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θ-continuity and D_θ-completion of posets

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We introduce a new concept of continuity of posets, called θ-continuity. Topological characterizations of θ-continuous posets are put forward. We also present two types of dcpo-completion of posets which are D_θ-completion and D_{s_2}-completion. Connections between these notions of continuity and dcpo-completions of posets are investigated. The main results are (1) a poset P is θ-continuous iff its θ-topology lattice is completely distributive iff it is a quasi θ-continuous and meet θ-continuous poset iff its D_θ-completion is a domain; (2) the D_θ-completion of a poset B is isomorphic to a domain L iff B is a θ-embedded basis of L; (3) if a poset P is θ-continuous, then the D_θ-completion $D_\theta(P)$ is isomorphic to the round ideal completion $RI(P, \ll_\theta)$.

1. Introduction

Domain theory, initiated by Dana Scott in the late 1960s, has been widely studied and applied to various areas of mathematics, logic and computer science (Gierz et al. 1980). For different purposes, the concept of domains has been generalized in different ways. Quasicontinuous domains were introduced as a common generalization of both generalized continuous lattices and domains by Gierz et al. (1983). They extended the way below relation between points to that of subsets of dcpos, and proved that quasicontinuous domains equipped with the Scott topologies are precisely the spectra of distributive hypercontinuous lattices. In Mao and Xu (2006), the concept of quasicontinuous posets was introduced using the Scott topology on posets instead of the way below relation on subsets of posets. The concept of quasicontinuous posets generalizes the spectral characterization of quasicontinuity from dcpos to posets. To avoid the requirement of the existence of directed joins, Erné introduced s_2-continuous posets, which allow to generalize important characterizations of continuity from complete lattices to arbitrary posets (Erné 1981). In the manner of Erné, Zhang and Xu came up with a new way below relation and used it to define s_2-quasicontinuous posets as a common generalization of both s_2-continuous posets and quasicontinuous domains (Zhang and Xu 2015). Recall that a complete lattice L is called meet continuous if it satisfies the distributive law that binary meets distribute over directed joins. Kou, Liu and Luo extended the definition of meet continuity to general dcpos and presented a purely topological characterization (Kou et al. 2003). A further generalization of meet continuity from dcpos to the setting of posets has been studied in the literature (Mao and Xu 2009). The study of domain theoretic concepts generalized from dcpos to posets is attracting more and more attention

(Huang et al. 2009; Keimel and Lawson 2009, 2012; Xu 2006; Zhao 2015). One orientation of the study is dcpo-completion of posets. In Zhao and Fan (2010), with the motivation to answer the question of whether posets and dcpos define the same class of Scott closed set lattices, Zhao and Fan introduced a new type of dcpo-completion of posets which is idempotent, called the D-completion. They showed that every poset and its D-completion have isomorphic Scott closed set lattices, which gave a positive answer to the problem. Similarly, a new question naturally arises: Do posets and dcpos have the same class of closed set lattices with respect to the s_2-topologies? We also observe that if there is an ideal without upper bounds in a poset, which is very common, then in the sense of s_2-approximation, all points can only be approximated by points in the ideal. To be s_2-continuous, there should be no points isolated from the ideal, i.e., every point should be directedly approximated by points in the ideal, in a precise sense explained in Section 2. This indicates that the concept of s_2-continuity is stronger than that of continuity.

In this paper, we introduce a new relation, called θ-approximation. It has the advantage that the existence of directed joins is not necessarily required and is weaker than s_2-approximation, which avoids the situation we mentioned above. The θ-continuous posets and the θ-topologies coincide with domains and Scott topologies in the case of dcpos. Two kinds of dcpo-completion of posets are put forward here, which we refer to as the D_θ-completion and D_{s_2}-completion. Every poset and its D_θ-completion (resp., D_{s_2}-completion) have isomorphic lattices of open sets with respect to the θ-topologies (resp., s_2-topologies). This gives a positive answer to the above question, and, moreover, establishes the same result for lattices of θ-open sets. Additionally, the D_θ-completion (resp., D_{s_2}-completion) can be extended to a reflector from the category $\mathbf{POS}_\theta$ (resp., $\mathbf{POS}_{s_2}$) of posets and continuous mappings with respect to the θ-topologies (resp., s_2-topologies) to the full subcategory $\mathbf{DCPO}$ of dcpos and Scott continuous mappings.

2. Preliminaries

The following are definitions of domain theory that will be used later, which can be found in the literature (Abramsky and Jung 1994; Gierz et al. 2003).

Let P and Q be posets. For $A \subseteq P$ and $x \in P$, we write: $\downarrow A = \{y \in P : y \leqslant a$ for some $a \in A\}$ and $\downarrow x = \downarrow\{x\}$; $A^{\downarrow} = \{y \in P : y \leqslant a$ for all $a \in A\}$. $\uparrow A$, $\uparrow x$, and $A^{\uparrow}$ are defined dually. We say that x is *way below* y, written $x \ll y$, if whenever $D \subseteq P$ is directed for which $\bigvee D$ exists (where $\bigvee D$ denotes the supremum of D), the relation $y \leqslant \bigvee D$ always implies $x \in \downarrow D$. We write $\twoheaddownarrow x = \{u \in P : u \ll x\}$, $\twoheaduparrow x = \{v \in P : x \ll v\}$. A subset U of P is *Scott open* if (i) $U = \uparrow U$; (ii) for each directed subset D, $D \bigcap U \neq \varnothing$ whenever $\bigvee D$ exists and $\bigvee D \in U$. Let $\sigma(P) = \{U \subseteq P : U$ is Scott open$\}$ denote the Scott topology of P and $\sigma(P)^c$ be the set of all Scott closed sets of P. Let $cl_\sigma(A)$ and $int_\sigma(A)$ denote the closure and interior of A with respect to the Scott topology. P is called continuous if $\twoheaddownarrow x$ is directed and $x = \bigvee \twoheaddownarrow x$ for all $x \in P$. A mapping $f : P \to Q$ is Scott continuous if $f(\bigvee D) = \bigvee f(D)$ holds for any directed subset D with existing $\bigvee D$.

Let $A^\delta = (A^\uparrow)^\downarrow$. We say that x *s_2-approximates* y, written $x \ll_{s_2} y$, if for each directed set $D \subseteq P$ with $y \in D^\delta$, there exists $d \in D$ with $x \leqslant d$. Recall the situation mentioned in the

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Introduction, if there is a directed subset D without upper bounds in P, i.e., $D^{\uparrow} = \varnothing$, then $D^{\delta} = P$ and thus $x \ll_{s_2} y$ always implies that $x \in {\downarrow}D$. We write $\Downarrow_{s_2} x = \{u \in P : u \ll_{s_2} x\}$ and $\Uparrow_{s_2} x$ is defined dually. P is called s_2-continuous if for all $x \in P$, $x = \bigvee \Downarrow_{s_2} x$, and $\Downarrow_{s_2} x$ is directed. A subset $U \subseteq P$ is called s_2-open if (i) $U = {\uparrow}U$; (ii) for every directed subset D, $D \bigcap U \neq \varnothing$ whenever $D^{\delta} \bigcap U \neq \varnothing$. Note that s_2-open is exactly the σ_2-open in Zhang and Xu (2015). In order to avoid confusion and misunderstanding, we use s_2 in place of σ_2 in this paper. Similar to the definition of Scott topology, the s_2-topology of P will be denoted by $s_2(P)$ and the set of all s_2-closed subsets of P will be denoted by $s_2(P)^c$. In the same way, we have an s_2-closure operator cl_{s_2} and an s_2-interior operator int_{s_2}.

