A MODIFIED FORM OF SIEGEL'S MEAN-VALUE THEOREM

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1. The Minkowski–Hlawka theorem† asserts that, if S is any n-dimensional star body, with the origin $\mathbf{o}$ as centre, and with volume less than $2\zeta(n)$, then there is a lattice of determinant 1 which has no point other than $\mathbf{o}$ in S. One of the methods used to prove this theorem splits up into three stages. (*a*) A function $\rho(\mathbf{x})$ is considered, and it is shown that some suitably defined mean value of the sum

$$\rho(\Lambda) = \sum_{\substack{\mathbf{x}\in\Lambda \\ \mathbf{x}\neq\mathbf{o}}} \rho(\mathbf{x}),$$

taken over a suitable set of lattices Λ of determinant 1, is equal, or approximately equal, to the integral

$$\int \rho(\mathbf{x})\, d\mathbf{x}$$

over the whole space. (*b*) By taking $\rho(\mathbf{x})$ to be equal, or approximately equal, to

$$\sum_{r=1}^{\infty} \mu(r)\, \sigma(r\mathbf{x}),$$

where $\sigma(\mathbf{x})$ is the characteristic function of S, and $\mu(r)$ is the Möbius function, it is shown that a corresponding mean value of the sum

$$\sigma(\Lambda^*) = \sum_{\mathbf{x}\in\Lambda^*} \sigma(\mathbf{x}),$$

where Λ^* is the set of primitive points of the lattice Λ, is equal, or approximately equal, to

$$\frac{1}{\zeta(n)}\int \sigma(\mathbf{x})\, d\mathbf{x}.$$

(*c*) The final result is deduced from this second mean-value result. Indeed, all the proofs of the Minkowski–Hlawka theorem with which we are familiar are essentially of this nature, although in many of the proofs two (or sometimes even all three) of the stages are condensed into a single stage.

In nearly all the proofs the mean values are taken over sets of lattices, which are chosen for sake of convenience in proving the required result‡; it is only in the proof given by Siegel (8) that the mean values are taken over all lattices Λ of determinant 1, and the lattices are all given equal weight in a certain sense.

† Minkowski (6), vol. 1, pp. 265, 270 and 277, states the result; the first published proof is due to Hlawka (4); for a simple proof see, for example, Cassels (1).

‡ But in special circumstances the set of lattices, over which the mean value is taken, is chosen for efficiency rather than convenience; see Mahler (5) and Davenport and Rogers (2).

In order to understand the nature of the results we obtain in this paper, and to appreciate their connexion with Siegel's results, it is necessary to consider the way in which Siegel's mean value is defined. Let Λ_0 denote the lattice of all points having integral coordinates. Then, for any linear transformation γ of determinant 1, the set $\Lambda = \gamma\Lambda_0$ is a lattice of determinant 1; and every lattice Λ of determinant 1 can be represented (in an infinite number of ways) in the form $\Lambda = \gamma\Lambda_0$, where γ is a linear transformation of determinant 1. Siegel introduces, into the group Γ of all linear transformations of determinant 1, a measure $\mu(\gamma)$, which is both left invariant and right invariant under the operations of the group. Since the total measure of this group Γ turns out to be infinite, it is not possible to define the mean value of $\rho(\Lambda)$ to be the ratio

$$\int_\Gamma \rho(\gamma\Lambda_0)\, d\mu(\gamma) \Big/ \int_\Gamma d\mu(\gamma).$$

Siegel overcomes this difficulty by using the Minkowski theory of the reduction of positive definite quadratic forms to define a fundamental region F of Γ such that

(i) F is measurable,

(ii) for almost all γ' in Γ, there is just one γ in F such that

$$\gamma\Lambda_0 = \gamma'\Lambda_0.$$

Then he is able to define his mean value to be

$$\int_F \rho(\gamma\Lambda_0)\, d\mu(\gamma) \Big/ \int_F d\mu(\gamma).$$

While the definition of F is to some extent arbitrary, it is clear from the invariance of the measure that the mean value will be the same for any set F satisfying the conditions (i) and (ii). Having introduced this mean value, Siegel shows that it is equal to the integral

$$\int \rho(\mathbf{x})\, d\mathbf{x}$$

taken over the whole space, provided $\rho(\mathbf{x})$ is integrable in the Riemann sense and vanishes outside a bounded domain. He also obtains the corresponding mean value of the sum, taken over the primitive points of a lattice of determinant 1.

In this paper we work with a mean value defined in a rather different way. We introduce a bound or norm $\|\gamma\|$ into the space Γ of linear transformations γ of determinant 1, by writing

$$\|\gamma\| = \sup_{|\mathbf{x}|\leqslant 1} |\gamma\mathbf{x}|,$$

where $|\mathbf{y}|$ is the distance of the point $\mathbf{y}$ from the origin $\mathbf{o}$. It will be clear that, for any $K > 1$, the set of γ of Γ with $\|\gamma\| \leqslant K$ is a compact measurable set with finite measure. We consider the mean value defined to be the limit

$$\lim_{K\to\infty} \int_{\|\gamma\|\leqslant K} \rho(\gamma\Lambda_0)\, d\mu(\gamma) \Big/ \int_{\|\gamma\|\leqslant K} d\mu(\gamma),$$

if it exists. We shall prove that it does exist, and that its value is $\int \rho(\mathbf{x})\, d\mathbf{x}$, provided $\rho(\mathbf{x})$ is Riemann integrable and vanishes outside a bounded region. Indeed, we shall prove more generally that this statement remains true not only if Λ_0 is the lattice of points with integral co-ordinates, but also if Λ_0 is any discrete set whose 'spherical density' (defined below) is unity.

