Proceedings of the Royal Society of Edinburgh, **134A**, 1041–1083, *2004*

Homogenization of the anisotropic heterogeneous linearized elasticity system in thin reticulated structures

J. Casado-Díaz and **M. Luna-Laynez**
Dpto. de Ecuaciones Diferenciales y Análisis Numérico,
Universidad de Sevilla, Fac. de Matemáticas, C/Tarfia s/n,
41012 Sevilla, Spain (jcasadod@us.es; mllaynez@us.es)

(MS received 12 November 2003; accepted 17 June 2004)

The aim of this paper is to study the asymptotic behaviour of the solutions of the linearized elasticity system, posed on thin reticulated structures involving several small parameters. We show that this behaviour depends on the relative size of the parameters. In each case, we obtain a limit system where the microstructure and macrostructure appear simultaneously. From it, we get a suitable approximation in L^2 of the displacements and the linearized strain tensor.

1. Introduction

In a previous paper (see [11]), we introduced a new method to study the asymptotic behaviour of the solutions of partial differential problems posed on thin reticulated structures, Ω_ε, depending on several small parameters. The method is an original adaptation of the Arbogast–Douglas–Hornung method in homogenization presented in [4] (see also [8] and [9] for other extensions). It is closely related to the two-scale convergence of Nguetseng and Allaire [1, 2, 20, 23]. The idea is to introduce an adequate change of variables that transforms Ω_ε in a fixed domain, depending on both the microscopic and the macroscopic variables. In [11], we studied the case of diffusion problems. In the present paper we consider the elasticity system. A simplified problem in dimension two has been considered in [10]. Here, we deal with two particular structures. The first one is shaped by the union of orthogonal beams, with thickness $\varepsilon d_\varepsilon$, disposed periodically, along all the directions, with period ε (see figures 1 and 2 for the two-dimensional case and figure 3 for the three-dimensional one). The second structure is obtained by taking the previous one in dimension two and adding oblique parallel bars (see figure 5) with cross-section $\varepsilon d_\varepsilon$. Here, ε and d_ε are two positive parameters that tend to zero. As in [21], the method applies to more general situations (bars not completely crossing the structure, plates instead of beams, tall structures, gridworks, etc.), but we prefer to consider the two reticulated structures mentioned above to simplify the exposition.

On both structures Ω_ε, we pose the elasticity problem

$$\left.\begin{aligned} -\operatorname{div}(A_\varepsilon e(u_\varepsilon) - H_\varepsilon) &= F_\varepsilon \quad \text{in } \Omega_\varepsilon,\\ u_\varepsilon &= 0 \quad \text{on } \Gamma_\varepsilon,\\ (A_\varepsilon e(u_\varepsilon) - H_\varepsilon)\nu_\varepsilon &= 0 \quad \text{on } \partial\Omega_\varepsilon \setminus \Gamma_\varepsilon, \end{aligned}\right\} \tag{1.1}$$

https://doi.org/10.1017/S0308210500003620 Published online by Cambridge University Press

where Γ_ε is the outside boundary of Ω_ε (see § 2) and ν_ε is the unit outward normal to Ω_ε. The exact hypotheses on A_ε, H_ε and F_ε are given in § 5. They allow us to consider materials that are non-homogeneous in microstructure and macrostructure (see remark 2.4) in each beam, both in the direction of its axis and in the transverse direction to it. For example, we can assume that the different bars that shape Ω_ε are made up of a core of a material surrounded by a different one. To our knowledge, other methods that deal with thin reticulated structures (see, for example, [5, 7, 17, 30]) do not allow us to consider this type of heterogeneity. Moreover, contrary to other related works, we do not assume any isotropy hypotheses on the elastic material that composes the structure. In particular, we do not suppose that it is orthotropic. In fact, the only assumption we make for the symmetry of the elasticity tensor A_ε is that it transforms the space of symmetric matrices onto itself. For a unique bar, these general hypotheses on A_ε were considered in [22].

In this paper we show that the asymptotic behaviour of the solutions u_ε of (1.1) is different according to the limit, ϑ, of $\varepsilon/d_\varepsilon$. There are three different situations, depending on whether ϑ is zero, a positive number or infinity. Our method allows us to study all the cases simultaneously. In particular, when $\vartheta = +\infty$, we prove that the deformations and the linearized strain tensor tend to infinity. Since the linearized elasticity model assumes small deformations, this shows that it can fail in this situation. In theorems 2.5 and 3.1, we give, for each value of ϑ, a strong approximation in $L^2(\Omega_\varepsilon)$ of u_ε and $e(u_\varepsilon)$ (corrector result). As in the classical two-scale convergence (see [1, 23]), these approximations are obtained by solving a partial differential system that contains all the scales together (see (2.12) and (3.2)). Contrary to other approaches, this result does not suppose any additional smoothness for the solutions of this system. Assuming them, we get, in fact, an asymptotic development z_ε of u_ε such that

$$\frac{d_\varepsilon^2}{(\varepsilon + d_\varepsilon)^2} \frac{1}{|\Omega_\varepsilon|} \int_{\Omega_\varepsilon} |e(u_\varepsilon - z_\varepsilon)|^2 \,\mathrm{d}x \to 0. \tag{1.2}$$

In order to prove these results, we first obtain a compactness theorem, which corresponds to the compactness theorem in the usual two-scale theory. It also applies to nonlinear problems, although, for the sake of simplicity, we prefer to remain in the linear case.

For the first structure considered in this paper (figures 1–3), assuming stronger homogeneity and isotropy hypotheses on the structure, it has been shown (see [5, 17]), by another method that passes to the limit first in ε and then in d_ε (which assumes that ε is much smaller than d_ε), that the limit problem of (1.1) is degenerate. This property of the limit problem seems to be the reason why the asymptotic behaviour of the solutions depends on the limit ϑ of $\varepsilon/d_\varepsilon$. However, we emphasize that the homogenization result for the second structure (figure 5) also depends on the value ϑ, although, in this case, passing to the limit first in ε and then in d_ε, it was proved in [12] that (1.1) has a limit problem that is non-degenerate. Our results also give this non-degenerate limit problem if $\vartheta = 0$, but, in the other cases, we prove that the limit problem is still degenerate (see remark 3.5).

We finish this introduction with some bibliographic notes. For the study of the elasticity system in thin domains, we refer, for example, to [13, 14, 19, 22, 27], where the structures considered are composed of a small number of elements. In the case of thin reticulated structures involving several small parameters, the classical method, to our knowledge, for dealing with this problem consists of passing to the limit first in one parameter, then in another one, and so on (see [5, 12, 15–18, 25, 26]). For diffusion equations, this approach provides, as in our case, the first three terms of the asymptotic development of the solutions of the corresponding problem, together with a corrector result (see [5]). However, as far as we know, this has not been carried out for the elasticity problem. In fact, the above-mentioned papers do not give a result, proving that the solution u_ε of (1.1) is close, in some sense, to the solution of the problem obtained by passing to the limit successively in the different parameters (i.e. it is not clear if the iterated limit is really a double limit). As we pointed out above, the limit behaviour depends on the ratio of ε to d_ε. However, passing to the limit successively in the parameters we are assuming that one of them is much smaller than the other ones. We mention that the fact that the limit problem depends on the chosen order in the parameters was first proved in [18].

Another approach used to study this type of problem is based on the two-scale method with respect to measures (see [6, 7, 28–30]). This method allows us to deal with very general structures. However, when it has been applied to the elasticity system, it has only given partial results. So, in [7], the unique case considered is $\vartheta = 0$, whereas in [28] the cases studied are $\vartheta = 0$ and $\vartheta = +\infty$. In this last work, an additional term is introduced to the equation, and thus the problem under study is not exactly the elasticity system; this additional term simplifies the problem because it avoids the estimation of the Korn constant in Ω_ε and the possibility of unbounded displacements. We also remark that these articles do not provide an asymptotic development z_ε of u_ε such that (1.2) holds. In fact, they only give the two-scale limit of the solutions u_ε of (1.1) and not of $e(u_\varepsilon)$, and, unlike our result (1.2), they do not provide any estimation of the error in an usual norm, such as the L^2 or H^1 norms. Both in [28] and [7], the case when the limit ϑ of $\varepsilon/d_\varepsilon$ is arbitrary is explicitly mentioned as an open problem.

2. A model structure: homogenization result

This section is devoted to the asymptotic analysis of the solutions of the linearized elasticity system (1.1) posed on a model sequence of reticulated structures in $\mathbb{R}^N$, $N \geqslant 2$. These structures are shaped by orthogonal thin bars disposed periodically along all the directions of the space. We assume that the bars are made of an anisotropic non-homogeneous elastic material. Both the size of the period and the ratio of the thickness of the bars over the period tend to zero. Mathematically, the problem can be formulated as follows.

We denote by $\{e_1, \dots, e_N\}$ the usual basis in $\mathbb{R}^N$. For $i \in \{1, \dots, N\}$ and $\zeta = (\zeta_1, \dots, \zeta_N) \in \mathbb{R}^N$, we write

$$\zeta_i' = \sum_{m \neq i} \zeta_m e_m,$$

so $\zeta = \zeta_i e_i + \zeta_i'$. We also denote by ζ_i' a generic point in $\mathbb{R}^N$ such that its ith coordinate is zero. Confusion is avoided by the context.

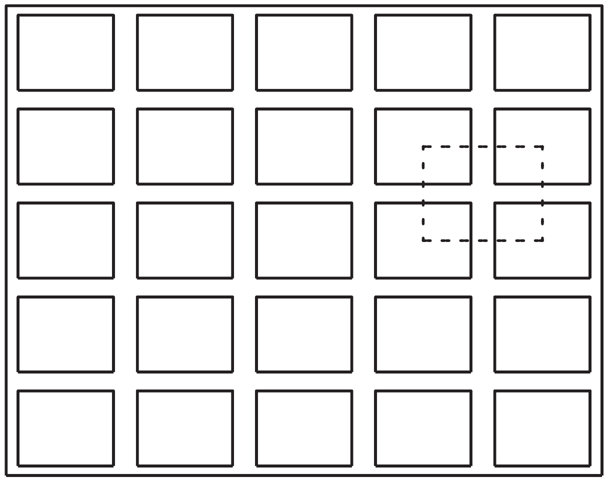

Figure 1.

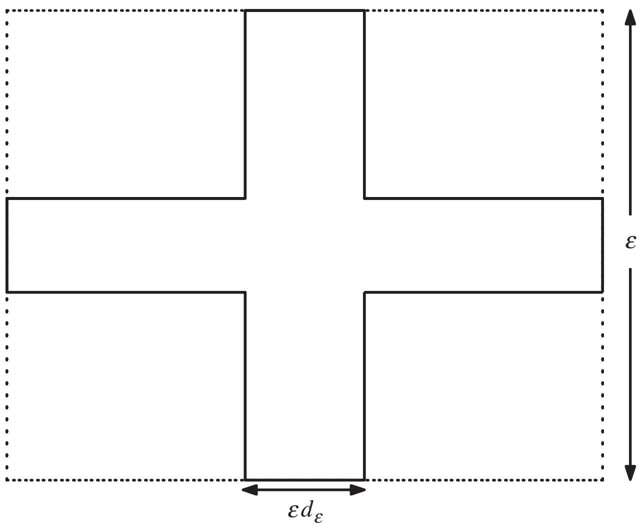

Figure 2.

For $\varepsilon > 0$, let $d_\varepsilon \in (0, 1)$ be a sequence that tends to zero as ε goes to zero. For $i \in \{1, \dots, N\}$, $k'_i \in \mathbb{Z}^N$ and $\varepsilon > 0$, L_ε^{i,k'_i} is the unbounded beam given by

$$L_\varepsilon^{i,k'_i} = \{x \in \mathbb{R}^N : |x'_i - \varepsilon k'_i|_\infty < \tfrac{1}{2}\varepsilon d_\varepsilon\}.$$

Then we define the open reticulated structure V_ε, $\varepsilon > 0$ (see figures 1 and 2 for $N = 2$ and figure 3 for $N = 3$), as

$$V_\varepsilon = \bigcup_{i=1}^{N} V_\varepsilon^i \quad \text{with } V_\varepsilon^i = \bigcup_{k'_i \in \mathbb{Z}^N} L_\varepsilon^{i,k'_i} \quad \forall i \in \{1, \dots, N\} \quad \forall \varepsilon > 0.$$

The intersection of the sets V_ε^i, $i \in \{1, \dots, N\}$, is denoted by ω_ε. For a smooth

https://doi.org/10.1017/S0308210500003620 Published online by Cambridge University Press

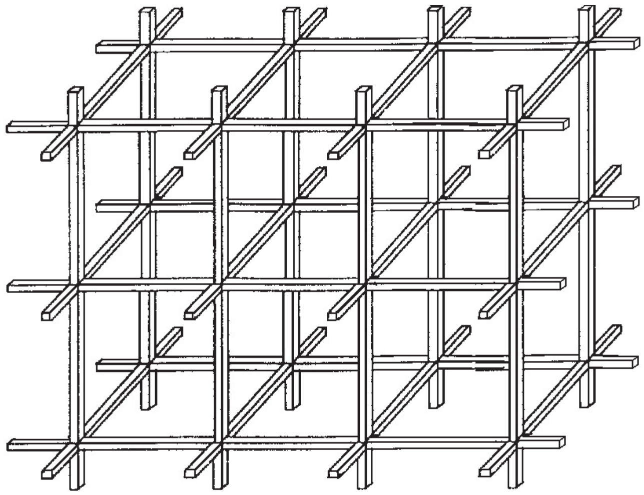

Figure 3.

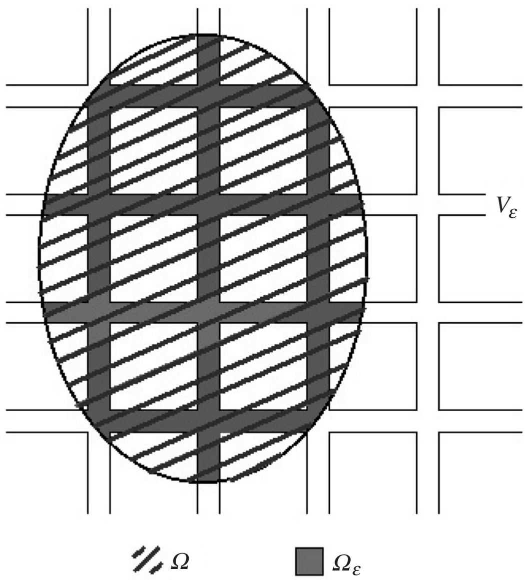

Figure 4.

bounded open set $\Omega \subset \mathbb{R}^N$, we define Ω_ε (see figure 4), Ω_ε^i and Γ_ε by

$$\Omega_\varepsilon = \Omega \cap V_\varepsilon, \quad \Omega_\varepsilon^i = \Omega \cap V_\varepsilon^i \qquad \forall i \in \{1, \dots, N\},$$
$$\Gamma_\varepsilon = \bar{\Omega}_\varepsilon \cap \partial\Omega \quad \forall \varepsilon > 0.$$

We denote by $H^1_{\Gamma_\varepsilon}(\Omega_\varepsilon)$ the functional space

$$H^1_{\Gamma_\varepsilon}(\Omega_\varepsilon) = \{u \in H^1(\Omega_\varepsilon) : u = 0 \text{ on } \Gamma_\varepsilon\}, \quad \varepsilon > 0.$$

We suppose that the elements of $H^1_{\Gamma_\varepsilon}(\Omega_\varepsilon)$ are defined on all V_ε by extending them by zero outside Ω_ε.

https://doi.org/10.1017/S0308210500003620 Published online by Cambridge University Press

Denote by $\mathcal{S}_N$ the space of N-dimensional symmetric matrices and by $\mathcal{L}(\mathcal{S}_N, \mathcal{S}_N)$ the space of linear maps of $\mathcal{S}_N$ into itself.

Let G be an open set of $\mathbb{R}^N$ and let $\phi : G \to \mathbb{R}^N$ be sufficiently smooth. We denote by $e(\phi) : G \to \mathcal{S}_N$ the symmetrized gradient (linearized strain tensor) of ϕ, i.e.

$$e(\phi)_{ij} = \tfrac{1}{2}(\partial_{x_i}\phi_j + \partial_{x_j}\phi_i) \quad \forall i, j \in \{1, \dots, N\}.$$

We denote by ν the unit outward normal to $\partial\Omega$ and by ν_ε, $\varepsilon > 0$, the unit outward normal to $\partial\Omega_\varepsilon$.

We consider $H_\varepsilon \in L^2(\Omega_\varepsilon; \mathcal{S}_N)$, $F_\varepsilon \in L^2(\Omega_\varepsilon)^N$ and $A_\varepsilon \in L^\infty(\Omega_\varepsilon; \mathcal{L}(\mathcal{S}_N, \mathcal{S}_N))$ such that there exist $\alpha, \beta > 0$ with (the exact hypotheses on F_ε, H_ε and A_ε are given below)

$$A_\varepsilon(x)\mathcal{M} : \mathcal{M} \geqslant \alpha|\mathcal{M}|^2, \quad |A_\varepsilon(x)\mathcal{M}| \leqslant \beta|\mathcal{M}| \quad \forall \mathcal{M} \in \mathcal{S}_N \quad \text{a.e. } x \in \Omega_\varepsilon \quad \forall \varepsilon > 0. \tag{2.1}$$

In the reticulated domain Ω_ε, let us consider the elasticity problem (1.1). It is well known that this problem has an unique solution u_ε in $H^1_{\Gamma_\varepsilon}(\Omega_\varepsilon)^N$ (see, for example, [14]). Our aim in this section is to describe the asymptotic behaviour of the solution u_ε and to give a corrector result for $e(u_\varepsilon)$ as ε tends to zero. The result exhibits three different regimes, depending on whether $\vartheta = \lim_{\varepsilon\to 0}(\varepsilon/d_\varepsilon)$ is zero, a positive number or infinity. In order to solve the homogenization problem and to express the result, we introduce some notations and definitions.

We set $Y = (-\frac{1}{2}, \frac{1}{2})$. For $i \in \{1, \dots, N\}$, we decompose Y^N as $Y^N = J^i + S^i$, where

$$J^i = \{y_i e_i : y_i \in Y\}, \qquad S^i = \{y \in Y^N : y_i = 0\} = \{y'_i \in Y^N\}.$$

Note that we can consider J^i and S^i as subsets of $\mathbb{R}$ and $\mathbb{R}^{N-1}$, respectively, identifying J^i with Y and S^i with Y^{N-1}.

For $\varepsilon > 0$ and $k \in \mathbb{Z}^N$, we define C^k_ε as the cube with centre εk and sides of length ε parallel to the coordinate axes, i.e. $C^k_\varepsilon = \varepsilon(k + Y^N)$. We also define P^k_ε as the cube with centre εk and sides of length $\varepsilon d_\varepsilon$ parallel to the coordinate axes. Thus $P^k_\varepsilon = C^k_\varepsilon \cap \omega_\varepsilon$. For $i \in \{1, \dots, N\}$, we write $B^{i,k}_\varepsilon = C^k_\varepsilon \cap V^i_\varepsilon$. We remark that $P^k_\varepsilon = B^{i,k}_\varepsilon \cap B^{j,k}_\varepsilon$ for every $k \in \mathbb{Z}^N$, every $\varepsilon > 0$ and every $i, j \in \{1, \dots, N\}$.

We define $\kappa : \mathbb{R}^N \to \mathbb{Z}^N$ by $\kappa(x) \in \mathbb{Z}^N$ and $x \in C^{\kappa(x)}_1$ for a.e. $x \in \mathbb{R}^N$. Then we set $C_\varepsilon(x) = C^{\kappa(x/\varepsilon)}_\varepsilon$ and $P_\varepsilon(x) = P^{\kappa(x/\varepsilon)}_\varepsilon$ a.e. $x \in \mathbb{R}^N$, $\varepsilon > 0$. We remark that x belongs to $C_\varepsilon(x)$ for every $\varepsilon > 0$ and a.e. $x \in \mathbb{R}^N$.

We now introduce suitable changes of variables, which transform each V^i_ε, $i \in \{1, \dots, N\}$, in a fixed domain, which depends on the microstructure and macrostructure. For $\varepsilon > 0$ and $i \in \{1, \dots, N\}$, we define $y^i_\varepsilon : V^i_\varepsilon \to Y^N$ by

$$y^i_\varepsilon(x) = \frac{x_i - \varepsilon\kappa_i(x/\varepsilon)}{\varepsilon} e_i + \frac{x'_i - \varepsilon\kappa'_i(x/\varepsilon)}{\varepsilon d_\varepsilon} \quad \text{a.e. } x \in V^i_\varepsilon. \tag{2.2}$$

We point out that, for fixed $k \in \mathbb{Z}^N$, $y^i_{\varepsilon_{|B^{i,k}_\varepsilon}}$ transforms $B^{i,k}_\varepsilon$ onto Y^N, for every $\varepsilon > 0$ and every $i \in \{1, \dots, N\}$.

