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Oscillation theory for Sturm–Liouville problems with indefinite coefficients

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Oscillation and related results are given for the problem

$$-(py')' + qy = \lambda ry \quad \text{on } [a_0, a_1] = I$$

under separated end conditions, assuming $1/p$, q and $r \in L_1(I)$. Attention focuses on the two cases (i) $p > 0$ with r indefinite, and (ii) p indefinite with $r > 0$.

1. Introduction

In this paper we shall discuss oscillation theory and related results for the regular Sturm–Liouville equation

$$-(py')' + qy = \lambda ry \quad \text{a.e. on } [a_0, a_1] = I, \tag{1.1}$$

given

$$(\cos\alpha_j)y(a_j) = (\sin\alpha_j)(py')(a_j), \quad j = 0, 1, \tag{1.2}$$

assuming that $p(x) \neq 0$ a.e. and $1/p, q, r \in L_1(I)$, where $-\infty \leqslant a_0 < a_1 \leqslant \infty$. Thus the problem is 'regular' in the sense of Zettl [17, § 4]. Our primary purpose is to compare the two basic cases where one of p and r is indefinite, i.e. takes both signs on sets of positive measure, and the other is definite, i.e. either positive a.e. or negative a.e.

For the 'right-definite case' where p and r are both definite, 'oscillation theory' goes back to Sturm (with more restricted coefficients). The theory under the L_1 conditions here can be found in Atkinson's book [2, § 8.4] (in fact, some semi-definiteness is permitted there). For such problems, there is a sequence of real eigenvalues λ_n $(n = 0, 1, 2, \dots)$, accumulating at either $-\infty$ or $+\infty$, and with oscillation count (i.e. number of zeros of a corresponding eigenfunction y_n in $]a_0, a_1[$) equal to n. We remark that the case where both p and r are indefinite admits examples of quite different nature where the spectrum is the whole complex plane (cf. [4, 9]).

There has been recent interest in the case where p is indefinite but $r > 0$. Atkinson and Mingarelli [4] give asymptotics, and Weidmann [16, p. 280] analyses specific examples (regular and singular) where p changes sign finitely often. General indefinite p is discussed by Moeller [14], who essentially shows that the minimal operator corresponding to $1/r$ times the left-hand side of (1.1) is not semi-bounded. Zettl [17, theorem 4.12(b)] states that

(A) the eigenvalues of (1.1), (1.2) can be listed, $\cdots < \lambda_{-2} < \lambda_{-1} < \lambda_0 < \lambda_1 < \lambda_2 < \dots$, where $\lambda_n \to \pm\infty$ as $n \to \pm\infty$.

Zettl cites [14], and the rest follows from [15, pp. 66, 90]. It should be noted that the cited operator theory approach does not give any special meaning to the subscripts of λ_n, in contrast with our labelling below. Fleige [10] discusses various notions of completeness of the corresponding eigenfunctions for the Dirichlet problem ($\alpha_0 = \alpha_1 = 0$) in the case $q = 0$, $r = 1$. Earlier, Allegretto and Mingarelli [1] established (A) for the Neumann problem ($\alpha_0 = \alpha_1 = \frac{1}{2}\pi$) in the case $q = 0$, $0 < 1/r \in L_\infty$. The cited results from [1] and [10] were obtained via the reciprocal transformation to a special 'left-definite' problem, and we shall discuss this below.

Continuing with the case that p is indefinite and $r > 0$, we interpret (1.1) in the sense of Carathéodory, so py' is absolutely continuous, but when p is discontinuous, y' can be also, and, moreover, the zeros of y and py' need not interlace each other. Thus the definition of 'oscillation count' is no longer clear. One possibility, examined in § 2, is to use the absolutely continuous Prüfer angle $\theta = \theta(\cdot, \lambda)$, defined by $\theta(a_0) = \alpha_0$ and

$$\theta' = \frac{1}{p}\cos^2\theta + (\lambda r - q)\sin^2\theta \quad \text{a.e. on } I. \tag{1.3}$$

It is well known that $\cot\theta = py'/y$, so the eigenvalues λ of (1.1), (1.2) are the solutions of $\theta(a_1, \lambda) = n\pi + \alpha_1$ for integers n. We shall prove that

(B) for each integer n (regardless of sign), there exists precisely one eigenvalue λ_n for which $\theta(a_1, \lambda_n) = n\pi + \alpha_1$.

In § 3 we shall connect these ideas with the zeros of the eigenfunctions y_n, and the associated py'_n, corresponding to λ_n of (B). It turns out that both y_n and py'_n can have infinitely many zeros, despite (B). Nevertheless, we do have the following result.

(C) If, for some $\lambda_+ \geqslant 0$,

$$\lambda_+ r - q \geqslant 0 \quad \text{a.e.}, \tag{1.4}$$

then py'_n has exactly n zeros in $]a_0, a_1[$ for all $\lambda_n > \lambda_+$,

provided α_j are chosen in the appropriate ranges at the outset. This result can fail if (1.4) does not hold. There is an analogous result if $\lambda_- r - q \leqslant 0$ for some $\lambda_- \leqslant 0$. In particular, we have the following.

