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Boundedness of integral operators on decreasing functions

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We continue the study of the boundedness of the operator

$$S_a f(t) = \int_0^\infty a(s) f(st)\,\mathrm{d}s$$

on the set of decreasing functions in $L^p(w)$. This operator was first introduced by Braverman and Lai and also studied by Andersen, and although the weighted weak-type estimate $S_a\colon L^p_{\mathrm{dec}}(w) \to L^{p,\infty}(w)$ was completely solved, the characterization of the weights w such that $S_a\colon L^p_{\mathrm{dec}}(w) \to L^p(w)$ is bounded is still open for the case in which $p > 1$. The solution of this problem will have applications in the study of the boundedness on weighted Lorentz spaces of important operators in harmonic analysis.

Keywords: weak L^p-spaces; Hardy–Littlewood maximal operator; Muckenhoupt weights

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1. Introduction and motivation

Let $L^0(\mu)$ be the set of μ-measurable functions on $\mathcal{M}$ and similarly for $L^0(\nu)$, where $(\mathcal{M}, \mu)$ and $(\mathcal{N}, \nu)$ are two σ-finite measure spaces, and let us consider such operators T that satisfy the inequality

$$(Tf)^*_\nu(t) \leqslant C \int_0^\infty a(s) f^*_\mu(st)\,\mathrm{d}s \tag{1.1}$$

for a certain positive and locally integrable function a and some positive constant C independent of t. We recall that f^*_μ is the decreasing rearrangement of f with respect to the measure μ [5],

$$f^*_\mu(t) = \inf\{s > 0;\ \mu(\{x \in \mathcal{M};\ |f(x)| > s\}) \leqslant t\},$$

and similarly for $(Tf)^*_\nu$. Let w be a positive and locally integrable function (that we call a weight) in $(0,\infty)$ and set

$$L^p_{\mathrm{dec}}(w) = \{f \in L^p(w)\colon f \text{ is positive and decreasing}\}$$

with

$$\|f\|_{L^p_{\mathrm{dec}}(w)} = \left(\int_0^\infty f(t)^p w(t)\,\mathrm{d}t\right)^{1/p}.$$

Let us also consider the weighted Lorentz space defined by [11, 12]:

$$\Lambda^p_\mu(w) = \left\{f \in L^0(\mu)\colon \|f\|_{\Lambda^p_\mu(w)} = \left(\int_0^\infty (f^*_\mu(t))^p w(t)\,\mathrm{d}t\right)^{1/p} < \infty\right\}.$$

These spaces include as particular cases the weighted Lebesgue spaces $L^p(u)$ and the classical Lorentz spaces $\Lambda^p(w)$ and are a unified framework to study weighted inequalities for many important operators in harmonic analysis (see, for example, [3, 7, 18] and references therein).

If not otherwise indicated, throughout this paper $0 < p < \infty$.

Now, if we define

$$S_a f(t) = \int_0^\infty a(s) f(st)\,\mathrm{d}s,$$

where a is a positive and locally integrable function, we clearly have the following result.

Lemma 1.1. *If T satisfies (1.1) and*

$$S_a\colon L^p_{\mathrm{dec}}(w) \to L^p(w)$$

is bounded, then so is

$$T\colon \Lambda^p_\mu(w) \to \Lambda^p_\nu(w).$$

Important examples of operators T as above are the following.

(I) The Hardy–Littlewood maximal operator

$$Mf(x) = \sup_{r>0} \frac{1}{|B(0,r)|}\int_{B(0,r)} |f(y)|\,\mathrm{d}y$$

satisfies (1.1) for $a(s) = \chi_{(0,1)}(s)$ whenever $\mathrm{d}\mu = \mathrm{d}\nu = u\,\mathrm{d}x$ with u a weight in the Muckenhoupt class A_1 [4, 15]. In this case, the corresponding operator S_a is the Hardy operator whose boundedness on $L^p_{\mathrm{dec}}(w)$ has been extensively studied and the characterization is the class of weights B_p [3]:

$$r^p \int_r^\infty \frac{w(t)}{t^p}\,\mathrm{d}t \leqslant C\int_0^r w(t)\,\mathrm{d}t$$

for some $C > 0$ independent of $r > 0$. For several properties concerning this class of weights we refer the reader to [19]. In particular, we shall use the characterization

$$w \in B_p \iff \exists \varepsilon > 0;\ \bar{W}(t) \leqslant Ct^{p-\varepsilon} \quad \forall t > 1,$$

where

$$\bar{W}(t) = \sup_{s>0} \frac{W(ts)}{W(s)}.$$

(II) If $T = H$ is the Hilbert transform

$$Hf(x) = \mathrm{P.\,V.} \int_{\mathbb{R}} \frac{f(x-y)}{y} \,\mathrm{d}y,$$

then it is known (see [4]) that, if $u \in A_1$,

$$(Tf)^*_u(t) \lesssim \frac{1}{t} \int_0^t f^*_u(s) \,\mathrm{d}s + \int_t^\infty f^*_u(s) \frac{\mathrm{d}s}{s} = \int_0^\infty \min\left(1, \frac{1}{u}\right) f^*(tu) \,\mathrm{d}u, \tag{1.2}$$

and hence T satisfies (1.1) for $a(u) = \min(1, 1/u)$. In this case, S_a is the so-called Calderón operator [5, ch. 3, pp. 141–142].

In fact, all operators T that are of joint weak type $(1, 1; \infty, \infty)$ with respect to the measures μ and ν (see [5, ch. 3, p. 143]) satisfy (1.2). In particular, if μ and ν are the Lebesgue measure, examples of such operators are the Riesz transform and some singular integral operators.

Now, if $a(t) = (1/t)\chi_{(1,\infty)}$, then

$$S_a f(t) = \int_t^\infty f(s) \frac{\mathrm{d}s}{s}$$

is the conjugate Hardy operator and it is known [16] that, in this case, the strong boundedness of S_a on $L^p_{\mathrm{dec}}(w)$ is characterized (for every $p > 0$) by the condition that $w \in B^*_\infty$; that is,

$$\sup_{r>0} \frac{1}{W(r)} \int_0^r \frac{W(s)}{s} \,\mathrm{d}s < \infty.$$

It follows that the boundedness of the Calderón operator is characterized by $w \in B_p \cap B^*_\infty$ [16, 18].