For a sequence of measurable functions $u_\varepsilon : V_\varepsilon \to \mathbb{R}^N$ and $i \in \{1, \dots, N\}$, we define $\hat{u}^i_\varepsilon : \mathbb{R}^N \times Y^N \to \mathbb{R}^N$ by

$$\hat{u}^i_\varepsilon(x, y) = u_\varepsilon\bigg(\varepsilon\kappa\bigg(\frac{x}{\varepsilon}\bigg) + \varepsilon y_i e_i + \varepsilon d_\varepsilon y'_i\bigg), \quad \varepsilon > 0. \tag{2.3}$$

Remark 2.1. We will use the functions $\hat{u}^i_\varepsilon$ to study the behaviour of u_ε in V^i_ε. Observe that, in $C^k_\varepsilon \times Y^N$, $k \in \mathbb{Z}^N$, $\varepsilon > 0$, $\hat{u}^i_\varepsilon(x, y)$ does not depend on the macroscopic variable x, and, as a function of the microscopic variable y, it is obtained from u_ε by the change of variables (2.2). So the variable x determines $k \in \mathbb{Z}^N$ such that x belongs to C^k_ε (i.e. $k = \kappa(x/\varepsilon)$), and then the variable y acts as a microscope, zooming the small beam $B^{i,k}_\varepsilon$ onto the fixed set Y^N. Therefore, the behaviour of u_ε in the small beam $B^{i,k}_\varepsilon$ can be deduced from the behaviour of $\hat{u}^i_\varepsilon$ with respect to the variable $y \in Y^N$.

The following definition will be useful when dealing with the change of variables (2.2). For $\varepsilon > 0$, $i \in \{1, \dots, N\}$ and $\hat{v} \in L^2(\mathbb{R}^N; H^1(Y^N))^N$, we define $e^i_\varepsilon(\hat{v}) \in L^2(\mathbb{R}^N \times Y^N; \mathcal{S}_N)$ by

$$e^i_\varepsilon(\hat{v})_{ii} = \frac{1}{\varepsilon}\partial_{y_i}\hat{v}_i, \quad 2e^i_\varepsilon(\hat{v})_{im} = \frac{1}{\varepsilon}\partial_{y_i}\hat{v}_m + \frac{1}{\varepsilon d_\varepsilon}\partial_{y_m}\hat{v}_i \quad \forall m \neq i,$$
$$2e^i_\varepsilon(\hat{v})_{mn} = \frac{1}{\varepsilon d_\varepsilon}(\partial_{y_n}\hat{v}_m + \partial_{y_m}\hat{v}_n) \quad \forall m, n \in \{1, \dots, N\} \setminus \{i\}.$$

Note that $e^i_\varepsilon(\hat{u}^i_\varepsilon)$ gives the strain tensor $e(u_\varepsilon)$ expressed in the variables $y = y^i_\varepsilon(x)$.

For a sufficiently smooth function $\phi = \phi(x, y)$ defined on $\mathbb{R}^N \times Y^N$, we write

$$e_y(\phi)_{ij} = \tfrac{1}{2}(\partial_{y_i}\phi_j + \partial_{y_j}\phi_i) \quad \forall i, j \in \{1, \dots, N\}.$$

For $i \in \{1, \dots, N\}$, we define the functional space $E^i = E^i_0 \times E^i_1 \times E^i_2 \times E^i_3$, with

$$E^i_0 = \{\hat{u}^i_0 \in L^2(\Omega) : \partial_{x_i}\hat{u}^i_0 \in L^2(\Omega),\ \hat{u}^i_0\nu_i = 0 \text{ on } \partial\Omega\},$$
$$\begin{aligned} E^i_1 = \{\hat{u}^i_1 \in L^2(\Omega; H^1(Y^N))^N : \hat{u}^i_1 \text{ is } y_i\text{-periodic}, \\ \hat{u}^i_{1,m}(x, z) = 0 \text{ a.e. } (x, z) \in \Omega \times S^i, \\ e_y(\hat{u}^i_1)_{in} = e_y(\hat{u}^i_1)_{mn} = 0 \\ \forall m, n \in \{1, \dots, N\} \setminus \{i\}\}, \end{aligned}$$
$$\begin{aligned} E^i_2 = \bigg\{\hat{u}^i_2 \in L^2(\Omega \times Y^N)^N : \hat{u}^i_{2,i} \in L^2(\Omega \times J^i; H^1(S^i)), \\ \hat{u}^i_{2,m} \in L^2(\Omega; H^1(Y^N)),\ \hat{u}^i_{2,m} \text{ is } y_i\text{-periodic}, \\ \int_{S^i}\hat{u}^i_{2,m}\,\mathrm{d}y'_i = 0 \text{ in } L^2(\Omega \times J^i),\ e_y(\hat{u}^i_2)_{nm} = 0 \\ \forall m, n \in \{1, \dots, N\} \setminus \{i\}\bigg\}, \end{aligned}$$
$$E^i_3 = \{\hat{u}^i_3 \in L^2(\Omega \times J^i; H^1(S^i))^N : \hat{u}^i_{3,i} = 0\}.$$

In these expressions, $\hat{u}^i_{1,m}$, $\hat{u}^i_{2,m}$ and $\hat{u}^i_{3,m}$ denote the mth component of $\hat{u}^i_1$, $\hat{u}^i_2$ and $\hat{u}^i_3$, respectively. The superscript 'i' means that these functional spaces will be used to describe the asymptotic behaviour of the solutions of (1.1) in Ω^i_ε. These conventions will be used throughout the paper.

Remark 2.2. The function $\hat{u}^i_1$ belongs to E^i_1 if and only if there exist

$$a^i \in L^2(\Omega; H^1(J^i)), \quad b^i_m \in L^2(\Omega; H^2(J^i)), \quad m \neq i,$$

such that a^i, b^i_m, $\partial_{y_i} b^i_m$ are y_i-periodic, $b^i_m(x,0)=0$ for a.e. $x\in\Omega$ and they satisfy

$$\hat u^i_{1,i}(x,y)=a^i(x,y_i)-\sum_{n\neq i}\partial_{y_i}b^i_n(x,y_i)y_n,$$

$$\hat u^i_{1,m}(x,y)=b^i_m(x,y_i)\quad \forall m\neq i$$

a.e. $(x,y)\in\Omega\times Y^N$.

Remark 2.3. The function $\hat u^i_2$ belongs to E^i_2 if and only if $\hat u^i_{2,i}$ belongs to $L^2(\Omega\times J^i;H^1(S^i))$ and, for every $m,n\in\{1,\dots,N\}\setminus\{i\}$, there exists $g^i_{mn}\in L^2(\Omega;H^1(J^i))$ such that g^i_{mn} is y_i-periodic, $g^i_{mn}=-g^i_{nm}$ and

$$\hat u^i_{2,m}(x,y)=\sum_{n\neq i}g^i_{mn}(x,y_i)y_n\quad \forall m\neq i\quad \text{a.e. } (x,y)\in\Omega\times Y^N.$$

For $(\hat u^i_0,\hat u^i_1,\hat u^i_2,\hat u^i_3)\in E^i$, we define $e^i_0(\hat u^i_0,\hat u^i_1,\hat u^i_2,\hat u^i_3)\in L^2(\mathbb{R}^N\times Y^N;\mathcal{S}_N)$ by

$$\left.\begin{aligned} e^i_0(\hat u^i_0,\hat u^i_1,\hat u^i_2,\hat u^i_3)_{ii}&=\partial_{x_i}\hat u^i_0+e_y(\hat u^i_1)_{ii},\\ e^i_0(\hat u^i_0,\hat u^i_1,\hat u^i_2,\hat u^i_3)_{in}&=e_y(\hat u^i_2)_{in},\\ e^i_0(\hat u^i_0,\hat u^i_1,\hat u^i_2,\hat u^i_3)_{mn}&=e_y(\hat u^i_3)_{mn}\end{aligned}\right\}\tag{2.4}$$

$\forall m,n\in\{1,\dots,N\}\setminus\{i\}$.

We also introduce $\mathcal{E}^i$ as the subspace of E^i defined by

$$\begin{aligned}\mathcal{E}^i=\{(\hat u^i_0,\hat u^i_1,\hat u^i_2,\hat u^i_3)\in E^i:\ &\hat u^i_0\in C^\infty_0(\Omega),\ \hat u^i_1\in C^\infty_0(\Omega;C^\infty(Y^N))^N,\\ &\hat u^i_{2,i}\in C^\infty_0(\Omega\times J^i;C^\infty(S^i)),\\ &\hat u^i_{2,m}\in C^\infty_0(\Omega;C^\infty(Y^N))\ \forall m\neq i,\\ &\hat u^i_3\in C^\infty_0(\Omega\times J^i;C^\infty(S^i))^N\\ &\text{and } \exists\delta>0 \text{ such that } \hat u^i_1=\hat u^i_2=\hat u^i_3=0\\ &\qquad\text{if } |y_i|<\tfrac12\delta\}.\end{aligned}$$

Let us now introduce the exact hypotheses that we are going to consider on F_ε, H_ε and A_ε. For every $i\in\{1,\dots,N\}$, we suppose that there exist

$$F^i:\Omega\times Y^N\to\mathbb{R}^N,\qquad H^i:\Omega\times Y^N\to\mathcal{S}_N,\qquad A^i\in L^\infty(\Omega\times Y^N;\mathcal{L}(\mathcal{S}_N,\mathcal{S}_N)),$$

$f^i,h^i\in L^2(Y^N)$ and $\rho^i\in C^0([0,+\infty))$ with $\rho^i(0)=0$ such that

$$F^i(\cdot,y),\ H^i(\cdot,y) \text{ are continuous in } \Omega\quad \text{a.e. } y\in Y^N,\tag{2.5}$$

$$|F^i(x,y)|\leqslant f^i(y),\quad |H^i(x,y)|\leqslant h^i(y)\quad \forall x\in\Omega\quad \text{a.e. } y\in Y^N,\tag{2.6}$$

$$|A^i(x,y)-A^i(\bar x,y)|\leqslant\rho^i(|x-\bar x|)\quad \forall x,\bar x\in\Omega\quad \text{a.e. } y\in Y^N,\tag{2.7}$$

$$A^i(x,y)\mathcal{M}:\mathcal{M}\geqslant\alpha|\mathcal{M}|^2\quad \forall\mathcal{M}\in\mathcal{S}_N\quad\forall x\in\Omega\quad \text{a.e. } y\in Y^N\tag{2.8}$$

and

$$A_\varepsilon(x)=A^i(x,y^i_\varepsilon(x)),\qquad H_\varepsilon(x)=H^i(x,y^i_\varepsilon(x)),\qquad F_\varepsilon(x)=F^i(x,y^i_\varepsilon(x))\tag{2.9}$$

a.e. $x\in\Omega^i_\varepsilon\setminus\omega_\varepsilon$. We also suppose that

$$\lim_{\varepsilon\to0}\frac{1}{|\Omega_\varepsilon|}\int_{\omega_\varepsilon}|F_\varepsilon|^2\,\mathrm{d}x=\lim_{\varepsilon\to0}\frac{1}{|\Omega_\varepsilon|}\int_{\omega_\varepsilon}|H_\varepsilon|^2\,\mathrm{d}x=0.\tag{2.10}$$

https://doi.org/10.1017/S0308210500003620 Published online by Cambridge University Press

REMARK 2.4. The assumptions on A_ε allow us to consider non-homogeneous elastic materials in macrostructure and microstructure (i.e. the elastic coefficients depend on the variables x and y), which may be different for every Ω_ε^i, $i \in \{1, \dots, N\}$. They can also be arbitrarily anisotropic. For instance, we can model structures that, along every direction, are shaped by composite beams, which are built with a core of a material surrounded by a different one. It is enough to take, for example,

$$A^i(x,y) = A_1^i \chi_{\{\|y_i'\|_\infty < r^i\}} + A_2^i \chi_{\{\|y_i'\|_\infty > r^i\}} \quad \forall i \in \{1, \dots, N\},$$

with $r^i \in (0, \frac{1}{2})$ and $A_1^i, A_2^i \in \mathcal{L}(\mathcal{S}_N, \mathcal{S}_N)$ for every $i \in \{1, \dots, N\}$. Furthermore, our hypotheses on F_ε and H_ε allow us to consider forces that depend on the microstructure. Observe that in (2.9) we do not make any assumption on the structure of A_ε, F_ε and H_ε in ω_ε. The measure of ω_ε is very small with respect to the measure of Ω_ε, and the limit behaviour of the solutions u_ε of (1.1) does not depend on how A_ε, F_ε and H_ε are in this set.

The main result of this paper is the following one (the proof of which is given in § 4).

THEOREM 2.5. *Let u_ε be the sequence of solutions of* (1.1) *and set* $\gamma_\varepsilon = d_\varepsilon/(\varepsilon + d_\varepsilon)$. *We suppose that there exists* $\lim_{\varepsilon \to 0} \gamma_\varepsilon = \gamma$ (*this always holds for a subsequence*). *Then the sequences $\hat{u}_\varepsilon^i$, $i \in \{1, \dots, N\}$, defined by (2.3) satisfy*

$$\gamma_\varepsilon e_\varepsilon^i(\hat{u}_\varepsilon^i) \to e_0^i(\hat{u}_0^i, \hat{u}_1^i, \hat{u}_2^i, \hat{u}_3^i) \quad \text{in } L^2(\mathbb{R}^N \times Y^N; \mathcal{S}_N), \tag{2.11}$$

where $(\hat{u}_0^i, \hat{u}_1^i, \hat{u}_2^i, \hat{u}_3^i) \in E^i$, $i \in \{1, \dots, N\}$, *is a solution of the variational problem*

$$\begin{aligned} \sum_{i=1}^N \int_{\Omega \times Y^N} (A^i e_0^i(\hat{u}_0^i, \hat{u}_1^i, \hat{u}_2^i, \hat{u}_3^i) - \gamma H^i) : e_0^i(\hat{v}_0^i, \hat{v}_1^i, \hat{v}_2^i, \hat{v}_3^i) \, \mathrm{d}y \, \mathrm{d}x \\ = \sum_{i=1}^N \int_{\Omega \times Y^N} \left(\gamma \sum_{j=1}^N F_j^i \hat{v}_0^j + (1-\gamma) \sum_{m \neq i} F_m^i \hat{v}_{1,m}^i \right) \mathrm{d}y \, \mathrm{d}x \\ \forall (\hat{v}_0^i, \hat{v}_1^i, \hat{v}_2^i, \hat{v}_3^i) \in E^i \quad \forall i \in \{1, \dots, N\}. \end{aligned} \tag{2.12}$$

Moreover, the sequences g_ε^i and G_ε^i, defined by

$$\left. \begin{aligned} g_{\varepsilon,i}^i(\cdot) &= \frac{1}{\varepsilon^N} \int_{C_\varepsilon(\cdot)} \hat{u}_0^i(\rho) \, \mathrm{d}\rho, \\ g_{\varepsilon,m}^i(\cdot) &= \frac{1}{\varepsilon^N} \int_{C_\varepsilon(\cdot)} \left[\hat{u}_0^m(\rho) + \frac{\varepsilon}{d_\varepsilon} \hat{u}_{1,m}^i(\rho, y_\varepsilon^i(\cdot)) \right] \mathrm{d}\rho \quad \forall m \neq i \end{aligned} \right\} \tag{2.13}$$

and

$$G_\varepsilon^i(\cdot) = \frac{1}{\varepsilon^N} \int_{C_\varepsilon(\cdot)} e_0^i(\hat{u}_0^i, \hat{u}_1^i, \hat{u}_2^i, \hat{u}_3^i)(\rho, y_\varepsilon^i(\cdot)) \, \mathrm{d}\rho, \tag{2.14}$$

give the following approximations to u_ε and $e(u_\varepsilon)$:

$$\lim_{\varepsilon\to 0}\frac{\gamma_\varepsilon^2}{|\Omega_\varepsilon^i|}\bigg[\int_{\Omega_\varepsilon^i}\left|u_{\varepsilon,i}(x)-\frac{1}{\gamma_\varepsilon}g^i_{\varepsilon,i}(x)\right|^2\mathrm{d}x + \gamma_\varepsilon^2\sum_{m\neq i}\int_{\Omega_\varepsilon^i}\left|u_{\varepsilon,m}(x)-\frac{1}{\gamma_\varepsilon}g^i_{\varepsilon,m}(x)\right|^2\mathrm{d}x + \int_{\Omega_\varepsilon^i}\left|e(u_\varepsilon)(x)-\frac{1}{\gamma_\varepsilon}G^i_\varepsilon(x)\right|^2\mathrm{d}x\bigg]=0 \quad \forall i\in\{1,\dots,N\}. \tag{2.15}$$

REMARK 2.6. If $(\hat u_0^i,\hat u_1^i,\hat u_2^i,\hat u_3^i)$ are strongly continuous in $x\in\mathbb{R}^N$, then (2.15) is equivalent to

$$\lim_{\varepsilon\to 0}\frac{\gamma_\varepsilon^2}{|\Omega_\varepsilon^i|}\int_{\Omega_\varepsilon^i}\bigg(\left|u_{\varepsilon,i}(x)-\frac{1}{\gamma_\varepsilon}\hat u_0^i(x)\right|^2 + \gamma_\varepsilon^2\sum_{m\neq i}\left|u_{\varepsilon,m}(x)-\frac{1}{\gamma_\varepsilon}\Big(\hat u_0^m(x)+\frac{\varepsilon}{d_\varepsilon}\hat u^i_{1,m}(x,y^i_\varepsilon(x))\Big)\right|^2\bigg)\mathrm{d}x=0,$$

$$\lim_{\varepsilon\to 0}\frac{\gamma_\varepsilon^2}{|\Omega_\varepsilon^i|}\int_{\Omega_\varepsilon^i}\left|e(u_\varepsilon)(x)-\frac{1}{\gamma_\varepsilon}e_0^i(\hat u_0^i,\hat u_1^i,\hat u_2^i,\hat u_3^i)(x,y^i_\varepsilon(x))\right|^2\mathrm{d}x=0.$$

In fact, if $(\hat u_0^i,\hat u_1^i,\hat u_2^i,\hat u_3^i)$, $i\in\{1,\dots,N\}$, are smooth enough, then, on defining $z^i_\varepsilon:\Omega^i_\varepsilon\to\mathbb{R}^N$ as

$$\left.\begin{aligned}
z^i_{\varepsilon,i}&=\frac{1}{\gamma_\varepsilon}\bigg[\hat u_0^i+\varepsilon\hat u^i_{1,i}(\cdot,y^i_\varepsilon) + \varepsilon d_\varepsilon\Big(\hat u^i_{2,i}(\cdot,y^i_\varepsilon)-\sum_{n\neq i}(\partial_{x_n}\hat u_0^i+\partial_{x_i}\hat u_0^n)y^i_{\varepsilon,n}\Big) - \varepsilon^2\sum_{n\neq i}\partial_{x_i}\hat u^i_{1,n}(\cdot,y^i_\varepsilon)y^i_{\varepsilon,n}\bigg],\\
z^i_{\varepsilon,m}&=\frac{1}{\gamma_\varepsilon}\bigg[\hat u_0^m+\frac{\varepsilon}{d_\varepsilon}\hat u^i_{1,m}(\cdot,y^i_\varepsilon)+\varepsilon\hat u^i_{2,m}(\cdot,y^i_\varepsilon) + \varepsilon d_\varepsilon\Big(\hat u^i_{3,m}(\cdot,y^i_\varepsilon)-\sum_{n\neq i}\partial_{x_n}\hat u_0^m y^i_{\varepsilon,n}\Big) - \varepsilon^2\sum_{n\neq i}\partial_{x_n}\hat u^i_{1,m}(\cdot,y^i_\varepsilon)y^i_{\varepsilon,n}\bigg] \quad \forall m\neq i,
\end{aligned}\right\} \tag{2.16}$$

we have

$$\lim_{\varepsilon\to 0}\frac{\gamma_\varepsilon^2}{|\Omega_\varepsilon^i|}\int_{\Omega_\varepsilon^i}\Big(|u_{\varepsilon,i}-z^i_{\varepsilon,i}|^2+\gamma_\varepsilon^2\sum_{m\neq i}|u_{\varepsilon,m}-z^i_{\varepsilon,m}|^2+|e(u_\varepsilon-z^i_\varepsilon)|^2\Big)\mathrm{d}x=0. \tag{2.17}$$

REMARK 2.7. Problem (2.12) is decoupled in N independent problems, one for every direction, i.e. one for every value of $i\in\{1,\dots,N\}$. In fact, for $i\in\{1,\dots,N\}$,

https://doi.org/10.1017/S0308210500003620 Published online by Cambridge University Press

$(\hat{u}_0^i, \hat{u}_1^i, \hat{u}_2^i, \hat{u}_3^i) \in E^i$ is a solution of the variational problem

$$\int_{\Omega\times Y^N} (A^i e_0^i(\hat{u}_0^i, \hat{u}_1^i, \hat{u}_2^i, \hat{u}_3^i) - \gamma H^i) : e_0^i(\hat{v}_0^i, \hat{v}_1^i, \hat{v}_2^i, \hat{v}_3^i)\,\mathrm{d}y\,\mathrm{d}x = \int_{\Omega\times Y^N} \biggl[\gamma\biggl(\sum_{j=1}^N F_i^j\biggr)\hat{v}_0^i + (1-\gamma)\sum_{m\neq i} F_m^i \hat{v}_{1,m}^i\biggr]\mathrm{d}y\,\mathrm{d}x \quad \forall(\hat{v}_0^i, \hat{v}_1^i, \hat{v}_2^i, \hat{v}_3^i) \in E^i. \tag{2.18}$$

An easy application of the Lax–Milgram theorem (in suitable quotient spaces) shows that, for every $i \in \{1,\dots,N\}$, problem (2.18) admits a solution. Although this solution is not unique, the functions $\hat{u}_0^i$, $\hat{u}_{1,m}^i$, with $m \neq i$, and $e_0^i(\hat{u}_0^i, \hat{u}_1^i, \hat{u}_2^i, \hat{u}_3^i)$ are defined univocally. Observe that these are precisely the terms that appear in the definition of the corrector for both u_ε and $e(u_\varepsilon)$.