(D) If $\pm q \leqslant 0$, then py'_n has exactly $|n|$ zeros in $]a_0, a_1[$ for all $\pm\lambda_n > 0$, respectively.

As indicated, there are various negative results, and we mention specifically that

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(E) there exists $1/p \in L_1$ so that, for any $q, r \in L_1$ with $r > 0$, all Dirichlet eigenfunctions for (1.1) have infinitely many zeros in $]a_0, a_1[$.

This may be contrasted with (D) in the case $q = 0$. Then the usual oscillation theorem holds for each py'_n but fails for each y_n.

Cases with $p > 0$ and r indefinite were discussed for continuous coefficients early this century (see [12, § § 10.6, 10.7] and the references therein). For integrable coefficients, the results are probably well known to experts, but although specific results are available we have not seen a complete theory in the published literature. In § 4 we present such a theory, partly to give an elementary approach independent of operator theory and completeness of eigenfunctions and partly to show the similarities and differences between this case and that of §§ 2 and 3. Specifically, (A) remains unchanged (for the real eigenvalues), while (B) is replaced by

(B′) for some integer k, there are exactly two eigenvalues λ for which $\theta(1, \lambda) = n\pi + \alpha_1$ whenever $n \geqslant k$.

Analogously, (C) may be replaced by

(C′) an eigenfunction y belonging to an eigenvalue λ satisfying $\theta(1, \lambda) = n\pi + \alpha_1$ has exactly n zeros in $]a_0, a_1[$,

again provided that the α_j are chosen in appropriate ranges. The case $k = 0$ may be realized by

(D′) if $q \geqslant 0$ and $0 \leqslant \alpha_0 < \frac{1}{2}\pi < \alpha_1 \leqslant \pi$, then, for every $n = 0, 1, 2, \ldots$, there is exactly one positive eigenvalue λ_n^+ and one negative eigenvalue λ_n^- for which $\theta(1, \lambda) = n\pi + \alpha_1$. Moreover, $\lambda_n^+ < \lambda_{n+1}^+$ and $\lambda_n^- > \lambda_{n+1}^-$.

This is (essentially) the so-called 'left-definite' case. Finally, in the case $q = 0$, we may apply the 'reciprocal transformation' (cf. [5]). Specifically, with $q = 0$ and $r \neq 0$ a.e., equation (1.1) transforms to

$$\left(\frac{1}{r} z'\right)' = \frac{\lambda}{p} z, \tag{1.5}$$

where $z = py'$. The Neumann problem for (1.1) transforms to the Dirichlet problem for (1.5) and vice versa, and this is the basis for the methods of [10, § 3] and [1, § 4] cited earlier. The example in (E) (with $q = 0$) transforms to

(E′) there exists $r \in L_1$ so that (for $q = 0 < p$ with $1/p \in L_1$), for all Neumann eigenfunctions for (1.1), py' has infinitely many zeros in $]a_0, a_1[$.

We remark that boundary conditions not of Dirichlet or Neumann type transform to λ-dependent boundary conditions, as noted by Fleige [10, p. 31]. The oscillation theory for such problems can be obtained from [7].

2. Prüfer angle

In §§ 2 and 3 we assume that $r > 0$. First we reduce the problem to the simpler-looking one, where $r = 1$, $a_0 = 0$ and $a_1 = 1$. This may be accomplished by writing

$$t(x) = c^{-1} \int_{a_0}^{x} r, \qquad c = \int_{a_0}^{a_1} r. \tag{2.1}$$

Then (1.1) becomes

$$-\frac{\mathrm{d}}{\mathrm{d}t}\left(c^{-2}rp\frac{\mathrm{d}y}{\mathrm{d}t}\right)+r^{-1}qy=\lambda y \quad \text{on } [0,1],$$

where the coefficients $(c^{-2}rp)^{-1}$, $r^{-1}q$ and 1 are all integrable over $[0,1]$ as functions of t.

In the sequel, we shall assume the above transformation to have been carried out initially, i.e. we assume $r=1$ and $[a_0,a_1]=[0,1]=I$. Our main tool for analysing θ of (1.3) as $|\lambda|\to\infty$ is the following result for a generalization of (1.3).

THEOREM 2.1. *Let $f,g,h,k\in L_1(I)$, with $g\geqslant 0$, $h>0$ a.e. and $f>0$ on a set of positive measure. For every $\lambda>0$, let $u=u(x,\lambda)$ be a solution of*

$$u'=f-\lambda^{-1}g+(\lambda h+k)\sin^2 u$$

such that $u(0,\lambda)$ is bounded as $\lambda\to+\infty$. Then $u(1,\lambda)\to+\infty$ as $\lambda\to+\infty$.