(III) Let

$$T_\varphi f(x) = \sup_{h>0} \frac{1}{h} \int_0^h \varphi\left(\frac{t}{h}\right) |f(x-t)| \,\mathrm{d}t,$$

where φ is a positive and integrable function with compact support in $(0, 1)$. Then, as was proved in [8], we have that

$$(T_\varphi f)^*(s) \leqslant C \int_0^1 \varphi^*(t) f^*(ts) \,\mathrm{d}t, \quad s \in (0, \infty).$$

For example, for every $0 < \alpha \leqslant 1$, the operators

$$M^+_\alpha f(x) = \sup_{r>x} \frac{1}{(r-x)^\alpha} \int_x^r \frac{|f(s)|}{(r-s)^{1-\alpha}} \,\mathrm{d}s$$

and

$$M^-_\alpha f(x) = \sup_{r<x} \frac{1}{(x-r)^\alpha} \int_r^x \frac{|f(s)|}{(s-r)^{1-\alpha}} \,\mathrm{d}s$$

are of this kind. These operators were studied in [8, 13, 17] in connection with the C_α summability criterion for the Lebesgue differentiation theorem.

(IV) Also, if we have a sublinear operator T that is bounded in L^∞ and satisfies a restricted weak-type inequality

$$T\colon L^{p,1}(\mu) \to L^{p,\infty}(\nu),$$

then standard techniques show that, for any $t \in (0,\infty)$,

$$(Tf)^*_\nu(t) \lesssim \int_0^1 s^{1/p-1} f^*_\mu(st)\,\mathrm{d}s.$$

(V) Given $1 \leqslant p,q \leqslant \infty$,the Lorentz maximal operator defined as

$$M_{p,q}f(x) = \sup_{x\in Q} \frac{\|f\chi_Q\|_{L^{p,q}}}{|Q|^{1/p}}$$

was considered in [10] and it was proved that, for any $t \in (0,\infty)$,

$$((M_{p,q}f)^*(t))^q \leqslant \int_0^1 f^*\left(\frac{st}{3^n}\right)^q s^{q/p-1}\,\mathrm{d}s.$$

(VI) More generally, given a rearrangement invariant space X on $\mathbb{R}^n$, the Hardy–Littlewood maximal operator M_X associated with the space X was considered in [14]:

$$M_X f(x) = \sup_{x\in Q} \|f\|_{X,Q},$$

where $\|f\|_{X,Q} = \|\tau_{l(Q)}(f\chi_Q)\|_X$ with $l(Q)$ the side length of the cube Q and $\tau_\delta f(x) = f(\delta x)$ is the dilation operator. It was also proved that

$$(M_X f)^*(t) \leqslant \int_0^1 f^*\left(\frac{st}{3^n}\right)\mathrm{d}\varphi_X(s), \quad t \in (0,\infty),$$

φ_X being the fundamental function of X. Using this inequality, Mastylo and Pérez prove that, under certain submultiplicity hypotheses on φ_X, M_X is bounded on L^p.

All these examples provide motivation to continue the investigation of the boundedness property of S_a on the cone of decreasing functions. This operator was first introduced by Braverman [6] and Lai [9] and was also studied by Andersen [2]. In particular, we mention that if $L^{p,\infty}_{\mathrm{dec}}(w)$ is the set of measurable decreasing functions such that

$$\|f\|_{L^{p,\infty}(w)} = \sup_{t>0} f(t)W(t)^{1/p} < \infty$$

and

$$L^{p,1}_{\mathrm{dec}}(w) = \left\{ f\downarrow;\ \|f\|_{L^{p,1}_{\mathrm{dec}}(w)} = \int_0^\infty f(t)W(t)^{1/p-1}w(t)\,\mathrm{d}t < \infty \right\},$$

then it is very easy to see that

$$S_a\colon L^{p,\infty}_{\mathrm{dec}}(w) \to L^{p,\infty}(w) \tag{1.3}$$

is bounded if and only if

$$\sup_{t>0} W^{1/p}(t) \int_0^\infty a(s)\frac{1}{W^{1/p}(st)}\,\mathrm{d}s < \infty, \tag{1.4}$$

and

$$S_a\colon L^{p,1}_{\mathrm{dec}}(w) \to L^{p,1}(w) \tag{1.5}$$

is bounded if and only if

$$\sup_{r>0} \frac{1}{W^{1/p}(r)} \int_0^\infty a(s) W^{1/p}\left(\frac{r}{s}\right) \mathrm{d}s < \infty. \tag{1.6}$$

To see this, just observe that in the first case it is enough to have that

$$\|S_a W^{-1/p}\|_{L^{p,\infty}(w)} < \infty,$$

while in the second case it is enough to check that S_a is bounded on characteristic decreasing functions; that is,

$$\sup_{r>0} \frac{\|S_a \chi_{(0,r)}\|_{L^{p,1}(w)}}{\|\chi_{(0,r)}\|_{L^{p,1}(w)}} < \infty.$$

Also, the complete characterization of the weights for which

$$S_a\colon L^p_{\mathrm{dec}}(w) \to L^{p,\infty}(w)$$

is bounded is known (see theorem 2.3), but the complete characterization of the weights w such that

$$S_a\colon L^p_{\mathrm{dec}}(w) \to L^p(w) \tag{1.7}$$

is bounded is still an open problem for the case in which $p > 1$; the continued investigation of this open case is the main goal of this paper.

From now on,

$$A(t) = \int_0^t a(s)\,\mathrm{d}s, \qquad W(t) = \int_0^t w(s)\,\mathrm{d}s$$

and, in general, $U(t) = \int_0^t u(s)\,\mathrm{d}s$ for any function u. We shall write simply $\|S_a\|$ to indicate the norm of the operator in (1.7); that is, $\|S_a\|_{L^p_{\mathrm{dec}}(w)\to L^p(w)}$.

Also, C will denote a constant independent of the parameters involved and, as usual, the symbol $\lesssim$ denotes that an inequality $\leqslant$ holds up to some constant C and, similarly, $\approx$ means that both $\lesssim$ and $\gtrsim$ hold.

2. Previously known results

In the study of (1.7), the following class of weights has a fundamental role (see theorems 2.4–2.6).