For $i \in \{1,\dots,N\}$, we can derive the problem that $\hat{u}_0^i$ satisfies by eliminating $\hat{u}_1^i$, $\hat{u}_2^i$ and $\hat{u}_3^i$ from (2.12). As an example, let us consider the case of an homogeneous isotropic elastic material. A simple but tedious calculus shows the following.

Proposition 2.8. *We suppose that there exists $\lambda \geqslant 0$ and $\mu > 0$ such that*

$$A_\varepsilon \mathcal{M} = \lambda\,\mathrm{trace}(\mathcal{M})\mathcal{I} + 2\mu\mathcal{M} \quad \forall \mathcal{M} \in \mathcal{S}_N \quad \forall\varepsilon > 0 \tag{2.19}$$

(by $\mathcal{I}$ we mean the N-dimensional identity matrix). We also suppose that there exist $F \in C^0(\bar{\Omega})^N$ and $H \in C^0(\bar{\Omega}; \mathcal{S}_N)$ such that $F_\varepsilon = F$ and $H_\varepsilon = H$, for every $\varepsilon > 0$. Let $(\hat{u}_0^i, \hat{u}_1^i, \hat{u}_2^i, \hat{u}_3^i) \in E^i$, $i \in \{1,\dots,N\}$, be a solution of (2.12). Then $\hat{u}_0^i$ satisfies

$$-\Lambda \partial^2_{x_i x_i} \hat{u}_0^i = N\gamma F_i - \gamma \partial_{x_i} H_{ii} + \frac{\lambda\gamma}{\lambda(N-1)+2\mu} \sum_{j\neq i} \partial_{x_i} H_{jj} \quad \text{in } \Omega, \tag{2.20}$$

where

$$\Lambda = \frac{2\mu(\lambda N + 2\mu)}{\lambda(N-1)+2\mu}. \tag{2.21}$$

The other terms satisfy the relations

$$\begin{aligned}
\partial_{y_i} \hat{u}_{1,i}^i &= -\frac{1-\gamma}{\Lambda}(6y_i^2 - 6|y_i| + 1)\sum_{j\neq i} F_j y_j, \\
\hat{u}_{1,m}^i &= \frac{1-\gamma}{2\Lambda} F_m y_i^2 (1-|y_i|)^2 \quad \forall m \neq i, \\
\tfrac{1}{2}(\partial_{y_i} \hat{u}_{2,m}^i + \partial_{y_m} \hat{u}_{2,i}^i) &= \frac{\gamma}{2\mu} H_{im} \quad \forall m \neq i, \\
\tfrac{1}{2}(\partial_{y_j} \hat{u}_{3,m}^i + \partial_{y_m} \hat{u}_{3,j}^i) &= \frac{\gamma}{2\mu} H_{jm} \quad \forall j, m \in \{1,\dots,N\} \setminus \{i\}, \quad j \neq m,
\end{aligned}$$

https://doi.org/10.1017/S0308210500003620 Published online by Cambridge University Press

$$\partial_{y_m}\hat{u}^i_{3,m} = \frac{-\lambda}{\lambda(N-1)+2\mu}\left(\partial_{x_i}\hat{u}^i_0 - \frac{1-\gamma}{\Lambda}(6y_i^2 - 6|y_i| + 1)\sum_{l=1}^N F_l y_l\right) + \frac{\gamma(\lambda(N-2)+2\mu)}{2\mu(\lambda(N-1)+2\mu)}H_{mm} - \frac{\lambda\gamma}{2\mu(\lambda(N-1)+2\mu)}\sum_{r\neq i,m} H_{rr} \quad \forall m \neq i.$$

REMARK 2.9. For $H = 0$, $N = 3$ and $\gamma = 1$ (i.e. $\lim_{\varepsilon\to 0}(\varepsilon/d_\varepsilon) = 0$), equation (2.20) coincides with the one obtained in [17] (as it was to be expected), where a problem strongly related to (1.1) (in the isotropic case) was studied. That problem consisted of fixing $d_\varepsilon = d$ and passing to the limit first in ε and then in d (clearly, this procedure assumes that ε is much smaller than d_ε). Nevertheless, to our knowledge, the results in [17] do not provide any convergence result for problem (1.1) where ε and d_ε are two arbitrary sequences tending to zero simultaneously.

REMARK 2.10. If $\lim_{\varepsilon\to 0}(\varepsilon/d_\varepsilon) = 0$ (i.e. $\gamma = 1$) and $\hat{u}_0 = (\hat{u}^1_0, \dots, \hat{u}^N_0)$ is sufficiently smooth (otherwise, we have to consider mean values on C^k_ε as in (2.13)), theorem 2.5 gives

$$\lim_{\varepsilon\to 0}\frac{1}{|\Omega_\varepsilon|}\int_{\Omega_\varepsilon}|u_\varepsilon(x) - \hat{u}_0(x)|^2\,\mathrm{d}x = 0,$$

which shows that $\hat{u}_0$ is the 'limit' of u_ε in the strong topology of L^2. The situation is different if $\lim_{\varepsilon\to 0}(\varepsilon/d_\varepsilon) \in (0, +\infty)$ (i.e. $\gamma \in (0,1)$). For the displacements along the direction of the bars, we still have that

$$\lim_{\varepsilon\to 0}\frac{1}{|\Omega^i_\varepsilon|}\int_{\Omega^i_\varepsilon}\left|u_{\varepsilon,i}(x) - \frac{1}{\gamma}\hat{u}^i_0(x)\right|^2\mathrm{d}x = 0 \quad \forall i \in \{1,\dots,N\}.$$

However, for the transverse displacements, what we have is (assuming sufficient smoothness)

$$\lim_{\varepsilon\to 0}\frac{1}{|\Omega^i_\varepsilon|}\int_{\Omega^i_\varepsilon}\left|u_{\varepsilon,m}(x) - \frac{1}{\gamma}\hat{u}^m_0(x) - \frac{1-\gamma}{\gamma^2}\hat{u}^i_{1,m}(x, y^i_\varepsilon)\right|^2\mathrm{d}x = 0 \quad \forall i,m \in \{1,\dots,N\}, \quad m \neq i.$$

Therefore, if we want a 'strong limit' of u_ε in L^2, we have to consider functions depending not only on x, but also on the microscopic variable y. In order to obtain a limit depending only on the macroscopic variable, we think it is more appropriate to look for a 'weak limit' u of u_ε, which can be defined by means of (see, for example, [7, 30], where this type of limit is considered)

$$\lim_{\varepsilon\to 0}\frac{1}{|\Omega_\varepsilon|}\int_{\Omega_\varepsilon}u_\varepsilon(x)\varphi(x)\,\mathrm{d}x = \frac{1}{|\Omega|}\int_\Omega u(x)\varphi(x)\,\mathrm{d}x \quad \forall\varphi \in C^\infty_0(\Omega)^N.$$

Using theorem 2.5, we immediately obtain that, for $\gamma \in (0,1)$, this function u is given by

$$u(x) = \frac{1}{\gamma}\hat{u}_0(x) + \frac{1-\gamma}{N\gamma^2}\sum_{j=1}^N\sum_{m\neq j}\int_{Y^N}\hat{u}^m_{1,j}(x,y)\,\mathrm{d}y\,e_j \quad \text{a.e. } x \in \Omega. \tag{2.22}$$

https://doi.org/10.1017/S0308210500003620 Published online by Cambridge University Press

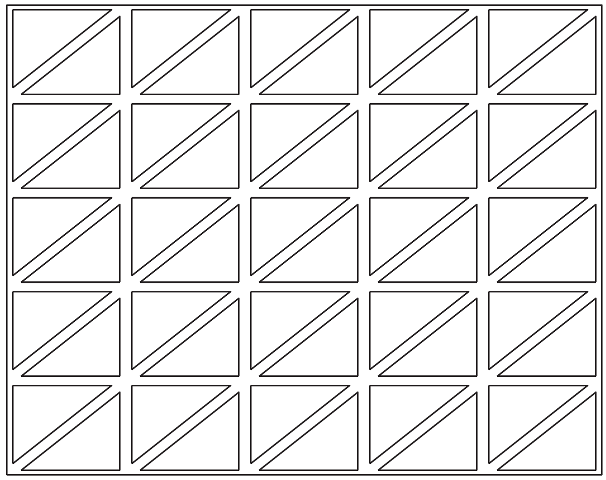

Figure 5.

So u is a linear combination of $\hat{u}_0$ and the mean values with respect to y of the functions $\hat{u}^m_{1,j}$. Finally, if $\lim_{\varepsilon\to 0}(\varepsilon/d_\varepsilon) = +\infty$ (i.e. $\gamma = 0$), u_ε is not bounded and then it does not have a limit. What we get is that the sequence $\gamma^2_\varepsilon u_\varepsilon$ converges in the sense stated above to

$$w(x) = \lim_{\varepsilon\to 0} \gamma^2_\varepsilon u_\varepsilon = \frac{1}{N} \sum_{j=1}^{N} \sum_{m\neq j} \int_{Y^N} \hat{u}^m_{1,j}(x, y)\, \mathrm{d}y\, e_j \quad \text{a.e. } x \in \Omega.$$

Observe that w does not depend on $\hat{u}_0$; it only depends on the functions $\hat{u}^m_{1,j}$.

3. A reinforced structure: homogenization result

For the structure considered in the previous section, the homogenized fourth-order tensor associated to the problem satisfied by $\hat{u}_0 = (\hat{u}^1_0, \dots, \hat{u}^N_0)$ is not strongly elliptic (see (2.20) for the case of an homogeneous isotropic material); in fact, we only have that $\partial_{x_i}\hat{u}^i_0$ is in $L^2(\Omega)$, $i \in \{1, \dots, N\}$, and not $\nabla\hat{u}_0 \in L^2(\Omega)^{N\times N}$. To obtain an elliptic problem, some authors (see [5,12]) propose the introduction of additional bars in the structures (reinforced structures). Passing to the limit first in ε and then in d_ε (which implies that $\varepsilon \ll d_\varepsilon$, or, equivalently, $\lim_{\varepsilon\to 0}(\varepsilon/d_\varepsilon) = 0$), they obtain a non-degenerate elliptic problem for the limit $\hat{u}_0$ of u_ε.

We see in this section that our method can also be applied to these structures with additional bars, which results in the limit behaviour still depending on the limit of $\varepsilon/d_\varepsilon$. Thus, although we prove that the corresponding function $\hat{u}_0$ that we obtain in this case satisfies an elliptic problem, we emphasize (see remark 2.10) that this function $\hat{u}_0$ is not, in general, the limit of u_ε (it gives the limit only when $\varepsilon/d_\varepsilon$ tends to zero). In fact, we will show that, even adding additional bars, the problem satisfied by the limit of u_ε (or $\gamma_\varepsilon u_\varepsilon$ if γ_ε tends to zero) is degenerate when $\varepsilon/d_\varepsilon$ does not converge to zero.

To simplify the exposition, we just consider a structure in dimension two. It is composed by the structure studied in § 2 with $N = 2$, and additional parallel oblique bars of cross-section $\varepsilon d_\varepsilon$, disposed periodically with period ε (see figure 5). This structure has been considered in [12] in the case of an isotropic elastic material.

We associate the superscript '$i = 1$' with the horizontal bars, '$i = 2$' with the vertical bars and '$i = 3$' with the oblique bars. For $i \in \{1, 2\}$, we keep the notation of the preceding sections (i.e. V_ε^1, V_ε^2, Ω_ε^1, Ω_ε^2, y_ε^1, y_ε^2, E^1, E^2, etc.). This notation refers to the horizontal and vertical bars that have been studied previously. For $i = 3$ (oblique bars), we need some extra notation.

Let $\{\tau, \zeta\}$ be the orthogonal basis in $\mathbb{R}^2$ given by

$$\tau = \frac{e_1 + e_2}{\sqrt{2}}, \qquad \zeta = \frac{-e_1 + e_2}{\sqrt{2}}.$$

We remark that, for $y \in \mathbb{R}^2$, $y\tau$, $y\zeta$ are the components of y in the basis $\{\tau, \zeta\}$. Analogously, for a symmetric tensor $\mathcal{M}$, $\mathcal{M}\tau\tau$, $\mathcal{M}\tau\zeta$, $\mathcal{M}\zeta\zeta$ are the components of $\mathcal{M}$ with respect to the basis $\{\tau, \zeta\}$. Thus it is that in the notation given below the vectors and tensors appear usually multiplied by τ and/or ζ.

Let V_ε^3 be the set

$$V_\varepsilon^3 = \bigcup_{q \in \mathbb{Z}} \{\varepsilon q e_1 + s_1\tau + s_2\zeta : s_1 \in \mathbb{R},\ -\tfrac{1}{2}\varepsilon d_\varepsilon < s_2 < \tfrac{1}{2}\varepsilon d_\varepsilon\} \quad \forall \varepsilon > 0.$$

Then we define the reticulated structure V_ε (see figure 5), $\varepsilon > 0$, by

$$V_\varepsilon = V_\varepsilon^1 \cup V_\varepsilon^2 \cup V_\varepsilon^3.$$

We denote by ω_ε the set $\omega_\varepsilon = V_\varepsilon^1 \cap V_\varepsilon^2 \cap V_\varepsilon^3$.

For a fixed smooth bounded open set Ω in $\mathbb{R}^2$, we define

$$\Omega_\varepsilon^3 = \Omega \cap V_\varepsilon^3, \quad \Omega_\varepsilon = \Omega_\varepsilon^1 \cup \Omega_\varepsilon^2 \cup \Omega_\varepsilon^3 \quad \text{and} \quad \Gamma_\varepsilon = \bar{\Omega}_\varepsilon \cap \partial\Omega.$$

We set $\mathfrak{D}^1 = \mathfrak{D}^2 = Y^2$ and $\mathfrak{D}^3 = J^3 + S^3$, where $J^3 = (\sqrt{2}Y)\tau$, $S^3 = Y\zeta$.

Analogously to y_ε^1, y_ε^2, we define a change of variables $y_\varepsilon^3 : V_\varepsilon^3 \to \mathfrak{D}^3$ for the oblique bars by

$$y_\varepsilon^3(x) = \frac{(x - \varepsilon k)\tau}{\varepsilon}\tau + \frac{(x - \varepsilon k)\zeta}{\varepsilon d_\varepsilon}\zeta \quad \text{a.e. } x \in \varepsilon k + \varepsilon J^3 + \varepsilon d_\varepsilon S^3 \quad \forall k \in \mathbb{Z}^2,$$

which, for every $k \in \mathbb{Z}^2$, transforms the oblique bar $\varepsilon k + \varepsilon J^3 + \varepsilon d_\varepsilon S^3$ onto $\mathfrak{D}^3$.

For a sequence $u_\varepsilon : V_\varepsilon \to \mathbb{R}^2$ of measurable functions, we define $\hat{u}_\varepsilon^3 : \mathbb{R}^2 \times \mathfrak{D}^3 \to \mathbb{R}^2$ by

$$\hat{u}_\varepsilon^3(x, y) = u_\varepsilon\left(\varepsilon\kappa\left(\frac{x}{\varepsilon}\right) + \varepsilon(y\tau)\tau + \varepsilon d_\varepsilon(y\zeta)\zeta\right) \quad \text{a.e. } (x, y) \in \mathbb{R}^2 \times \mathfrak{D}^3. \tag{3.1}$$

We will use the function $\hat{u}_\varepsilon^3$ to describe the behaviour of u_ε on the oblique bars V_ε^3.

For $\varepsilon > 0$ and $\hat{v} \in L^2(\mathbb{R}^2; H^1(\mathfrak{D}^3))^2$, we define $e_\varepsilon^3(\hat{v}) \in L^2(\mathbb{R}^2 \times \mathfrak{D}^3; \mathcal{S}_2)$ by the equalities

$$\begin{aligned} e_\varepsilon^3(\hat{v})\tau\tau &= \frac{1}{\varepsilon}\nabla_y(\hat{v}\tau)\tau, \\ e_\varepsilon^3(\hat{v})\tau\zeta &= \frac{1}{\varepsilon}\nabla_y(\hat{v}\zeta)\tau + \frac{1}{\varepsilon d_\varepsilon}\nabla_y(\hat{v}\tau)\zeta, \\ e_\varepsilon^3(\hat{v})\zeta\zeta &= \frac{1}{\varepsilon d_\varepsilon}\nabla_y(\hat{v}\zeta)\zeta. \end{aligned}$$

Observe that, for $u_\varepsilon : V_\varepsilon \to \mathbb{R}^2$, $e^3_\varepsilon(\hat{u}^3_\varepsilon)$ gives the strain tensor $e(u_\varepsilon)$ expressed in the variables $y = y^3_\varepsilon(x)$.

By E^3 we denote the functional space $E^3 = E^3_0 \times E^3_1 \times E^3_2 \times E^3_3$, where ($\nu$ is the unit outward normal vector to $\partial\Omega$)

$$
\begin{aligned}
E^3_0 &= \{\hat{u}^3_0 \in L^2(\Omega) : \nabla\hat{u}^3_0\tau \in L^2(\Omega),\ \hat{u}^3_0(\nu\tau) = 0 \text{ on } \partial\Omega\},\\
E^3_1 &= \{\hat{u}^3_1 \in L^2(\Omega; H^1(\mathfrak{D}^3))^2 : \hat{u}^3_1(x, \tfrac{1}{2}\sqrt{2}\tau + z) = \hat{u}^3_1(x, -\tfrac{1}{2}\sqrt{2}\tau + z)\\
&\qquad\qquad \text{a.e. } (x, z) \in \Omega \times S^3,\\
&\qquad \hat{u}^3_1(x, z)\zeta = 0 \text{ a.e. } (x, z) \in \Omega \times S^3,\\
&\qquad\qquad e_y(\hat{u}^3_1)\tau\zeta = e_y(\hat{u}^3_1)\zeta\zeta = 0\},\\
E^3_2 &= \{\hat{u}^3_2 \in L^2(\Omega \times J^3; H^1(S^3))^2 : \hat{u}^3_2\zeta = 0\},\\
E^3_3 &= \{\hat{u}^3_3 \in L^2(\Omega \times J^3; H^1(S^3))^2 : \hat{u}^3_3\tau = 0\}.
\end{aligned}
$$

For $(\hat{u}^3_0, \hat{u}^3_1, \hat{u}^3_2, \hat{u}^3_3) \in E^3$, we define $e^3_0(\hat{u}^3_0, \hat{u}^3_1, \hat{u}^3_2, \hat{u}^3_3) \in L^2(\mathbb{R}^2 \times \mathfrak{D}^3; \mathcal{S}_2)$ by

$$
\begin{aligned}
e^3_0(\hat{u}^3_0, \hat{u}^3_1, \hat{u}^3_2, \hat{u}^3_3)\tau\tau &= (\nabla\hat{u}^3_0)\tau + \nabla_y(\hat{u}^3_1\tau)\tau,\\
e^3_0(\hat{u}^3_0, \hat{u}^3_1, \hat{u}^3_2, \hat{u}^3_3)\tau\zeta &= \nabla_y(\hat{u}^3_2\zeta)\tau,\\
e^3_0(\hat{u}^3_0, \hat{u}^3_1, \hat{u}^3_2, \hat{u}^3_3)\zeta\zeta &= \nabla_y(\hat{u}^3_3\zeta)\zeta.
\end{aligned}
$$

We are interested in the asymptotic behaviour of (1.1) for the current choice of Ω_ε and Γ_ε. We assume that hypotheses (2.1)–(2.9) are satisfied for $i \in \{1, 2, 3\}$ (when $i = 3$, Y^2 must be replaced by $\mathfrak{D}^3$). The following theorem (whose proof we give in § 4.4) describes the asymptotic behaviour of u_ε and provides a corrector result for both u_ε and $e(u_\varepsilon)$.