Proof. It is easy to see that it is enough to consider the case that $u(0,\lambda)$ is constant and then we may assume $u(0,\lambda)=0$. From a standard differential inequality [13, theorem 1.10.2], $u(x,\lambda)$ is non-decreasing in λ for each fixed $x\in I$, so we can define

$$v(x)=\lim_{\lambda\to\infty}u(x,\lambda)\in\,]-\infty,\infty].$$

For any $c<d\in I$, we have

$$\begin{aligned}u(d,\lambda)-u(c,\lambda)&=\int_c^d(f-\lambda^{-1}g+(\lambda h+k)\sin^2 u)\\&\geqslant-\int_c^d(|f|+g+|k|),\end{aligned}$$

provided $\lambda\geqslant 1$. It follows that

$$v(d)\geqslant v(c)-\int_c^d(|f|+g+|k|),$$

so

$$\limsup_{s\to x-}v(s)\leqslant v(x)\leqslant\liminf_{t\to x+}v(t)\quad\text{for all } x\in[0,1]. \tag{2.2}$$

Now suppose that

$$u(1,\lambda)\text{ is bounded as }\lambda\to\infty. \tag{2.3}$$

Then v is a bounded function and

$$\lambda^{-1}u(1,\lambda)=\lambda^{-1}\int_0^1(f-\lambda^{-1}g+k\sin^2 u)+\int_0^1 h\sin^2 u,$$

so letting $\lambda\to\infty$ and using Lebesgue's dominated convergence theorem, we obtain $\int_0^1 h\sin^2 v=0$. Since $h>0$ a.e., it follows that $v(x)$ is a multiple of π a.e. Note that v is lower semicontinuous because it is the limit of a non-decreasing sequence of continuous functions. Together with (2.2), this implies that v is left-continuous. Hence

$$v(x)\text{ is a multiple of }\pi\text{ for every }x\in[0,1]. \tag{2.4}$$

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Appealing to (2.2) again, we now see that v is locally non-decreasing, i.e. for every $x \in I$, there is $\delta > 0$ such that $v(s) \leqslant v(x) \leqslant v(t)$ whenever $s, t \in I$, $x - \delta < s < x < t < x + \delta$. A standard compactness argument yields that v is non-decreasing and so v is piecewise constant on I. Choose an interval $[c, d]$ with $c < d$ so that $\int_c^d f > 0$, which is possible because f is positive on a set of positive measure. Since v is piecewise constant, we can find such an interval on which v is constant. Moreover,

$$u(d, \lambda) - u(c, \lambda) \geqslant \int_c^d (f - \lambda^{-1} g + k \sin^2 u),$$

so letting $\lambda \to \infty$ and using (2.4) and Lebesgue's dominated convergence theorem again, we obtain

$$0 = v(d) - v(c) \geqslant \int_c^d f > 0,$$

a contradiction. It follows that (2.3) must fail. □

Although we shall use the following constructions sparingly, it will be convenient to have the notation available. We write D for the set of $y \in AC(I)$ such that $py' \in AC(I)$ and the boundary conditions (1.2) are satisfied. Then $A : D \to L_1(I)$ is defined by

$$Ay = -(py')' + qy. \tag{2.5}$$

We also introduce two bilinear forms on D^2, denoted by

$$(y, z) = \int_0^1 y\bar{z} \quad \text{and} \quad a(y, z) = \int_0^1 (Ay)\bar{z}. \tag{2.6}$$

Integration by parts and (1.2) yield

$$a(y, z) = y(0)\bar{z}(0) \cot^* \alpha_0 - y(1)\bar{z}(1) \cot^* \alpha_1 + \int_0^1 (py'\bar{z}' + qy\bar{z}), \tag{2.7}$$

where $\cot^* \beta = \cot \beta$ if $\sin \beta \neq 0$ and $\cot^* \beta = 0$ if $\sin \beta = 0$.

We are now ready to prove (A) of § 1.

THEOREM 2.2. *Let p be indefinite. The eigenvalues of* (1.1), (1.2) *are all real and may be indexed in increasing order as λ_n, n any integer, where $\theta(1, \lambda_n) = n\pi + \alpha_1$. We have $\lambda_n \to +\infty$ as $n \to +\infty$ and $\lambda_n \to -\infty$ as $n \to -\infty$.*

Proof. Multiplying (1.1) by $\bar{y}$ and integrating, we obtain $a(y, y) = \lambda(y, y)$, so $\lambda = (y, y)^{-1} a(y, y)$ is real by (2.6), (2.7).

Setting $f = 1/p$, $g = 0$, $h = 1$ and $k = -q - 1/p$, we see from (1.3) and theorem 2.1 that $\theta(1, \lambda) \to +\infty$ as $\lambda \to +\infty$. Similar analysis with θ, λ, p and q multiplied by -1 shows that $\theta(1, \lambda) \to -\infty$ as $\lambda \to -\infty$. Since $\theta(1, \lambda)$ is continuous and strictly increasing in λ, by the standard theory of dependence of (1.3) on the parameter λ, the results follow directly. □

3. Oscillation theory when $r > 0$

We continue to assume $r = 1$ via the transformation (2.1) and we write $\hat{I} =]0, 1[$. Following theorem 2.2, we shall consider only real eigenfunctions. We start with

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some instructive examples that will make use of the following inequality for solutions of equations of the form (1.3).