Definition 2.1. We say that a weight w in $(0,\infty)$ is in B^a_p if

$$\|w\|^p_{B^a_p} = \sup_{r>0} \frac{1}{W(r)} \int_0^\infty A^p\left(\frac{r}{t}\right) w(t)\,\mathrm{d}t < \infty. \tag{2.1}$$

First of all, we observe that since

$$\int_0^\infty A^p\left(\frac{r}{t}\right) w(t)\,\mathrm{d}t \geqslant \int_0^r A^p\left(\frac{r}{t}\right) w(t)\,\mathrm{d}t \geqslant A^p(1) W(r)$$

we have that, if $A(1) \neq 0$, $w \in B_p^a$ if and only if

$$\int_0^\infty A^p\left(\frac{r}{t}\right) w(t)\,\mathrm{d}t \approx W(r). \tag{2.2}$$

In fact, if $A(1) = 0$ but W satisfies the Δ_2-condition, that is,

$$W(2t) \lesssim W(t) \quad \forall t > 0, \tag{2.3}$$

then one can also see that $w \in B_p^a$ if and only if (2.2) holds. Moreover, if w satisfies the doubling property, we can assume, without loss of generality, that $A(1) \neq 0$, since if $A(1) = 0$, we can consider $a(s) + \chi_{(1/2,1)}(s)$ and the boundedness on $L^p_{\mathrm{dec}}(w)$ of this new operator is the same as that of the original one. Let us also mention, for the sake of completeness and to answer a question from the referee, that, for an arbitrary a, we do not know how (1.4), (1.6) and (2.1) relate to each other.

Finally, we have to mention that, in general, $w \in B_p^a$ does not imply that W satisfies the Δ_2-condition (2.3) since there are weights in the class B_∞^* whose primitives are not doubling, as the example $w(t) = \mathrm{e}^t$ shows.

However, the following result proves that, in many other cases, we do have the following property.

Proposition 2.2. *Let us assume that there exists $r_0 < 1$ such that $A(r_0) \neq 0$. Then, if $w \in B_p^a$, we have that W satisfies the doubling property.*

Proof. We have that, for every $r > 0$,

$$A^p(r_0) W\left(\frac{r}{r_0}\right) \leqslant \int_0^{r/r_0} A^p\left(\frac{r}{t}\right) w(t)\,\mathrm{d}t \leqslant \int_0^\infty A^p\left(\frac{r}{t}\right) w(t)\,\mathrm{d}t \lesssim W(r),$$

and since $1/r_0 > 1$ the result follows. □

Also, observe that if $w \in B_p^a \cap L^1$, we have that, for every $r > 0$,

$$\int_0^\infty \left(\int_0^{r/t} a(s)\,\mathrm{d}s\right)^p w(t)\,\mathrm{d}t \lesssim \int_0^\infty w(t)\,\mathrm{d}t,$$

and hence it follows by the monotone convergence theorem that

$$\int_0^\infty a(s)\,\mathrm{d}s < \infty.$$

Consequently,

$$B_p^a \cap L^1 \neq \emptyset \quad \Longrightarrow \quad a \in L^1.$$

The following result gives the complete characterization of the weak-type boundedness.

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THEOREM 2.3 (Andersen [2]).

(i) *If* $0 < p \leqslant 1$,

$$S_a \colon L^p_{\mathrm{dec}}(w) \to L^{p,\infty}(w) \tag{2.4}$$

is bounded if and only if, for every $0 < s, r < \infty$,

$$A^p\left(\frac{s}{r}\right) W(r) \lesssim W(s).$$

(ii) *If* $p > 1$, *(2.4) holds if and only if*

$$\sup_{r>0} W^{1/p}(r) \left(\int_0^\infty A^{p'}\left(\frac{s}{r}\right) W^{-p'}(s) w(s) \,\mathrm{d}s \right)^{1/p'} < \infty, \tag{2.5}$$

or, equivalently,

$$\sup_{r>0} W^{1/p}(r) \left(\int_0^\infty A^{p'-1}(s) W^{1-p'}(sr) a(s) \,\mathrm{d}s \right)^{1/p'} < \infty.$$

With respect to (1.7), the following results are already known.

THEOREM 2.4 (Lai [9]). *If* $0 < p < \infty$ *and (1.7) holds, then* $w \in B^a_p$ *and* $\|w\|_{B^a_p} \leqslant \|S_a\|$.

Proof. It is enough to apply the hypothesis to the case $f = \chi_{(0,r)}$. □

THEOREM 2.5 (Lai [9]). *For every* $0 < p \leqslant 1$, *(1.7) holds if and only if* $w \in B^a_p$.

THEOREM 2.6 (Lai [9]). *Let* $p > 1$ *and let* a *be such that* A *is quasi-submultiplicative in* $(0, 1)$ *and in* $(1, \infty)$*; that is, there exists* $c > 0$*, such that, for every* $0 < r, s \leqslant 1$ *or* $1 \leqslant r, s < \infty$,

$$A(rs) \leqslant cA(r)A(s).$$

Then (1.7) holds if and only if $w \in B^a_p$.

REMARK 2.7. Let $p > 1$. Then, clearly

$$1 \in B^a_p \quad \Longleftrightarrow \quad \int_0^\infty A^p(s) \frac{\mathrm{d}s}{s^2} < \infty, \tag{2.6}$$

while by theorem 2.3 we have that if $w = 1$, $S_a \colon L^p_{\mathrm{dec}} \to L^{p,\infty}$ is bounded if and only if

$$\int_0^\infty A^{p'}(s) \frac{1}{s^{p'}} \,\mathrm{d}s < \infty. \tag{2.7}$$

Therefore, it is clear that if $p = 2$, then condition (2.6) matches condition (2.7). But given $p \neq 2$, we can find a function a satisfying one and only one of these two conditions; that is, in general $w \in B^a_p$ and (2.4) are independent. This has two important consequences:

(i) the weak-type boundedness (2.4) does not imply (1.7) in the case $p > 1$;

(ii) in general, $w \in B^a_p$ is not sufficient to have (1.7).

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Another way of proving this last observation is the following: if $w \in B_p^a$, we have that $g(t) = A(1/t) \in L^p(w)$, and hence if (1.7) holds, we would have that $S_a g(t) < \infty$ for every $t > 0$, which implies that

$$\int_0^\infty a(s) A\left(\frac{1}{s}\right) \mathrm{d}s < \infty. \tag{2.8}$$

Now let us take, for $\alpha > 1$,

$$a(s) = \frac{s^{1/p-1}}{(1 + |\log s|)^{\alpha/p}}.$$

One can then see that $A(s) \approx s^{1/p}/(1 + |\log s|)^{\alpha/p}$ and, by (2.6), we obtain that $1 \in B_p^a$ but (2.8) does not hold if $p \geqslant 2\alpha$.