A subset A of P is called D-closed if for all directed subsets $D \subseteq A$, if $\bigvee D$ exists, then $\bigvee D \in A$. The set of complements of all D-closed sets of P forms a topology, which will be called the D-topology of P. Let cl_d, called D-closure, be the closure operator with respect to the D-topology. A D-completion of a poset P is a dcpo L together with a Scott continuous mapping $\eta : P \to L$, such that for any Scott continuous mapping $f : P \to M$ into a dcpo M, there exists a unique Scott continuous mapping $\hat{f} : L \to M$ satisfying $f = \hat{f} \circ \eta$.

Lemma 2.1 (Zhao and Fan 2010). If $f : P \to Q$ is a Scott continuous function between posets, then for any $X \subseteq P$, $f(cl_d(X)) \subseteq cl_d(f(X))$.

Lemma 2.2 (Zhao and Fan 2010). If X is a subset of a poset P and $f, g : cl_d(X) \to Q$ are Scott continuous mappings into a poset Q such that $f|_X = g|_X$, then $f = g$.

For a topological space (X, τ), a binary relation $\leqslant_\tau$ is defined as follows: $x \leqslant_\tau y \Leftrightarrow x \in cl_\tau(y)$. Let ${\uparrow}_\tau A = \{x \in X : a \leqslant_\tau x \text{ for some } a \in A\}$. A topological space (X, τ) is called *locally finitary compact* if for each $U \in \tau$ and $x \in U$, there exists a finite subset F such that $x \in int_\tau({\uparrow}_\tau F) \subseteq {\uparrow}_\tau F \subseteq U$.

Lemma 2.3 (Xu and Yang 2009). For a topological space (X, τ), the following conditions are equivalent:

1. (X, τ) is locally finitary compact.
2. $(\tau, \subseteq)$ is a hypercontinuous lattice.

3. θ-continuous posets

Recall that in a dcpo L, we say $x \ll y$ if $y \leqslant \bigvee D$ always implies $x \leqslant d$ for some $d \in D$ where D is directed. The idea behind the notion of θ-approximation comes from the fact that the condition $y \leqslant \bigvee D$ is equivalent to say that D has an upper bound and y is below every upper bound of D. In this manner, the directed completeness is not necessarily required. Before giving the definition of θ-approximation, we introduce the following map to formalize the idea. For any poset P, let $\Theta : 2^P \to 2^P$ be defined by $\Theta(A) = {\downarrow}A$, if $A^{\uparrow} = \varnothing$, and $\Theta(A) = A^{\delta}$, if $A^{\uparrow} \neq \varnothing$. In the case of dcpos, if D is directed in P, then $\Theta(D) = D^{\delta} = {\downarrow}\bigvee D$.

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Definition 3.1. Let P be a poset and $x, y \in P$.

i. We say that x *θ-approximates* y, in symbols $x \ll_\theta y$, if for all directed subsets $D \subseteq P$, $y \in \Theta(D)$ always implies $x \in \downarrow D$. An element satisfying $x \ll_\theta x$ is said to be *θ-compact*. We write $\Downarrow_\theta x = \{a \in P : a \ll_\theta x\}$ and $\Uparrow_\theta x = \{a \in P : x \ll_\theta a\}$.
ii. P is called *θ-continuous* if for all $x \in P$, the set $\Downarrow_\theta x$ is directed and $x = \bigvee \Downarrow_\theta x$.

Proposition 3.1. Let P be a poset and $x, y, u, v \in P$.

1. $x \ll_\theta y$ implies $x \leqslant y$.
2. $u \leqslant x \ll_\theta y \leqslant v$ implies $u \ll_\theta v$.
3. $u \ll_\theta x$, $v \ll_\theta x$ and $u \vee v$ exists imply $u \vee v \ll_\theta x$.
4. If a smallest element $\perp$ exists, then $\perp \ll_\theta x$.
5. $x \ll_{s_2} y$ implies $x \ll_\theta y$.
6. $x \ll_\theta y$ implies $x \ll y$.
7. If P is a dcpo, then $x \ll_{s_2} y \Leftrightarrow x \ll_\theta y \Leftrightarrow x \ll y$.
8. If every directed subset in P has at least one upper bound, then P is θ-continuous iff P is s_2-continuous.

Proof. (1)–(4) are straightforward.

For (5), suppose $x \ll_{s_2} y$ and D a directed set such that $y \in \Theta(D)$. Then, $y \in D^\delta$ by $\Theta(D) \subseteq D^\delta$, thus $x \in \downarrow D$ by $x \ll_{s_2} y$ and hence $x \ll_\theta y$.

For (6) and (7), given any directed set D with existing sup, we have $D^\delta = \Theta(D) = \downarrow \bigvee D$. Hence, $y \leqslant \bigvee D \Leftrightarrow y \in \Theta(D) \Leftrightarrow y \in D^\delta$, which completes the proof.

For (8), if every directed subset D has an upper bound, then we always have $\Theta(D) = D^\delta$. Thus, $x \ll_\theta y$ iff $x \ll_{s_2} y$, the proof is complete. □

By Proposition 3.1, we know $\ll_\theta$ is stronger than $\ll$ and weaker than $\ll_{s_2}$. The following examples show differences among these three relations.

Example 3.1. The constructions below are illustrated in Figure 1.

1. Let $P_1 = \{p\} \bigcup [0, 1) \bigcup (1, 2]$, endow $[0, 1) \bigcup (1, 2]$ with the natural order, $p \leqslant x$ iff $1 < x$ or $x = p$, and no other relations. Then, $p \ll p$ but $p \not\ll_\theta p$ since $\Theta([0, 1)) = [0, 1)^\delta = \{p\} \bigcup [0, 1)$. We conclude that P_1 is a continuous poset, however, not θ-continuous.
2. Let $P_2 = \{q\} \bigcup [0, 1] \times \{0\} \bigcup [0, 1) \times \{1\}$, the partial order $\leqslant$ on P_2 is defined by $(x_1, y_1) \leqslant (x_2, y_2)$ iff $x_1 \leqslant x_2$ and $y_1 = y_2$, and q has no order relations with other points except itself. Then, P_2 is a poset, $q \ll_\theta q$, $(x_0, y_0) \ll_\theta (x_1, y_1)$ iff $x_0 < x_1$ and $y_0 = y_1$. However, $([0, 1) \times \{1\})^\delta = P_2$, thus $q \not\ll_{s_2} q$, $(x_0, 0) \not\ll_{s_2} (x_1, 0)$ for any $x_0, x_1 \in [0, 1]$. Then, P_2 is a θ-continuous poset, but it is not s_2-continuous.

Definition 3.2. Let P be a poset. A subset $U \subseteq P$ is called *θ-open* if it satisfies

i. $U = \uparrow U$;
ii. $\Theta(D) \bigcap U \neq \varnothing$ implies $D \bigcap U \neq \varnothing$ for all directed sets $D \subseteq P$.

For any poset P, the set $\theta(P) = \{U \subseteq P : U \text{ is } \theta\text{-open}\}$ forms a topology, called the *θ-topology* of P. The set of all θ-closed sets of P is denoted by $\theta(P)^c = \{P \backslash U : U \in \theta(P)\}$. Recall that a subset F of P is a filter if every finite subset of F has a lower bound in F

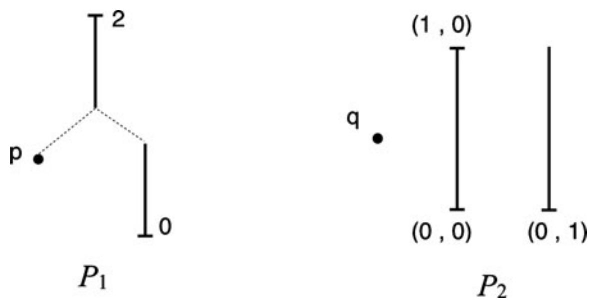

Fig. 1. Example 3.1.

and $F = \uparrow F$. Let $\Theta Filt(P) = \{F : F$ is a θ-open filter of $P\}$ denote all θ-open filters. For any subset $A \subseteq P$, let $cl_\theta(A)$ denote the closure of A and $int_\theta(A)$ denote the interior of A with respect to the θ-topology.