For example, by taking Λ_0 to be the set of all primitive points of the lattice of points with integral coordinates, and by homogeneity considerations, we are able to deduce the corresponding mean value result for the sum taken over the primitive points of a lattice of determinant 1. The Minkowski–Hlawka theorem is an immediate consequence of this second mean-value result.

2. In this section we introduce the concept of the spherical density of a set of points, we explain the method Siegel uses to introduce the measure $\mu(\gamma)$ in the space Γ, and we state our main result formally in terms of these concepts and the norm introduced in § 1.

If Λ_0 is any discrete set of points, let $N(r)$ denote the number of points of Λ_0 in the closed sphere $S(r)$ with centre $\mathbf{o}$ and radius r, and let $V(r)$ denote the volume of this sphere. Then we say that Λ_0 has *spherical density d*, if

$$N(r)/V(r) \to d \quad \text{as} \quad r \to \infty.$$

In order to introduce the *measure* $\mu(\gamma)$, it is convenient to regard the space Γ as a subset of the space Γ_+ of all linear transformations of positive determinant. Further, we regard each linear transformation γ as a matrix (γ_{ij}) and as a point $(\gamma_{11}, \gamma_{12}, \ldots, \gamma_{nn})$ in n^2-dimensional Euclidean space R^{n^2}, so that Γ and Γ_+ are subsets of this Euclidean space. Let m denote the ordinary Lebesgue measure in R^{n^2}. It is easy to verify that this measure is both left and right invariant under the operations of the group Γ; for, if c is a fixed matrix of Γ, it is easy to see that the transformations $\gamma \to c\gamma$ and $\gamma \to \gamma c$ are linear transforms of R^{n^2} with determinant

$$(\det c)^n = 1,$$

so that these transformations leave the measure of any set of Γ_+ invariant. If G is any Borel set of Γ, we take the measure $\mu(G)$ of G in Γ to be

$$\mu(G) = m(G_+),$$

where G_+ is the 'cone' in Γ_+ with base G and vertex the origin, given by

$$G_+ = \bigcup_{0<\lambda\leqslant 1} \lambda G.$$

It is immediately clear that the function μ, defined in this way, is a countably additive measure function defined on the Borel sets of Γ. Further, it is clear from the invariance properties of the measure m that μ is both left and right invariant under the operations of Γ.

With these definitions we can state our main theorem.

THEOREM 1. *Let Λ_0 be a discrete set of points with spherical density d. Let $\rho(\mathbf{x})$ be a Riemann integrable function vanishing outside a bounded region. Then*

$$\int_{\|\gamma\|\leqslant K} \rho(\gamma\Lambda_0)\, d\mu(\gamma) \Big/ \int_{\|\gamma\|\leqslant K} d\mu(\gamma) \to d \int \rho(\mathbf{x})\, d\mathbf{x}, \tag{1}$$

as $K \to \infty$, the integral on the right being over the whole space.

3. In this section we state and prove a lemma, which will form the basis of our proof of Theorem 1. We first introduce some notation. Let Θ denote the space of all points $\boldsymbol{\theta}$ in n-dimensional space with $|\boldsymbol{\theta}| = 1$; so that Θ is the surface of the unit sphere. Let $\mu(\boldsymbol{\theta})$ denote the ordinary solid-angle measure† on this spherical surface, normalized so that

$$\int_{\Theta} d\mu(\boldsymbol{\theta}) = 1.$$

LEMMA 1. *Let Λ_0 be a discrete set of points, and let $\rho(\mathbf{x})$ be a continuous function vanishing outside a bounded domain. Then, for any $K > 0$,*

$$\int_{\|\gamma\| \leqslant K} \rho(\gamma\Lambda_0)\, d\mu(\gamma) = \int_{\|\gamma\| \leqslant K} \left\{\Sigma' \int_{\Theta} \rho(\gamma|\,\mathbf{x}\,|\,\boldsymbol{\theta})\, d\mu(\boldsymbol{\theta})\right\} d\mu(\gamma), \tag{2}$$

where‡ Σ' denotes summation over the points $\mathbf{x}$ other than $\mathbf{o}$ of Λ_0.

Before proving this lemma, it seems appropriate to explain the part it plays in the proof of Theorem 1. It enables us, in the integral over the points γ of Γ with $\|\gamma\| \leqslant K$, to replace the integrand $\rho(\gamma\Lambda_0)$, about which we know little, by the function

$$\Sigma' \int_{\Theta} \rho(\gamma\,|\,\mathbf{x}\,|\,\boldsymbol{\theta})\, d\mu(\boldsymbol{\theta}),$$

which we may regard as a smoothed form of $\rho(\gamma\Lambda_0)$, and which we will be able to show is, if $\|\gamma\|$ is large, sensibly constant and equal approximately to the integral

$$d\int \rho(\mathbf{x})\, d\mathbf{x}. \tag{3}$$

This will show that the ratio

$$\int_{\|\gamma\| \leqslant K} \rho(\gamma\Lambda_0)\, d\mu(\gamma) \Big/ \int_{\|\gamma\| \leqslant K} d\mu(\gamma)$$

is approximately equal to the integral (3), when K is large, and will enable us to prove Theorem 1.