Corollary 2.8. *If S_a satisfies (1.7) and $p > 1$, then*

$$\sup_{r>0} \left(\int_0^\infty A^p\left(\frac{r}{t}\right) w(t) \,\mathrm{d}t\right)^{1/p} \left(\int_0^\infty A^{p'}\left(\frac{s}{r}\right) W^{-p'}(s) w(s) \,\mathrm{d}s\right)^{1/p'} < \infty.$$

Proof. The proof follows immediately by writing the conditions $w \in B_p^a$ and (2.5). □

Before going on, for the sake of completeness let us now state some other important examples, which can also be found in [2, 9].

(i) If $a(t) = \mathrm{e}^{-t} \in L^1(\mathbb{R})$, we obtain that

$$S_a f(x) = \int_0^\infty \mathrm{e}^{-t} f(xt) \,\mathrm{d}t = \frac{1}{x} \mathcal{L} f\left(\frac{1}{x}\right), \tag{2.9}$$

where $\mathcal{L}$ is the Laplace transform. In this case $A(t) = 1 - \mathrm{e}^{-t}$.

(ii) If $a(t) = (1-t)^\alpha \chi_{(0,1)}(t)$ with $\alpha > -1$, we obtain the Riemann–Liouville operator

$$R_\alpha f(x) = \int_0^1 (1-s)^\alpha f(xs) \,\mathrm{d}s = \frac{1}{x^{\alpha+1}} \int_0^x (x-t)^\alpha f(t) \,\mathrm{d}t. \tag{2.10}$$

In this case,

$$A(t) = \int_0^t (1-s)^\alpha \chi_{(0,1)}(s) \,\mathrm{d}s = \int_0^{\min(t,1)} (1-s)^\alpha \,\mathrm{d}s \approx (1-t)_+^{\alpha+1} - 1.$$

3. The B_p^a class of weights

Henceforth, we shall assume that $p > 1$.

Theorem 3.1. *$w \in B_p^a$ if and only if*

$$S_a \colon L^{p,1}_{\mathrm{dec}}(w) \to L^p(w)$$

is bounded.

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Proof. To obtain the sufficient condition, it is enough to apply the boundedness hypothesis to $f = \chi_{(0,r)}$. Conversely, writing $S_a f$ in terms of the distribution function $\lambda_f(y) = |\{x;\ |f(x)| > y\}|$, we have that

$$S_a f(t) = \int_0^\infty A\biggl(\frac{\lambda_f(y)}{t}\biggr)\,\mathrm{d}y \tag{3.1}$$

and using Minkowski's inequalities we have that

$$\begin{aligned}\|S_a f\|_{L^p(w)} &= \biggl(\int_0^\infty S_a f(t)^p w(t)\,\mathrm{d}t\biggr)^{1/p}\\ &= \biggl(\int_0^\infty \biggl(\int_0^\infty A\biggl(\frac{\lambda_f(y)}{t}\biggr)\,\mathrm{d}y\biggr)^p w(t)\,\mathrm{d}t\biggr)^{1/p}\\ &\leqslant \int_0^\infty \biggl(\int_0^\infty A\biggl(\frac{\lambda_f(y)}{t}\biggr)^p w(t)\,\mathrm{d}t\biggr)^{1/p}\,\mathrm{d}y\\ &\lesssim \int_0^\infty W(\lambda_f(y))^{1/p}\,\mathrm{d}y\\ &\approx \|f\|_{L^{p,1}(w)},\end{aligned}$$

and the result follows. □

Proposition 3.2. *If $w \in B_p^a$ satisfies (2.2), then, for every decreasing function,*

$$\|f\|_{L^p(w)} \lesssim \|S_a f\|_{L^p(w)}.$$

Proof. If $w \in B_p^a$, then $\int_0^\infty A^p(r/t)w(t)\,\mathrm{d}t \lesssim W(r)$, and hence

$$\int_0^\infty A^p\biggl(\frac{r}{t}\biggr)w(t)\,\mathrm{d}t < \infty.$$

Now, by (2.2),

$$\begin{aligned}W(r) \lesssim \int_0^r A^p\biggl(\frac{r}{t}\biggr)w(t)\,\mathrm{d}t &= \int_0^\infty \biggl(\int_0^r A^{p-1}\biggl(\frac{s}{t}\biggr)a\biggl(\frac{s}{t}\biggr)\frac{\mathrm{d}s}{t}\biggr)w(t)\,\mathrm{d}t\\ &= \int_0^r \biggl(\int_0^\infty A^{p-1}\biggl(\frac{s}{t}\biggr)a\biggl(\frac{s}{t}\biggr)w(t)\frac{\mathrm{d}t}{t}\biggr)\,\mathrm{d}s,\end{aligned}$$

and thus, for every decreasing function f,

$$\int_0^\infty f^p(s)w(s)\,\mathrm{d}s \lesssim \int_0^\infty \biggl(\int_0^\infty f^p(s)A^{p-1}\biggl(\frac{s}{t}\biggr)a\biggl(\frac{s}{t}\biggr)\frac{\mathrm{d}s}{t}\biggr)w(t)\,\mathrm{d}t.$$

Now, in the inner expression, if we write $h_t(y) = f(ty)$, we obtain that

$$\begin{aligned}\int_0^\infty f^p(s)A^{p-1}\biggl(\frac{s}{t}\biggr)a\biggl(\frac{s}{t}\biggr)\frac{\mathrm{d}s}{t} &\approx \int_0^\infty y^{p-1}A^p(\lambda_{h_t}(y))\,\mathrm{d}y \lesssim \biggl(\int_0^\infty A(\lambda_{h_t}(y))\,\mathrm{d}y\biggr)^p\\ &= \biggl(\int_0^\infty a(y)h_t(y)\,\mathrm{d}s\biggr)^p = S_a f(t)^p\end{aligned}$$

and the result follows. □

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As a consequence, if (1.7) and (2.2) hold, then

$$\|S_a f\|_{L^p(w)} \approx \|f\|_{L^p(w)}.$$

Proposition 3.3. *If $w \in B_p^a$, then*

$$\sup_{t>0} A^p\left(\frac{1}{t}\right)\bar{W}(t) < \infty,$$

and in fact

$$\bar{W}^{1/p}(t) \lesssim \frac{1}{\sup_s[A(s/t)A(1/s)]}. \tag{3.2}$$

Proof. We have that, for every $s > 0$,

$$A^p\left(\frac{r}{s}\right)W(s) = A^p\left(\frac{r}{s}\right)\int_0^s w(t)\,\mathrm{d}t \leqslant \int_0^\infty A^p\left(\frac{r}{t}\right)w(t)\,\mathrm{d}t \lesssim W(r),$$

and hence

$$\frac{W(s)}{W(r)} \lesssim \frac{1}{A^p(r/s)}.$$

Therefore, for every $t > 0$,

$$\bar{W}(1/t) = \sup_{r>0}\frac{W(r/t)}{W(r)} \lesssim \frac{1}{A^p(t)}. \tag{3.3}$$