Proposition 3.2. Let P be a poset, $\forall x, y, z \in P$, $\forall A, U \subseteq P$.

1. A is θ-closed iff $D \subseteq A$ implies $\Theta(D) \subseteq A$ for any directed subset D of P.
2. $s_2(P) \subseteq \theta(P) \subseteq \sigma(P)$.
3. $cl_\sigma(A) \subseteq cl_\theta(A) \subseteq cl_{s_2}(A)$.
4. $int_{s_2}(A) \subseteq int_\theta(A) \subseteq int_\sigma(A)$.
5. U is a co-prime in $\theta(P)$ iff $U \in \Theta$Filt(P).
6. If $y \in int_\theta(\uparrow x)$, then $x \ll_\theta y$.

Proof.

1.By the definition of θ-closed sets.

2.Let U be an s_2-open set in P, for every directed subset D, $\Theta(D) \bigcap U \neq \varnothing \Rightarrow D^\delta \bigcap U \neq \varnothing \Rightarrow D \bigcap U \neq \varnothing$, therefore U is θ-open. In a similar way, one has that U is θ-open implies U is Scott open. Thus, $s_2(P) \subseteq \theta(P) \subseteq \sigma(P)$.

3.and 4. are the consequences of (2).

5.$\Rightarrow$: It suffices to show that U is a filter. Suppose $x, y \in U$. By (1), we have $\downarrow x$ and $\downarrow y$ are θ-closed. Thus, $P \backslash \downarrow x$ and $P \backslash \downarrow y$ are θ-open and $U \nsubseteq (P \backslash \downarrow x) \bigcup (P \backslash \downarrow y) = P \backslash (\downarrow x \bigcap \downarrow y)$ since U is a co-prime in $\theta(P)$. Hence, $\exists z \in U$ such that $z \in \downarrow x \bigcap \downarrow y$.
$\Leftarrow$: Suppose that U is not a co-prime in $\theta(P)$. Then, $\exists V, W \in \theta(P)$ such that $U \subseteq V \bigcup W$ and $\exists x \in U \backslash V$ and $\exists y \in U \backslash W$. There exists $z \in U$ such that $z \leqslant x$ and $z \leqslant y$ since U is a filter. As θ-open sets are upper sets, we have $z \notin V \bigcup W$, a contradiction to $U \subseteq V \bigcup W$.

6.For every directed set D such that $y \in \Theta(D)$, we have $\Theta(D) \bigcap int_\theta(\uparrow x) \neq \varnothing$ since $y \in int_\theta(\uparrow x)$. By the Definition 3.2, $D \bigcap int_\theta(\uparrow x) \neq \varnothing$. Then, $x \in \downarrow D$ and thus $x \ll_\theta y$.

□

Remark 3.1. Note that for any directed subset $D \subseteq P$, D^δ is always s_2-closed. However, this is not the case with $\Theta(D)$ in the θ-topology. For example, let $\mathbb{R}$ be the set of all real numbers and $\mathbb{N}$ all natural numbers, $\mathcal{P} = \{A : A \subseteq \mathbb{R}\} \backslash \{\mathbb{R}\}$ be a poset with the partial order of inclusion. Then, $\mathcal{D} = \{F \subseteq \mathbb{R} : F$ is finite$\}$ and $\mathcal{C} = \{F \subseteq \mathbb{N} : F$ is finite$\}$ are directed subsets of $\mathcal{P}$. We have $\Theta(\mathcal{D}) = \mathcal{D}$ and $\mathcal{C} \subseteq \mathcal{D}$, but $\Theta(\mathcal{C}) = \{A : A \subseteq \mathbb{N}\} \notin \Theta(\mathcal{D})$. Thus, $\Theta(\mathcal{D})$ is not θ-closed by Proposition 3.2(1).

The following theorem is a generalization of Theorem II –1.14 in Gierz et al. (2003).

Theorem 3.1. For any poset P, the following conditions are equivalent:

1. P is a θ-continuous poset.
2. P is continuous and $x \ll y$ implies $x \ll_\theta y$ for all $x, y \in P$.
3. Each $\Uparrow_\theta x$ is θ-open, and if $U \in \theta$(P), then $U = \bigcup\{\Uparrow_\theta x : x \in U\}$.
4. P is continuous and $\sigma(P) = \theta(P)$.
5. ΘFilt(P) is a basis of θ(P) and θ(P) is a continuous lattice.
6. $\theta(P)$ has enough co-primes and is a continuous lattice.
7. $\theta(P)$ is completely distributive.
8. both $\theta(P)$ and $\theta(P)^c$ are continuous.

Proof. (1) $\Rightarrow$ (2): Suppose x $\ll$ y, then $x \leqslant d$ for some $d \in \Downarrow_\theta y$ since $\Downarrow_\theta y$ is directed and $y = \bigvee \Downarrow_\theta y$, hence $x \ll_\theta y$ by Proposition 3.1(2). Thus, x $\ll$ y iff $x \ll_\theta y$ and P is continuous by Proposition 3.1(6).

(2) $\Rightarrow$ (1): Straightforward.

(2) $\Rightarrow$ (3): For every directed set D with $\Theta(D) \bigcap \Uparrow_\theta x \neq \emptyset$, $\exists y \in \Uparrow_\theta x \bigcap \Theta(D)$. Then, there exists $z \in P$ such that $x \ll_\theta z \ll_\theta y$ by (2) and the interpolation property of continuous posets. Thus, $z \in \downarrow D \bigcap \Uparrow_\theta x$ and $\Uparrow_\theta x$ is θ-open. It is clear that $\bigcup\{\Uparrow_\theta x : x \in U\} \subseteq U$. If $u \in U$, then $\Theta(\Downarrow_\theta u) = \downarrow u$ by θ-continuity. Thus, $\Downarrow_\theta u \bigcap U \neq \emptyset$ and $\exists v \in U$ such that $u \in \Uparrow_\theta v$, the proof is complete.

(3) $\Rightarrow$ (1): Let $x \in P$, obviously, $\Downarrow_\theta x$ is directed and not empty. For each $y \in (\Downarrow_\theta x)^\uparrow$, if $y \notin \uparrow x$, then $L \setminus \downarrow y$ is θ-open and contains x. By (3), there exists $z \in L \setminus \downarrow y$ such that $x \in \Uparrow_\theta z$, hence $z \leqslant y$, a contradiction. And $x \in (\Downarrow_\theta x)^\uparrow$, therefore, $x = \bigvee \Downarrow_\theta x$.

(3) $\Rightarrow$ (4): (4) is clear from (1), (2) and (3).

(4) $\Rightarrow$ (7): This is obvious.

(6) $\Leftrightarrow$ (7) $\Leftrightarrow$ (8) : See Theorem I –3.16 in Gierz et al. (2003).

(6) $\Leftrightarrow$ (5): Consequence of Proposition 3.2(5).