Theorem 3.1. *Let u_ε be the sequence of solutions of (1.1) and set $\gamma_\varepsilon = d_\varepsilon/(\varepsilon + d_\varepsilon)$. We suppose there exists* $\lim_{\varepsilon\to0}\gamma_\varepsilon = \gamma$ (*this always holds for a subsequence*). *Then the sequences $\hat{u}^i_\varepsilon$, $i \in \{1, 2, 3\}$, defined by (2.3) and (3.1) satisfy*

$$\gamma_\varepsilon e^i_\varepsilon(\hat{u}^i_\varepsilon) \to e^i_0(\hat{u}^i_0, \hat{u}^i_1, \hat{u}^i_2, \hat{u}^i_3) \quad \text{in } L^2(\mathbb{R}^2 \times \mathfrak{D}^i; \mathcal{S}_2),$$

where $(\hat{u}^i_0, \hat{u}^i_1, \hat{u}^i_2, \hat{u}^i_3) \in E^i$, $i \in \{1, 2, 3\}$, with $\hat{u}^3_0 = \hat{u}_0\tau$, $\hat{u}_0 = (\hat{u}^1_0, \hat{u}^2_0)$, is a solution of the variational problem

$$
\begin{aligned}
\sum_{i=1}^{3}\int_{\Omega\times\mathfrak{D}^i} &(A^i e^i_0(\hat{u}^i_0, \hat{u}^i_1, \hat{u}^i_2, \hat{u}^i_3) - \gamma H^i) : e^i_0(\hat{v}^i_0, \hat{v}^i_1, \hat{v}^i_2, \hat{v}^i_3)\,\mathrm{d}y\,\mathrm{d}x\\
&= \gamma\sum_{i=1}^{3}\int_{\Omega\times\mathfrak{D}^i} F^i\hat{v}_0\,\mathrm{d}y\,\mathrm{d}x\\
&\quad + (1-\gamma)\biggl(\int_{\Omega\times\mathfrak{D}^1} F^1_2\hat{v}^1_{1,2}\,\mathrm{d}y\,\mathrm{d}x + \int_{\Omega\times\mathfrak{D}^2} F^2_1\hat{v}^2_{1,1}\,\mathrm{d}y\,\mathrm{d}x\\
&\qquad\qquad + \int_{\Omega\times\mathfrak{D}^3}(F^3\zeta)(\hat{v}^3_1\zeta)\,\mathrm{d}y\,\mathrm{d}x\biggr)\\
&\qquad \forall(\hat{v}^i_0, \hat{v}^i_1, \hat{v}^i_2, \hat{v}^i_3) \in E^i, \quad i \in \{1, 2, 3\},\\
&\qquad \textit{such that } \hat{v}^3_0 = \hat{v}_0\tau,\ \hat{v}_0 = (\hat{v}^1_0, \hat{v}^2_0).
\end{aligned}
\tag{3.2}
$$

https://doi.org/10.1017/S0308210500003620 Published online by Cambridge University Press

Moreover, the sequences $g^i_\varepsilon : \Omega^i_\varepsilon \to \mathbb{R}^2$ *and* $G^i_\varepsilon : \Omega^i_\varepsilon \to \mathcal{S}_2$, $i \in \{1,2,3\}$, *defined by (2.13)–(2.14) and*

$$\left.\begin{aligned} g^3_{\varepsilon,1}(\cdot) &= \frac{1}{\varepsilon^2}\int_{C_\varepsilon(\cdot)} \hat{u}_0(\rho)\tau \,\mathrm{d}\rho, \\ g^3_{\varepsilon,2}(\cdot) &= \frac{1}{\varepsilon^2}\int_{C_\varepsilon(\cdot)} \left[\hat{u}_0(\rho)\zeta + \frac{\varepsilon}{d_\varepsilon}\hat{u}^3_1(\rho, y^3_\varepsilon(\cdot))\zeta\right] \mathrm{d}\rho, \end{aligned}\right\} \tag{3.3}$$
$$G^3_\varepsilon(\cdot) = \frac{1}{\varepsilon^2}\int_{C_\varepsilon(\cdot)} e^3_0(\hat{u}^3_0, \hat{u}^3_1, \hat{u}^3_2, \hat{u}^3_3)(\rho, y^3_\varepsilon(\cdot))\,\mathrm{d}\rho,$$

satisfy (2.15) and

$$\lim_{\varepsilon\to 0}\frac{\gamma^2_\varepsilon}{|\Omega^3_\varepsilon|}\left[\int_{\Omega^3_\varepsilon}\left|u_\varepsilon(x)\tau - \frac{1}{\gamma_\varepsilon}g^3_{\varepsilon,1}(x)\right|^2 \mathrm{d}x + \gamma^2_\varepsilon\int_{\Omega^3_\varepsilon}\left|u_\varepsilon(x)\zeta - \frac{1}{\gamma_\varepsilon}g^3_{\varepsilon,2}(x)\right|^2 \mathrm{d}x \right.$$
$$\left. + \int_{\Omega^3_\varepsilon}\left|e(u_\varepsilon)(x) - \frac{1}{\gamma_\varepsilon}G^3_\varepsilon(x)\right|^2 \mathrm{d}x\right] = 0. \tag{3.4}$$

Remark 3.2. As in remark 2.6, if $(\hat{u}^i_0, \hat{u}^i_1, \hat{u}^i_2, \hat{u}^i_3)$ are sufficiently smooth, then theorem 3.1 provides an asymptotic development of u_ε such that (2.17) holds.

Remark 3.3. The main difference between (2.12) and (3.2) is due to the conditions $\hat{u}^3_0 = \hat{u}_0\tau$, $\hat{u}_0 = (\hat{u}^1_0, \hat{u}^2_0)$, which means that the limit system (3.2) is not decoupled, i.e. we cannot decompose (3.2) into three independent problems, one for every direction given by the index $i \in \{1,2,3\}$.

By the definition of E^i_0, $i \in \{1,2\}$, we have that $\partial_{x_1}\hat{u}^1_0 = e(\hat{u}^0)_{11}$ and $\partial_{x_2}\hat{u}^2_0 = e(\hat{u}^0)_{22}$ belong to $L^2(\Omega)$. Since $\hat{u}^3_0 \in E^3_0$, we also have that $(\nabla\hat{u}^3_0)\tau$ belongs to $L^2(\Omega)$, and taking into account $\hat{u}^3_0 = \hat{u}_0\tau$, this gives

$$\begin{aligned} \tfrac{1}{2}(e(\hat{u}_0)_{11} + 2e(\hat{u}_0)_{12} + e(\hat{u}_0)_{22}) &= \tfrac{1}{2}(\partial_{x_1}\hat{u}^1_0 + \partial_{x_1}\hat{u}^2_0 + \partial_{x_2}\hat{u}^1_0 + \partial_{x_2}\hat{u}^2_0) \\ &= (\nabla\hat{u}^3_0)\tau \in L^2(\Omega), \end{aligned}$$

which shows that $e(\hat{u}_0)_{12}$ is also in $L^2(\Omega)$. Thus $e(\hat{u}_0)$ belongs to $L^2(\Omega;\mathcal{S}_2)$. Since $\hat{u}_0$ is also in $L^2(\Omega)^2$ and vanishes on $\partial\Omega$, we conclude that $\hat{u}_0$ belongs to $H^1_0(\Omega)^2$. Thus it is that we can now prove that $\hat{u}_0$ is a solution of an elliptic problem, as we will see in proposition 3.4, where we consider the case of an homogeneous isotropic elastic material.

Similarly to proposition 2.8, it is easy to prove the following result.

Proposition 3.4. *We suppose that there exists* $\lambda \geqslant 0$ *and* $\mu > 0$ *such that* A_ε *is defined by (2.19). We also suppose* $F_\varepsilon = F$ *and* $H_\varepsilon = H$ *for every* $\varepsilon > 0$, *with* $F \in C^0(\bar{\Omega})$ *and* $H \in C^0(\bar{\Omega};\mathcal{S}_2)$ *fixed. Let* $(\hat{u}^i_0, \hat{u}^i_1, \hat{u}^i_2, \hat{u}^i_3) \in E^i$, $i \in \{1,2,3\}$, *be a solution of (3.2). Then* $\hat{u}_0 = (\hat{u}^1_0, \hat{u}^2_0) \in H^1_0(\Omega)^2$ *is the unique solution of the variational problem*

$$\int_\Omega [\hat{A}e(\hat{u}_0) - \gamma\hat{H}] : e(\hat{v}_0)\,\mathrm{d}x = (2+\sqrt{2})\gamma\int_\Omega F\hat{v}_0\,\mathrm{d}x \quad \forall\hat{v}_0 \in H^1_0(\Omega)^2, \tag{3.5}$$

https://doi.org/10.1017/S0308210500003620 Published online by Cambridge University Press

where $\hat{A}$ is the fourth-order tensor defined by

$$\hat{A}_{1111} = \varLambda(1 + \tfrac{1}{4}\sqrt{2}), \quad \hat{A}_{2222} = \varLambda(1 + \tfrac{1}{4}\sqrt{2}), \quad \hat{A}_{ijkl} = \tfrac{1}{4}\varLambda\sqrt{2} \text{ in the other cases,}$$

with $\varLambda$ given by (2.21), $N = 2$, and $\hat{H} \in C^0(\bar{\Omega}; \mathcal{S}_2)$ defined by

$$\hat{H}_{11} = H_{11} + \frac{H_{11} + 2H_{12} + H_{22}}{2\sqrt{2}} - \frac{\lambda}{\lambda + 2\mu}\left(H_{22} + \frac{H_{11} - 2H_{12} + H_{22}}{2\sqrt{2}}\right),$$

$$\hat{H}_{22} = H_{22} + \frac{H_{11} + 2H_{12} + H_{22}}{2\sqrt{2}} - \frac{\lambda}{\lambda + 2\mu}\left(H_{11} + \frac{H_{11} - 2H_{12} + H_{22}}{2\sqrt{2}}\right),$$

$$\hat{H}_{12} = \frac{H_{11} + 2H_{12} + H_{22}}{2\sqrt{2}} - \frac{\lambda}{\lambda + 2\mu}\frac{H_{11} - 2H_{12} + H_{22}}{2\sqrt{2}}.$$

Remark 3.5. As expected, when $H = 0$ and $\gamma = 1$ (i.e. $\lim_{\varepsilon\to 0}(\varepsilon/d_\varepsilon) = 0$), equation (3.5) is the same problem obtained in [12], where problem (1.1) (in the isotropic case) is studied by passing to the limit first in ε and then in d_ε.

Contrary to the structure considered in the previous section, we now see that $\hat{u}_0$ satisfies a non-degenerate elliptic problem in $H_0^1(\Omega)^2$, but we recall that $\hat{u}_0$ is the limit of u_ε only if $\gamma = 1$ (see remark 2.10). For $\gamma \in (0, 1)$, analogously to (2.22), we have

$$\begin{aligned} u(x) &= \lim_{\varepsilon\to 0} u_\varepsilon \\ &= \frac{1}{\gamma}\hat{u}_0(x) + \frac{1-\gamma}{(1+\sqrt{2})\gamma^2}\biggl[\int_{\mathfrak{D}^1} \hat{u}^1_{1,2}(x, y)\,\mathrm{d}y\, e_2 \\ &\qquad + \int_{\mathfrak{D}^2} \hat{u}^2_{1,1}(x, y)\,\mathrm{d}y\, e_1 + \int_{\mathfrak{D}^3} \hat{u}^3_1(x, y)\zeta\,\mathrm{d}y\,\zeta\biggr] \\ &= \frac{1}{\gamma}\hat{u}_0(x) + \frac{1}{\gamma^2}w(x) \quad \text{a.e. } x \in \Omega, \end{aligned}$$

with $w = BF$, where B is the matrix

$$B = \frac{(1-\gamma)^2}{30(2+\sqrt{2})\varLambda}\begin{pmatrix} \frac{1}{2}(2\sqrt{2}+1) & -\sqrt{2} \\ -\sqrt{2} & \frac{1}{2}(2\sqrt{2}+1) \end{pmatrix}.$$

Then w is the solution of the degenerate problem $B^{-1}w = F$ (in fact, it is not a partial differential problem), and therefore u does not satisfy an elliptic problem. For $\gamma = 0$, we can prove that

$$\lim_{\varepsilon\to 0} \gamma_\varepsilon^2 u_\varepsilon = w.$$

In conclusion, when $\gamma \neq 1$, the limit problem for the macrostructure is degenerate even with the extra oblique bars.

4. Proof of the results

This section demonstrates the homogenization results stated in the previous sections (i.e. theorems 2.5 and 3.1). In order to prove theorem 2.5, we begin, in § 4.1, by proving estimates of Korn type for functions in $H^1_{\Gamma_\varepsilon}(\Omega_\varepsilon)^N$. Afterwards, in § 4.2, we consider a sequence u_ε in $H^1_{\Gamma_\varepsilon}(\Omega_\varepsilon)^N$, which satisfies

$$\frac{1}{|\Omega_\varepsilon|}\int_{\Omega_\varepsilon}|e(u_\varepsilon)|^2\,\mathrm{d}x \leqslant C \quad \forall \varepsilon > 0.$$

Using the change of variables y^i_ε, $i \in \{1,\dots,N\}$, given by (2.2), we transform u_ε into new sequences of functions $\hat{u}^i_\varepsilon$, $i \in \{1,\dots,N\}$, (see (2.3)), which are defined on a fixed domain (independent of ε). Then we prove a compactness result for the new sequences of functions. We emphasize that, in the compactness result, we do not use u_ε as a solution of (1.1); we only assume that u_ε satisfies the above inequality. Finally, in § 4.3, by means of the compactness result, we pass to the limit in (1.1). In § 4.4, we prove theorem 3.1, following the same lines of theorem 2.5.

Throughout this section, C denotes a generic positive constant that can change from one line to another one and does not depend on ε; by O_ε we denote a generic real sequence that can change from one line to another one and converges to zero as ε tends to zero.

4.1. *A priori* estimates

We obtain some inequalities of Korn type for the model structure introduced in § 2. We use the following version of Korn's inequality.

Lemma 4.1. *Let G be a bounded connected Lipschitz open set of $\mathbb{R}^N$. Then there exists $C > 0$ such that*

$$\int_G\left|\partial_{x_n}u_m(x) - \frac{1}{|G|}\int_G \partial_{x_n}u_m(z)\,\mathrm{d}z\right|^2\mathrm{d}x \leqslant C\int_G|e(u)|^2\,\mathrm{d}x \quad \forall m,n \in \{1,\dots,N\}, \tag{4.1}$$

for every $u \in H^1(G)^N$.

Proof. Given $u \in H^1(G)^N$, by Korn's inequality, there exist $C > 0$, independent of u, and an N-dimensional skew-symmetric matrix $P = (P_{mn})$ such that

$$\int_G|\partial_{x_n}u_m(x) - P_{mn}|^2\,\mathrm{d}x \leqslant C\int_G|e(u)|^2\,\mathrm{d}x \quad \forall m,n \in \{1,\dots,N\}. \tag{4.2}$$

Using

$$\int_G\left|\partial_{x_n}u_m(x) - \frac{1}{|G|}\int_G \partial_{x_n}u_m(z)\,\mathrm{d}z\right|^2\mathrm{d}x = \min_{s\in\mathbb{R}}\int_G|\partial_{x_n}u_m(x) - s|^2\,\mathrm{d}x \quad \forall m,n \in \{1,\dots,N\},$$

we obtain (4.1). □

The main result of this subsection is theorem 4.3. To prove it, we use the following lemma.

Lemma 4.2. *For $a \in (0,\frac{1}{8})$ and $\varepsilon > 0$, we take*

$$L_\varepsilon(a) = \{z \in \varepsilon Y^N : |z'_1|_\infty < \tfrac{1}{2}\varepsilon a\}$$

https://doi.org/10.1017/S0308210500003620 Published online by Cambridge University Press

and

$$L^q_\varepsilon(a) = \{z \in L_\varepsilon(a) : |z_1 - \tfrac{1}{2}(-1)^q\varepsilon| < \tfrac{1}{2}\varepsilon a\}, \quad q \in \{1,2\}.$$

Then there exists $C > 0$, which does not depend on a, such that, for every $u \in H^1(L_\varepsilon(a))^N$ and every $m \neq 1$, we have

$$\int_{L_\varepsilon(a)} |\partial_{z_1} u_m|^2 \,\mathrm{d}z \leqslant C\biggl(\frac{1}{a^2}\int_{L_\varepsilon(a)} |e(u)|^2 \,\mathrm{d}z + \frac{1}{\varepsilon^2 a}\sum_{q=1}^{2}\int_{L^q_\varepsilon(a)} |u_m|^2 \,\mathrm{d}z\biggr). \tag{4.3}$$

Proof. It is enough to prove the result for $\varepsilon = 1$; the general case then follows using the change of variables $y = \varepsilon z$, which transforms $L_\varepsilon(a)$ in $L_1(a)$ and $L^q_\varepsilon(a)$ in $L^q_1(a)$, $q \in \{1,2\}$.

We write

$$L(a) = L_1(a), \quad L^q(a) = L^q_1(a), \quad q \in \{1,2\}.$$

For $u = (u_1,\dots,u_N) \in H^1(L(a))^N$, we define $w = (w_1,\dots,w_N) \in H^1(Y^N)^N$ by

$$w_1(y) = u_1(y_1e_1 + ay'_1), \quad w_m(y) = au_m(y_1e_1 + ay'_1) \quad \forall m \in \{2,\dots,N\}.$$

For $m > 1$, inequality (4.1) applied to w_m gives

$$\begin{aligned}\int_{L(a)}\biggl|\partial_{z_1}u_m - \frac{1}{|L(a)|}\int_{L(a)}\partial_{z_1}u_m \,\mathrm{d}r\biggr|^2 \mathrm{d}z \\ &= a^{N-3}\int_{Y^N}\biggl|\partial_{y_1}w_m - \int_{Y^N}\partial_{y_1}w_m \,\mathrm{d}s\biggr|^2 \mathrm{d}y \\ &\leqslant Ca^{N-3}\int_{Y^N}|e(w)|^2\,\mathrm{d}y \\ &\leqslant \frac{C}{a^2}\int_{L(a)}|e(u)|^2\,\mathrm{d}z.\end{aligned}$$

Thus we deduce

$$\begin{aligned}&\int_{L(a)}|\partial_{z_1}u_m|^2\,\mathrm{d}z \\ &\quad\leqslant \frac{C}{a^2}\int_{L(a)}|e(u)|^2\,\mathrm{d}z + \frac{2}{a^{N-1}}\biggl|\int_{\{z_1=1/2\}}u_m\,\mathrm{d}z'_1 - \int_{\{z_1=-1/2\}}u_m\,\mathrm{d}z'_1\biggr|^2.\end{aligned} \tag{4.4}$$

Taking into account the estimates

$$\begin{aligned}&\biggl|\int_{\{z_1=1/2\}}u_m\,\mathrm{d}z'_1 - \int_{\{z_1=-1/2\}}u_m\,\mathrm{d}z'_1\biggr|^2 \\ &\qquad\leqslant 2a^{N-1}\biggl(\int_{\{z_1=1/2\}}|u_m|^2\,\mathrm{d}z'_1 + \int_{\{z_1=-1/2\}}|u_m|^2\,\mathrm{d}z'_1\biggr)\end{aligned}$$

and

$$\int_{\{z_1=(-1)^q/2\}}|u_m|^2\,\mathrm{d}z'_1 \leqslant \frac{4}{a}\int_{L^q(a)}|u_m|^2\,\mathrm{d}z + a\int_{L^q(a)}|\partial_{z_1}u_m|^2\,\mathrm{d}z \quad \forall q \in \{1,2\},$$

https://doi.org/10.1017/S0308210500003620 Published online by Cambridge University Press

we derive, from (4.4),

$$\int_{L(a)} |\partial_{z_1} u_m|^2\,\mathrm{d}z \leqslant \frac{C}{a^2}\int_{L(a)} |e(u)|^2\,\mathrm{d}z + \sum_{q=1}^{2}\biggl(\frac{C}{a}\int_{L^q(a)} |u_m|^2\,\mathrm{d}z + 4a\int_{L^q(a)} |\partial_{z_1} u_m|^2\,\mathrm{d}z\biggr),$$

for every $u \in H^1(L(a))^N$. Since $a < \frac{1}{8}$, we conclude that (4.3) holds for $\varepsilon = 1$. □

THEOREM 4.3. *There exists $C > 0$ such that, for every $u \in H^1_{\Gamma_\varepsilon}(\Omega_\varepsilon)^N$ and every $\varepsilon > 0$, we have*

$$\int_{\omega_\varepsilon} |u|^2\,\mathrm{d}x \leqslant d_\varepsilon \int_{\Omega_\varepsilon} |e(u)|^2\,\mathrm{d}x, \tag{4.5}$$

$$\int_{\Omega^i_\varepsilon} |u_i|^2\,\mathrm{d}x \leqslant C\int_{\Omega_\varepsilon} |e(u)|^2\,\mathrm{d}x \quad \forall i \in \{1,\dots,N\}, \tag{4.6}$$

$$\int_{\Omega_\varepsilon} |u|^2\,\mathrm{d}x \leqslant C\biggl(1 + \frac{\varepsilon^2}{d_\varepsilon^2}\biggr)\int_{\Omega_\varepsilon} |e(u)|^2\,\mathrm{d}x, \tag{4.7}$$

$$\int_{\Omega^i_\varepsilon} |\partial_{x_i} u_j|^2\,\mathrm{d}x \leqslant C\biggl(\frac{1}{\varepsilon^2} + \frac{1}{d_\varepsilon^2}\biggr)\int_{\Omega_\varepsilon} |e(u)|^2\,\mathrm{d}x \quad \forall i,j \in \{1,\dots,N\}. \tag{4.8}$$

Proof. Estimates (4.5) and (4.6) follow immediately from the Poincaré inequalities

$$\int_{P^k_\varepsilon} |u_i|^2\,\mathrm{d}x \leqslant C\varepsilon d_\varepsilon \int_{L^{i,k'_i}_\varepsilon} |\partial_{x_i} u_i|^2\,\mathrm{d}x \quad \forall i \in \{1,\dots,N\},$$

$$\int_{L^{i,k'_i}_\varepsilon} |u_i|^2\,\mathrm{d}x \leqslant C\int_{L^{i,k'_i}_\varepsilon} |\partial_{x_i} u_i|^2\,\mathrm{d}x \qquad \forall i \in \{1,\dots,N\},$$

for every $k \in \mathbb{Z}^N$, every $\varepsilon > 0$ and every $u \in H^1_{\Gamma_\varepsilon}(\Omega_\varepsilon)^N$.

Since the constant that appears in lemma 4.2 is invariant by translations and rotations, for every $i \in \{1,\dots,N\}$, $m \in \{1\dots,N\} \setminus \{i\}$ and $\varepsilon > 0$ (small enough), we get

$$\int_{B^{i,k}_\varepsilon + \varepsilon e_i/2} |\partial_{x_i} u_m|^2\,\mathrm{d}x \leqslant C\biggl(\frac{1}{d_\varepsilon^2}\int_{B^{i,k}_\varepsilon + \varepsilon e_i/2} |e(u)|^2\,\mathrm{d}x + \frac{1}{\varepsilon^2 d_\varepsilon}\int_{P^k_\varepsilon \cup P^{k+e_i}_\varepsilon} |u_m|^2\,\mathrm{d}x\biggr),$$

for every $u \in H^1_{\Gamma_\varepsilon}(\Omega_\varepsilon)^N$. Adding these inequalities in $k \in \mathbb{Z}^N$, we obtain

$$\int_{\Omega^i_\varepsilon} |\partial_{x_i} u_m|^2\,\mathrm{d}x \leqslant C\biggl(\frac{1}{d_\varepsilon^2}\int_{\Omega^i_\varepsilon} |e(u)|^2\,\mathrm{d}x + \frac{1}{\varepsilon^2 d_\varepsilon}\int_{\omega_\varepsilon} |u_m|^2\,\mathrm{d}x\biggr) \quad \forall u \in H^1_{\Gamma_\varepsilon}(\Omega_\varepsilon)^N \quad \forall \varepsilon > 0. \tag{4.9}$$

Then, by (4.5), we deduce (4.8).