Lemma 3.1. *Let ψ be a real solution of the differential equation*

$$\psi' = g(x) + h(x)\sin^2\psi,$$

where $g, h \in L_1(I)$. If $\psi(x_0) = 0$, then

$$\left|\psi(x) - \int_{x_0}^{x} g\right| \leqslant H\mathrm{e}^{2H} G(x)^2$$

for $x \geqslant x_0$, where

$$H = \int_0^1 |h|, \qquad G(x) = \max_{t\in[x_0,x]}\left|\int_{x_0}^{t} g\right|.$$

Proof. We have

$$\psi(x) = \int_{x_0}^{x} g + \int_{x_0}^{x} h\sin^2\psi. \tag{3.1}$$

Since $\sin^2\psi \leqslant |\sin\psi| \leqslant |\psi|$, we obtain

$$|\psi(x)| \leqslant G(x) + \int_{x_0}^{x} |h||\psi|.$$

The function $G(x)$ is monotonically non-decreasing. Hence, by Gronwall's inequality,

$$|\psi(x)| \leqslant \mathrm{e}^{H} G(x).$$

From (3.1) we now find

$$\left|\psi(x) - \int_{x_0}^{x} g\right| \leqslant \int_{x_0}^{x} |h|\psi^2 \leqslant H\mathrm{e}^{2H} G(x)^2.$$

□

Theorem 3.2. *Let*

$$\frac{1}{p(x)} = 2x\cos\frac{1}{x} + \sin\frac{1}{x}$$

and $\alpha_0 = 0$. Then, for any $q \in L_1(I)$, every eigenfunction of (1.1), (1.2) has infinitely many zeros in $\hat{I}$, accumulating at 0.

Proof. For fixed λ, we write (1.3) in the form

$$\theta' = \frac{1}{p} + h\sin^2\theta,$$

where $h = \lambda - q - 1/p \in L_1(I)$. It follows from lemma 3.1 that

$$\left|\theta(x) - x^2\cos\frac{1}{x}\right| \leqslant H\mathrm{e}^{2H} x^4, \tag{3.2}$$

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with $H = \int_0^1 |h|$. For $x_m = (m\pi)^{-1}$, where m is any positive integer, we obtain

$$|\theta(x_m) - (-1)^m (m\pi)^{-2}| \leqslant H e^{2H} (m\pi)^{-4}.$$

Thus, if $(m\pi)^2 > He^{2H}$, $(-1)^m \theta(x_m) > 0$. It follows that for any λ, θ has infinitely many zeros, accumulating at 0, and thus the same is true for any eigenfunction. □

We remark that the above example works equally well for any $r \in L_1$. It is a development of the remark in [4] that the function $f : x \to \int_0^x 1/p$ can have infinitely many zeros.

Further examples can be built from theorem 3.2, as the following shows.

Corollary 3.3. *Let* $1/p \in L_1(I)$ *and*

$$-q(x) = 2x \cos \frac{1}{x} + \sin \frac{1}{x}.$$

Then, for appropriate α_j, *(1.1), (1.2) has eigenvalue* 0 *and if* y *is the corresponding eigenfunction, then* py' *has infinitely many zeros in* $\hat{I}$.

Proof. From (1.3), $\theta' = \lambda - q + (1/p + q - \lambda)\cos^2\theta$, so if $\phi = \theta - \frac{1}{2}\pi$ and $\lambda = 0$, then

$$\phi' = \lambda - q + \left(\frac{1}{p} + q - \lambda\right)\sin^2\phi \tag{3.3}$$

$$= -q + \left(\frac{1}{p} + q\right)\sin^2\phi. \tag{3.4}$$

Since this is of the same form as (3.1), ϕ (and hence py') has infinitely many zeros in $\hat{I}$ if $\phi(0) = 0$, i.e. $\alpha_0 = \frac{1}{2}\pi$. □

We may combine the above results as follows. Let

$$\frac{1}{p(x)} = -q(x) = \beta\left(2x\cos\frac{1}{x} + \sin\frac{1}{x}\right) \quad \text{on } [0, \tfrac{1}{2}],$$

with $\lambda = 0$ and $\alpha_0 = \frac{1}{2}\pi$. Then (3.4) shows that $\phi(x) = \beta x^2 \cos(1/x)$ and we now choose β so that $\phi(\frac{1}{2}) = -\frac{1}{4}\pi$. Continuing p and q over $]\frac{1}{2}, 1]$ by their even extensions about $x = \frac{1}{2}$, we see that $\theta(\frac{1}{2}) = \frac{1}{4}\pi$, and so $\theta(1) = 0$ since $\phi(x, 0)$ is odd about $\frac{1}{2}$. Thus, if we choose $\alpha_1 = 0$, then we obtain an eigenfunction y for which both y and py' have infinitely many zeros. Although these examples make the prospect for oscillation theory look bleak, we do have the following positive result.