(5) $\Rightarrow$ (1): For any $x \in P$, let $D = \{y \in P : x \in int_\theta(\uparrow y)\}$. Then, $\forall y \in D$, $y \ll_\theta x$ by Proposition 3.2(6). We claim D is directed as for any $U \in \theta(P)$ containing x, there exists $y \in U$ such that $x \in int_\theta(\uparrow y)$. Suppose not, there exist $V \in \theta(P)$ and $F \in \Theta Filt(P)$ such that $x \in F \subseteq V \ll U$ since $\theta(P)$ is a continuous lattice and $\Theta Filt(P)$ is a basis of $\theta(P)$. Then $\forall y \in U$, we have $y \in F_y \subseteq P \setminus \downarrow z \in \theta(P)$ for some $z \in F$ and $F_y \in \Theta Filt(P)$. Thus, $F \subseteq V \subseteq \bigcup_{i \in G} F_{y_i}$ for some finite set G and $y_i \in F_{y_i} \subseteq P \setminus \downarrow z_i$. Then, $\exists z_0 \in F$ such that $z_0 \leqslant z_i$ for all $i \in G$. However, $z_0 \notin F_{y_i}$ for all $i \in G$, a contradiction. This proves the claim. If $z \in D^\uparrow$, we can show $x \leqslant z$ in the way that given in (3) $\Rightarrow$ (1). Thus, $x = \bigvee D$ and $x \in \Theta(D)$. Then, $\Downarrow_\theta x = \downarrow D = D$ and hence P is a θ-continuous poset. □

Theorem 3.2. Let P be a poset. Then the following statements are equivalent.

1. P is s_2-continuous.
2. P is θ-continuous and $x \ll_\theta y$ implies $x \ll_{s_2} y$ for all $x, y \in P$.
3. P is continuous and $x \ll y$ implies $x \ll_{s_2} y$ for all $x, y \in P$.

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4. P is θ-continuous and $\theta(P) = s_2(P)$.
5. P is continuous and $\sigma(P) = s_2(P)$.

Proof. (1) $\Rightarrow$ (2): By s_2-continuity, we have $\Downarrow_{s_2} x$ is directed and $x \in \Theta(\Downarrow_{s_2} x)$. Thus, $\Downarrow_\theta x \subseteq \Downarrow_{s_2} x$. Hence, $\Downarrow_\theta x = \Downarrow_{s_2} x$ by Proposition 3.1(5).

(2) $\Rightarrow$ (3): By the equivalence of (1) and (2) in Theorem 3.1.

(3) $\Rightarrow$ (1): Straightforward.

(3) $\Rightarrow$ (4): We have $\Uparrow_\theta x = \Uparrow_{s_2} x$, and $\{\Uparrow_\theta x : x \in P\}$ is a base of $\theta(P)$ by Theorem 3.1. Moreover, $\{\Uparrow_{s_2} x : x \in P\}$ is a base of $s_2(P)$ since P is s_2-continuous, this can be proved in the way of (2) $\Rightarrow$ (3) in Theorem 3.1. Thus, $\theta(P) = s_2(P)$.

(4) $\Rightarrow$ (5): By the equivalence of (1) and (4) in Theorem 3.1.

(5) $\Rightarrow$ (1): All we need to show is that $x \ll y$ implies $x \ll_{s_2} y$. For any directed D with $y \in D^\delta$, we have $D^\delta \bigcap \Uparrow x \neq \emptyset$. Thus, $D \bigcap \Uparrow x \neq \emptyset$ since $\Uparrow x \in \sigma(P) = s_2(P)$. Hence, $x \in \downarrow D$. Therefore, $x \ll y$ implies $x \ll_{s_2} y$ as desired. □

4. Quasi θ-continuous posets

Recall that for a dcpo L, the following two conditions are equivalent: (1) L is a quasicontinuous domain; (2) For all $x \in L$ and $U \in \sigma(L)$, $x \in U$ implies that there is a non-empty finite $F \subseteq L$ such that $x \in int_{\sigma(L)} \uparrow F \subseteq \uparrow F \subseteq U$ (Heckmann 1992). The concept of quasicontinuous posets was introduced in the manner of the condition (2) (Mao and Xu 2006). And a poset is s_2-quasicontinuous iff the s_2-topology is locally finitary compact (Zhang and Xu 2015).

Definition 4.1. A poset P is called a *quasi θ-continuous* poset if for all $x \in P$ and $U \in \theta(P)$, $x \in U$ implies that there is a non-empty finite subset $F \subseteq P$ such that $x \in int_{\theta(P)}(\uparrow F) \subseteq \uparrow F \subseteq U$.

Proposition 4.1. Let P be a poset.

1. If P is a θ-continuous poset, then P is quasi θ-continuous.
2. P is s_2-quasicontinuous $\Rightarrow$ P is quasi θ-continuous $\Rightarrow$ P is quasicontinuous.

Proof.

1. If $x \in U \in \theta(P)$, then $x \in \Uparrow_\theta y \in \theta(P)$ for some $y \in U$ by Theorem 3.1. Thus, $x \in int_{\theta(P)}(\uparrow\{y\}) \subseteq \uparrow\{y\} \subseteq U$.
2. Consequence of $x \in int_{s_2(P)}(\uparrow F) \Rightarrow x \in int_{\theta(P)}(\uparrow F) \Rightarrow x \in int_{\sigma(P)}(\uparrow F)$ by Proposition 3.2(4).

□

Theorem 4.1. A poset P is quasi θ-continuous iff the latice $\theta(P)$ of all θ-open subsets with inclusion order is a hypercontinuous lattice.

Proof. P is quasi θ-continuous iff $(P, \theta(P))$ is locally finitary compact by Definition 4.1. Now apply Lemma 2.3. □

Definition 4.2. A poset P is called *meet θ-continuous* if for any $x \in P$ and any directed subset $D \subseteq P$ with $x \in \Theta(D)$, then $x \in cl_\theta(\downarrow x \bigcap \downarrow D)$.

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Proposition 4.2.

1. If P is a θ-continuous poset, then P is meet θ-continuous.
2. If P is meet s_2-continuous, then P is meet θ-continuous.

Proof.

1. Let $x \in P$ and D be a directed set with $x \in \Theta(D)$. Clearly, $\Downarrow_\theta x \subseteq \downarrow D$. Thus, $cl_\theta(\Downarrow_\theta x) \subseteq cl_\theta(\downarrow x \bigcap \downarrow D)$ and $cl_\theta(\Downarrow_\theta x) = \downarrow x$ by θ-continuity, as required.
2. Recall that if P is a meet s_2-continuous poset, then for any $x \in P$ and any directed set D, $x \in D^\delta$ always implies $x \in cl_{s_2}(\downarrow x \bigcap \downarrow D)$ (see Zhang and Xu 2015). For any $A \in \theta(P)^c$ with $\downarrow x \bigcap \downarrow D \subseteq A \subseteq \downarrow x$, and any directed subset $B \subseteq A$. Then, $B^\uparrow \neq \emptyset$. Thus, $B^\delta = \Theta(B) \subseteq A$. Hence, $A \in s_2(P)^c$ and $cl_\theta(\downarrow x \bigcap \downarrow D) = cl_{s_2}(\downarrow x \bigcap \downarrow D)$. Moreover, $x \in \Theta(D)$ always implies $x \in D^\delta$. Therefore, P is meet θ-continuous. □

Theorem 4.2. Let P be a poset. Then the following conditions are equivalent:

1. P is a meet θ-continuous poset.
2. For all $U \in \theta(P)$ and all $x \in P$, one has $\uparrow(U \bigcap \downarrow x) \in \theta(P)$.
3. $\theta(P)^c$ is a complete Heyting algebra.

Proof. (1) ⇒ (2): Let D be a directed set such that $\Theta(D) \bigcap \uparrow(U \bigcap \downarrow x) \neq \emptyset$, then $\exists y \in \Theta(D) \bigcap \uparrow(U \bigcap \downarrow x)$. By meet θ-continuity, we have $y \in cl_\theta(\downarrow y \bigcap \downarrow D)$. Thus, $\downarrow D \bigcap \downarrow y \bigcap U \neq \emptyset$. Since $\downarrow D \bigcap \downarrow y \bigcap U \subseteq \downarrow D \bigcap \downarrow x \bigcap U$, we have $D \bigcap \uparrow(U \bigcap \downarrow x) \neq \emptyset$ which shows that $\uparrow(U \bigcap \downarrow x)$ is θ-open.