Proof of Lemma 1. For any point $\mathbf{x}$ other than $\mathbf{o}$ of n-dimensional space, consider the function

$$\rho(\gamma\,|\,\mathbf{x}\,|\,\boldsymbol{\theta})$$

defined on the Cartesian product space $\Gamma \times \Theta$. Since ρ is a continuous function, it is clear that $\rho(\gamma\,|\,\mathbf{x}\,|\,\boldsymbol{\theta})$ is continuous on $\Gamma \times \Theta$. We introduce the product measure $\mu(\gamma \times \boldsymbol{\theta})$ in $\Gamma \times \Theta$ (see Halmos (3), §35, pp. 143–5), and consider the integral

$$\int_{\|\gamma\| \leqslant K} \rho(\gamma\,|\,\mathbf{x}\,|\,\boldsymbol{\theta})\, d\mu(\gamma \times \boldsymbol{\theta}). \tag{4}$$

Since the set of points γ of Γ with $\|\gamma\| \leqslant K$ is a compact set of Γ, and Θ is a compact space, it is clear that the integral (4) is over a compact subset of $\Gamma \times \Theta$. It follows, from

† Although we use the same letter to denote the measures in the spaces Γ and Θ, the space in which the measure is defined will always be clear from the argument of μ; further, there should be no confusion with the Möbius function.

‡ Here, in the integrand on the right, the linear transformation γ operates on the product of the scalar $|\mathbf{x}|$ with the vector $\boldsymbol{\theta}$.

the general form of Fubini's theorem†, that the integral (4) is equal to both the repeated integrals

$$\int_{\Theta}\left\{\int_{\|\gamma\|\leqslant K}\rho(\gamma\,|\,\mathbf{x}\,|\,\boldsymbol{\theta})\,d\mu(\gamma)\right\}d\mu(\boldsymbol{\theta}) \tag{5}$$

and

$$\int_{\|\gamma\|\leqslant K}\left\{\int_{\Theta}\rho(\gamma\,|\,\mathbf{x}\,|\,\boldsymbol{\theta})\,\mathrm{d}\mu(\boldsymbol{\theta})\right\}d\mu(\gamma). \tag{6}$$

We show that the integral (5) can be expressed in a simpler form. For any fixed point $\boldsymbol{\theta}$ of Θ, it is possible to choose a transformation ω of Γ having an orthogonal matrix such that

$$\omega\mathbf{x} = |\,\mathbf{x}\,|\,\boldsymbol{\theta}.$$

Thus

$$\int_{\|\gamma\|\leqslant K}\rho(\gamma\,|\,\mathbf{x}\,|\,\boldsymbol{\theta})\,d\mu(\gamma) = \int_{\|\gamma\|\leqslant K}\rho(\gamma\omega\mathbf{x})\,d\mu(\gamma).$$

Now changing the variable of integration from γ to $\delta = \gamma\omega$, and using the invariance of the measure $\mu(\gamma)$, we have

$$\int_{\|\gamma\|\leqslant K}\rho(\gamma\omega\mathbf{x})\,d\mu(\gamma) = \int_{\|\delta\omega^{-1}\|\leqslant K}\rho(\delta\mathbf{x})\,d\mu(\delta).$$

But, since ω^{-1} has an orthogonal matrix, it is clear, from the definition of the norm, that

$$\|\,\delta\omega^{-1}\,\| = \|\,\delta\,\|,$$

for all δ of Γ. Hence

$$\int_{\|\delta\omega^{-1}\|\leqslant K}\rho(\delta\mathbf{x})\,d\mu(\delta) = \int_{\|\gamma\|\leqslant K}\rho(\gamma\mathbf{x})\,d\mu(\gamma).$$

Combining these results, we see that the integral (5) is equal to

$$\int_{\Theta}\left\{\int_{\|\gamma\|\leqslant K}\rho(\gamma\mathbf{x})\,d\mu(\gamma)\right\}d\mu(\boldsymbol{\theta}) = \int_{\|\gamma\|\leqslant K}\rho(\gamma\mathbf{x})\,d\mu(\gamma)\int_{\Theta}d\mu(\boldsymbol{\theta}).$$

Using the normalization of the measure $\mu(\boldsymbol{\theta})$, and the equalities of the integrals (5) and (6), we obtain the result

$$\int_{\|\gamma\|\leqslant K}\rho(\gamma\mathbf{x})\,d\mu(\gamma) = \int_{\|\gamma\|\leqslant K}\left\{\int_{\Theta}\rho(\gamma\,|\,\mathbf{x}\,|\,\boldsymbol{\theta})\,d\mu(\boldsymbol{\theta})\right\}d\mu(\gamma), \tag{7}$$

for any point $\mathbf{x}$.