Now, since $\bar{W}$ is submultiplicative, i.e. $\bar{W}(uv) \leqslant \bar{W}(u)\bar{W}(v)$, we obtain that

$$\bar{W}(t) \leqslant \bar{W}\left(\frac{t}{s}\right)\bar{W}(s) \lesssim \frac{1}{A^p(s/t)A^p(1/s)}$$

and the result follows by taking the infimum in $s > 0$. □

Corollary 3.4. *If $a \notin L^1$, then*

$$B_p^a \subset B_\infty^*.$$

Proof. We have that $A(\infty) = \infty$, and hence (3.2) implies that $\bar{W}(0^+) = 0$, which is equivalent (see [1]) to $w \in B_\infty^*$. □

Remark 3.5. If $a \in L^1$, it is immediate to see that the B_p^a condition reads

$$\int_r^\infty A^p\left(\frac{r}{t}\right)w(t)\,\mathrm{d}t \lesssim W(r). \tag{3.4}$$

Moreover, in this case, if we write

$$S_a f(t) = \int_0^\infty a(s)f(st)\,\mathrm{d}s = \int_0^1 a(s)f(st)\,\mathrm{d}s + \int_1^\infty a(s)f(st)\,\mathrm{d}s,$$

we have that

$$\int_1^\infty a(s)f(st)\,\mathrm{d}s \leqslant f(t)\int_1^\infty a(s)\,\mathrm{d}s,$$

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and hence we only have to study the first part of the operator; that is,

$$\tilde{S}_a f(t) = \int_0^1 a(s) f(st)\,\mathrm{d}s.$$

PROPOSITION 3.6. *If (1.7) holds for some $w \neq 0$, then*

$$\inf_{t>0} A(t) A\left(\frac{1}{t}\right) = 0$$

and

$$\sup_{t>0} A(t) A\left(\frac{1}{t}\right) < \infty.$$

In particular, A cannot be quasi-submultiplicative in $(0,\infty)$.

Proof. Let us assume that $\inf_{t>0} A(t)A(1/t) = C > 0$. Then, since $w \in B_p^a$, using (2.8),

$$\int_0^\infty a(s) \frac{1}{A(s)}\,\mathrm{d}s \leqslant \frac{1}{C} \int_0^\infty a(s) A\left(\frac{1}{s}\right) \mathrm{d}s < \infty,$$

which is a contradiction since the integral on the left is clearly not finite.

To prove the second part, we observe that if $\sup_{t>0} A(t)A(1/t) = \infty$, then necessarily $\lim_{t\to 0} A(t)A(1/t) = \infty$ and we arrive at the same contradiction. □

REMARK 3.7. By (3.4), it follows that if $a \in L^1$ and there exists $\alpha > 0$ such that $A(x) \approx x^\alpha$ for every $x \in (0,1)$, then $B_p^a = B_{p\alpha}$.

In particular (see also [9]), the following hold.

(1) If $a_{\mathcal{L}} = \mathrm{e}^{-t}$ is the function associated with the Laplace transform as in (2.9), we have that, for every $p > 0$, $B_{a_{\mathcal{L}}} = B_p$.

(2) If $a_{\mathcal{RL}}$ is the function associated with the Riemann–Liouville operator (2.10), we have that, for every $p > 0$, $B_{a_{\mathcal{RL}}} = B_p$.

4. Sufficient conditions

PROPOSITION 4.1. *If*

$$\int_0^\infty a(s) \bar{W}^{1/p}\left(\frac{1}{s}\right) \mathrm{d}s < +\infty, \tag{4.1}$$

then S_a is bounded from $L^p_{\mathrm{dec}}(w)$ to $L^p(w)$.

Proof. We have that

$$\begin{aligned}\|S_a f\|_{L^p(w)} = \left\| \int_0^\infty a(s) f(st)\,\mathrm{d}t \right\|_{L^p(w)} &\leqslant \int_0^\infty a(s) \|f(s\cdot)\|_{L^p_{\mathrm{dec}}(w)}\,\mathrm{d}s \\ &\leqslant \|f\|_{L^p_{\mathrm{dec}}(w)} \int_0^\infty a(s) \|D_{1/s}\|_{L^p(w)}\,\mathrm{d}s,\end{aligned}$$

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where

$$\begin{aligned}\|D_{1/s}\|_{L^p(w)} &= \sup_{f\downarrow} \frac{(\int_0^\infty f(st)^p w(t)\,dt)^{1/p}}{(\int_0^\infty f(t)^p w(t)\,dt)^{1/p}} \\ &= \sup_{f\downarrow} \frac{(\int_0^\infty f(y)^p w(y/s)(dy/s))^{1/p}}{(\int_0^\infty f(t)^p w(t)\,dt)^{1/p}} \\ &= \sup_{r>0} \frac{(\int_0^r w(y/s)(dy/s))^{1/p}}{(\int_0^r w(t)\,dt)^{1/p}} \\ &= \left[\sup_{r>0} \frac{W(r/s)}{W(r)}\right]^{1/p} \\ &= \bar{W}\left(\frac{1}{s}\right)^{1/p},\end{aligned}$$

and the result follows. □

REMARK 4.2. If (4.1) holds, then

$$\sup_{t>0} W^{1/p}(t) \int_0^\infty a(s) \frac{1}{W^{1/p}(st)}\,ds \leqslant \int_0^\infty a(s)\bar{W}^{1/p}\left(\frac{1}{s}\right) ds < +\infty$$

and similarly

$$\sup_{r>0} \frac{1}{W^{1/p}(r)} \int_0^\infty a(s) W^{1/p}\left(\frac{r}{s}\right) ds \leqslant \int_0^\infty a(s)\bar{W}^{1/p}\left(\frac{1}{s}\right) ds < +\infty,$$

and thus, by (1.4) and (1.6), we have that S_a satisfies both (1.3) and (1.5).