(2) ⇒ (3): Suppose $A, B_i \in \theta(P)^c$ $(i \in I)$. Then we immediately have $\bigvee_{i \in I}(A \wedge B_i) \subseteq A \wedge \bigvee_{i \in I} B_i$. In order to show $A \wedge \bigvee_{i \in I} B_i \subseteq \bigvee_{i \in I}(A \wedge B_i)$, let $x \in A \wedge \bigvee_{i \in I} B_i$ and $U \in \theta(P)$ with $x \in U$, we conclude $x \in \uparrow(U \bigcap A) \in \theta(P)$ since A is obvious a downset and $\uparrow(U \bigcap A) = \bigcup_{a \in A} \uparrow(U \bigcap \downarrow a)$. Then, $\uparrow(U \bigcap A) \bigcap (\bigcup_{i \in I} B_i) \neq \emptyset$ because $x \in \bigvee_{i \in I} B_i = cl_\theta(\bigcup_{i \in I} B_i)$. Thus, $U \bigcap (A \bigcap \bigcup_{i \in I} B_i) = U \bigcap (\bigcup_{i \in I} A \bigcap B_i) \neq \emptyset$. Hence, $x \in \bigvee_{i \in I}(A \wedge B_i)$ and therefore $\bigvee_{i \in I}(A \wedge B_i) = A \wedge \bigvee_{i \in I} B_i$.

(3) ⇒ (1): For any $x \in \Theta(D)$, where D is a directed subset. If $D^\uparrow = \emptyset$, then $\Theta(D) = \downarrow D$ and $x \in cl_\theta(\downarrow x) = cl_\theta(\downarrow x \bigcap \downarrow D)$. Else then $\Theta(D) = D^\delta \in \theta(P)^c$ by Proposition 3.2(2). Thus, $x \in \downarrow x \bigcap \Theta(D) = \downarrow x \bigcap \bigvee_{d \in D} \downarrow d = \bigvee_{d \in D} (\downarrow x \bigcap \downarrow d) = cl_\theta(\downarrow x \bigcap \downarrow D)$ by (3). This shows that P is meet θ-continuous. □

Theorem 4.3. P is a θ-continuous poset iff P is a meet θ-continuous and quasi θ-continuous poset.

Proof. ⇒ : By Proposition 4.1, 4.2.

⇐ : By Theorem 4.1 and 4.2, we have $\theta(P)$ is a hypercontinuous lattice and $\theta(P)^c$ is a complete Heyting algebra. Thus, $\theta(P)$ is completely distributive (see Theorem 5.6 in Mao and Xu (2006)). We obtain that P is θ-continuous by Theorem 3.1. □

5. D_θ-completion and invariant properties

Definition 5.1. Let P and Q be posets, a map $f : P \to Q$ is called *θ-continuous* (resp., *s_2-continuous*), if f is continuous with respect to the θ-topologies (resp., the s_2-topologies).

Proposition 5.1. Let P and Q be posets, a map $f : P \to Q$. Then,

1. f is θ-continuous iff f is monotone and for any directed subset $D \subseteq P$, $\Theta(f(\Theta(D))) = \Theta(f(D))$;
2. f is s_2-continuous iff f is monotone and for any directed subset $D \subseteq P$, $f(D^\delta)^\delta = f(D)^\delta$;
3. f is s_2-continuous $\Rightarrow$ f is θ-continuous $\Rightarrow$ f is Scott continuous;
4. if P and Q are dcpos, then f is s_2-continuous $\Leftrightarrow$ f is θ-continuous $\Leftrightarrow$ f is Scott continuous.

Proof.

1. $\Rightarrow$: For all $x, y \in P$ with $x \leqslant y$, we have $x \in \downarrow y \subseteq f^{-1}(\downarrow f(y))$ since $f^{-1}(\downarrow f(y))$ is θ-closed by θ-continuity of f. Thus, $f(x) \in \downarrow f(y)$. By arbitrariness of x and y, we obtain that f is monotone. For any directed set D, if $D^\uparrow = \varnothing$, then $\Theta(D) = \downarrow D$, hence $\Theta(f(\Theta(D))) = \Theta(f(\downarrow D)) = \Theta(f(D))$; else $D^\uparrow \neq \varnothing$, then $f(D)^\uparrow \neq \varnothing$ by monotonicity of f that we have proved. Thus, $\Theta(D)$ and $\Theta(f(D))$ are θ-closed. To show that $\Theta(f(\Theta(D))) = \Theta(f(D))$, we only need to prove $f(\Theta(D)) \subseteq \Theta(f(D))$. Again by θ-continuity of f, we have $D \subseteq f^{-1}(\Theta(f(D))) \in \theta(P)^c$. Thus, $\Theta(D) \subseteq f^{-1}(\Theta(f(D)))$. Hence, $f(\Theta(D)) \subseteq f(f^{-1}(\Theta(f(D)))) \subseteq \Theta(f(D))$, as required.
   $\Leftarrow$: For any $U \in \theta(Q)$, $f^{-1}(U) \subseteq P$ is an upper set since f is monotone. Suppose D is a directed subset of P such that $\Theta(D) \bigcap f^{-1}(U) \neq \varnothing$. Then, $f(\Theta(D)) \bigcap U \neq \varnothing$. Hence, $\Theta(f(\Theta(D))) \bigcap U \neq \varnothing$. By the condition, we have $\Theta(f(D)) \bigcap U \neq \varnothing$. Therefore, $f(D) \bigcap U \neq \varnothing$ and then $D \bigcap f^{-1}(U) \neq \varnothing$, which shows $f^{-1}(U)$ is θ-open.
2. The proof is similar to (1). Note that for any directed set D, we always have that D^δ and $f(D)^\delta$ are s_2-closed.
3. Since $f(D^\delta)^\delta = f(D)^\delta$ implies $\Theta(f(\Theta(D))) = \Theta(f(D))$ for any directed subsets D, we conclude f is s_2-continuous $\Rightarrow$ f is θ-continuous. And if $\bigvee D$ exists, then $\Theta(D) = \downarrow \bigvee D$. Thus, if f is θ-continuous, then $\downarrow f(\bigvee D) = \Theta(f(\Theta(D))) = \Theta(f(D))$. Hence, $f(\bigvee D) = \bigvee f(D)$ and f is Scott continuous.
4. Since $s_2(L) = \theta(L) = \sigma(L)$ for all dcpos L. □

Definition 5.2. A *D_θ-completion* (L, η_θ) of a poset P is a dcpo L together with a θ-continuous map η_θ, such that for any dcpo B and θ-continuous map $f : P \to B$, there exists a unique θ-continuous map $\hat{f}$ satisfying $f = \hat{f} \circ \eta_\theta$.

Theorem 5.1. Let P be a poset, $cl_d(PI(P))$ be the D-closure of PI(P) $= \{\downarrow x : x \in P\}$ in $\theta(P)^c$, and define η_θ : P $\to cl_d(PI(P))$ by $\eta_\theta(x) = \downarrow x$ for all $x \in P$. Then, $(cl_d(PI(P)), \eta_\theta)$ is a D_θ-completion of P.