The result (2) will follow from the equation (7) once we can show that, when the integrals in (7) are summed over all the points $\mathbf{x}$ other than $\mathbf{o}$ of Λ_0, the order of the summations and integrations can be changed. To do this, it is sufficient to show that the integrals in (7) vanish at all but a finite number of the points $\mathbf{x}$ of Λ_0. Since $\rho(\mathbf{x})$ vanishes outside a bounded region, we may choose R so large that $\rho(\mathbf{x}) = 0$ for all $\mathbf{x}$ with $|\,\mathbf{x}\,| > R$. So, if one of the integrals in (7) is non-zero, there will be a linear transformation γ in Γ, such that

$$|\,\gamma\mathbf{x}\,| \leqslant R \quad \text{and} \quad \|\,\gamma\,\| \leqslant K.$$

† See Halmos (3), §36, pp. 145–8. Note that it would be possible to avoid the use of the general theory of product spaces by expressing the integrals in the spaces Θ, Γ and $\Gamma\times\Theta$ as ordinary Lebesgue integrals over appropriate cones in Euclidean spaces of dimensions n, n^2 and $n(n+1)$, and by using the classical Fubini theorem.

Now, as γ is a linear transformation of determinant 1, it can be expressed† in the form $\gamma = \omega_1 \xi \omega_2$, where ω_1 and ω_2 are orthogonal matrices and ξ is a diagonal matrix with diagonal elements $\xi_1, \xi_2, \ldots, \xi_n$ satisfying

$$\xi_1 \geqslant \xi_2 \geqslant \ldots \geqslant \xi_n > 0,$$

and $$\xi_1 \xi_2 \ldots \xi_n = 1.$$

So $$[\inf_{|\mathbf{y}|=1} |\gamma \mathbf{y}|][\sup_{|\mathbf{z}|=1} |\gamma \mathbf{z}|]^{n-1} = [\inf_{|\mathbf{y}|=1} |\xi \mathbf{y}|][\sup_{|\mathbf{z}|=1} |\xi \mathbf{z}|]^{n-1}$$
$$= \xi_n \xi_1^{n-1} \geqslant \xi_n \xi_{n-1} \ldots \xi_1 = 1.$$

Hence $$\frac{R}{|\mathbf{x}|} K^{n-1} \geqslant \left[\left|\gamma \frac{\mathbf{x}}{|\mathbf{x}|}\right|\right] \|\gamma\|^{n-1}$$
$$\geqslant [\inf_{|\mathbf{y}|=1} |\gamma \mathbf{y}|][\sup_{|\mathbf{z}|=1} |\gamma \mathbf{z}|]^{n-1} \geqslant 1,$$

and $|\mathbf{x}| \leqslant RK^{n-1}$. Since Λ_0 is discrete, there will only be a finite number of points $\mathbf{x}$ of Λ_0 for which the integrals in (7) are non-zero. This completes the proof of the lemma.

4. In this section we prove two lemmas about the function

$$f(r) = \int_\Theta \rho(\gamma r \boldsymbol{\theta})\, d\mu(\boldsymbol{\theta}),$$

on the assumption that $\rho(\mathbf{x})$ is continuous and vanishes outside a bounded region. The first lemma summarizes some of the properties of this function; while the second lemma shows that these properties are sufficient to ensure that the sum

$$\Sigma' f(|\mathbf{x}|),$$

taken over the points other than $\mathbf{o}$ of a set of spherical density d is, for large values of $\|\gamma\|$, approximately equal to the integral

$$d \int_0^\infty f(r)\, dJ_n r^n,$$

where J_n is the volume of the n-dimensional unit sphere. For the sequel, it is important to note that all the constants introduced in this section are independent of the linear transformation γ.

LEMMA 2. *The function $f(r)$ satisfies the identity*

$$\int_0^\infty f(r)\, dJ_n r^n = \int \rho(\mathbf{x})\, d\mathbf{x}. \tag{8}$$

† It follows from the theory of the orthogonal reduction of positive definite quadratic forms that such a representation is always possible. For, as $\gamma'\gamma$ is the matrix of a positive-definite quadratic form, we have $\gamma'\gamma = \omega_2' \xi \xi \omega_2$, where ξ is a diagonal matrix of the required type and ω_2 is an orthogonal matrix. Then the matrix ω_1, defined by the equation $\gamma = \omega_1 \xi \omega_2$, is automatically orthogonal.

There are constants c_1 and c_2, such that

$$\int_0^\infty |f(r)| \, dJ_n r^n \leqslant c_1. \tag{9}$$

and

$$|f(r)| \leqslant \frac{c_2}{r\|\gamma\|}, \tag{10}$$

for $r > 0$. For every constant $\epsilon > 0$, there is a constant $\lambda > 1$, such that

$$\int_0^\infty \max_{r/\lambda \leqslant s \leqslant \lambda r} |f(r) - f(s)| \, dJ_n r^n \leqslant \epsilon. \tag{11}$$

Proof. In the first place

$$\int_0^\infty f(r) \, dJ_n r^n = \int_0^\infty \left[\int_\Theta \rho(\gamma r \boldsymbol{\theta}) \, nJ_n r^{n-1} d\mu(\boldsymbol{\theta})\right] dr. \tag{12}$$