Theorem 3.4. *Suppose* $-\frac{1}{2}\pi < \alpha_j < \frac{1}{2}\pi$, $j = 0, 1$, *and that* y_n *is an eigenfunction of* (1.1), (1.2) *corresponding to* λ_n *of theorem 2.2.*

(i) *For each integer* n, py_n' *has at least* $|n|$ *zeros in* $\hat{I}$.

(ii) *If* $q < \lambda_{n_+}$ *a.e. for some* n_+, *then* py_n' *has exactly* n *zeros in* $\hat{I}$ *for each* $n \geqslant n_+$.

(iii) *If* $q > \lambda_{n_-}$ *a.e. for some* n_-, *then* py_n' *has exactly* $-n$ *zeros in* $\hat{I}$ *for each* $n \leqslant n_-$.

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Proof. (i) Since $\theta(x,\lambda_n)$ moves continuously from α_0 to $n\pi+\alpha_1$ as x moves from 0 to 1, θ must take each value $j\pi+\frac{1}{2}\pi$ $(0\leqslant j<n)$ at least once if $n>0$. The argument for $n<0$ is similar.

(ii) Suppose $\lambda\geqslant\lambda_{n_+}$, so $g=\lambda-q>0$ a.e. We claim that

$$\theta=\theta(\cdot,\lambda)\text{ increases through odd multiples of }\tfrac{1}{2}\pi \tag{3.5}$$

in the sense that $\theta(x_0)=(k+\frac{1}{2})\pi$ implies $\theta(x)>\theta(x_0)$ (less than $\theta(x_0)$) in a right (left) neighbourhood of x_0. This is similar to [2, theorem 8.4.1], but for completeness we give a proof. We set $\phi=\theta-(k+\frac{1}{2})\pi$. Then, applying lemma 3.1 to (3.3) (together with a corresponding inequality to the left of x_0), we obtain

$$\left|\phi(x)-\int_{x_0}^{x}g\right|\leqslant H\mathrm{e}^{2H}\left(\int_{x_0}^{x}g\right)^2$$

for all $x\in I$, where

$$H=\int_0^1\left|\frac{1}{p}-\lambda+q\right|.$$

This establishes (3.5).

Thus the argument of (i) holds with 'at least' replaced by 'precisely', for any $\lambda_n\geqslant\lambda_{n_+}$. (iii) is proved similarly. □

Remark 3.5. If $\alpha_0=-\frac{1}{2}\pi<\alpha_1<\frac{1}{2}\pi$, then the oscillation count remains unchanged for $n\geqslant 0$, but for $n<0$ in (i) and (iii), $|n|$ and $-n$ should be replaced by $-1-n$. Similarly, if $\alpha_1=-\frac{1}{2}\pi<\alpha_0<\frac{1}{2}\pi$, then the count for $n>0$ in (i) and (iii) is $n-1$ instead. Finally, for the Neumann problem $\alpha_0=-\frac{1}{2}\pi=\alpha_1$, the count must be changed for all $n\neq 0$ to $|n-1|$ in (i), $n-1$ in (ii) and $1-n$ in (iii). See [1, theorem 4.1] for the special case $q=0$, $n=\pm 1$. Moreover, the situation for $n=0$ is somewhat special, since py' can be identically zero (this occurs when q is the constant function λ_0).

Corollary 3.6. *If $q\in L_\infty(I)$, then theorem 3.4 (amended if necessary as in remark 3.5) applies whenever $\lambda_{n_+}>\|q\|_\infty$ and $\lambda_{n_-}<-\|q\|_\infty$.*

We note that this result applies to the example in corollary 3.3, where $\|q\|_\infty\leqslant 3$. Thus the oscillation counts for py_n' are quite regular for $|\lambda_n|>3$, despite the irregularity at the eigenvalue 0.

The final example of this section shows what can happen when the conditions on q in theorem 3.4 (ii) are removed.

Example 3.7. Let q in corollary 3.3 be replaced by

$$q(x)=\tfrac{1}{2}x^{-1/2}\cos x^{-1/4}+\tfrac{1}{4}x^{-3/4}\sin x^{-1/4}.$$

If $\alpha_0=\frac{1}{2}\pi$ again, then py' has infinitely many zeros in $\hat{I}$ for any eigenfunction y. To see this, we apply lemma 3.1 to (3.3) and obtain

$$|\phi(x)-\lambda x+x^{1/2}\cos x^{-1/4}|\leqslant H\mathrm{e}^{2H}(|\lambda|x+x^{1/2})^2,$$

where

$$H=\int_0^1\left|\frac{1}{p}+q-\lambda\right|,$$

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as the analogue of (3.2). If we now pick $x_m = (m\pi)^{-4}$, then we can find $m(\lambda)$ so that $(-1)^m \phi(x_m) > 0$ for all $m > m(\lambda)$. Thus ϕ, and hence py', has infinitely many zeros in $\hat{I}$ for any λ.

4. Sturm theory for $p > 0$

In this case, we can make a substitution, as in § 2, to ensure $p = 1$. Indeed,

$$t(x) = \left(\int_{a_0}^{a_1} \frac{1}{p}\right)^{-1} \int_{a_0}^{x} \frac{1}{p}$$

suffices. We remark that substitutions of this type, and the reciprocal transformation mentioned in § 1, are far from new (cf. [5]). From now on, we assume $p = 1$, $[a_0, a_1] = [0, 1] = I$ and that r is indefinite.