Proof. To show η_θ is θ-continuous, we only need to prove that for every directed set D, $\eta_\theta(\Theta(D)) \subseteq \Theta(\eta_\theta(D))$ by Proposition 5.1(1). If $D^\uparrow = \varnothing$, then $\Theta(D) = \downarrow D$, thus $\eta_\theta(\Theta(D)) = \{\downarrow x : x \in \downarrow D\} \subseteq \Theta(\{\downarrow d : d \in D\}) = \Theta(\eta_\theta(D))$. Else $D^\uparrow \neq \varnothing$, then $\Theta(D) = D^\delta$

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is θ-closed and is the supremum of $\{\downarrow x : x \in \Theta(D)\}$. Hence, $\eta_\theta(\Theta(D)) = \{\downarrow x : x \in \Theta(D)\} \subseteq \{A \in \theta(P)^c : A \subseteq \Theta(D)\} = \Theta(\{\downarrow d : d \in D\}) = \Theta(\eta_\theta(D))$. Therefore, η_θ is θ-continuous.

Now consider a θ-continuous map $f : P \to B$, where B is a dcpo. Define $g : \theta(P)^c \to \theta(B)^c$ by $g(A) = cl_\theta(f(A))$. For proving that g is θ-continuous, it suffices to show $g(\bigvee_{i\in I} A_i) \subseteq \bigvee_{i\in I} g(A_i)$, $A_i \in \theta^c(P)$ for all $i \in I$. Since f is θ-continuous, we have $f^{-1}(\bigvee_{i\in I} g(A_i)) = f^{-1}(cl_\theta(f(\bigcup_{i\in I} A_i))) \supseteq \bigvee_{i\in I} A_i$. Thus, $g(\bigvee_{i\in I} A_i) = cl_\theta(f(\bigvee_{i\in I} A_i)) \subseteq cl_\theta(f(f^{-1}(\bigvee_{i\in I} g(A_i)))) \subseteq cl_\theta(\bigvee_{i\in I} g(A_i)) = \bigvee_{i\in I} g(A_i)$. Hence, g preserves arbitrary joins between dcpos, and therefore g is Scott continuous and θ-continuous. By Lemma 2.1, $g(cl_d(PI(P))) \subseteq cl_d(g(PI(P))) = cl_d(\{\downarrow f(x) : x \in P\}) \subseteq PI(B)$ since B is a dcpo. Let $\hat{f} = \bigvee g|_{cl_d(PI(P))}$. Then, $\hat{f} : cl_d(PI(P)) \to B$ is a θ-continuous map and $f = \hat{f} \circ \eta_\theta$. By Lemma 2.2, we know for any $h : cl_d(PI(P)) \to B$ such that $f = h \circ \eta_\theta$, then $h = \hat{f}$. Therefore, $(cl_d(PI(P)), \eta_\theta)$ is a D_θ-completion of P. □

We shall use $D_\theta(P)$ to denote the dcpo of a D_θ-completion of a poset P. Clearly, $D_\theta(P)$ is unique up to isomorphism and idempotent, i.e., if P is a dcpo, then $P \cong D_\theta(P)$. Let $D_\theta : POS_\theta \to DCPO$ be a functor defined by the following:

$$\begin{array}{ccc} P & \xrightarrow{\eta_{\theta(P)}} & D_\theta(P) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle D_\theta(f)} \\ Q & \xrightarrow[\eta_{\theta(Q)}]{} & D_\theta(Q) \end{array}$$

where $D_\theta(f) = \widehat{\eta_{\theta(Q)} \circ f}$ is the unique θ-continuous mapping such that the diagram commutes. Then, D_θ is a reflector and thus the full subcategory $DCPO$ of POS_θ is reflective in POS_θ.

Lemma 5.1. If A is a subset of a poset P with $P = cl_d(A)$. Then for any $B \in \theta(P)^c$, $B = cl_\theta(B \bigcap A)$.

Proof. Since $B \in \theta(P)^c$, we have $B \in \sigma(P)^c$. And by Lemma 7 in Zhao and Fan (2010), $B = cl_\sigma(B \bigcap A)$. But $cl_\sigma(B \bigcap A) \subseteq cl_\theta(B \bigcap A) \subseteq B$, hence $B = cl_\theta(B \bigcap A)$. □

Theorem 5.2. If $D_\theta(P)$ is the dcpo of a D_θ-completion of a poset P, then $\theta(P) \cong \theta(D_\theta(P))$.

Proof. By Theorem 5.1, it is equivalent to show $\theta(P)^c \cong \theta(cl_d(PI(P)))^c$. Let $\Phi : \theta(P)^c \to \theta(cl_d(PI(P)))^c$ be defined by $\Phi(X) = cl_\theta(\{\downarrow x : x \in X\})$ for all $X \in \theta(P)^c$. Then, $\Phi(X) \subseteq cl_d(PI(P)) \bigcap \{A \in \theta(P)^c : A \subseteq X\}$ since $cl_d(PI(P))$ is a subdcpo of $\theta(P)^c$ and $\{A \in \theta(P)^c : A \subseteq X\} \in \theta(\theta(P)^c)^c$. For any $X, Y \in \theta(P)^c$ such that $X \neq Y$, without loss of generality, let $y \in Y \setminus X$. Then, $\downarrow y \notin \{A \in \theta(P)^c : A \subseteq X\}$ and thus $\downarrow y \notin \Phi(X)$. But $\downarrow y \in \Phi(Y)$. Therefore, $\Phi(X) \neq \Phi(Y)$ and hence Φ is injective.

Next, assume that $\mathcal{B} \in \theta(cl_d(PI(P)))^c$. Let $C = \bigcup_{B\in\mathcal{B}} B$, we claim that $C \in \theta(P)^c$. Suppose $D \subseteq C \subseteq P$ is directed, if $D^\uparrow = \emptyset$, then $\Theta(D) = \downarrow D \subseteq C$ since it is obvious that $C = \downarrow C$; else $D^\uparrow \neq \emptyset$, then $\Theta(D) \in \theta(P)^c$, we obtain $\Theta(D) \in \mathcal{B}$ since

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; $\Theta(D) = \bigvee\{\downarrow d : d \in D\} \in cl_d(PI(P))$ and $\{\downarrow d : d \in D\} \subseteq \mathcal{B}$. This proves the claim. Then, $\Phi(C) = cl_\theta(\{\downarrow c : c \in C\}) = cl_\theta(\mathcal{B} \bigcap PI(P)) = \mathcal{B}$ by Lemma 5.1. Hence, Φ is surjective. And clearly, Φ preserves inclusion order. Therefore, Φ is an isomorphism and thus $\theta(P)^c \cong \theta(cl_d(PI(P)))^c$. □

The following Corollary is an immediate consequence of Theorems 5.2, 3.1, 4.1 and 4.2.

Corollary 5.1. For every poset P and consider the following statements:

1. P is a θ-continuous poset.
2. $\theta(D_\theta(P))$ is a completely distributive lattice.
3. $D_\theta(P)$ is a domain.
4. P is a meet θ-continuous poset.
5. $\theta(D_\theta(P))^c$ is a complete Heyting algebra.
6. $D_\theta(P)$ is a meet continuous dcpo.
7. P is a quasi θ-continuous poset.
8. $\theta(D_\theta(P))$ is a hypercontinuous lattice.
9. $D_\theta(P)$ is a quasicontinuous domain.

Then, (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3), (4) $\Leftrightarrow$ (5) $\Leftrightarrow$ (6), (7) $\Leftrightarrow$ (8) $\Leftrightarrow$ (9) and (1) $\Leftrightarrow$ (4) + (7).

6. Comparisons of several dcpo-completions

Definition 6.1. A D_{s_2}*-completion* (L, η_{s_2}) of a poset P is a dcpo L together with a s_2-continuous map η_{s_2}, such that for any dcpo B and s_2-continuous map $f : P \to B$, there exists a unique s_2-continuous map $\hat{f}$ satisfying $f = \hat{f} \circ \eta_{s_2}$. The dcpo of a D_{s_2}-completion of a poset P is denoted by $D_{s_2}(P)$.