But on writing $r\boldsymbol{\theta} = \mathbf{x}$, the integral on the right of (12) will be recognized as the integral of $\rho(\gamma\mathbf{x})$ over the whole space, expressed in terms of generalized spherical polar coordinates. Since γ is a linear transformation of determinant 1, this integral is

$$\int \rho(\gamma \mathbf{x}) \, d\mathbf{x} = \int \rho(\mathbf{x}) \, d\mathbf{x}.$$

This proves (8). Similarly

$$\int_0^\infty |f(r)| \, dJ_n r^n \leqslant \int_0^\infty \left[\int_\Theta |\rho(\gamma r \boldsymbol{\theta})| \, nJ_n r^{n-1} d\mu(\boldsymbol{\theta})\right] dr$$

$$= \int |\rho(\gamma\mathbf{x})| \, d\mathbf{x} = \int |\rho(\mathbf{x})| \, d\mathbf{x},$$

which proves (9), as the integral on the right-hand side is finite and independent of γ.

We now obtain the inequality (10). As in the proof of Lemma 1, we may choose $R > 0$, so large that $\rho(\mathbf{x}) = 0$ for all $\mathbf{x}$ with $|\mathbf{x}| > R$. Let $\sigma(\mathbf{x})$ be a characteristic function of the set of points $\mathbf{x}$ with $|\mathbf{x}| \leqslant R$; and let P be the upper bound of $|\rho(\mathbf{x})|$, for all points $\mathbf{x}$. Then

$$|f(r)| \leqslant \int_\Theta |\rho(\gamma r \boldsymbol{\theta})| \, d\mu(\boldsymbol{\theta}) \leqslant P \int_\Theta \sigma(\gamma r \boldsymbol{\theta}) \, d\mu(\boldsymbol{\theta}).$$

Now, as above, γ can be expressed in the form $\gamma = \omega_1 \xi \omega_2$, where ω_1 and ω_2 are orthogonal matrices and ξ is a diagonal matrix with diagonal elements $\xi_1, \ldots, \xi_n$ satisfying

$$\xi_1 \geqslant \xi_2 \geqslant \ldots \geqslant \xi_n > 0.$$

Clearly

$$\|\gamma\| = \sup_{|\mathbf{x}|=1} |\gamma\mathbf{x}| = \sup_{|\mathbf{y}|=1} |\xi\mathbf{y}| = \xi_1.$$

Further, by the symmetry of the function σ, and the invariance of the solid-angle measure $\mu(\boldsymbol{\theta})$, we have

$$\int_\Theta \sigma(\gamma r \boldsymbol{\theta}) \, d\mu(\boldsymbol{\theta}) = \int_\Theta \sigma(\omega_1 \xi r \omega_2 \boldsymbol{\theta}) \, d\mu(\boldsymbol{\theta})$$

$$= \int_\Theta \sigma(\xi r \boldsymbol{\theta}) \, d\mu(\boldsymbol{\theta}).$$

But, if $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_n)$ and

$$|\theta_1| > \frac{R}{r\|\gamma\|},$$

then $$|\xi_1 r\theta_1| > R,$$

so that $$|\xi r\boldsymbol{\theta}| > R$$

and $$\sigma(\xi r\boldsymbol{\theta}) = 0.$$

Hence, writing $\phi = R/(r\|\gamma\|)$, we have

$$\int_\Theta \sigma(\gamma r\boldsymbol{\theta})\, d\mu(\boldsymbol{\theta}) \leqslant \int_{|\theta_1|\leqslant\phi} d\mu(\boldsymbol{\theta}).$$

Expressing this surface integral as an n-dimensional integral, we have

$$\int_{|\theta_1|\leqslant\phi} d\mu(\boldsymbol{\theta}) = \frac{1}{nJ_n}\int_S d\mathbf{x},$$

where S is the set of points $\mathbf{x}$ with

$$x_1^2 + x_2^2 + \ldots + x_n^2 \leqslant 1, \quad x_1^2 \leqslant \phi^2\{x_1^2 + x_2^2 + \ldots + x_n^2\}.$$

So, if C is the set of points $\mathbf{x}$ with

$$x_1^2 \leqslant \phi^2, \quad x_2^2 + \ldots + x_n^2 \leqslant 1,$$

then C contains S, and $$\int_{|\theta_1|\leqslant\phi} d\mu(\boldsymbol{\theta}) \leqslant \frac{1}{nJ_n}\int_C d\mathbf{x} = \frac{2\phi J_{n-1}}{nJ_n}.$$

Our inequalities now show that

$$|f(r)| \leqslant \frac{2J_{n-1}RP}{nJ_n r\|\gamma\|}.$$

This proves (10).