PROPOSITION 4.3. *If*

$$\int_0^\infty \left(\int_0^\infty A\left(\frac{s}{t}\right)^{p'} W(s)^{-p'} w(s)\,ds\right)^{p/p'} w(t)\,dt < +\infty, \tag{4.2}$$

then $S_a f$ is bounded from $L^p_{\mathrm{dec}}(w)$ to $L^p(w)$.

Proof. We have that

$$\|S_a f\|^p_{L^p(w)} = \int_0^\infty \left(\int_0^\infty a(s) f(st)\,ds\right)^p w(t)\,dt \leqslant \|f\|^p_{L^p(w)} \int_0^\infty H(t)^p w(t)\,dt,$$

where

$$H(t) = \sup_{f\downarrow} \frac{\int_0^\infty a(s) f(st)\,ds}{(\int_0^\infty f(s)^p w(s)\,ds)^{1/p}} = \sup_{f\downarrow} \frac{\int_0^\infty (1/t) a(s/t) f(s)\,ds}{(\int_0^\infty f(s)^p w(s)\,ds)^{1/p}},$$

and the result follows using Sawyer's formula [18]. □

By considering the case of the Hardy operator and $w = 1$, we see that (4.2) is not a necessary condition for the boundedness of S_a from $L^p_{\mathrm{dec}}(w)$ to $L^p(w)$.

REMARK 4.4. Another expression equivalent to (4.2) (see [18]) is

$$\int_0^\infty \left[\int_0^\infty \left(\int_y^\infty \frac{a(s)}{W(st)} \, \mathrm{d}s \right)^{p'/p} a(y) \, \mathrm{d}y \right]^{p/p'} w(t) \, \mathrm{d}t < \infty.$$

PROPOSITION 4.5.

(a) *If $a \in L^1$ and there exists $\varepsilon > 0$ such that $w \in B^a_{p-\varepsilon}$, then S_a satisfies (1.7).*

(b) *If $a \notin L^1$, $\operatorname{supp} a \subset (r, +\infty)$ for some $r > 0$ and, for every $t > 1$ and some $\alpha > 1$,*

$$\bar{W}^{1/p}\left(\frac{1}{t}\right) \lesssim \frac{1}{A(t)(1 + \log^+ A(t))^\alpha}, \tag{4.3}$$

then S_a satisfies (1.7). In particular, this is the case if there exists $\varepsilon > 0$ such that $w \in B^a_{p+\varepsilon}$.

Proof. (a) By (3.3), we have that

$$\bar{W}^{1/p}\left(\frac{1}{t}\right) \lesssim \frac{1}{A(t)^{(p-\varepsilon)/p}}.$$

Hence,

$$\int_0^1 a(s) \bar{W}\left(\frac{1}{s}\right)^{1/p} \mathrm{d}s \leqslant \int_0^1 a(s) A^{(\varepsilon-p)/p}(s) \, \mathrm{d}s \approx A^{\varepsilon/p}(1) < \infty.$$

On the other hand, since $a \in L^1$,

$$\int_1^\infty a(s) \bar{W}\left(\frac{1}{s}\right)^{1/p} \mathrm{d}s \leqslant \bar{W}(1)^{1/p} \int_1^\infty a(s) \, \mathrm{d}s < \infty,$$

and hence (4.1) holds and the result follows.

Similarly, to prove (b) we observe that in this case, if N is such that $A(N) \neq 0$,

$$\begin{aligned} \int_0^\infty a(s) \bar{W}\left(\frac{1}{s}\right)^{1/p} \mathrm{d}s &= \int_r^\infty a(s) \bar{W}\left(\frac{1}{s}\right)^{1/p} \mathrm{d}s \\ &\lesssim 1 + \int_N^\infty \frac{a(s)}{A(s)(1 + \log^+ A(s))^\alpha} \, \mathrm{d}s < \infty \end{aligned}$$

and the result follows. □

5. Self-improving properties of B^a_p

A well-known fact of the B_p class is the so-called $p - \varepsilon$ property that says that, for every $w \in B_p$, there exists $\varepsilon > 0$ such that $w \in B_{p-\varepsilon}$. Since $B_{p-\varepsilon} \subset B_p$, we say that the weights in the class B_p satisfy a self-improving property.

In this section, we study conditions on the function a such that the class B^a_p satisfies certain self-improving properties. Before that, we mention that in the proof of theorem 2.6, Lai decomposes the operator S_a into two parts that are treated separately, namely,

$$S_a f(t) = \int_0^1 a(s)f(st)\,\mathrm{d}s + \int_1^\infty a(s)f(st)\,\mathrm{d}s := S_a^1 f(t) + S_a^2 f(t).$$

In fact, what is proved in [2, 9] is the following result. We shall present in this paper a new proof of it.

THEOREM 5.1.

(a) *If A is quasi-submultiplicative in $(0,1)$, then S_a^1 is bounded from $L^p_{\mathrm{dec}}(w)$ to $L^p(w)$ if and only if $w \in B_p^a$.*

(b) *If A is quasi-submultiplicative in $(1,\infty)$, then S_a^2 is bounded from $L^p_{\mathrm{dec}}(w)$ to $L^p(w)$ if and only if $w \in B_p^a$.*

Our proof will be an immediate consequence of proposition 4.5 and the following and more general result. But first we need to recall an easy lemma concerning submultiplicative functions [1].

LEMMA 5.2.

(i) *If $\varphi\colon (0,1] \to [0,1]$ is an increasing submultiplicative function, then*

$$\varphi(\lambda) < 1 \quad \textit{for some } \lambda \in (0,1)$$

if and only if

$$\varphi(x) \lesssim \frac{1}{(1+\log(1/x))^\alpha} \quad \forall \alpha > 0,\ 0 < x < 1.$$

(ii) *For every submultiplicative increasing function φ defined in $[1,\infty)$,*

$$\varphi(\lambda) < \lambda \quad \textit{for some } \lambda > 1 \quad \iff \quad \exists \gamma < 1\colon \varphi(x) \lesssim x^\gamma \quad \forall x > 1.$$

PROPOSITION 5.3.

(a) *If a is supported in $(0,1)$ and A is quasi-submultiplicative in $(0,1)$, then B_p^a satisfies the $p-\varepsilon$ property; that is,*

$$\forall w \in B_p^a\ \exists \varepsilon > 0\colon w \in B_{p-\varepsilon}^a.$$

(b) *If a is supported in $(1,\infty)$ and A is quasi-submultiplicative in $(1,\infty)$, then, for every $t > 1$, B_p^a satisfies (4.3).*

Proof. First of all, we can assume without lost of generality (just by changing A to cA) that A is submultiplicative in $(0,1)$ or $(1,\infty)$ and also that A is strictly increasing.