There are many parallel results to the D_θ-completion of posets. We only exhibit the following two key theorems here.

Theorem 6.1. Let P be a poset, $cl_d(PI(P))$ be the D-closure of PI(P) $= \{\downarrow x : x \in P\}$ in $s_2(P)^c$, define $\eta_{s_2} : P \to cl_d(PI(P))$ by $\eta_{s_2}(x) = \downarrow x$. Then, $(cl_d(PI(P)), \eta_{s_2})$ is a D_{s_2}-completion of P.

Proof. The proof is analogous to Theorem 5.1 and thus is omitted here. □

Theorem 6.2. If $D_{s_2}(P)$ is the dcpo of a D_{s_2}-completion of a poset P, then $s_2(P) \cong s_2(D_{s_2}(P))$.

Proof. This can be verified by the way of Theorem 5.2. In fact, analogous to Lemma 5.1, we have if A is a subset of a poset P with $P = cl_d(A)$, then for any $B \in s_2(P)^c$, $B = cl_{s_2}(B \bigcap A)$. □

Proposition 6.1. For every poset P, we have the following:

1. If P is θ-continuous, then $D_\theta(P) \cong D(P)$.
2. If P is s_2-continuous, then $D_{s_2}(P) \cong D_\theta(P) \cong D(P)$.

Proof.

1.If P is θ-continuous, then $\theta(P) = \sigma(P)$ by Theorem 3.1. Thus, $cl_d(PI(P))$ in $\theta(P)$ and in $\sigma(P)$ are the same. Since $cl_d(PI(P))$ in $\sigma(P)$ is a D-completion of P (see Theorem 1 in Zhao and Fan (2010)) and Theorem 5.1, we obtain $D_\theta(P) \cong D(P)$.
2.By (1), Theorems 3.2 6.1. □

Now we have three types of dcpo-completion of posets including the D-completion given in Zhao and Fan (2010). The following examples show differences and connections among the D-completion, D_θ-completion and D_{s_2}-completion based on Example 3.1.

Example 6.1. Please refer to Figure 2 for better understanding. Let $D(P)$ denote the dcpo of a D-completion of P.

1. The poset P_1 is a continuous poset but not θ-continuous as we have showed in Example 3.1(2). We have $D(P_1) = \{\downarrow x : x \in P_1\} \bigcup \{[0, 1)\}$ is a domain; however, $D_\theta(P_1) = \{\downarrow x : x \in P_1\} \bigcup \{\{p\} \bigcup [0, 1)\}$ which is not a continuous dcpo.
2. The poset P_2 is a continuous poset and also θ-continuous which is not s_2-continuous in Example 3.1(3). It can be verified that $D(P_2) = D_\theta(P_2) = \{\downarrow x : x \in P_2\} \bigcup \{[0, 1) \times \{1\}\}$ and thus both the D-completion and D_θ-completion of P_2 are continuous dcpos. However, $D_{s_2}(P_2) = \{\downarrow x : x \in P_2\} \bigcup \{P_2\}$ is not a domain as expected since P_2 is not an s_2-continuous poset.

In theoretical computer science, domains are usually wanted to be objects suitable for computation. The notion of a basis is presented to find a proper notion of a recursive or recursively enumerable domain. Recall that a subset B of a domain L is called a *basis* iff (i) $\twoheaddownarrow x \bigcap B$ is directed for all $x \in L$, and (ii) $x = \bigvee(\twoheaddownarrow x \bigcap B)$ (Gierz et al. 2003). In Xu (2006), Xu introduced the following concept of an embedded basis. Let B and L be posets. If there is an order-embedding map $j : B \to L$ preserving existing directed sups such that $j(B)$ is a basis for L, then (B, j) is called an *embedded basis* of L. For convenience, if (B, j) is an embedded basis of L, then we also say that B is an embedded basis of L and take B as a subset of L with j as the inclusion map. Obviously, B is an embedded basis of L iff B is a basis of L and for every directed subset D of B with existing $\bigvee_B D$, one always has $\bigvee_B D = \bigvee_L D$. Xu proved that if a poset B is an embedded basis for a dcpo L, then L is isomorphic to the round ideal completion $RI(B, \ll)$ (see Theorem 3.8 in Xu (2006)).

Definition 6.2. Let B be a poset and L a domain. If there is an order-embedding, θ-continuous (resp., s_2-continuous) map $j : B \to L$ such that $j(B)$ is a basis for L, then (B, j) is called a(n) θ-embedded (resp., s_2-embedded) basis of L.

Proposition 6.2. Let B be a poset and L a domain, a map $j : B \to L$.

1. (B, j) is an s_2-embedded basis $\Rightarrow$ (B, j) is a θ-embedded basis $\Rightarrow$ (B, j) is an embedded basis.
2. If (B, j) is a θ-embedded basis of L, then for any $x, y \in B$, we have $x \ll_\theta y$ in B iff $x \ll_L y$ in L, and thus B is a θ-continuous poset.

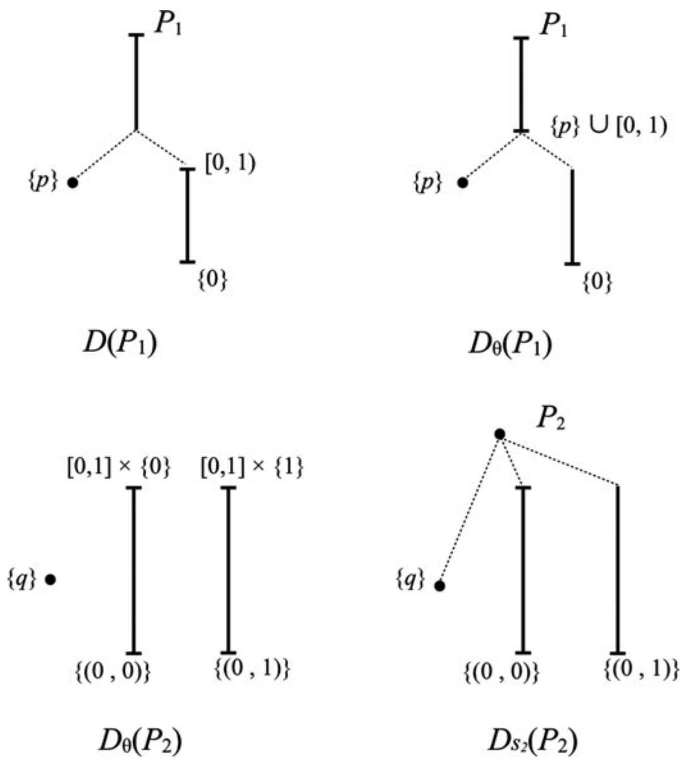

Fig. 2. Example 6.1.

3. If (B, j) is an s_2-embedded basis of L, then for any $x, y \in B$, we have $x \ll_{s_2} y$ in B iff $x \ll_L y$ in L, and thus B is an s_2-continuous poset.

Proof.

1. By Proposition 5.1(3).
2. For any directed subset $D \subseteq B$, we have $\Theta_L(\Theta_B(D)) = \Theta_L(D)$ since we take j as an inclusion map. If $x \ll_L y$ with $y \in \Theta_B(D)$ $(\subseteq \Theta_L(\Theta_B(D)))$, then $y \in \Theta_L(D)$. Thus, $x \leqslant d$ for some $d \in D$ and hence $x \ll_\theta y$ in B. If $x \ll_\theta y$ in B, then $y = \bigvee_L(\downarrow\!\!\!\downarrow_L y \bigcap B) = \bigvee_B(\downarrow\!\!\!\downarrow_L y \bigcap B)$ since B is a basis of L and $y \in B$. Thus, $y \in \Theta_B(\downarrow\!\!\!\downarrow_L y \bigcap B)$ and hence $x \leqslant d \ll_L y$ for some $d \in \downarrow\!\!\!\downarrow_L y \bigcap B$.
3. Similar to the proof of (2). $\square$

Theorem 6.3. Let L be a domain and B a poset, then

1. $D(B) \cong L$ iff B is an embedded basis of L;
2. $D_\theta(B) \cong L$ iff B is a θ-embedded basis of L;
3. $D_{s_2}(B) \cong L$ iff B is an s_2-embedded basis of L.