To complete the proof of the lemma, suppose that a positive ϵ is given. Since $\rho(\mathbf{x})$ is continuous, and vanishes outside a bounded region, it is uniformly continuous. So, for any $\eta > 0$, it is possible to choose a positive number δ so small that

$$|\rho(\mathbf{x}) - \rho(\mathbf{y})| \leqslant \eta$$

for all points $\mathbf{x}, \mathbf{y}$ with $$|\mathbf{x} - \mathbf{y}| \leqslant \delta.$$

Choose λ, with $1 < \lambda < 2$, so close to 1 that

$$(\lambda - 1)\, 2R \leqslant \delta.$$

Now consider any point $\mathbf{x}$, and any scalar ν with

$$1/\lambda \leqslant \nu \leqslant \lambda.$$

If $|\mathbf{x}| \leqslant 2R$, we have

$$|\mathbf{x} - \nu\mathbf{x}| = |1 - \nu|\,|\mathbf{x}| \leqslant (\lambda - 1)\, 2R \leqslant \delta,$$

and so $$|\rho(\mathbf{x}) - \rho(\nu\mathbf{x})| \leqslant \eta = \eta\sigma(\tfrac{1}{2}\mathbf{x}).$$

If $|\mathbf{x}| > 2R$, then $$|\mathbf{x}| > R, \quad |\nu\mathbf{x}| > R,$$

so that $$|\rho(\mathbf{x}) - \rho(\nu\mathbf{x})| = 0 = \eta\sigma(\tfrac{1}{2}\mathbf{x}).$$

Thus, provided $1/\lambda \leqslant \nu \leqslant \lambda$, we have

$$|\rho(\mathbf{x}) - \rho(\nu\mathbf{x})| \leqslant \eta\sigma(\tfrac{1}{2}\mathbf{x})$$

for all $\mathbf{x}$. Hence

$$\begin{aligned}\int_0^\infty \max_{r/\lambda\leqslant s\leqslant \lambda r} |f(r)-f(s)|\, dJ_n r^n \\ &= \int_0^\infty \max_{1/\lambda\leqslant \nu\leqslant\lambda} |f(r)-f(\nu r)|\, dJ_n r^n \\ &\leqslant \int_0^\infty \left[\int_\Theta \max_{1/\lambda\leqslant\nu\leqslant\lambda} |\rho(\gamma r\boldsymbol{\theta})-\rho(\gamma\nu r\boldsymbol{\theta})|\, d\mu(\boldsymbol{\theta})\right] dJ_n r^n \\ &\leqslant \int_0^\infty \left[\int_\Theta \eta\sigma(\tfrac{1}{2}\gamma r\boldsymbol{\theta})\, d\mu(\boldsymbol{\theta})\right] dJ_n r^n \\ &= \eta\int \sigma(\tfrac{1}{2}\mathbf{x})\, d\mathbf{x} = \eta 2^n J_n R^n.\end{aligned}$$

The lemma follows on choosing η so that

$$\eta 2^n J_n R^n < \epsilon.$$

LEMMA 3. *Let Λ_0 be a discrete set of points with spherical density d. Then there is a constant c_4 such that*

$$|\Sigma' f(|\mathbf{x}|)| < c_4, \tag{13}$$

the sum being taken over all points $\mathbf{x}$ other than $\mathbf{o}$ of Λ_0. Further, for every constant $\epsilon > 0$ there is a constant $G = G(\epsilon)$ such that

$$\left|\Sigma' f(|\mathbf{x}|) - d\int_0^\infty f(r)\, dJ_n r^n\right| < \epsilon, \tag{14}$$

for all γ of Γ with $\|\gamma\| \geqslant G$.

Proof. We arrange the proof so that the first result turns up incidentally during the proof of the second result. Suppose that $\epsilon > 0$ is a given constant. Write

$$\epsilon_1 = \frac{\epsilon}{4(d+1)}.$$

Let c_1, c_2 and λ be the constants provided by Lemma 2, when the ϵ of Lemma 2 is taken to be ϵ_1. Choose a constant η, with $0 < \eta < 1$, such that

$$\eta < \frac{\epsilon}{4c_1}.$$

Consider the points $\mathbf{x}$, other than $\mathbf{o}$, of Λ_0. Suppose that these points are $\mathbf{x}_1, \mathbf{x}_2, \ldots$, when enumerated in some order. It is convenient to write $r(t) = |\mathbf{x}_t|$ for $t = 1, 2, \ldots$. Let $S(a)$ and $A(a)$ denote the sphere of points $\mathbf{x}$ with $|\mathbf{x}| \leqslant a$, and the annular set of points $\mathbf{x}$ with $a < |\mathbf{x}| \leqslant \lambda a$. For any set E, let $N(E)$, and $V(E)$, denote the number of points $\mathbf{x}$ other than $\mathbf{o}$ of Λ_0 in E, and the volume of E. Since Λ_0 has spherical density d, it follows that, as $a \to \infty$,

$$\begin{aligned} N(A(a)) &= N(S(\lambda a)) - N(S(a)) \\ &= dV(S(\lambda a)) - dV(S(a)) + o(V(S(\lambda a))) \\ &= dJ_n(\lambda^n - 1)\, a^n + o(a^n) \\ &= dV(A(a)) + o(V(A(a))).\end{aligned}$$

So we may choose a so large that

$$|N(A(b))-dV(A(b))|<\eta V(A(b))$$

for all $b \geqslant a$.