(a) In this case, we can assume that

$$A\colon [0,1] \to [0, \|a\|_1]$$

is bijective. Then one can easily see that $1/A^{-1}$ is also submultiplicative and by (3.3) we obtain that, for every $y > 1$,

$$\bar{W}^{1/p}\left(\frac{1}{A^{-1}(1/y)}\right) \lesssim y.$$

We then observe that there are two options: either, for every $y > 1$,

$$y < \bar{W}^{1/p}\left(\frac{1}{A^{-1}(1/y)}\right),$$

and hence, for every $t < 1$,

$$\bar{W}^{1/p}\left(\frac{1}{t}\right) \approx \frac{1}{A(t)}, \tag{5.1}$$

or there exists $y > 1$ such that

$$\bar{W}^{1/p}\left(\frac{1}{A^{-1}(1/y)}\right) < y.$$

In this last case, by lemma 5.2(ii), we obtain that there exists $\gamma < 1$ such that

$$\bar{W}^{1/p}\left(\frac{1}{A^{-1}(1/y)}\right) \lesssim y^{\gamma},$$

or equivalently, for every $t < 1$,

$$\bar{W}\left(\frac{1}{t}\right) \lesssim \frac{1}{A(t)^{\gamma p}},$$

from which the result follows by (4.1).

Finally, if (5.1) holds, then one can easily check that $A(x) \approx x^{\alpha}$ for some α and every $x \in (0,1)$, and hence by remark 3.7 we obtain that $w \in B_{p\alpha}$. Consequently, $w \in B_{p\alpha-\varepsilon}$ for some ε, which implies that

$$\bar{W}\left(\frac{1}{t}\right) \lesssim \frac{1}{t^{p\alpha-\varepsilon}},$$

and this contradicts (5.1).

(b) To prove this part, we consider

$$A\colon [1,+\infty) \to [0,+\infty)$$

to be bijective. Then, as before, we have that $1/A^{-1}$ is submultiplicative and by (3.2), for every $y > 0$,

$$\bar{W}^{1/p}\left(\frac{1}{A^{-1}(1/y)}\right) \lesssim y.$$

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Then, since $A^{-1}(0) = 1$, there exists $y > 1$ such that

$$\frac{1}{y}\bar{W}^{1/p}\left(\frac{1}{A^{-1}(1/y)}\right) < 1.$$

In this case, by lemma 5.2(i), we obtain, for example, that

$$\bar{W}^{1/p}\left(\frac{1}{A^{-1}(1/y)}\right) \lesssim \frac{y}{(1+\log(1/y))^2},$$

or equivalently, for every $t > 1$,

$$\bar{W}^{1/p}\left(\frac{1}{t}\right) \lesssim \frac{1}{A(t)(1+\log^+ A(t))^2},$$

and the result follows. □

Proof of theorem 5.1. The proof is an immediate consequence of propositions 5.3 and 4.5. □

Remark 5.4. If

$$A(x) = x^{1/p}\frac{1}{\log x}\chi_{(0,1/2)}(x),$$

we have that

$$\int_0^1 A^p(x)\,\frac{\mathrm{d}x}{x^2} = \int_0^{1/2}\frac{\mathrm{d}x}{x\log^p x} < \infty,$$

and hence $1 \in B_p^a$ but $1 \notin B_q^a$ for any $q < p$. Hence, in general, property $p-\varepsilon$ does not hold.

For the case in which $A \notin L^\infty$ we also have the following result.

Theorem 5.5. *If $A \notin L^\infty$ and (1.7) holds, then there exists $\delta > 0$ such that*

$$S_{a_\delta}\colon L^p_{\mathrm{dec}}(w) \to L^p(w)$$

is bounded, where

$$a_\delta(s) = \frac{1}{s^{1-\delta}}\int_0^s a(y)\,\frac{\mathrm{d}y}{y^\delta}.$$

Proof. By proposition 3.4, $w \in B_\infty^*$, and hence, if $Qf(t) = \int_t^\infty f(s)(\mathrm{d}s/s)$, we have that, for every h decreasing,

$$\|Qh\|_{L^p(w)} \lesssim \|h\|_{L^p(w)},$$

and hence, in particular,

$$\|QS_af\|_{L^p(w)} \lesssim \|S_af\|_{L^p(w)} \lesssim \|f\|_{L^p(w)}.$$

Now,

$$QS_af(t) = \int_t^\infty S_af(s)\,\frac{\mathrm{d}s}{s} = \int_t^\infty\int_0^\infty a(y)f(sy)\,\mathrm{d}y\,\frac{\mathrm{d}s}{s} = \int_0^\infty f(st)\frac{A(s)}{s}\,\mathrm{d}s,$$

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and hence we obtain that S_{b_1} is bounded on $L^p(w)$ with constant less than or equal to $K = \|S_a\|\|Q\|$, where $b_1(s) = A(s)/s$. If we repeat the argument, we obtain that S_{b_n} is bounded with constant less than or equal to K^n, where

$$b_n(s) = \frac{B_{n-1}(s)}{s} = \cdots = \frac{1}{(n-1)!s}\int_0^s a(y)\biggl(\log^+\frac{s}{y}\biggr)^{n-1}\mathrm{d}y.$$

Now, if we take $B > K$, we construct the operator

$$\sum_{n=1}^{\infty}\frac{1}{B^n}S_{b_n}f(t) = \int_0^\infty \frac{1}{s}\biggl(\int_0^s\biggl(\frac{s}{y}\biggr)^{1/B}a(y)\,\mathrm{d}y\biggr)f(st)\,\mathrm{d}s,$$

and the result follows taking $\delta = 1/B$. □

6. Iterative operators

In this section we consider the iteration operators

$$S_a^{(n)}f(t) = S_a(S_a^{(n-1)}f)(t)$$

and we assume, for simplicity, that

$$A(1) = 1.$$

LEMMA 6.1. *For every $n \in \mathbb{N}$, it holds that $S_a^{(n)}f(t) = S_{a_n}f(t)$, where*

$$A_n(t) = \int_0^\infty a(s)A_{n-1}\biggl(\frac{t}{s}\biggr)\mathrm{d}s \tag{6.1}$$

with $A_1 = A$.