Finally, in order to demonstrate (4.7), we use the fact that, for i, m as above and $\varepsilon > 0$, we have

$$\int_{B^{i,k}_\varepsilon} |u_m|^2\,\mathrm{d}x \leqslant C\biggl(\varepsilon^2\int_{B^{i,k}_\varepsilon} |\partial_{x_i} u_m|^2\,\mathrm{d}x + \frac{1}{d_\varepsilon}\int_{P^k_\varepsilon} |u_m|^2\,\mathrm{d}x\biggr).$$

Adding in $k \in \mathbb{Z}^N$ and using (4.5) and (4.8), we conclude that

$$\int_{\Omega_\varepsilon^i} |u_m|^2 \,\mathrm{d}x \leqslant C\left(1 + \frac{\varepsilon^2}{d_\varepsilon^2}\right) \int_{\Omega_\varepsilon} |e(u)|^2 \,\mathrm{d}x,$$

which implies that (4.7). □

Remark 4.4. In theorem 4.3, we consider homogeneous Dirichlet boundary conditions everywhere on the outer boundary. However, the same proof shows that the result holds true if we suppose, for every $i \in \{1, \dots, N\}$, that $u_{\varepsilon,i} = 0$ on $\partial\Omega_\varepsilon \cap V_\varepsilon^i$.

4.2. Compactness result

In this subsection we consider a sequence u_ε that satisfies

$$\frac{1}{|\Omega_\varepsilon|} \int_{\Omega_\varepsilon} |e(u_\varepsilon)|^2 \,\mathrm{d}x \leqslant C \quad \forall \varepsilon > 0, \tag{4.10}$$

with Ω_ε the model structure defined in § 2, and we obtain a compactness result (theorem 4.7) for the sequences $\hat{u}_\varepsilon^i$, $i \in \{1, \dots, N\}$, defined by (2.3). This result will be applied later to the sequence of solutions of (1.1). We start with the following lemma.

Lemma 4.5. *Let u_ε be a sequence in $H^1_{\Gamma_\varepsilon}(\Omega_\varepsilon)^N$ such that* (4.10) *holds and define $\bar{u}_\varepsilon : \mathbb{R}^N \to \mathbb{R}^N$ by*

$$\bar{u}_\varepsilon(x) = \frac{1}{|P_\varepsilon(x)|} \int_{P_\varepsilon(x)} u_\varepsilon(\eta) \,\mathrm{d}\eta. \tag{4.11}$$

Then, for every $i \in \{1, \dots, N\}$, there exists $\hat{u}_0^i \in E_0^i$ such that, up to a subsequence, we have

$$\bar{u}_{\varepsilon,i} \rightharpoonup \hat{u}_0^i \quad \text{in } L^2(\mathbb{R}^N),$$

$$\frac{\bar{u}_{\varepsilon,i}(\cdot + \varepsilon e_i) - \bar{u}_{\varepsilon,i}(\cdot)}{\varepsilon} \rightharpoonup \partial_{x_i} \hat{u}_0^i \quad \text{in } L^2(\mathbb{R}^N).$$

Proof. Taking into account that $\bar{u}_\varepsilon$ is constant on each cube C_ε^k, $k \in \mathbb{Z}^N$, and using Hölder's inequality, we obtain

$$\int_{\mathbb{R}^N} |\bar{u}_\varepsilon(x)|^2 \,\mathrm{d}x = \sum_{k \in \mathbb{Z}^N} \left(\frac{1}{\varepsilon^N d_\varepsilon^N}\right)^2 \int_{C_\varepsilon^k} \left| \int_{P_\varepsilon^k} u_\varepsilon(\eta) \,\mathrm{d}\eta \right|^2 \mathrm{d}x \leqslant \frac{1}{d_\varepsilon^N} \int_{\omega_\varepsilon} |u_\varepsilon(x)|^2 \,\mathrm{d}x.$$

Using (4.5) and (4.10), we deduce that $\bar{u}_\varepsilon$ is bounded in $L^2(\mathbb{R}^N)^N$, and then, for every $i \in \{1, \dots, N\}$, there exists $\hat{u}_0^i$ in $L^2(\mathbb{R}^N)$ such that, up to a subsequence, $\bar{u}_{\varepsilon,i}$ converges to $\hat{u}_0^i$ in the weak topology of $L^2(\mathbb{R}^N)$. On the other hand, the sequence $\bar{v}_\varepsilon$, defined by

$$\bar{v}_{\varepsilon,i} = \frac{\bar{u}_{\varepsilon,i}(\cdot + \varepsilon e_i) - \bar{u}_{\varepsilon,i}(\cdot)}{\varepsilon}, \quad i \in \{1, \dots, N\}, \quad \varepsilon > 0,$$

satisfies

$$\int_{\mathbb{R}^N} |\bar{v}_{\varepsilon,i}|^2 \, \mathrm{d}x = \sum_{k\in\mathbb{Z}^N} \int_{C_\varepsilon^k} |\bar{v}_{\varepsilon,i}|^2 \, \mathrm{d}x = \frac{\varepsilon^N}{(\varepsilon^N d_\varepsilon^N)^2} \sum_{k\in\mathbb{Z}^N} \left| \int_{P_\varepsilon^k} \frac{u_{\varepsilon,i}(\eta+\varepsilon e_i) - u_{\varepsilon,i}(\eta)}{\varepsilon} \, \mathrm{d}\eta \right|^2. \tag{4.12}$$

By (4.10), the estimate

$$\begin{aligned}\left| \int_{P_\varepsilon^k} [u_{\varepsilon,i}(\eta+\varepsilon e_i) - u_{\varepsilon,i}(\eta)] \, \mathrm{d}\eta \right|^2 &\leqslant \left| \int_0^\varepsilon \int_{P_\varepsilon^k} |\partial_{x_i} u_{\varepsilon,i}(\eta + t e_i)| \, \mathrm{d}\eta \, \mathrm{d}t \right|^2 \\ &\leqslant \varepsilon^2 d_\varepsilon^2 \left| \int_{B_\varepsilon^{i,k} \cup B_\varepsilon^{i,k+e_i}} |\partial_{x_i} u_{\varepsilon,i}(\eta)| \, \mathrm{d}\eta \right|^2 \\ &\leqslant C(\varepsilon d_\varepsilon)^2 \varepsilon^N d_\varepsilon^{N-1} \int_{B_\varepsilon^{i,k} \cup B_\varepsilon^{i,k+e_i}} |\partial_{x_i} u_{\varepsilon,i}(\eta)|^2 \, \mathrm{d}\eta\end{aligned}$$

and (4.12), we derive

$$\|\bar{v}_{\varepsilon,i}\|^2_{L^2(\mathbb{R}^N)} \leqslant \frac{C}{d_\varepsilon^{N-1}} \int_{\Omega_\varepsilon^i} |\partial_{x_i} u_{\varepsilon,i}(\eta)|^2 \, \mathrm{d}\eta \leqslant C.$$

Then there exists $\bar{v} \in L^2(\mathbb{R}^N)^N$ such that, up to a subsequence, $\bar{v}_\varepsilon$ converges weakly to $\bar{v}$ in $L^2(\mathbb{R}^N)^N$.

Now, for $\varphi \in C_0^\infty(\mathbb{R}^N)$, we have

$$\begin{aligned}\int_{\mathbb{R}^N} \bar{v}_i(x)\varphi(x) \, \mathrm{d}x &= \int_{\mathbb{R}^N} \bar{v}_{\varepsilon,i}(x)\varphi(x) \, \mathrm{d}x + O_\varepsilon \\ &= \int_{\mathbb{R}^N} \bar{u}_{\varepsilon,i}(x) \frac{\varphi(x-\varepsilon e_i) - \varphi(x)}{\varepsilon} \, \mathrm{d}x + O_\varepsilon \\ &= -\int_{\mathbb{R}^N} \hat{u}_0^i(x) \partial_{x_i}\varphi(x) \, \mathrm{d}x + O_\varepsilon \quad \forall i \in \{1,\dots,N\}.\end{aligned}$$

This implies that $\bar{v}_i = \partial_{x_i}\hat{u}_0^i$ for every $i \in \{1,\dots,N\}$. Furthermore, since $\bar{u}_\varepsilon = 0$ in $\{x \in \mathbb{R}^N : \mathrm{dist}(x,\Omega) > \varepsilon\sqrt{N}\}$ and Ω is smooth, we conclude that $\hat{u}_0^i$ belongs to E_0^i for every $i \in \{1,\dots,N\}$. □

Remark 4.6. The definition (4.11) of $\bar{u}_\varepsilon$ is closely related to the operator P_ε considered in [3].

Theorem 4.7. *We assume that there exists*

$$\lim_{\varepsilon\to 0} \left(\frac{\varepsilon}{d_\varepsilon}\right) = \vartheta \in [0,+\infty]$$

(*this always holds for a subsequence*). *Let u_ε be a sequence in $H^1_{\Gamma_\varepsilon}(\Omega_\varepsilon)^N$ that satisfies (4.10) and define $\hat{u}_\varepsilon^i$, $i \in \{1,\dots,N\}$, by (2.3). Then, for every $i \in \{1,\dots,N\}$,* *there exists a subsequence of* ε*, still denoted by* ε*, and* $(\hat{u}_0^i, \hat{u}_1^i, \hat{u}_2^i, \hat{u}_3^i) \in E^i$ *such that*

$$\hat{u}_{\varepsilon,i}^i \rightharpoonup \hat{u}_0^i \quad \text{in } L^2(\Omega \times Y^N), \tag{4.13}$$

$$\left.\begin{aligned} \hat{u}_{\varepsilon,m}^i &\rightharpoonup \hat{u}_0^m && \text{if } \vartheta = 0, \\ \hat{u}_{\varepsilon,m}^i &\rightharpoonup \hat{u}_0^m + \vartheta \hat{u}_{1,m}^i && \text{if } \vartheta \in (0,+\infty) \quad \text{in } L^2(\Omega \times Y^N) \quad \forall m \neq i, \\ \frac{d_\varepsilon}{\varepsilon}\hat{u}_{\varepsilon,m}^i &\rightharpoonup \hat{u}_{1,m}^i && \text{if } \vartheta = +\infty, \end{aligned}\right\} \tag{4.14}$$

$$e_\varepsilon^i(\hat{u}_\varepsilon^i) \rightharpoonup e_0^i(\hat{u}_0^i, \hat{u}_1^i, \hat{u}_2^i, \hat{u}_3^i) \quad \text{in } L^2(\Omega \times Y^N; \mathcal{S}_N), \tag{4.15}$$

where $e_0^i(\hat{u}_0^i, \hat{u}_1^i, \hat{u}_2^i, \hat{u}_3^i)$ *is defined by (2.4).*

Proof. We fix $i \in \{1, \dots, N\}$ and we proceed in several steps.

Step 1. Using the change of variables (2.2) and the fact that, for every $k \in \mathbb{Z}^N$, $\hat{u}_\varepsilon^i(x, y)$ does not depend on x in $C_\varepsilon^k \times Y^N$, we obtain

$$\begin{aligned} \int_{\Omega_\varepsilon^i} |u_{\varepsilon,m}(x)|^2 \,\mathrm{d}x &= \sum_{k\in\mathbb{Z}^N} \int_{B_\varepsilon^{i,k}} |u_{\varepsilon,m}(x)|^2 \,\mathrm{d}x \\ &= d_\varepsilon^{N-1} \sum_{k\in\mathbb{Z}^N} \varepsilon^N \int_{Y^N} |\hat{u}_{\varepsilon,m}^i(\varepsilon k, y)|^2 \,\mathrm{d}y \\ &= d_\varepsilon^{N-1} \sum_{k\in\mathbb{Z}^N} \int_{C_\varepsilon^k} \int_{Y^N} |\hat{u}_{\varepsilon,m}^i(x, y)|^2 \,\mathrm{d}y \,\mathrm{d}x \\ &= d_\varepsilon^{N-1} \int_{\mathbb{R}^N\times Y^N} |\hat{u}_{\varepsilon,m}^i(x, y)|^2 \,\mathrm{d}y \,\mathrm{d}x \quad \forall m \in \{1, \dots, N\}. \end{aligned}$$

Then, from (4.6), (4.7) and (4.10), we derive

$$\int_{\mathbb{R}^N\times Y^N} |\hat{u}_{\varepsilon,i}^i(x, y)|^2 \,\mathrm{d}y \,\mathrm{d}x \leqslant C, \tag{4.16}$$

$$\int_{\mathbb{R}^N\times Y^N} |\hat{u}_{\varepsilon,m}^i(x, y)|^2 \,\mathrm{d}y \,\mathrm{d}x \leqslant C\left(1 + \frac{\varepsilon^2}{d_\varepsilon^2}\right) \quad \forall m \in \{1, \dots, N\} \setminus \{i\}. \tag{4.17}$$

Reasoning analogously with $\partial_{x_i} u_{\varepsilon,m}$, $m \neq i$, and $e(u_\varepsilon)$, and taking into account (4.8) and (4.10), we deduce that

$$\int_{\mathbb{R}^N\times Y^N} |\partial_{y_i} \hat{u}_{\varepsilon,m}^i(x, y)|^2 \,\mathrm{d}y \,\mathrm{d}x \leqslant C\left(1 + \frac{\varepsilon^2}{d_\varepsilon^2}\right) \quad \forall m \neq i \tag{4.18}$$

and

$$\int_{\mathbb{R}^N\times Y^N} |e_\varepsilon^i(\hat{u}_\varepsilon^i)|^2 \,\mathrm{d}y \,\mathrm{d}x \leqslant C. \tag{4.19}$$

From (4.18) and (4.19), it immediately follows that

$$\int_{\mathbb{R}^N\times Y^N} |\partial_{y_m} \hat{u}_{\varepsilon,i}^i(x, y)|^2 \,\mathrm{d}y \,\mathrm{d}x \leqslant C(\varepsilon^2 + d_\varepsilon^2) \quad \forall m \neq i. \tag{4.20}$$

https://doi.org/10.1017/S0308210500003620 Published online by Cambridge University Press

Taking into account

$$\frac{1}{d_\varepsilon}\int_{|y_i|<d_\varepsilon/2}\hat{u}^i_{\varepsilon,i}(\cdot,y)\,\mathrm{d}y=\bar{u}_{\varepsilon,i}(\cdot), \tag{4.21}$$

with $\bar{u}_{\varepsilon,i}$ defined by (4.11), and using (4.19) and (4.20), we obtain

$$\|\hat{u}^i_{\varepsilon,i}-\bar{u}_{\varepsilon,i}\|^2_{L^2(\mathbb{R}^N\times Y^N)}\leqslant C\|\nabla_y\hat{u}^i_{\varepsilon,i}\|^2_{L^2(\mathbb{R}^N\times Y^N)^N}\leqslant C(\varepsilon^2+d_\varepsilon^2)\quad\forall\varepsilon>0.$$

Then, from lemma 4.5, we deduce that there exists $\hat{u}^i_0\in E^i_0$ such that, up to a subsequence, equation (4.13) holds.

From (4.19), we derive that there exists a subsequence of ε, still denoted by ε, such that $e^i_\varepsilon(\hat{u}^i_\varepsilon)$ converges weakly in $L^2(\mathbb{R}^N\times Y^N;\mathcal{S}_N)$. The following steps of the proof are devoted to characterizing this limit.

Step 2. We define $\hat{w}^i_\varepsilon:\mathbb{R}^N\times Y^N\to\mathbb{R}^N$ by

$$\begin{aligned}&\hat{w}^i_{\varepsilon,i}(x,y)\\&\quad=\frac{1}{\varepsilon}\biggl(\hat{u}^i_{\varepsilon,i}(x,y)-\frac{1}{d_\varepsilon}\int_{|\eta_i|<d_\varepsilon/2}\hat{u}^i_{\varepsilon,i}(x,\eta)\,\mathrm{d}\eta-\sum_{n\neq i}\int_{Y^N}\partial_{y_n}\hat{u}^i_{\varepsilon,i}(x,\eta)\,\mathrm{d}\eta\,y_n\biggr),\\&\hat{w}^i_{\varepsilon,m}(x,y)\\&\quad=\frac{d_\varepsilon}{\varepsilon}\biggl(\hat{u}^i_{\varepsilon,m}(x,y)-\frac{1}{d_\varepsilon}\int_{-d_\varepsilon/2}^{d_\varepsilon/2}\hat{u}^i_{\varepsilon,m}(x,\eta_ie_i+y_i')\,\mathrm{d}\eta_i-\int_{Y^N}\partial_{y_i}\hat{u}^i_{\varepsilon,m}(x,\eta)\,\mathrm{d}\eta\,y_i\biggr),\\&\qquad\qquad m\neq i,\end{aligned}$$

for every $\varepsilon>0$ and a.e. $(x,y)\in\mathbb{R}^N\times Y^N$. By (4.19), $e_y(\hat{w}^i_\varepsilon)_{ii}$ is bounded in $L^2(\mathbb{R}^N\times Y^N)$ and $e_y(\hat{w}^i_\varepsilon)_{in}$, $e_y(\hat{w}^i_\varepsilon)_{mn}$ converge strongly to zero in $L^2(\mathbb{R}^N\times Y^N)$ for every $m,n\in\{1,\dots,N\}\setminus\{i\}$. Since

$$\int_{Y^N}\partial_{y_m}\hat{w}^i_{\varepsilon,i}(x,\eta)\,\mathrm{d}\eta=\int_{Y^N}\partial_{y_i}\hat{w}^i_{\varepsilon,m}(x,\eta)\,\mathrm{d}\eta=0\quad\text{a.e. }x\in\mathbb{R}^N\quad\forall m\in\{1,\dots,N\}\setminus\{i\},$$

lemma 4.1 implies that, for every $m\neq i$, $\partial_{y_m}\hat{w}^i_{\varepsilon,i}$ and $\partial_{y_i}\hat{w}^i_{\varepsilon,m}$ are bounded in $L^2(\mathbb{R}^N\times Y^N)$. Moreover, since

$$\int_{|\eta_i|<d_\varepsilon/2}\hat{w}^i_{\varepsilon,i}(x,\eta)\,\mathrm{d}\eta=0\quad\text{a.e. }x\in\mathbb{R}^N \tag{4.22}$$

and

$$\int_{-d_\varepsilon/2}^{d_\varepsilon/2}\hat{w}^i_{\varepsilon,m}(x,\eta_ie_i+y_i')\,\mathrm{d}\eta_i=0\quad\forall m\neq i\quad\text{a.e. }(x,y_i')\in\mathbb{R}^N\times S^i, \tag{4.23}$$

the Poincaré–Wirtinger inequality gives

$$\int_{Y^N}|\hat{w}^i_{\varepsilon,i}(x,\eta)|^2\,\mathrm{d}\eta\leqslant C\int_{Y^N}|\nabla_y\hat{w}^i_{\varepsilon,i}(x,\eta)|^2\,\mathrm{d}\eta\quad\text{a.e. }x\in\mathbb{R}^N$$

and

$$\int_Y |\hat{w}^i_{\varepsilon,m}(x,\eta_i e_i + y_i')|^2 \,\mathrm{d}\eta_i \leqslant C \int_Y |\partial_{y_i}\hat{w}^i_{\varepsilon,m}(x,\eta_i e_i + y_i')|^2 \,\mathrm{d}\eta_i \qquad \forall m \neq i \quad \text{a.e. } (x,y_i') \in \mathbb{R}^N \times S^i.$$

Integrating these inequalities, we derive that $\hat{w}^i_\varepsilon$ is bounded in $L^2(\mathbb{R}^N \times Y^N)^N$, and, as $e_y(\hat{w}^i_\varepsilon)$ is also bounded in $L^2(\mathbb{R}^N \times Y^N;\mathcal{S}_N)$, from Korn's inequality, we conclude that $\hat{w}^i_\varepsilon$ is really bounded in $L^2(\mathbb{R}^N;H^1(Y^N))^N$. Thus there exist a subsequence of ε, still denoted by ε, and $\hat{w}^i \in L^2(\mathbb{R}^N;H^1(Y^N))^N$ such that

$$\hat{w}^i_\varepsilon \rightharpoonup \hat{w}^i \quad \text{in } L^2(\mathbb{R}^N;H^1(Y^N))^N.$$

Clearly,

$$e_y(\hat{w}^i)_{in} = e_y(\hat{w}^i)_{mn} = 0 \quad \text{for every } m,n \in \{1,\dots,N\} \setminus \{i\}.$$

Furthermore, by (4.23), $\hat{w}^i_m(x,y_i') = 0$, for every $m \neq i$ and a.e. $(x,y_i') \in \mathbb{R}^N \times S^i$.