In order to simplify the notation, let $I(0)$ denote the half-open half-closed interval $0<r\leqslant a$, and let $I(s)$, for $s\geqslant 1$, denote the interval $a\lambda^{s-1}<r\leqslant a\lambda^s$. Let $N(s)$ denote the number of positive integers t for which the number $r(t) = |x_t|$ lies in $I(s)$, and write

$$V(s) = \int_{I(s)} dJ_n r^n.$$

Then, if $s\geqslant 1$, we have

$$N(s) = N(A(a\lambda^{s-1})), \quad V(s) = V(A(a\lambda^{s-1}));$$

so that

$$|N(s)-dV(s)|<\eta V(s), \tag{15}$$

provided $s\geqslant 1$.

Now rearranging the sum over the points $\mathbf{x}$, other than $\mathbf{o}$, of Λ_0, and dividing up the range of integration of the integral, we have the identity

$$\begin{aligned}\Sigma' f(|\mathbf{x}|) - d\int_0^\infty f(r)\,dJ_n r^n &= \sum_{s=0}^{\infty}\left[\sum_{r(t)\in I(s)} f(r(t)) - d\int_{I(s)} f(r)\,dJ_n r^n\right]\\ &= \sum_{r(t)\in I(0)} f(r(t)) - d\int_{I(0)} f(r)\,dJ_n r^n\\ &\quad+\sum_{s=1}^{\infty}\left[\frac{N(s)}{V(s)}\int_{I(s)}\frac{1}{N(s)}\sum_{r(t)\in I(s)}\{f(r(t))-f(r)\}\,dJ_n r^n + \frac{N(s)-dV(s)}{V(s)}\int_{I(s)} f(r)\,dJ_n r^n\right],\end{aligned}$$

the division by $N(s)$ being possible, for $s\geqslant 1$, since (15) ensures that $N(s)$ is positive, if $s\geqslant 1$. Hence, using (15) and the inequalities provided by Lemma 2, we have

$$\begin{aligned}&\left|\Sigma' f(|\mathbf{x}|) - d\int_0^\infty f(r)\,dJ_n r^n\right|\\ &\leqslant \sum_{r(t)\in I(0)}\frac{c_2}{r(t)\,\|\gamma\|} + d\int_0^a \frac{c_2}{r\,\|\gamma\|}\,dJ_n r^n\\ &\qquad+\sum_{s=1}^{\infty}\left[(d+\eta)\int_{I(s)}\max_{t\in I(s)}|f(t)-f(r)|\,dJ_n r^n + \eta\int_{I(s)}|f(r)|\,dJ_n r^n\right]\\ &\leqslant \frac{1}{\|\gamma\|}\left[\sum_{r(t)\in I(0)}\frac{c_2}{r(t)} + \frac{c_2 n J_n d}{n-1}a^{n-1}\right]\\ &\qquad+(d+\eta)\int_0^\infty \max_{r/\lambda\leqslant t\leqslant\lambda r}|f(t)-f(r)|\,dJ_n r^n + \eta\int_0^\infty |f(r)|\,dJ_n r^n\\ &\leqslant \frac{c_3}{\|\gamma\|} + (d+1)\,\epsilon_1 + \eta c_1\\ &\leqslant \frac{c_3}{\|\gamma\|} + \tfrac{1}{2}\epsilon,\end{aligned} \tag{16}$$

where
$$c_3 = \sum_{0<r(t)\leqslant a} \frac{c_2}{r(t)} + \frac{c_2 n J_n d}{n-1} a^{n-1}$$
is a constant independent of γ.

It follows immediately, from (16) and the result (8) of Lemma 2, since $\|\gamma\| \geqslant 1$ for all γ of Γ, that the result (13) holds with
$$c_4 = c_3 + \tfrac{1}{2}\epsilon + \left|\int \rho(\mathbf{x})\, d\mathbf{x}\right|,$$
for any chosen ϵ. Further, it follows from (16) that the result (14) holds with
$$G = 2c_3/\epsilon.$$
This completes the proof of the lemma.

5. *Proof of Theorem* 1. We first prove the theorem on the assumption that $\rho(\mathbf{x})$ is continuous. Let $\epsilon > 0$ be given. For any G and K, with $0 < G < K$, it follows from Lemma 1 that
$$\begin{aligned}\int_{\|\gamma\|\leqslant K} \rho(\gamma\Lambda_0)\, d\mu(\gamma) - d\int \rho(\mathbf{x})\, d\mathbf{x} \int_{\|\gamma\|\leqslant K} d\mu(\gamma) \\ = \int_{\|\gamma\|\leqslant K} \left[\Sigma' f(|\mathbf{x}|) - d\int \rho(\mathbf{x})\, d\mathbf{x}\right] d\mu(\gamma) \\ = \int_{\|\gamma\|\leqslant G} \Sigma' f(|\mathbf{x}|)\, d\mu(\gamma) - d\int \rho(\mathbf{x})\, d\mathbf{x} \int_{\|\gamma\|\leqslant G} d\mu(\gamma) \\ + \int_{G<\|\gamma\|\leqslant K} \left[\Sigma' f(|\mathbf{x}|) - d\int \rho(\mathbf{x})\, d\mathbf{x}\right] d\mu(\gamma).\end{aligned} \tag{17}$$
Provided only that G is sufficiently large, it follows from Lemma 2 that the modulus of the two sides of (17) does not exceed
$$\left[c_4 + d\left|\int \rho(\mathbf{x})\, d\mathbf{x}\right|\right] \int_{\|\gamma\|\leqslant G} d\mu(\gamma) + \tfrac{1}{2}\epsilon \int_{\|\gamma\|\leqslant K} d\mu(\gamma).$$
Now
$$\int_{\|\gamma\|\leqslant K} d\mu(\gamma) \to +\infty$$
as† $K \to \infty$. So provided only that K is sufficiently large, the modulus of the two sides of (17) does not exceed
$$\epsilon \int_{\|\gamma\|\leqslant K} d\mu(\gamma).$$
This proves the required result, on the assumption that $\rho(\mathbf{x})$ is continuous.