Proof. (1) $\Leftarrow$: It is evident that $\downarrow x \bigcap B$ is Scott closed in B for all $x \in L$ by the definition of an embedded basis. If $x \leqslant y$ in L, then $\downarrow x \bigcap B \subseteq \downarrow y \bigcap B$, and vice versa, since $x = \bigvee(\downarrow x \bigcap B) \leqslant \bigvee(\downarrow y \bigcap B) = y$. Thus, $L \cong \{\downarrow x \bigcap B : x \in L\}$. Let $cl_d(PI(B))$ be the D-closure of $PI(B)$ in $\sigma(B)$. We shall show that $cl_d(PI(B)) = \{\downarrow x \bigcap B : x \in L\}$. Let $cl_{\sigma(B)}$ be the Scott closure operator of poset B. We claim that $cl_{\sigma(B)}(\twoheaddownarrow x \bigcap B) = \downarrow x \bigcap B$. Clearly, $cl_{\sigma(B)}(\twoheaddownarrow x \bigcap B) \subseteq cl_{\sigma(B)}(\downarrow x \bigcap B) = \downarrow x \bigcap B$. Assume that $\exists y \in (\downarrow x \bigcap B) \backslash cl_{\sigma(B)}(\twoheaddownarrow x \bigcap B)$. Then, $\twoheaddownarrow y \bigcap B \subseteq \twoheaddownarrow x \bigcap B$. Thus, $y \in cl_{\sigma(B)}(\twoheaddownarrow x \bigcap B)$ since $\twoheaddownarrow y \bigcap B$ is directed with supremum y in B, a contradiction which proves the claim. Applying the claim, we have $\bigvee\{\downarrow b \bigcap B : b \in \twoheaddownarrow x \bigcap B\} = \downarrow x \bigcap B$ in $\sigma(B)^c$. Therefore, $PI(B) \subseteq \{\downarrow x \bigcap B : x \in L\} \subseteq cl_d(PI(B))$. Now we only need to prove that $\{\downarrow x \bigcap B : x \in L\}$ is a subdcpo of $\sigma(B)^c$. For any directed set $D \subseteq L$, we immediately have $\bigvee\{\downarrow d \bigcap B : d \in D\} \subseteq \downarrow(\bigvee D) \bigcap B$ in $\sigma(B)^c$. Let $c \in \downarrow(\bigvee D) \bigcap B$. Then, $\twoheaddownarrow c \bigcap B \subseteq \bigcup\{\downarrow d \bigcap B : d \in \downarrow D\} \subseteq \bigvee\{\downarrow d \bigcap B : d \in D\}$. Thus, $c \in cl_{\sigma(B)}(\twoheaddownarrow c \bigcap B) \subseteq \bigvee\{\downarrow d \bigcap B : d \in D\}$. Therefore, $\bigvee\{\downarrow d \bigcap B : d \in D\} = \downarrow(\bigvee D) \bigcap B$, and $cl_d(PI(B)) = \{\downarrow x \bigcap B : x \in L\}$. We obtain $D(B) \cong L$.

$\Rightarrow$: Clearly, B is an embedded basis of L iff $PI(B)$ is an embedded basis of $cl_d(PI(B))$. By Theorem 4 in Zhao and Fan (2010), we have that B is a continuous poset. Then, $D(B) \cong cl_d(PI(B)) = Spec(\sigma(B)^c) = \{cl_{\sigma(B)}(D) : D$ is a directed subset of $B\}$ by Remark 3 and Lemma 12 in Zhao and Fan (2010). All we need to prove is that $PI(B)$ is an embedded basis of $\Psi := \{cl_{\sigma(B)}(D) : D$ is a directed subset of $B\}$. Here we consider posets B and Ψ. Let $x, y \in B$ with $x \ll y$. For any directed subset $\{cl_{\sigma(B)}(D_i) : i \in I\}$ of Ψ with $\downarrow y \leqslant \bigvee\limits_{i \in I} cl_{\sigma(B)}(D_i)$ implies $\downarrow y \subseteq cl_{\sigma(B)}(\bigcup\limits_{i \in I} D_i) = B \backslash \bigcup\{\uparrow\!\!\uparrow d : d \in B \backslash \downarrow \bigcup\limits_{i \in I} D_i\}$ since B is a continuous poset. Thus, $x \in \downarrow \bigcup\limits_{i \in I} D_i$ and hence $x \ll y$ in B implies $\downarrow x \ll \downarrow y$ in Ψ. For any directed subset D of B, we claim that $\twoheaddownarrow cl_{\sigma(B)}(D) \bigcap PI(B) = \{\downarrow b : b \in \bigcup\{\twoheaddownarrow d : d \in D\}\}$, where $\twoheaddownarrow cl_{\sigma(B)}(D) = \{A \in \Psi : A \ll cl_{\sigma(B)}(D)$ in $\Psi\}$. If $b \ll d \in D$, then $\downarrow b \ll \downarrow d \leqslant cl_{\sigma(B)}(D)$. Thus, $\{\downarrow b : b \in \bigcup\{\twoheaddownarrow d : d \in D\}\} \subseteq \twoheaddownarrow cl_{\sigma(B)}(D) \bigcap PI(B)$. But $\{\downarrow b : b \in \bigcup\{\twoheaddownarrow d : d \in D\}\}$ is directed and $\bigvee\{\downarrow b : b \in \bigcup\{\twoheaddownarrow d : d \in D\}\} = cl_{\sigma(B)}(\bigcup\{\downarrow b : b \in \bigcup\{\twoheaddownarrow d : d \in D\}\}) = cl_{\sigma(B)}(D)$ since B is continuous. Thus, $\twoheaddownarrow cl_{\sigma(B)}(D) \bigcap PI(B) = \{\downarrow b : b \in \bigcup\{\twoheaddownarrow d : d \in D\}\}$ and hence $PI(B)$ is a basis of Ψ. For any directed $D \subseteq B$ with $\bigvee_B D$, obviously we have $\bigvee_{PI(B)}\{\downarrow x : x \in D\} = \downarrow \bigvee_B D = \bigvee_\Psi\{\downarrow x : x \in D\}$. Therefore, $PI(B)$ is an embedded basis of Ψ, the proof is complete.

(2) $\Leftarrow$: We have B is a θ-continuous poset by Proposition 6.2(2). Then, it is straightforward from (1) and Propositions 6.1(1).

$\Rightarrow$: By Corollary 5.1, we have B is θ-continuous. Thus, B is a θ-embedded basis since (1) and Proposition 6.1(1).

(3) This can be verified directly by (1), Theorem 6.2, Proposition 6.1(2) and Proposition 6.2(3). □

Corollary 6.1. If a poset P is continuous, resp. θ-continuous, resp. s_2-continuous, then $D(P) \cong RI(P, \ll)$, resp. $D_\theta(P) \cong RI(P, \ll_\theta)$, resp. $D_{s_2}(P) \cong RI(P, \ll_{s_2})$.

Proof. A direct consequence of Theorem 3.8 in Xu (2006) and Theorem 6.3. □

From Corollary 6.1, we know that the D-completion of a poset P is exactly the round ideal completion $RI(P, \ll)$ in the continuous case. However, the D_θ-completion and D_{s_2}-completion provide another two different ways of dcpo-completion of continuous posets.

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