By the definition of $\hat{u}^i_\varepsilon$, we have that

$$\hat{u}^i_{\varepsilon,j}(x+\varepsilon e_i,\cdot) = \hat{u}^i_{\varepsilon,j}(x,\cdot+e_i) \quad \text{in } L^2(\{y_i = -\tfrac{1}{2}\}) \quad \forall j \in \{1,\dots,N\} \quad \text{a.e. } x \in \mathbb{R}^N. \tag{4.24}$$

From (4.24) with $j = i$, it follows that, for a.e. $y \in \{y_i = -\frac{1}{2}\}$ and a.e. $x \in \mathbb{R}^N$,

$$\begin{aligned}
&\hat{w}^i_{\varepsilon,i}(x,y+e_i) - \hat{w}^i_{\varepsilon,i}(x+\varepsilon e_i,y)\\
&\quad = \frac{1}{\varepsilon d_\varepsilon}\int_{|\eta_i|<d_\varepsilon/2} (\hat{u}^i_{\varepsilon,i}(x+\varepsilon e_i,\eta) - \hat{u}^i_{\varepsilon,i}(x,\eta))\,\mathrm{d}\eta\\
&\qquad + \frac{1}{\varepsilon}\sum_{n\neq i}\int_{Y^N} (\partial_{y_n}\hat{u}^i_{\varepsilon,i}(x+\varepsilon e_i,\eta) - \partial_{y_n}\hat{u}^i_{\varepsilon,i}(x,\eta))\,\mathrm{d}\eta\, y_n.
\end{aligned} \tag{4.25}$$

From (4.21) and lemma 4.5, we obtain

$$\frac{1}{\varepsilon d_\varepsilon}\int_{|\eta_i|<d_\varepsilon/2} (\hat{u}^i_{\varepsilon,i}(x+\varepsilon e_i,\eta) - \hat{u}^i_{\varepsilon,i}(x,\eta))\,\mathrm{d}\eta \rightharpoonup \partial_{x_i}\hat{u}^i_0 \quad \text{in } L^2(\mathbb{R}^N).$$

We now consider $\varphi \in C_0^\infty(\mathbb{R}^N)$ and $n \in \{1,\dots,N\} \setminus \{i\}$. By (4.20), we have

$$\begin{aligned}
&\frac{1}{\varepsilon}\int_{\mathbb{R}^N}\int_{Y^N} (\partial_{y_n}\hat{u}^i_{\varepsilon,i}(x+\varepsilon e_i,\eta) - \partial_{y_n}\hat{u}^i_{\varepsilon,i}(x,\eta))\,\mathrm{d}\eta\varphi(x)\,\mathrm{d}x\\
&\qquad = \int_{\mathbb{R}^N}\int_{Y^N} \partial_{y_n}\hat{u}^i_{\varepsilon,i}(x,\eta)\,\mathrm{d}\eta\frac{\varphi(x-\varepsilon e_i)-\varphi(x)}{\varepsilon}\,\mathrm{d}x + O_\varepsilon = O_\varepsilon.
\end{aligned}$$

Thus the second term on the right-hand side of (4.25) tends to zero in the sense of the distributions. So, passing to the limit in (4.25), we get

$$\hat{w}^i_i(x,\cdot+e_i) - \hat{w}^i_i(x,\cdot) = \partial_{x_i}\hat{u}^i_0(x) \quad \text{in } L^2(\{y_i = -\tfrac{1}{2}\}) \quad \text{a.e. } x \in \mathbb{R}^N.$$

https://doi.org/10.1017/S0308210500003620 Published online by Cambridge University Press

On the other hand, using (4.24) for $j = m \in \{1,\dots,N\}\setminus\{i\}$, we deduce that, for a.e. $y \in \{y_i = -\frac{1}{2}\}$ and a.e. $x \in \mathbb{R}^N$,

$$\begin{aligned}\hat{w}^i_{\varepsilon,m}(x, y+e_i) - \hat{w}^i_{\varepsilon,m}(x+\varepsilon e_i, y) &= \frac{1}{\varepsilon}\int_{-d_\varepsilon/2}^{d_\varepsilon/2}(\hat{u}^i_{\varepsilon,m}(x+\varepsilon e_i, \eta_i e_i + y_i') - \hat{u}^i_{\varepsilon,m}(x, \eta_i e_i + y_i'))\,\mathrm{d}\eta_i \\ &\quad - \frac{d_\varepsilon}{2\varepsilon}\int_{Y^N}(\partial_{y_i}\hat{u}^i_{\varepsilon,m}(x+\varepsilon e_i, \eta) + \partial_{y_i}\hat{u}^i_{\varepsilon,m}(x,\eta))\,\mathrm{d}\eta. \qquad (4.26)\end{aligned}$$

We study the two terms on the right-hand side of (4.26). To calculate the limit in $\mathcal{D}'(\mathbb{R}^N \times S^i)$ of the first term, we take $\varphi \in C_0^\infty(\mathbb{R}^N \times S^i)$. We have

$$\begin{aligned}&\int_{\mathbb{R}^N}\int_{S^i}\frac{1}{\varepsilon}\int_{-d_\varepsilon/2}^{d_\varepsilon/2}(\hat{u}^i_{\varepsilon,m}(x+\varepsilon e_i, \eta_i e_i + y_i') - \hat{u}^i_{\varepsilon,m}(x, \eta_i e_i + y_i'))\,\mathrm{d}\eta_i\varphi(x,y_i')\,\mathrm{d}y_i'\,\mathrm{d}x \\ &\qquad = \int_{\mathbb{R}^N}\int_{S^i}\int_{-d_\varepsilon/2}^{d_\varepsilon/2}\hat{u}^i_{\varepsilon,m}(x, \eta_i e_i + y_i')\,\mathrm{d}\eta_i\frac{\varphi(x-\varepsilon e_i, y_i') - \varphi(x,y_i')}{\varepsilon}\,\mathrm{d}y_i'\,\mathrm{d}x. \qquad (4.27)\end{aligned}$$

An easy calculation shows that

$$\begin{aligned}&\frac{1}{d_\varepsilon}\int_{-d_\varepsilon/2}^{d_\varepsilon/2}\hat{u}^i_{\varepsilon,m}(x,\eta_i e_i + y_i')\,\mathrm{d}\eta_i \\ &\qquad = \frac{1}{d_\varepsilon}\int_{-d_\varepsilon/2}^{d_\varepsilon/2}u_{\varepsilon,m}\bigg(\varepsilon\kappa\bigg(\frac{x}{\varepsilon}\bigg) + \varepsilon d_\varepsilon\frac{\eta_i}{d_\varepsilon}e_i + \varepsilon(d_\varepsilon y_m)e_m + \varepsilon d_\varepsilon\sum_{n\neq i,m}y_n e_n\bigg)\,\mathrm{d}\eta_i \\ &\qquad = \int_{-1/2}^{1/2}\hat{u}^m_{\varepsilon,m}\bigg(x, \tau_i e_i + d_\varepsilon y_m e_m + \sum_{n\neq i,m}y_n e_n\bigg)\,\mathrm{d}\tau_i,\end{aligned}$$

for every $m \neq i$ and a.e. $(x, y_i') \in \mathbb{R}^N \times S^i$. Taking into account (4.21) and the corresponding estimates to (4.19) and (4.20) for $\hat{u}^m_\varepsilon$, we derive

$$\begin{aligned}\int_\Omega\int_{S^i}\bigg|\bar{u}_{\varepsilon,m}(x) - \frac{1}{d_\varepsilon}\int_{-d_\varepsilon/2}^{d_\varepsilon/2}\hat{u}^i_{\varepsilon,m}(x,\eta_i e_i + y_i')d\eta_i\bigg|^2\,\mathrm{d}y_i'\,\mathrm{d}x &\leqslant C\|\nabla_y\hat{u}^m_{\varepsilon,m}\|_{L^2(\Omega\times Y^N)^N} \\ &\leqslant C(\varepsilon^2 + d_\varepsilon^2).\end{aligned}$$

Thus, up to a subsequence, we have

$$\frac{1}{d_\varepsilon}\int_{-d_\varepsilon/2}^{d_\varepsilon/2}\hat{u}^i_{\varepsilon,m}(\cdot,\eta_i e_i + \cdot)\,\mathrm{d}\eta_i \rightharpoonup \hat{u}^m_0 \quad \text{in } L^2(\Omega\times S^i) \quad \forall m \in \{1,\dots,N\}\setminus\{i\}. \qquad (4.28)$$

So, from (4.27) and (4.28), we get

$$\frac{1}{\varepsilon}\int_{-d_\varepsilon/2}^{d_\varepsilon/2}(\hat{u}^i_{\varepsilon,m}(x+\varepsilon e_i, \eta_i e_i + y_i') - \hat{u}^i_{\varepsilon,m}(x, \eta_i e_i + y_i'))\,\mathrm{d}\eta_i \to 0 \quad \text{in } \mathcal{D}'(\mathbb{R}^N \times S^i). \qquad (4.29)$$

For the second term on the right-hand side of (4.26), we take $\varphi \in C_0^\infty(\mathbb{R}^N)$, and, using the change of variables (2.2) and the estimates (4.8) and (4.10), we get

$$
\begin{aligned}
\frac{d_\varepsilon}{2\varepsilon}\int_{\mathbb{R}^N}\int_{Y^N}(\partial_{y_i}\hat{u}^i_{\varepsilon,m}(x+\varepsilon e_i,\eta)+\partial_{y_i}\hat{u}^i_{\varepsilon,m}(x,\eta))\,\mathrm{d}\eta\varphi(x)\,\mathrm{d}x \\
&= \frac{d_\varepsilon}{\varepsilon}\sum_{k\in\mathbb{Z}^N}\frac{1}{\varepsilon^N d_\varepsilon^{N-1}}\int_{B_\varepsilon^{i,k}}\varepsilon\partial_{x_i}u_{\varepsilon,m}(\rho)\int_{C_\varepsilon^k}\tfrac{1}{2}(\varphi(x-\varepsilon e_i)+\varphi(x))\,\mathrm{d}x\,\mathrm{d}\rho \\
&= \frac{d_\varepsilon}{d_\varepsilon^{N-1}}\sum_{k\in\mathbb{Z}^N}\int_{B_\varepsilon^{i,k}}\partial_{x_i}u_{\varepsilon,m}(\rho)\varphi(\rho)\,\mathrm{d}\rho + O_\varepsilon \\
&= \frac{d_\varepsilon}{d_\varepsilon^{N-1}}\int_{\Omega_\varepsilon^i}\partial_{x_i}u_{\varepsilon,m}(\rho)\varphi(\rho)\,\mathrm{d}\rho + O_\varepsilon \\
&= -\frac{d_\varepsilon}{d_\varepsilon^{N-1}}\int_{\Omega_\varepsilon^i}u_{\varepsilon,m}(\rho)\partial_{x_i}\varphi(\rho)\,\mathrm{d}\rho + O_\varepsilon. \qquad (4.30)
\end{aligned}
$$

By (4.7) and (4.10), we have

$$
\left|\frac{d_\varepsilon}{d_\varepsilon^{N-1}}\int_{\Omega_\varepsilon^i}u_{\varepsilon,m}(\rho)\partial_{x_i}\varphi(\rho)\,\mathrm{d}\rho\right|^2 \leqslant C\frac{d_\varepsilon^2}{d_\varepsilon^{N-1}}\int_{\Omega_\varepsilon^i}|u_{\varepsilon,m}(\rho)|^2\,\mathrm{d}\rho \leqslant C(\varepsilon^2+d_\varepsilon^2).
$$

So (4.30) gives

$$
\frac{d_\varepsilon}{2\varepsilon}\int_{Y^N}(\partial_{y_i}\hat{u}^i_{\varepsilon,m}(x+\varepsilon e_i,\eta)+\partial_{y_i}\hat{u}^i_{\varepsilon,m}(x,\eta))\,\mathrm{d}\eta \to 0 \quad \text{in } \mathcal{D}'(\mathbb{R}^N). \tag{4.31}
$$

Using (4.29) and (4.31), we can pass to the limit in (4.26) to conclude that

$$
\hat{w}^i_m(x,\cdot+e_i)=\hat{w}^i_m(x,\cdot) \quad \text{in } L^2(\{y_i=-\tfrac{1}{2}\}) \quad \text{a.e. } x\in\mathbb{R}^N \quad \forall m\neq i.
$$

Hence the function

$$
\hat{u}^i_1(x,y)=\hat{w}^i(x,y)-\partial_{x_i}\hat{u}^i_0(x)y_ie_i, \quad (x,y)\in\mathbb{R}^N\times Y^N,
$$

belongs to E^i_1 and satisfies

$$
e^i_\varepsilon(\hat{u}^i_\varepsilon)_{ii} \rightharpoonup \partial_{x_i}\hat{u}^i_0+\partial_{y_i}\hat{u}^i_{1,i} \quad \text{in } L^2(\mathbb{R}^N\times Y^N).
$$

In order to prove (4.14), we distinguish between two cases, depending on the ratio of ε to d_ε. First, we suppose that $\vartheta\in[0,+\infty)$. From (4.28), the definition of $\hat{w}^i_{\varepsilon,m}$ and its convergence to $\hat{u}^i_{1,m}$ in the weak topology of $L^2(\mathbb{R}^N\times Y^N)$, $m\neq i$, we have

$$
\hat{u}^i_{\varepsilon,m}(x,y)-\hat{u}^m_0(x)-\int_{Y^N}\partial_{y_i}\hat{u}^i_{\varepsilon,m}(x,\eta)\,\mathrm{d}\eta\,y_i-\frac{\varepsilon}{d_\varepsilon}\hat{u}^i_{1,m}(x,y) \rightharpoonup 0 \quad \text{in } L^2(\mathbb{R}^N\times Y^N). \tag{4.32}
$$

By (4.18), we also have that $\partial_{y_i}\hat{u}^i_{\varepsilon,m}$, $m\neq i$, is bounded in $L^2(\mathbb{R}^N\times Y^N)$. We consider φ in $C_0^\infty(\mathbb{R}^N)$. Using the change of variables (2.2) and the estimates (4.7) and (4.10), we obtain

$$\begin{aligned}\left|\int_{\mathbb{R}^N}\int_{Y^N}\partial_{y_i}\hat{u}^i_{\varepsilon,m}(x,\eta)\,\mathrm{d}\eta\varphi(x)\,\mathrm{d}x\right| &= \left|\frac{\varepsilon}{d_\varepsilon^{N-1}}\int_{\Omega^i_\varepsilon}\partial_{x_i}u_{\varepsilon,m}(z)\varphi(z)\,\mathrm{d}z\right| + O_\varepsilon \\ &= \left|\frac{\varepsilon}{d_\varepsilon^{N-1}}\int_{\Omega^i_\varepsilon}u_{\varepsilon,m}(z)\partial_{x_i}\varphi(z)\,\mathrm{d}z\right| + O_\varepsilon \\ &\leqslant C\varepsilon\left(1+\frac{\varepsilon^2}{d_\varepsilon^2}\right)^{1/2} + O_\varepsilon \\ &\leqslant C\varepsilon + O_\varepsilon \\ &= O_\varepsilon, \end{aligned} \tag{4.33}$$

which, joined to (4.18), implies that

$$\int_{Y^N}\partial_{y_i}\hat{u}^i_{\varepsilon,m}(x,\eta)\,\mathrm{d}\eta\, y_i \rightharpoonup 0 \quad \text{in } L^2(\mathbb{R}^N\times Y^N) \quad \forall m\neq i.$$

Hence, from (4.32), we derive

$$\hat{u}^i_{\varepsilon,m} \rightharpoonup \hat{u}^m_0 + \vartheta\hat{u}^i_{1,m} \quad \text{in } L^2(\mathbb{R}^N\times Y^N) \quad \forall m\neq i \quad \text{if } \vartheta\in[0,+\infty). \tag{4.34}$$

Let us now suppose that $\vartheta = \lim_{\varepsilon\to 0}(\varepsilon/d_\varepsilon) = +\infty$. In this case, from (4.28) and the definition of $\hat{w}^i_{\varepsilon,m}$, $m\neq i$, we obtain

$$\frac{d_\varepsilon}{\varepsilon}\hat{u}^i_{\varepsilon,m}(x,y) - \frac{d_\varepsilon}{\varepsilon}\int_{Y^N}\partial_{y_i}\hat{u}^i_{\varepsilon,m}(x,\eta)\,\mathrm{d}\eta\, y_i - \hat{u}^i_{1,m}(x,y) \rightharpoonup 0 \quad \text{in } L^2(\mathbb{R}^N\times Y^N) \quad \forall m\neq i. \tag{4.35}$$

By virtue of (4.18), $(d_\varepsilon/\varepsilon)\partial_{y_i}\hat{u}^i_{\varepsilon,m}$ is bounded in $L^2(\Omega\times Y^N)$ and, reasoning as in (4.33), we get

$$\begin{aligned}\left|\frac{d_\varepsilon}{\varepsilon}\int_{\mathbb{R}^N}\int_{Y^N}\partial_{y_i}\hat{u}^i_{\varepsilon,m}(x,\eta)\,\mathrm{d}\eta\varphi(x)\,\mathrm{d}x\right| &= \left|\frac{d_\varepsilon}{\varepsilon}\frac{\varepsilon}{d_\varepsilon^{N-1}}\int_{\Omega^i_\varepsilon}u_{\varepsilon,m}(z)\partial_{x_i}\varphi(z)\,\mathrm{d}z\right| + O_\varepsilon \\ &\leqslant Cd_\varepsilon\left(1+\frac{\varepsilon^2}{d_\varepsilon^2}\right)^{1/2} + O_\varepsilon \\ &= O_\varepsilon \quad \forall\varphi\in C_0^\infty(\mathbb{R}^N), \end{aligned}$$

which, together with (4.35), implies

$$\frac{d_\varepsilon}{\varepsilon}\hat{u}^i_{\varepsilon,m} \rightharpoonup \hat{u}^i_{1,m} \quad \text{in } L^2(\mathbb{R}^N\times Y^N) \quad \forall m\neq i \quad \text{if } \vartheta = +\infty.$$

This completes the proof of (4.14).

Step 3. Let us now characterize the weak limit of $e^i_\varepsilon(\hat{u}^i_\varepsilon)_{im}$ in $L^2(\mathbb{R}^N\times Y^N)$, $m\neq i$. We start by extending $\hat{u}^i_\varepsilon$ to $\mathbb{R}^N\times(\mathbb{R}e_i+S^i)$ by taking

$$\hat{u}^i_\varepsilon(x,y) = u_\varepsilon\left(\varepsilon\kappa\left(\frac{x}{\varepsilon}\right) + \varepsilon y_i e_i + \varepsilon d_\varepsilon y'_i\right) \quad \forall i\in\{1,\dots,N\} \quad \forall\varepsilon>0.$$

https://doi.org/10.1017/S0308210500003620 Published online by Cambridge University Press

In this way, $\hat{u}^i_\varepsilon$ belongs to $L^2(\mathbb{R}^N; H^1(\mathbb{R}e_i + S^i))^N$ and satisfies

$$\hat{u}^i_\varepsilon(x,y) = \hat{u}^i_\varepsilon(x + n\varepsilon e_i, (y_i - n)e_i + y'_i) \quad \forall n \in \mathbb{Z}, \quad i \in \{1,\dots,N\}, \quad \varepsilon > 0. \tag{4.36}$$

Using this extension, we construct a regularization $\tilde{u}^i_\varepsilon : \mathbb{R}^N \times Y^N \to \mathbb{R}^N$ of $\hat{u}^i_\varepsilon$ by

$$\tilde{u}^i_\varepsilon(x,y) = \frac{1}{d_\varepsilon}\int_{-d_\varepsilon/2}^{d_\varepsilon/2} \hat{u}^i_\varepsilon(x, (y_i + \eta_i)e_i + y'_i)\,\mathrm{d}\eta_i, \quad \varepsilon > 0.$$

Then $\partial_{y_i}\tilde{u}^i_\varepsilon$ belongs to $L^2(\mathbb{R}^N; H^1(Y^N))^N$ for every $\varepsilon > 0$.