Proof. For $n = 2$,

$$\begin{aligned}S_a(S_af)(t) &= \int_0^\infty a(s)S_af(st)\,\mathrm{d}s\\ &= \int_0^\infty a(s)\int_0^\infty a(y)f(sty)\,\mathrm{d}y\,\mathrm{d}s\\ &= \int_0^\infty f(zt)\int_0^\infty a(s)a\biggl(\frac{z}{s}\biggr)\frac{\mathrm{d}s}{s}\,\mathrm{d}z\\ &= \int_0^\infty a_2(z)f(zt)\,\mathrm{d}z\end{aligned}$$

and

$$A_2(z) = \int_0^z a_2(u)\,\mathrm{d}u = \int_0^\infty a(s)A\biggl(\frac{z}{s}\biggr)\mathrm{d}s.$$

The result then follows by induction since, by (3.1),

$$\begin{aligned}S_a(S_{a_{n-1}}f)(t) &= \int_0^\infty a(s)\biggl(\int_0^\infty A_{n-1}\biggl(\frac{\lambda_f(y)}{st}\biggr)\mathrm{d}y\biggr)\mathrm{d}s\\ &= \int_0^\infty A_n\biggl(\frac{\lambda_f(y)}{t}\biggr)\mathrm{d}y.\end{aligned}$$

□

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From here it follows that, for every $t > 0$ and every $n \in \mathbb{N}$,

$$A_n(t) \geqslant A_{n-1}(t) \geqslant A(t).$$

Therefore,

$$B_p^{a_n} \subset B_p^{a_{n-1}} \subset \cdots \subset B_p^a$$

and we also have the following proposition.

PROPOSITION 6.2. *For every* $n \in \mathbb{N}$,

$$S_a \colon L^p_{\text{dec}}(w) \to L^p(w) \quad \Longleftrightarrow \quad S_{a_n} \colon L^p_{\text{dec}}(w) \to L^p(w).$$

COROLLARY 6.3. *If (1.7) holds, then, for every* $n \in \mathbb{N}$, $w \in B_p^{a_n}$, *and in fact*

$$\|w\|_{B_p^{a_n}} \leqslant \|S_a\|^n.$$

As in proposition 3.6, we obtain the following proposition.

PROPOSITION 6.4. *If* a *satisfies that, for some* $n \in \mathbb{N}$,

$$\inf_{t>0} A(t) A_n\left(\frac{1}{t}\right) > 0$$

or

$$\sup_{t>0} A(t) A_n\left(\frac{1}{t}\right) = +\infty,$$

then there is no weight $w \neq 0$ *such that* S_a *is bounded from* $L^p_{\text{dec}}(w)$ *to* $L^p(w)$.

THEOREM 6.5. *Let us assume that* S_a *satisfies (1.7). Then, for every* $\lambda > \|S_a\|$, *there exists a locally integrable function in* $(0, \infty)$, $a_{\infty,\lambda}$, *such that*

$$A(t) + \frac{1}{\lambda} \int_0^\infty a(s) A_{\infty,\lambda}\left(\frac{t}{s}\right) \mathrm{d}s = A_{\infty,\lambda}(t), \quad t > 0, \tag{6.2}$$

and

$$S_{a_{\infty,\lambda}} \colon L^p_{\text{dec}}(w) \to L^p(w)$$

is bounded.

Proof. Set $a_{\infty,\lambda}$ such that

$$A_{\infty,\lambda}(t) = \sum_{n=0}^\infty \frac{A_n(t)}{\lambda^n}. \tag{6.3}$$

Using lemma 6.1 we obtain that

$$S_{a_{\infty,\lambda}} f(t) = \sum_{n=0}^\infty \frac{S_a^{(n)} f(t)}{\lambda^n},$$

and hence, since $\lambda > \|S_a\|$, the result follows immediately. Finally, (6.2) follows easily from (6.1) and the definition of $A_{\infty,\lambda}$. □

We observe that, for every $n \in \mathbb{N}$,

$$B_p^{a_{\infty,\lambda}} \subset B_p^{a_n} \subset B_p^a.$$

Corollary 6.6. *If (1.7) holds, then, for every $\lambda > \|S_a\|$:*

(i) $w \in B_p^{a_{\infty,\lambda}}$;

(ii) $\displaystyle\sup_{r>0} W^{1/p}(r)\biggl(\int_0^\infty A_{\infty,\lambda}^{p'}\Bigl(\frac{s}{r}\Bigr)W^{-p'}(s)w(s)\,\mathrm{d}s\biggr)^{1/p'} < \infty.$

Equivalently (in the same way as in corollary 2.8),

$$\sup_{r>0}\biggl(\int_0^\infty A_{\infty,\lambda}^{p}\Bigl(\frac{r}{t}\Bigr)w(t)\,\mathrm{d}t\biggr)^{1/p}\biggl(\int_0^\infty A_{\infty,\lambda}^{p'}\Bigl(\frac{s}{r}\Bigr)W^{-p'}(s)w(s)\,\mathrm{d}s\biggr)^{1/p'} < \infty. \tag{6.4}$$

Remark 6.7. If $a(s) = \chi_{(0,1)}$, then one can see (solving the corresponding differential equation) that the solution to (6.2) is given by

$$A_{\infty,\lambda}(t) = \begin{cases} \dfrac{\lambda}{\lambda-1}t^{1-1/\lambda} & \text{if } 0 < t < 1,\\[2mm] \dfrac{\lambda}{\lambda-1} & \text{if } t \geqslant 1. \end{cases}$$

Moreover, one can also check that the condition $w \in B_p^{a_{\infty,\lambda}}$ corresponds to the $p-\varepsilon$ property of the B_p weights.

Remark 6.8. We make the final remark that, from corollary 2.8 and proposition 6.2, (1.7) implies that the quantity

$$K = \sup_{r,n}\biggl(\int_0^\infty A_n^{p}\Bigl(\frac{r}{t}\Bigr)w(t)\,\mathrm{d}t\biggr)^{1/np}\biggl(\int_0^\infty A_n^{p'}\Bigl(\frac{s}{r}\Bigr)W^{-p'}(s)w(s)\,\mathrm{d}s\biggr)^{1/np'}$$

is finite, and thus one can easily see that we can define $A_{\infty,\lambda}$ for every $\lambda > K$ as in (6.3) and we have that $w \in B_p^{a_{\infty,\lambda}}$ and both (6.2) and (6.4) hold.

A final question: is (6.4) sufficient to have (1.7) whenever $\lambda > K$?

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