Reality of the eigenvalues is no longer guaranteed, and so extra preparation is needed. We first look at the behaviour of the Prüfer angle $\theta(1, \lambda)$ as $\lambda \to \pm\infty$.

LEMMA 4.1. *We have $\theta(1, \lambda) \to +\infty$ as $\lambda \to +\infty$ and as $\lambda \to -\infty$.*

Proof. Suppose $\lambda > 0$. We introduce a modified Prüfer angle $\psi = \psi(x, \lambda)$ by

$$\cos\theta = \rho\cos\psi, \qquad \lambda\sin\theta = \rho\sin\psi, \qquad |\theta(0,\lambda) - \psi(0,\lambda)| < \tfrac{1}{2}\pi, \tag{4.1}$$

and set $\chi = \psi - \frac{1}{2}\pi$. We claim $\chi(1, \lambda) \to +\infty$ as $\lambda \to +\infty$, and since $|\theta - \chi| < \pi$, this will give $\theta(1, \lambda) \to +\infty$. A simple calculation gives

$$\chi' = \lambda\sin^2\chi + (r - \lambda^{-1}q)\cos^2\chi,$$

so by the differential inequality theory [13, theorem 1.10.2], it suffices to prove the claim with $q\cos^2\chi$ replaced by $|q|$. But this follows from theorem 2.1 with $f = r$, $g = |q|$, $h = 1$ and $k = -r$. If we replace λ by $-\lambda$ and r by $-r$, we obtain the statement for $\lambda \to -\infty$. □

LEMMA 4.2. *Eigenfunctions of* (1.1), (1.2) *corresponding to finitely many distinct eigenvalues are linearly independent.*

Proof. Lemma 4.1 shows that there is a real number τ that is not an eigenvalue of (1.1), (1.2). Therefore, replacing A by $A - \tau r$ and λ by $\lambda - \tau$, we assume that

$$Ay = 0 \quad \Rightarrow \quad y = 0, \tag{4.2}$$

without loss of generality. Now suppose (λ_j, y_j) are eigenpairs of (1.1), (1.2), with the λ_j distinct for $j = 1, \ldots, \ell$, so

$$Ay_j = \lambda_j r y_j, \tag{4.3}$$

and (4.2) shows that each $\lambda_j \neq 0$.

Suppose ℓ is minimal, so that $y_1, \ldots, y_\ell$ are linearly dependent. Then

$$y_\ell = \sum_{j=1}^{\ell-1} c_j y_j$$

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for some c_j not all zero. Now, from (4.3),

$$\sum_{j=1}^{\ell-1} c_j\lambda_j^{-1}Ay_j = ry_\ell = \lambda_\ell^{-1}Ay_\ell = \sum_{j=1}^{\ell-1} c_j\lambda_\ell^{-1}Ay_j,$$

so

$$A\sum_{j=1}^{\ell-1} c_j(\lambda_j^{-1} - \lambda_\ell^{-1})y_j = 0.$$

Thus (4.2) gives

$$\sum_{j=1}^{\ell-1} c_j(\lambda_j^{-1} - \lambda_\ell^{-1})y_j = 0,$$

contradicting the minimality of ℓ. □

The following lemma on quadratic forms will be needed to prove theorem 4.4 below.

LEMMA 4.3. *Let $g \in L_1(I)$ be real valued and $c, d \in \mathbb{R}$. Define, for $f \in C^1(I)$,*

$$G(f) = c|f(0)|^2 + d|f(1)|^2 + \int_0^1 |f'|^2 + \int_0^1 g|f|^2.$$

For every infinite-dimensional linear subspace V of $C^1(I)$, there is a function $h \in V$ such that $G(h) > 0$.

Proof. Let $f \in C^1(I)$ and $\epsilon > 0$. Then

$$|f(x)|^2 \leqslant |f(t)|^2 + 2\left|\int_t^x ff'\right| \leqslant |f(t)|^2 + \epsilon^{-1}\int_0^1 |f|^2 + \epsilon\int_0^1 |f'|^2.$$

Integrating with respect to t, we obtain

$$|f(x)|^2 \leqslant (\epsilon^{-1} + 1)\int_0^1 |f|^2 + \epsilon\int_0^1 |f'|^2. \tag{4.4}$$