We take $\tilde{s}^i_\varepsilon$ as $\tilde{s}^i_{\varepsilon,i} = \tilde{u}^i_{\varepsilon,i}/(\varepsilon d_\varepsilon)$, $\tilde{s}^i_{\varepsilon,m} = \tilde{u}^i_{\varepsilon,m}/\varepsilon$ if $m \neq i$. Using

$$\partial^2_{y_n y_m}\tilde{s}^i_{\varepsilon,i} = \partial_{y_m} e_y(\tilde{s}^i_\varepsilon)_{in} + \partial_{y_n} e_y(\tilde{s}^i_\varepsilon)_{im} - \partial_{y_i} e_y(\tilde{s}^i_\varepsilon)_{nm} \qquad \forall m,n \in \{1,\dots,N\}\setminus\{i\}, \quad \varepsilon > 0,$$

and the fact that, for every $m,n \in \{1,\dots,N\}\setminus\{i\}$, $\partial_{y_m} e_y(\tilde{s}^i_\varepsilon)_{in}$, $\partial_{y_n} e_y(\tilde{s}^i_\varepsilon)_{im}$ and $\partial_{y_i} e_y(\tilde{s}^i_\varepsilon)_{mn}$ belong to $L^2(\mathbb{R}^N \times J^i; H^{-1}(S^i))$, we get

$$\nabla_{y'_i}(\partial_{y_m}\tilde{s}^i_{\varepsilon,i})(x, y_i e_i + \cdot) \in H^{-1}(S^i)^{N-1} \quad \text{for a.e. } (x,y_i) \in \mathbb{R}^N \times Y.$$

Then, by a known result (see, for instance, [24]), $\partial_{y_m}\tilde{s}^i_{\varepsilon,i}(x, y_i e_i + \cdot)$ belongs to $L^2(S^i)$ and satisfies

$$\begin{aligned}\bigg\|\partial_{y_m}\tilde{s}^i_{\varepsilon,i}(x, y_i e_i + \cdot) - \int_{S^i}\partial_{y_m}\tilde{s}^i_{\varepsilon,i}(x, y_i e_i + \eta'_i)\,\mathrm{d}\eta'_i\bigg\|^2_{L^2(S^i)} &\leqslant C\sum_{n\neq i}(\|\partial_{y_m} e_y(\tilde{s}^i_\varepsilon)_{in}(x, y_i e_i + \cdot)\|^2_{H^{-1}(S^i)} \\ &\qquad + \|\partial_{y_n} e_y(\tilde{s}^i_\varepsilon)_{im}(x, y_i e_i + \cdot)\|^2_{H^{-1}(S^i)}) \\ &\quad + C\sum_{n\neq i}\|\partial_{y_i} e_y(\tilde{s}^i_\varepsilon)_{mn}(x, y_i e_i + \cdot)\|^2_{H^{-1}(S^i)}\end{aligned}$$

for every $m \in \{1\dots,N\}\setminus\{i\}$, every $\varepsilon > 0$ and a.e. $(x,y_i) \in \mathbb{R}^N \times Y$. Integrating this inequality in $(x,y_i) \in \mathbb{R}^N \times Y$, we get

$$\begin{aligned}&\bigg\|\partial_{y_m}\tilde{s}^i_{\varepsilon,i} - \int_{S^i}\partial_{y_m}\tilde{s}^i_{\varepsilon,i}d\eta'_i\bigg\|^2_{L^2(\mathbb{R}^N\times Y^N)} \\ &\quad\leqslant C\sum_{n\neq i}(\|\partial_{y_m} e_y(\tilde{s}^i_\varepsilon)_{in}\|^2_{L^2(\mathbb{R}^N\times J^i;H^{-1}(S^i))} + \|\partial_{y_n} e_y(\tilde{s}^i_\varepsilon)_{im}\|^2_{L^2(\mathbb{R}^N\times J^i;H^{-1}(S^i))}) \\ &\qquad + C\sum_{n\neq i}\|\partial_{y_i} e_y(\tilde{s}^i_\varepsilon)_{mn}\|^2_{L^2(\mathbb{R}^N\times J^i;H^{-1}(S^i))} \\ &\quad\leqslant C\sum_{n\neq i}\|e_y(\tilde{s}^i_\varepsilon)_{in}\|^2_{L^2(\mathbb{R}^N\times Y^N)} + C\sum_{n\neq i}\|\partial_{y_i} e_y(\tilde{s}^i_\varepsilon)_{mn}\|^2_{L^2(\mathbb{R}^N\times Y^N)}.\end{aligned} \tag{4.37}$$

Now we use

$$\begin{aligned}\|e_y(\tilde{s}^i_\varepsilon)_{in}\|^2_{L^2(\mathbb{R}^N\times Y^N)} &= \int_{\mathbb{R}^N\times Y^N}\bigg|\frac{1}{d_\varepsilon}\int_{-d_\varepsilon/2}^{d_\varepsilon/2} e^i_\varepsilon(\hat{u}^i_\varepsilon)_{in}(x, (y_i + \eta_i)e_i + y'_i)\,\mathrm{d}\eta_i\bigg|^2\,\mathrm{d}y\,\mathrm{d}x \\ &\leqslant C\|e^i_\varepsilon(\hat{u}^i_\varepsilon)\|^2_{L^2(\mathbb{R}^N\times Y^N)}\end{aligned} \tag{4.38}$$

https://doi.org/10.1017/S0308210500003620 Published online by Cambridge University Press

and

$$\|\partial_{y_i} e_y(\tilde{s}^i_\varepsilon)_{mn}\|^2_{L^2(\mathbb{R}^N\times Y^N)} = \int_{\mathbb{R}^N\times Y^N} |e^i_\varepsilon(\hat{u}^i_\varepsilon)_{mn}(x,(y_i+\tfrac{1}{2}d_\varepsilon)e_i+y'_i) - e^i_\varepsilon(\hat{u}^i_\varepsilon)_{mn}(x,(y_i-\tfrac{1}{2}d_\varepsilon)e_i+y'_i)|^2\,\mathrm{d}y\,\mathrm{d}x \leqslant C\|e^i_\varepsilon(\hat{u}^i_\varepsilon)\|^2_{L^2(\mathbb{R}^N\times Y^N)}. \tag{4.39}$$

So, from (4.37)–(4.39) and (4.19) and the definition of $\tilde{s}^i_{\varepsilon,i}$, we deduce that

$$\left\|\frac{1}{\varepsilon d_\varepsilon}\left(\partial_{y_m}\tilde{u}^i_{\varepsilon,i} - \int_{S^i}\partial_{y_m}\tilde{u}^i_{\varepsilon,i}\,\mathrm{d}\tau'_i\right)\right\|^2_{L^2(\mathbb{R}^N\times Y^N)} \leqslant C \quad \forall m\neq i. \tag{4.40}$$

From (4.19) and (4.40), it easily follows that

$$\left\|\frac{1}{\varepsilon}\left(\partial_{y_i}\tilde{u}^i_{\varepsilon,m} - \int_{S^i}\partial_{y_i}\tilde{u}^i_{\varepsilon,m}\,\mathrm{d}\tau'_i\right)\right\|^2_{L^2(\mathbb{R}^N\times Y^N)} \leqslant C \quad \forall m\neq i. \tag{4.41}$$

We define $\tilde{t}^i_\varepsilon : \mathbb{R}^N\times Y^N \to \mathbb{R}^N$ by

$$\tilde{t}^i_{\varepsilon,i}(x,y) = \frac{1}{\varepsilon d_\varepsilon}\left(\tilde{u}^i_{\varepsilon,i}(x,y) - \int_{S^i}\tilde{u}^i_{\varepsilon,i}(x,y_ie_i+\tau'_i)\,\mathrm{d}\tau'_i - \sum_{n\neq i}\left(\int_{S^i}\partial_{y_n}\tilde{u}^i_{\varepsilon,i}(x,y_ie_i+\tau'_i)\,\mathrm{d}\tau'_i\right)y_n\right),$$

$$\tilde{t}^i_{\varepsilon,m}(x,y) = \frac{1}{\varepsilon}\left(\tilde{u}^i_{\varepsilon,m}(x,y) - \int_Y \tilde{u}^i_{\varepsilon,m}(x,\tau_ie_i+y'_i)\,\mathrm{d}\tau_i - \int_{S^i}\tilde{u}^i_{\varepsilon,m}(x,y_ie_i+\tau'_i)\,\mathrm{d}\tau'_i + \int_{Y^N}\tilde{u}^i_{\varepsilon,m}(x,\tau)\,\mathrm{d}\tau\right) \quad \text{if } m\neq i.$$

From (4.40) and (4.41), we obtain

$$\|\partial_{y_m}\tilde{t}^i_{\varepsilon,i}\|_{L^2(\mathbb{R}^N\times Y^N)} \leqslant C, \qquad \|\partial_{y_i}\tilde{t}^i_{\varepsilon,m}\|_{L^2(\mathbb{R}^N\times Y^N)} \leqslant C \tag{4.42}$$

for every $\varepsilon>0$ and every $i,m\in\{1,\dots,N\}$ with $m\neq i$. Moreover, since

$$\int_{S^i}\tilde{t}^i_{\varepsilon,i}(x,y_ie_i+\eta'_i)\,\mathrm{d}\eta'_i = 0 \quad \text{a.e. } (x,y_i)\in\mathbb{R}^N\times Y \tag{4.43}$$

and

$$\int_Y \tilde{t}^i_{\varepsilon,m}(x,\eta_ie_i+y'_i)\,\mathrm{d}\eta_i = 0 \quad \text{a.e. } (x,y'_i)\in\mathbb{R}^N\times S^i \quad \forall m\neq i, \tag{4.44}$$

the Poincaré–Wirtinger inequality gives

$$\int_{S^i}|\tilde{t}^i_{\varepsilon,i}(x,y_ie_i+\eta'_i)|^2\,\mathrm{d}\eta'_i \leqslant C\sum_{m\neq i}\int_{S^i}|\partial_{y_m}\tilde{t}^i_{\varepsilon,i}(x,y_ie_i+\eta'_i)|^2\,\mathrm{d}\eta'_i \quad \text{a.e. } (x,y_i)\in\mathbb{R}^N\times Y$$

and

$$\int_Y |\tilde{t}^i_{\varepsilon,m}(x,\eta_i e_i + y_i')|^2 \, d\eta_i \leqslant C \int_Y |\partial_{y_i} \tilde{t}^i_{\varepsilon,m}(x,\eta_i e_i + y_i')|^2 \, d\eta_i, \quad \text{a.e. } (x,y_i') \in \mathbb{R}^N \times S^i.$$

So $\tilde{t}^i_{\varepsilon,i}$ and $\tilde{t}^i_{\varepsilon,m}$, $m \neq i$, are, respectively, bounded in

$$L^2(\mathbb{R}^N \times J^i; H^1(S^i)) \quad \text{and} \quad L^2(\mathbb{R}^N \times S^i; H^1(J^i)).$$

Extracting a subsequence of ε if necessary, we deduce that there exists

$$\hat{t}^i = (\hat{t}^i_1, \dots, \hat{t}^i_N)$$

such that

$$\begin{aligned} \tilde{t}^i_{\varepsilon,i} &\rightharpoonup \hat{t}^i_i \quad \text{in } L^2(\mathbb{R}^N \times J^i; H^1(S^i)), \\ \tilde{t}^i_{\varepsilon,m} &\rightharpoonup \hat{t}^i_m \quad \text{in } L^2(\mathbb{R}^N \times S^i; H^1(J^i)) \quad \forall m \neq i. \end{aligned}$$

We remark that

$$\begin{aligned} e_y(\tilde{t}^i_\varepsilon)_{mn}(x,y) &= \int_{-d_\varepsilon/2}^{d_\varepsilon/2} \bigg(e^i_\varepsilon(\hat{u}^i_\varepsilon)_{mn}(x,(y_i+\eta_i)e_i + y_i') \\ &\qquad - \int_Y e^i_\varepsilon(\hat{u}^i_\varepsilon)_{mn}(x,(\tau_i+\eta_i)e_i + y_i') \, d\tau_i \bigg) \, d\eta_i, \\ e_y(\tilde{t}^i_\varepsilon)_{im}(x,y) &= \frac{1}{d_\varepsilon} \int_{-d_\varepsilon/2}^{d_\varepsilon/2} \bigg(e^i_\varepsilon(\hat{u}^i_\varepsilon)_{im}(x,(y_i+\eta_i)e_i + y_i') \\ &\qquad - \int_{S^i} e^i_\varepsilon(\hat{u}^i_\varepsilon)_{im}(x,(y_i+\eta_i)e_i + \tau_i') \, d\tau_i' \bigg) \, d\eta_i, \end{aligned}$$

for every $m, n \in \{1,\dots,N\} \setminus \{i\}$ and a.e. $(x,y) \in \mathbb{R}^N \times Y^N$. So, by (4.19), we can now pass to the limit in ε to deduce

$$\begin{aligned} e_y(\hat{t}^i)_{mn} &= 0 \quad \forall m,n \in \{1,\dots,N\} \setminus \{i\}, \\ e_y(\hat{t}^i)_{im}(x,y) &= \hat{\mu}^i_m(x,y) - \int_{S^i} \hat{\mu}^i_m(x, y_i e_i + \tau_i') \, d\tau_i' \quad \forall m \in \{1,\dots,N\} \setminus \{i\}, \end{aligned} \tag{4.45}$$

where we denote by $\hat{\mu}^i_m$ the weak limit in $L^2(\mathbb{R}^N \times Y^N)$ of $e^i_\varepsilon(\hat{u}^i_\varepsilon)_{im}$, $m \neq i$.

We define $\hat{u}^i_2 = (\hat{u}^i_{2,1}, \dots, \hat{u}^i_{2,N})$ by

$$\begin{aligned} \hat{u}^i_{2,i}(x,y) &= \hat{t}^i_i(x,y) + 2 \sum_{r \neq i} \int_{S^i} \hat{\mu}^i_r(x, y_i e_i + \tau_i') \, d\tau_i' y_r, \\ \hat{u}^i_{2,m}(x,y) &= \hat{t}^i_m(x,y) \quad \forall m \neq i. \end{aligned}$$

Then $\hat{u}^i_{2,i}$ belongs to $L^2(\mathbb{R}^N \times J^i; H^1(S^i))$, $\hat{u}^i_{2,m}$ belongs to $L^2(\mathbb{R}^N \times S^i; H^1(J^i))$ for $m \neq i$ and

$$e^i_\varepsilon(\hat{u}^i_\varepsilon)_{im} \rightharpoonup \hat{\mu}^i_m = e_y(\hat{u}^i_2)_{im} \quad \text{in } L^2(\mathbb{R}^N \times Y^N).$$

From the definition of $\hat{u}^i_{2,m}$ and (4.45), $\hat{u}^i_2$ satisfies

$$\int_{S^i} \hat{u}^i_{2,m}(x, y_i e_i + \eta_i') \, d\eta_i' = 0 \quad \text{a.e. } (x, y_i) \in \mathbb{R}^N \times Y \tag{4.46}$$

https://doi.org/10.1017/S0308210500003620 Published online by Cambridge University Press

and

$$e_y(\hat{u}_2^i)_{mn} = 0 \quad \forall m, n \in \{1, \dots, N\} \setminus \{i\}. \tag{4.47}$$

We prove that, for every $m \neq i$, $\hat{u}_{2,m}^i$ is periodic in the variable y_i. For this purpose, we remark that, since $\hat{u}_{2,m}^i$ belongs to $L^2(\mathbb{R}^N \times S^i; H^1(J^i))$, equations (4.46) and (4.47) imply that, for every m, $n \neq i$, there exists $g_{mn}^i \in L^2(\mathbb{R}^N; H^1(J^i))$ such that $g_{mn}^i = -g_{nm}^i$ and $\hat{u}_{2,m}^i$ satisfies

$$\hat{u}_{2,m}^i(x, y) = \sum_{n \neq i} g_{mn}^i(x, y_i) y_n \quad \forall m \neq i \quad \text{a.e. } (x, y) \in \mathbb{R}^N \times Y^N.$$

In particular, this shows that $\hat{u}_{2,m}^i \in L^2(\mathbb{R}^N; H^1(Y^N))$ for every $m \neq i$. Moreover, in order to prove that $\hat{u}_{2,m}^i$ is periodic in y_i, it suffices to prove this property for the functions g_{mn}^i, $m, n \in \{1, \dots, N\} \setminus \{i\}$. Now

$$\begin{aligned} g_{mn}^i(x, y_i) &= 12 \int_{S^i} \hat{u}_{2,m}^i(x, y_i e_i + \eta_i') \eta_n \, \mathrm{d}\eta_i' \\ &= \lim_{\varepsilon \to 0} 12 \int_{S^i} \tilde{t}_{\varepsilon,m}^i(x, y_i e_i + \eta_i') \eta_n \, \mathrm{d}\eta_i' \quad \text{in } L^2(\mathbb{R}^N; H^1(J^i)). \end{aligned}$$

From the definition of $\tilde{t}_{\varepsilon,m}^i$ and the equality

$$\tilde{u}_{\varepsilon,m}^i(x + \varepsilon e_i, \cdot) = \tilde{u}_{\varepsilon,m}^i(x, \cdot + e_i) \quad \text{in } L^2(\{y_i = -\tfrac{1}{2}\}) \quad \text{a.e. } x \in \mathbb{R}^N,$$

we derive

$$\begin{aligned} & g_{mn}^i(x, \tfrac{1}{2}) - g_{mn}^i(x, -\tfrac{1}{2}) \\ &\quad = \lim_{\varepsilon \to 0} \frac{12}{\varepsilon} \int_{Y^N} (\tilde{u}_{\varepsilon,m}^i(x + \varepsilon e_i, \eta) - \tilde{u}_{\varepsilon,m}^i(x, \eta)) \eta_n \, \mathrm{d}\eta \quad \text{in } L^2(\mathbb{R}^N). \end{aligned} \tag{4.48}$$

For $\varphi \in C_0^\infty(\mathbb{R}^N)$, we have

$$\begin{aligned} & \frac{1}{\varepsilon} \int_{\mathbb{R}^N} \int_{Y^N} (\tilde{u}_{\varepsilon,m}^i(x + \varepsilon e_i, \eta) - \tilde{u}_{\varepsilon,m}^i(x, \eta)) \eta_n \, \mathrm{d}\eta \varphi(x) \, \mathrm{d}x \\ &\quad = \int_{\mathbb{R}^N} \int_{Y^N} \tilde{u}_{\varepsilon,m}^i(x, \eta) \eta_n \, \mathrm{d}\eta \frac{\varphi(x - \varepsilon e_i) - \varphi(x)}{\varepsilon} \, \mathrm{d}x \\ &\quad = \int_{\mathbb{R}^N} \int_{Y^N} (\tilde{r}_{\varepsilon,m}^i(x, \eta) + \tilde{u}_{\varepsilon,m}^i(x, \eta_i')) \eta_n \, \mathrm{d}\eta \frac{\varphi(x - \varepsilon e_i) - \varphi(x)}{\varepsilon} \, \mathrm{d}x, \end{aligned}$$

where we denote

$$\tilde{r}_{\varepsilon,m}^i(x, y) = \tilde{u}_{\varepsilon,m}^i(x, y) - \int_{S^i} \tilde{u}_{\varepsilon,m}^i(x, y_i e_i + \tau_i') \, \mathrm{d}\tau_i' - \tilde{u}_{\varepsilon,m}^i(x, y_i') + \int_{S^i} \tilde{u}_{\varepsilon,m}^i(x, \tau_i') \, \mathrm{d}\tau_i'.$$

From $\tilde{r}_{\varepsilon,m}^i(x, y_i') = 0$, a.e. $(x, y_i') \in \mathbb{R}^N \times S^i$, $\partial_{y_i} \tilde{r}_{\varepsilon,m}^i = \varepsilon \partial_{y_i} \tilde{t}_{\varepsilon,m}^i$, equation (4.42) and the inequality

$$\begin{aligned} & \int_Y |\tilde{r}_{\varepsilon,m}^i(x, \eta_i e_i + y_i') - \tilde{r}_{\varepsilon,m}^i(x, y_i')|^2 \, \mathrm{d}\eta_i \\ &\qquad \leqslant C \int_Y |\partial_{y_i} \tilde{r}_{\varepsilon,m}^i(x, \eta_i e_i + y_i')|^2 \, \mathrm{d}\eta_i \quad \text{a.e. } (x, y_i') \in \mathbb{R}^N \times S^i, \end{aligned}$$

https://doi.org/10.1017/S0308210500003620 Published online by Cambridge University Press

we deduce that

$$\int_{\mathbb{R}^N}\int_{Y^N}\tilde r^i_{\varepsilon,m}(x,\eta)\eta_n\,\mathrm{d}\eta\frac{\varphi(x-\varepsilon e_i)-\varphi(x)}{\varepsilon}\,\mathrm{d}x=O_\varepsilon\quad\forall m\neq i. \tag{4.49}$$

On the other hand, by the definition of $\tilde u^i_{\varepsilon,m}$ and the change of variables (2.2), we have

$$\begin{aligned}\int_{Y^N}\tilde u^i_{\varepsilon,m}(x,\eta_i')\eta_n\,\mathrm{d}\eta&=\frac{1}{d_\varepsilon}\int_{S^i}\int_{-d_\varepsilon/2}^{d_\varepsilon/2}\hat u^i_{\varepsilon,m}(x,\eta_ie_i+\eta_i')\eta_n\,\mathrm{d}\eta_i\,\mathrm{d}\eta_i'\\&=\frac{1}{\varepsilon^Nd_\varepsilon^N}\int_{P_\varepsilon(x)}u_{\varepsilon,m}(z)\frac{z_n-\varepsilon\kappa_n(x/\varepsilon)}{\varepsilon d_\varepsilon}\,\mathrm{d}z\\&=\frac{1}{d_\varepsilon}\int_{S^m}\int_{-d_\varepsilon/2}^{d_\varepsilon/2}\hat u^m_{\varepsilon,m}(x,\eta_me_m+\eta_m')\eta_n\,\mathrm{d}\eta_m\,\mathrm{d}\eta_m'\end{aligned}$$

for a.e. $x\in\Omega$. Then

$$\begin{aligned}&\int_{\mathbb{R}^N}\int_{Y^N}\tilde u^i_{\varepsilon,m}(x,\eta_i')\eta_n\,\mathrm{d}\eta\frac{\varphi(x-\varepsilon e_i)-\varphi(x)}{\varepsilon}\,\mathrm{d}x\\&\quad=\int_{\mathbb{R}^N}\int_{S^m}\biggl(\frac{1}{d_\varepsilon}\int_{-d_\varepsilon/2}^{d_\varepsilon/2}\hat u^m_{\varepsilon,m}(x,\eta_me_m+\eta_m')\,\mathrm{d}\eta_m\biggr)\eta_n\frac{\varphi(x-\varepsilon e_i)-\varphi(x)}{\varepsilon}\,\mathrm{d}\eta_i'\,\mathrm{d}x.\end{aligned}$$

Due to the fact that $\hat u^m_{\varepsilon,m}$ converges weakly in $L^2(\mathbb{R}^N\times Y^N)$ to $\hat u^m_0$, equation (4.19) (with i replaced by m) and the inequality

$$\begin{aligned}\int_{Y^N}\biggl|\hat u^m_{\varepsilon,m}(x,y)-\frac{1}{d_\varepsilon}\int_{-d_\varepsilon/2}^{d_\varepsilon/2}\hat u^m_{\varepsilon,m}(x,\eta_me_m+y_m')\,\mathrm{d}\eta_m\biggr|^2\mathrm{d}y\\\leqslant C\int_{Y^N}|\partial_{y_m}\hat u^m_{\varepsilon,m}(x,y)|^2\,\mathrm{d}y,\end{aligned}$$

we derive

$$\begin{aligned}&\int_{\mathbb{R}^N}\int_{Y^N}\tilde u^i_{\varepsilon,m}(x,\eta_i')\eta_n\,\mathrm{d}\eta\frac{\varphi(x-\varepsilon e_i)-\varphi(x)}{\varepsilon}\,\mathrm{d}x\\&\qquad=-\int_{\mathbb{R}^N}\int_{Y^N}\eta_n\,\mathrm{d}\eta\,\hat u^m_0(x)\partial_{x_i}\varphi(x)\,\mathrm{d}x+O_\varepsilon=O_\varepsilon,\end{aligned}$$

which, together with (4.48) and (4.49), gives that g^i_{mn} is y_i-periodic, for every m, $n\in\{1,\dots,N\}\setminus\{i\}$.