The general result, when $\rho(\mathbf{x})$ is integrable in the Riemann sense, follows from the result when $\rho(\mathbf{x})$ is continuous; since, if $\rho(\mathbf{x})$ is integrable in the Riemann sense and vanishes outside a bounded region, then, for any $\epsilon > 0$, there will be continuous functions $\kappa(\mathbf{x})$, $\tau(\mathbf{x})$, vanishing outside a bounded region, and satisfying
$$\kappa(\mathbf{x}) \leqslant \rho(\mathbf{x}) \leqslant \tau(\mathbf{x}) \quad \text{for all } \mathbf{x}$$

† This result is well known. It may be easily proved by exhibiting an infinite sequence of disjoint compact subsets of Γ, each set having the same positive volume. For example, take S to be the set of all γ of Γ with $\|\gamma\| \leqslant 2$, and consider the sets, obtained by repeatedly applying to S a diagonal transformation ξ, with a norm sufficiently large to ensure that the sets S, ξS are disjoint.

and
$$\int \tau(\mathbf{x})\,d\mathbf{x} - \epsilon \leqslant \int \rho(\mathbf{x})\,d\mathbf{x} \leqslant \int \kappa(\mathbf{x})\,d\mathbf{x} + \epsilon.$$

For the truth of the result for these functions $\kappa(\mathbf{x})$ and $\sigma(\mathbf{x})$ implies that

$$\limsup_{K\to\infty} \int_{\|\gamma\|\leqslant K} \rho(\gamma\Lambda_0)\,d\mu(\gamma) \Big/ \int_{\|\gamma\|\leqslant K} d\mu(\gamma) \leqslant d\int \tau(\mathbf{x})\,d\mathbf{x} \leqslant d\int \rho(\mathbf{x})\,d\mathbf{x} + d\epsilon,$$

$$\liminf_{K\to\infty} \int_{\|\gamma\|\leqslant K} \rho(\gamma\Lambda_0)\,d\mu(\gamma) \Big/ \int_{\|\gamma\|\leqslant K} d\mu(\gamma) \geqslant d\int \kappa(\mathbf{x})\,d\mathbf{x} \geqslant d\int \rho(\mathbf{x})\,d\mathbf{x} - d\epsilon.$$

Since ϵ may be arbitrarily small this completes the proof of the theorem.

6. In this section we show that Theorem 1 essentially contains the following mean-value result, which differs from Theorem 1 in that the sum is taken over the primitive points of a lattice.

THEOREM 2. *Let Λ be a lattice with determinant Δ. Let $\rho(\mathbf{x})$ be a Riemann integrable function vanishing outside a bounded region. Let Σ^* denote summation over the primitive points of the lattice Λ. Then*

$$\int_{\|\gamma\|\leqslant K} \Sigma^* \rho(\gamma\mathbf{x})\,d\mu(\gamma) \Big/ \int_{\|\gamma\|\leqslant K} d\mu(\gamma) \to \frac{1}{\zeta(n)\,\Delta}\int \rho(\mathbf{x})\,d\mathbf{x},$$

as $K\to\infty$.

Proof. The lattice Λ is a set of spherical density $1/\Delta$. It is well known† that the set of primitive points of Λ is a set with spherical density $1/(\Delta\zeta(n))$. So the result follows immediately from Theorem 1.

Note added in proof. The authors have recently established that the result proved in this paper holds for a *Lebesgue integrable* function $\rho(x)$, provided only that it vanishes outside a bounded set. It is hoped to publish an outline of the proof of this slight extension later.

REFERENCES

(1) CASSELS, J. W. S. *Proc. Camb. phil. Soc.* 49 (1953), 165–6.
(2) DAVENPORT, H. and ROGERS, C. A. *Duke math. J.* 14 (1947), 367–75.
(3) HALMOS, P. R. *Measure theory* (New York, 1950).
(4) HLAWKA, E. *Math. Z.* 49 (1944), 285–312.
(5) MAHLER, K. *Duke math. J.* 13 (1946), 611–21.
(6) MINKOWSKI, H. *Gesammelte Abhandlungen* (Leipzig, 1911).
(7) ROGERS, C. A. *Ann. Math., Princeton* (2), 48 (1947), 994–1002.
(8) SIEGEL, C. L. *Ann. Math., Princeton* (2), 46 (1945), 340–7.

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† The proof of this result is a simple exercise in the use of the Möbius inversion formulae; see, for example, the lemma in Rogers (7), p. 998.