Hence

$$\int_0^1 g|f|^2 \geqslant -\int_0^1 |g|\left((\epsilon^{-1} + 1)\int_0^1 |f|^2 + \epsilon\int_0^1 |f'|^2\right),$$

and with this and (4.4) again we have

$$G(f) \geqslant \frac{1}{2}\int_0^1 |f'|^2 - K\int_0^1 |f|^2, \tag{4.5}$$

where $K > 0$ is a constant independent of f. The proof of the lemma is now by contradiction. If the lemma is not true, there is an infinite sequence $f_n \in C^1(I)$, $n = 1, 2, 3, \dots$, such that

$$\int_0^1 f_m\bar{f}_n = \delta_{mn}$$

and $G(f_n) \leqslant 0$ for all n. By (4.5),

$$\int_0^1 |f_n'|^2 \leqslant 2K$$

and so, for $t < x$,

$$|f_n(x) - f_n(t)| \leqslant \int_t^x |f_n'| \leqslant \sqrt{2K}\sqrt{x-t}.$$

This shows that the sequence $\{f_n\}$ is uniformly equicontinuous. By (4.4), the sequence is also uniformly bounded. By the theorem of Arzela and Ascoli, there is a subsequence of $\{f_n\}$, which we may again denote be $\{f_n\}$, such that f_n converges uniformly to a function f. Then

$$0 = \lim_{n\to\infty} \int_0^1 f_n \bar{f}_{n+1} = \int_0^1 |f|^2 = \lim_{n\to\infty} \int_0^1 |f_n|^2 = 1,$$

which is the desired contradiction. □

The next result, which is well known for more restricted coefficients, has several implications for what follows.

THEOREM 4.4. *For at most finitely many (complex) eigenvalues* λ *of* (1.1), (1.2), *the corresponding eigenfunctions* y *satisfy*

$$\lambda \int_0^1 r|y|^2 \leqslant 0.$$

Proof. Let (λ_j, y_j), $j \in J$, be the set of eigenpairs of (1.1), (1.2) for which

$$\lambda_j \int_0^1 r|y_j|^2 \leqslant 0.$$

From (4.3), we obtain, in a well-known way,

$$a(y_j, y_\ell) = 0 \quad \text{for } j, \ell \in J, \quad j \neq \ell$$

and

$$a(y_j, y_j) = \lambda_j \int_0^1 r|y_j|^2 \leqslant 0 \quad \text{for } j \in J. \tag{4.6}$$

Hence the quadratic form $a(f,f)$ is non-positive on the linear span V of the eigenvectors y_j with $j \in J$. By (2.7), $a(f,f)$ is of the form considered in lemma 4.3. Hence we conclude that V is finite dimensional. By lemma 4.2, we find that J is a finite set. □

COROLLARY 4.5. *At most finitely many eigenvalues of* (1.1), (1.2) *are non-real.*

Proof. Standard manipulations (cf. (4.6)) give

$$a(y, y) = \lambda \int_0^1 r|y|^2 = 0$$

for any eigenpair (λ, y) with $\lambda \notin \mathbb{R}$. Now apply theorem 4.4. □

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We turn now to oscillation theory for real eigenvalues. Example 3.7, which holds regardless of the sign of p, shows that we cannot count zeros of py' in general. Hence, for a real eigenpair (λ, y) of (1.1), (1.2), we shall define the oscillation count $\omega(\lambda)$ as the number of zeros of y in $\hat{I}$. We claim that this number is finite. We choose the angle α_0 in the range $[0,\pi[$ and α_1 in the range $]0,\pi]$. If λ is a real eigenvalue of (1.1), (1.2), then there is an integer n such that $\theta(1,\lambda) = n\pi + \alpha_1$. Now an application of lemma 3.1 to (1.3) shows that, for fixed λ, $\theta(x,\lambda)$ increases through multiples of π. By the choice of α_0, α_1, we see that $n = \omega(\lambda)$, justifying our claim. In particular, n is non-negative.

THEOREM 4.6.

(i) *The real eigenvalues can be listed,*

$$\cdots < \lambda_2^- < \lambda_1^- < \lambda_0^- < 0 \leqslant \lambda_0^+ < \lambda_1^+ < \lambda_2^+ \ldots,$$

where $\lambda_n^\pm \to \pm\infty$ *as* $n \to \infty$.

(ii) *Let M be the minimal oscillation count of the eigenvalues of* (1.1), (1.2). *For every integer $n > M$, at least two eigenvalues λ exist with $\omega(\lambda) = n$.*

Proof. (i) Lemma 4.1 establishes the existence of an infinite sequence of eigenvalues accumulating at both $\pm\infty$. The discreteness of the spectrum follows from analyticity of $\theta(1,\lambda)$.

(ii) Continuity and the asymptotics of $\theta(1,\lambda)$ from lemma 4.1 show that $\theta(1,\lambda)$ attains a minimum value, say $\theta(1,\lambda_0)$. The result follows because M is the smallest integer satisfying $M\pi + \alpha_1 \geqslant \theta(1,\lambda_0)$. □

Since $\theta(1,\lambda)$ can be non-monotonic on the intervals $]-\infty,\lambda_0[$ and $]\lambda_0,\infty[$, there is no obvious analogue of theorem 2.2 via Prüfer methods. It is possible, however, to give an analogue for large enough eigenvalues via theorem 4.4.

