar{v}_n \forall \bar{w}_n \neg\rho(a, s, \bar{v}_1, \dots, \bar{v}_n)$ where $s \in N$. Hence $\mathcal{N}_R \models \neg\exists y \delta(a, y)$ which means that T is not an axiomatisation of T_R , which is a contradiction. \dashv

As an immediate application of Theorem 8.2, we give another proof of Proposition 7.13 which states that if E is the exponential differential equation, i.e., it is given by $Dy = yDx$, then D is not definable in \mathcal{F}_E .⁷ Indeed, Kirby gives an $\forall\exists$ -axiomatisation of the first-order theory of the exponential differential equation ([6]). It is not model complete however, hence D cannot be definable due to Theorem 8.2. Of course it is the Ax-Schanuel inequality that is responsible for this. As Kirby proved, it is an adequate predimension inequality.

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⁷Note that Kirby [6] gives yet another proof of this fact by considering a lattice of reducts for exponential differential equations of some collections of semiabelian varieties, and showing that each of these can be expanded properly inside DCF_0 .

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