THEOREM 4.7. *There exist $n_\pm$ so that $\omega(\lambda_{n+1}^+) = \omega(\lambda_n^+) + 1$ whenever $n \geqslant n_+$ and $\omega(\lambda_{n+1}^-) = \omega(\lambda_n^-) + 1$ whenever $n \geqslant n_-$.*

Proof. It suffices to consider n_+. We first compute the derivative θ_λ of the function θ with respect to λ. Let $y(x) = y(x,\lambda)$ be the solution of (1.1), (1.2), with $y(0) = \sin\alpha_0$, $y'(0) = \cos\alpha_0$. Then $(\sec^2\theta)\theta_\lambda = (y')^{-2}d$, where $d = y'y_\lambda - yy'_\lambda$, so

$$\theta_\lambda = (y^2 + {y'}^2)^{-1}d. \tag{4.7}$$

Now $-y''_\lambda = (\lambda r - q)y_\lambda + ry$, so with (1.1) we obtain

$$d' = y''y_\lambda - yy''_\lambda = ry^2.$$

Since $y_\lambda(0) = y'_\lambda(0) = 0$, $d(0) = 0$, so

$$d(1) = \int_0^1 ry^2$$

and (4.7) gives

$$\theta_\lambda(1) = (y(1)^2 + y'(1)^2)^{-1}\int_0^1 ry^2. \tag{4.8}$$

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Thus, by theorem 4.4, all but finitely many $\theta_\lambda(1, \lambda_n^+)$ are positive, which implies the desired result. □

In the 'left definite' case, one can take $n_- = n_+ = 0$.

COROLLARY 4.8. *Suppose $q \geqslant 0$ and $0 \leqslant \alpha_0 \leqslant \frac{1}{2}\pi \leqslant \alpha_1 \leqslant \pi$. Then $\omega(\lambda_n^+) = n$. In the 'exceptional case', $q = 0$ and $\alpha_0 = \frac{1}{2}\pi = \alpha_1$, $\omega(\lambda_n^-) = n + 1$; otherwise, $\omega(\lambda_n^-) = n$.*

Proof. For an eigenpair (λ, y) of (1.1), (1.2), we have

$$a(y, y) = \lambda \int_0^1 ry^2 \tag{4.9}$$

(cf. (4.6)). From (2.7), we obtain $a(y, y) \geqslant 0$, with equality precisely in the exceptional case when $\lambda = 0$ and y is a constant. Thus, by (4.8), $\lambda\theta_\lambda(1) > 0$, provided $\lambda \neq 0$. This proves that $n_- = n_+ = 0$. In the exceptional case, $\lambda_0^+ = 0$ and $\omega(\lambda_0^+) = 0$. In the other case, we set $\lambda = 0$, which is not an eigenvalue by (4.9). By equation (1.3), $\theta' \leqslant \cos^2\theta$. Since $\alpha_0 \in [0, \frac{1}{2}\pi]$, this implies $\theta(1, 0) \leqslant \frac{1}{2}\pi$, with equality only if $q = 0$ and $\alpha_0 = \frac{1}{2}\pi$. Hence $\theta(1, 0) < \alpha_1$, and therefore the minimal oscillation count M is zero. In each case, then, ω increases by one for each successive eigenvalue away from 0. □

4.1. Remarks

(i) Most of the results of this section are well known under additional conditions. For continuous coefficients, see Haupt [11] and the papers of Richardson (1910–1918) cited in [8]. For $r \in L_\infty$, two-parameter eigencurve methods in L_2 suffice [6]. If r is a.e. non-zero, then one can use an appropriate restriction of $r^{-1}A$ (cf. [9]). See [1] for lemma 4.2 when r is a.e. non-zero on an open set.

(ii) In special cases, theorem 4.4 has been proved using the minimum–maximum principle, or some other means based on the completeness of eigenfunctions of the operator A. For example, Haupt [11] uses Green's function for $Ay = \mu y$. Our proof based on lemma 4.3 is more elementary.

(iii) For the conditions used here, theorem 4.6 follows from the asymptotics of [4], which is a sophisticated analysis of an equation like (1.3). Our method is simpler, but we obtain no asymptotics.

(iv) Equation (4.8) can be found in [3]. It would be interesting to prove theorems 4.4 and 4.7 entirely by Prüfer angle techniques. Such a proof is possible (via ψ of (4.1)) for corollary 4.8 (cf. [7]).

(vi) An alternative proof of theorem 4.7 can be derived via eigencurves of the two-parameter equation

$$Ay = \lambda ry - \mu y, \tag{4.10}$$

to which theorem 2.2 applies for each real λ. Let $\mu_n(\lambda)$, $y_n(\lambda)$ be a real eigenpair for (4.10) with $\omega(\mu_n(\lambda)) = n$. By a standard result (see [8, eqn (2.5)]),

$$\mu_n'(\lambda) = \int_0^1 ry_n(\lambda)^2 \bigg/ \int_0^1 y_n(\lambda)^2. \tag{4.11}$$

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Thus, by theorem 4.4, all but finitely many $\mu_n'(\lambda_n^+)$ are positive, so all but finitely many eigenvalue functions μ_n have precisely one zero (i.e. λ_n^+). The result now follows because $\omega(\mu_{n+1}(\lambda)) = \omega(\mu_n(\lambda)) + 1$.

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