STEP 4. We define $\hat z^i_\varepsilon:\mathbb{R}^N\times Y^N\to\mathbb{R}^N$ as $\hat z^i_{\varepsilon,i}(x,y)=0$ and

$$\begin{aligned}\hat z^i_{\varepsilon,m}(x,y)=\frac{1}{\varepsilon d_\varepsilon}\biggl(\hat u^i_{\varepsilon,m}(x,y)-\int_{S^i}\hat u^i_{\varepsilon,m}(x,y_ie_i+\tau_i')\,\mathrm{d}\tau_i'\\-\sum_{n\neq i}\int_{S^i}\partial_{y_n}\hat u^i_{\varepsilon,m}(x,y_ie_i+\tau_i')\,\mathrm{d}\tau_i'y_n\biggr),\end{aligned}$$

for every $m\neq i$, every $\varepsilon>0$ and a.e. $(x,y)\in\Omega\times Y^N$.

https://doi.org/10.1017/S0308210500003620 Published online by Cambridge University Press

For every $m,n \in \{1,\dots,N\}\setminus\{i\}$, lemma 4.1 and (4.19) imply that $\partial_{y_n}\hat{z}^i_{\varepsilon,m}$ is bounded in $L^2(\mathbb{R}^N\times Y^N)$. Poincaré–Wirtinger's inequality also gives $\hat{z}^i_{\varepsilon,m}$ is bounded in $L^2(\mathbb{R}^N\times Y^N)$. So $\hat{z}^i_{\varepsilon,m}$ is bounded in $L^2(\mathbb{R}^N\times J^i;H^1(S^i))$. Then there exists $\hat{z}^i_m\in L^2(\Omega\times J^i;H^1(S^i))$ such that, for a subsequence, $\hat{z}^i_{\varepsilon,m}$ converges weakly to $\hat{z}^i_m$ in $L^2(\Omega\times J^i;H^1(S^i))$. Moreover, since

$$e_y(\hat{z}^i_\varepsilon)_{mn} = e^i_\varepsilon(\hat{u}^i_\varepsilon)_{mn} - \int_{S^i} e^i_\varepsilon(\hat{u}^i_\varepsilon)_{mn}\,\mathrm{d}\tau'_i \quad \forall m,n\in\{1,\dots,N\}\setminus\{i\} \quad \forall\varepsilon>0,$$

we deduce that

$$\hat{\sigma}^i_{mn}(x,y) = e_y(\hat{z}^i)_{mn}(x,y) + \int_{S^i}\hat{\sigma}^i_{mn}(x,y_ie_i+\tau'_i)\,\mathrm{d}\tau'_i \quad \forall m,n\in\{1,\dots,N\}\setminus\{i\}$$

and a.e. $(x,y)\in\Omega\times Y^N$, where $\hat{\sigma}^i_{mn}$ denotes the weak limit in $L^2(\mathbb{R}^N\times Y^N)$ of $e^i_\varepsilon(\hat{u}^i_\varepsilon)_{mn}$. Defining then $\hat{u}^i_3=(\hat{u}^i_{3,1},\dots,\hat{u}^i_{3,N})\in E^i_3$ as $\hat{u}^i_{3,i}=0$ and

$$\hat{u}^i_{3,m}(x,y) = \hat{z}^i_m(x,y) + \sum_{n\neq i}\int_{S^i}\hat{\sigma}^i_{mn}(x,y_ie_i+\tau'_i)\,\mathrm{d}\tau'_i y_n \qquad \forall m\neq i \quad \text{a.e. } (x,y)\in\Omega\times Y^N,$$

we have

$$e^i_\varepsilon(\hat{u}^i_\varepsilon)_{mn} \rightharpoonup e_y(\hat{u}^i_3)_{mn} \quad \text{in } L^2(\mathbb{R}^N\times Y^N) \quad \forall m,n\in\{1,\dots,N\}\setminus\{i\}.$$

□

The following result gives a converse of theorem 4.7.

PROPOSITION 4.8. *Let $(\hat{u}^i_0,\hat{u}^i_1,\hat{u}^i_2,\hat{u}^i_3)$ be in $\mathcal{E}^i$, $i\in\{1,\dots,N\}$. Then there exists a sequence $u_\varepsilon\in H^1_{\Gamma_\varepsilon}(\Omega_\varepsilon)^N$ such that $\hat{u}^i_\varepsilon$, $i\in\{1,\dots,N\}$, defined by (2.3), satisfy*

$$\hat{u}^i_{\varepsilon,i}\to\hat{u}^i_0 \quad \text{in } L^2(\mathbb{R}^N\times Y^N), \tag{4.50}$$

$$\hat{u}^i_{\varepsilon,m}-\hat{u}^m_0-\frac{\varepsilon}{d_\varepsilon}\hat{u}^i_{1,m}\to 0 \quad \text{in } L^2(\mathbb{R}^N\times Y^N) \quad \forall m\neq i, \tag{4.51}$$

$$e^i_\varepsilon(\hat{u}^i_\varepsilon)\to e^i_0(\hat{u}^i_0,\hat{u}^i_1,\hat{u}^i_2,\hat{u}^i_3) \quad \text{in } L^2(\mathbb{R}^N\times Y^N;\mathcal{S}_N). \tag{4.52}$$

Proof. We consider $\theta_\varepsilon\in C^1(\mathbb{R})$ such that

$$\begin{aligned} 0\leqslant\theta_\varepsilon(t)&\leqslant 1 && \forall t\in\mathbb{R},\\ \theta_\varepsilon(t)&=0 && \text{if } |t|<\tfrac12 d_\varepsilon,\\ \theta_\varepsilon(t)&=1 && \text{if } |t|>\tfrac12\sqrt{d_\varepsilon},\\ \left|\frac{\mathrm{d}\theta_\varepsilon}{\mathrm{d}t}\right|&\leqslant\frac{C}{\sqrt{d_\varepsilon}} && \forall t\in\mathbb{R}. \end{aligned}$$

Then, for y^i_ε given by (2.2), we define $u_\varepsilon:\Omega_\varepsilon\to\mathbb{R}^N$ by

$$\begin{aligned} u_{\varepsilon,i|\Omega^i_\varepsilon} = \hat{u}^i_0 + \varepsilon\hat{u}^i_{1,i}(\cdot,y^i_\varepsilon) &+ \varepsilon d_\varepsilon\bigg(\hat{u}^i_{2,i}(\cdot,y^i_\varepsilon) - \theta_\varepsilon(y^i_{\varepsilon,i})\sum_{n\neq i}(\partial_{x_n}\hat{u}^i_0+\partial_{x_i}\hat{u}^n_0)y^i_{\varepsilon,n}\bigg)\\ &- \varepsilon^2\sum_{n\neq i}\partial_{x_i}\hat{u}^i_{1,n}(\cdot,y^i_\varepsilon)y^i_{\varepsilon,n}, \end{aligned}$$

https://doi.org/10.1017/S0308210500003620 Published online by Cambridge University Press

$$u_{\varepsilon,m|\Omega^i_\varepsilon} = \hat{u}^m_0 + \frac{\varepsilon}{d_\varepsilon}\hat{u}^i_{1,m}(\cdot, y^i_\varepsilon) + \varepsilon\hat{u}^i_{2,m}(\cdot, y^i_\varepsilon) + \varepsilon d_\varepsilon\biggl(\hat{u}^i_{3,m}(\cdot, y^i_\varepsilon) - \theta_\varepsilon(y^i_{\varepsilon,i})\sum_{n\neq i}\partial_{x_n}\hat{u}^m_0 y^i_{\varepsilon,n}\biggr) - \varepsilon^2\sum_{n\neq i}\partial_{x_n}\hat{u}^i_{1,m}(\cdot, y^i_\varepsilon)y^i_{\varepsilon,n} \quad \forall m\neq i.$$

Then u_ε belongs to $H^1_{\Gamma_\varepsilon}(\Omega_\varepsilon)^N$ and it is easy to see that $\hat{u}^i_\varepsilon$, $i\in\{1,\dots,N\}$, defined by (2.3), satisfy (4.50), (4.51) and (4.52). □

REMARK 4.9. It is easy to prove that proposition 4.8 holds for $(\hat{u}^i_0, \hat{u}^i_1, \hat{u}^i_2, \hat{u}^i_3)\in E^i$, $i\in\{1,\dots,N\}$. Thus theorem 4.7 is optimal. We do not prove this general result because we will not use it in the following (see [21] for a proof).

Using proposition 4.8, we can now prove the following corrector result for a sequence u_ε in $H^1_{\Gamma_\varepsilon}(\Omega_\varepsilon)^N$ such that the convergence in (4.15) is strong.

THEOREM 4.10. *Let u_ε be in $H^1_{\Gamma_\varepsilon}(\Omega_\varepsilon)^N$, define $\hat{u}^i_\varepsilon$, $i\in\{1,\dots,N\}$, by (2.3) and take $\gamma_\varepsilon = d_\varepsilon/(\varepsilon+d_\varepsilon)$. Let us suppose that, for every $i\in\{1,\dots,N\}$, there exists $(\hat{u}^i_0, \hat{u}^i_1, \hat{u}^i_2, \hat{u}^i_3)\in E^i$ such that (4.52) holds. Then the sequences $g^i_\varepsilon\in L^2(\mathbb{R}^N)^N$ and $G^i_\varepsilon\in L^2(\mathbb{R}^N;\mathcal{S}_N)$, $i\in\{1,\dots,N\}$, defined by*

$$\left.\begin{aligned} g^i_{\varepsilon,i}(\cdot) &= \frac{1}{\varepsilon}^N\int_{C_\varepsilon(\cdot)}\hat{u}^i_0(\rho)\,\mathrm{d}\rho,\\ g^i_{\varepsilon,m}(\cdot) &= \frac{1}{\varepsilon^N}\int_{C_\varepsilon(\cdot)}\biggl[\hat{u}^m_0(\rho) + \frac{\varepsilon}{d_\varepsilon}\hat{u}^i_{1,m}(\rho, y^i_\varepsilon(\cdot))\biggr]\mathrm{d}\rho \quad \forall m\neq i \end{aligned}\right\} \tag{4.53}$$

and

$$G^i_\varepsilon(\cdot) = \frac{1}{\varepsilon^N}\int_{C_\varepsilon(\cdot)} e^i_0(\hat{u}^i_0, \hat{u}^i_1, \hat{u}^i_2, \hat{u}^i_3)(\rho, y^i_\varepsilon(\cdot))\,\mathrm{d}\rho, \tag{4.54}$$

satisfy

$$\lim_{\varepsilon\to 0}\frac{1}{|\Omega^i_\varepsilon|}\biggl[\int_{\Omega^i_\varepsilon}|u_{\varepsilon,i}(x) - g^i_{\varepsilon,i}(x)|^2\,\mathrm{d}x + \gamma^2_\varepsilon\sum_{m\neq i}\int_{\Omega^i_\varepsilon}|u_{\varepsilon,m}(x) - g^i_{\varepsilon,m}(x)|^2\,\mathrm{d}x + \int_{\Omega^i_\varepsilon}|e(u_\varepsilon)(x) - G^i_\varepsilon(x)|^2\,\mathrm{d}x\biggr] = 0 \quad \forall i\in\{1,\dots,N\}. \tag{4.55}$$

Proof. For $i\in\{1,\dots,N\}$, let $(\varphi^{i,n}_0, \varphi^{i,n}_1, \varphi^{i,n}_2, \varphi^{i,n}_3)$ be a sequence in $\mathcal{E}^i$ such that

$$\lim_{n\to\infty}\varphi^{i,n}_0 = \hat{u}^i_0 \quad \text{in } L^2(\Omega), \tag{4.56}$$

$$\lim_{n\to\infty}\varphi^{i,n}_{1,m} = \hat{u}^i_{1,m} \quad \text{in } L^2(\Omega\times Y^N) \quad \forall m\neq i, \tag{4.57}$$

$$\lim_{n\to\infty} e^i_0(\varphi^{i,n}_0, \varphi^{i,n}_1, \varphi^{i,n}_2, \varphi^{i,n}_3) = e^i_0(\hat{u}^i_0, \hat{u}^i_1, \hat{u}^i_2, \hat{u}^i_3) \quad \text{in } L^2(\Omega\times Y^N;\mathcal{S}_N). \tag{4.58}$$

https://doi.org/10.1017/S0308210500003620 Published online by Cambridge University Press

From proposition 4.8, for every $n \in \mathbb{N}$, there exists $w_\varepsilon^n \in H^1_{\Gamma_\varepsilon}(\Omega_\varepsilon)^N$, $\varepsilon > 0$, which satisfies

$$\hat{w}^{i,n}_{\varepsilon,i} \to \varphi_0^{i,n} \quad \text{in } L^2(\mathbb{R}^N \times Y^N), \tag{4.59}$$

$$\hat{w}^{i,n}_{\varepsilon,m} - \varphi_0^{m,n} - \frac{\varepsilon}{d_\varepsilon}\varphi^{i,n}_{1,m} \to 0 \quad \text{in } L^2(\mathbb{R}^N \times Y^N) \quad \forall m \neq i, \tag{4.60}$$

$$e^i_\varepsilon(\hat{w}^{i,n}_\varepsilon) \to e^i_0(\varphi_0^{i,n}, \varphi_1^{i,n}, \varphi_2^{i,n}, \varphi_3^{i,n}) \quad \text{in } L^2(\mathbb{R}^N \times Y^N; \mathcal{S}_N). \tag{4.61}$$

Using the change of variables (2.2) and the estimate (4.6), we get

$$\begin{aligned}
\|\hat{u}^i_{\varepsilon,i} - \hat{w}^{i,n}_{\varepsilon,i}\|^2_{L^2(\mathbb{R}^N\times Y^N)} &= \frac{1}{d_\varepsilon^{N-1}}\|u_{\varepsilon,i} - w^n_{\varepsilon,i}\|^2_{L^2(\Omega^i_\varepsilon)} \\
&\leqslant \frac{C}{d_\varepsilon^{N-1}}\|e(u_\varepsilon - w^n_\varepsilon)\|^2_{L^2(\Omega^i_\varepsilon;\mathcal{S}_N)} \\
&= C\|e^i_\varepsilon(\hat{u}^i_\varepsilon - \hat{w}^{i,n}_\varepsilon)\|^2_{L^2(\mathbb{R}^N\times Y^N;\mathcal{S}_N)} \quad \forall n \in \mathbb{N}, \quad \varepsilon > 0.
\end{aligned}$$

Passing to the limit first in ε and then in n, by (4.61) and (4.58), we conclude that

$$\lim_{n\to\infty}\limsup_{\varepsilon\to 0}\|\hat{u}^i_{\varepsilon,i} - \hat{w}^{i,n}_{\varepsilon,i}\|_{L^2(\mathbb{R}^N\times Y^N)} = 0, \tag{4.62}$$

which, by (4.59) and (4.56), implies that

$$\hat{u}^i_{\varepsilon,i} \to \hat{u}^i_0 \quad \text{in } L^2(\mathbb{R}^N \times Y^N). \tag{4.63}$$

On the other hand, since $\hat{u}^i_\varepsilon(x,y)$ does not depend on x in C^k_ε, $k \in \mathbb{Z}^N$, $\varepsilon > 0$, the change of variables (2.2) gives

$$\begin{aligned}
\frac{1}{d_\varepsilon^{N-1}}\int_{V^i_\varepsilon}|u_{\varepsilon,i}(x) - g^i_{\varepsilon,i}(x)|^2\,\mathrm{d}x &= \frac{1}{d_\varepsilon^{N-1}}\sum_{k\in\mathbb{Z}^N}\int_{B^{i,k}_\varepsilon}\left|u_{\varepsilon,i}(x) - \frac{1}{\varepsilon^N}\int_{C^k_\varepsilon}\hat{u}^i_0(\rho)\,\mathrm{d}\rho\right|^2\mathrm{d}x \\
&= \frac{1}{\varepsilon^N}\sum_{k\in\mathbb{Z}^N}\int_{Y^N}\left|\int_{C^k_\varepsilon}(\hat{u}^i_{\varepsilon,i}(\rho,y) - \hat{u}^i_0(\rho))\,\mathrm{d}\rho\right|^2\mathrm{d}y \\
&\leqslant \int_{\mathbb{R}^N\times Y^N}|\hat{u}^i_{\varepsilon,i}(\rho,y) - \hat{u}^i_0(\rho)|^2\,\mathrm{d}y\,\mathrm{d}\rho.
\end{aligned}$$

By (4.63), we deduce that the first term in (4.55) tends to zero.

Using (4.7), and reasoning similarly as we did to deduce (4.62), we get

$$\lim_{n\to\infty}\limsup_{\varepsilon\to 0}\gamma_\varepsilon\|\hat{u}^i_{\varepsilon,m} - \hat{w}^{i,n}_{\varepsilon,m}\|_{L^2(\mathbb{R}^N\times Y^N)} = 0 \quad \forall m \neq i.$$

The inequality

$$\begin{aligned}
&\gamma_\varepsilon\left\|\hat{u}^i_{\varepsilon,m} - \hat{u}^m_0 - \frac{\varepsilon}{d_\varepsilon}\hat{u}^i_{1,m}\right\|_{L^2(\mathbb{R}^N\times Y^N)} \\
&\quad\leqslant \gamma_\varepsilon\|\hat{u}^i_{\varepsilon,m} - \hat{w}^{i,n}_{\varepsilon,m}\|_{L^2(\mathbb{R}^N\times Y^N)} + \gamma_\varepsilon\left\|\hat{w}^{i,n}_{\varepsilon,m} - \varphi_0^{m,n} - \frac{\varepsilon}{d_\varepsilon}\varphi^{i,n}_{1,m}\right\|_{L^2(\mathbb{R}^N\times Y^N)} \\
&\qquad + \gamma_\varepsilon\left\|\varphi_0^{m,n} + \frac{\varepsilon}{d_\varepsilon}\varphi^{i,n}_{1,m} - \hat{u}^m_0 - \frac{\varepsilon}{d_\varepsilon}\hat{u}^i_{1,m}\right\|_{L^2(\mathbb{R}^N\times Y^N)}
\end{aligned}$$

https://doi.org/10.1017/S0308210500003620 Published online by Cambridge University Press

and equations (4.60), (4.56) and (4.57) then give

$$\gamma_\varepsilon\biggl(\hat u^i_{\varepsilon,m} - \hat u^m_0 - \frac{\varepsilon}{d_\varepsilon}\hat u^i_{1,m}\biggr) \to 0 \quad \text{in } L^2(\mathbb{R}^N \times Y^N) \quad \forall m \neq i,$$

which, as above, implies that the second term in (4.55) tends to zero.

The convergence to zero of the third term in (4.55) is easily deduced using the change of variables (2.2) and (4.52). □

4.3. Homogenization result

We already have the suitable tools to prove the homogenization result for (1.1) in the case of the model structure defined in § 2.

Proof of theorem 2.5. Taking $\gamma_\varepsilon^2 u_\varepsilon$ as a test function in (1.1) and using (4.7), we deduce that

$$\frac{1}{|\Omega_\varepsilon|}\int_{\Omega_\varepsilon} |e(\gamma_\varepsilon u_\varepsilon)|^2\,\mathrm{d}x \leqslant \quad \forall\varepsilon > 0.$$

Then, by theorem 4.7 applied to the sequence $\gamma_\varepsilon u_\varepsilon$, for every $i \in \{1,\dots,N\}$, there exists $(\hat u^i_0, \hat u^i_1, \hat u^i_2, \hat u^i_3) \in E^i$ such that, up to a subsequence, equations (4.13), (4.14) and (4.15) hold with $\hat u^i_\varepsilon$ replaced by $\gamma_\varepsilon \hat u^i_\varepsilon$. For $(\hat v^i_0, \hat v^i_1, \hat v^i_2, \hat v^i_3)$ in $\mathcal{E}^i$, $i \in \{1,\dots,N\}$, we consider the sequence v_ε in $H^1_{\Gamma_\varepsilon}(\Omega_\varepsilon)^N$ given by proposition 4.8 applied to $(\hat v^i_0, \hat v^i_1, \hat v^i_2, \hat v^i_3)$, $i \in \{1,\dots,N\}$. Taking $\gamma_\varepsilon v_\varepsilon$ as a test function in (1.1) and using the continuity with respect to x of F^i, H^i and A^i, we get

$$\begin{aligned}
\sum_{i=1}^N \int_{\mathbb{R}^N\times Y^N} & A^i e^i_\varepsilon(\gamma_\varepsilon \hat u^i_\varepsilon) : e^i_\varepsilon(\hat v^i_\varepsilon)\,\mathrm{d}y\,\mathrm{d}x + O_\varepsilon \\
&= \frac{1}{d_\varepsilon^{N-1}}\int_{\Omega_\varepsilon} A_\varepsilon e(\gamma_\varepsilon u_\varepsilon) : e(v_\varepsilon)\,\mathrm{d}x \\
&= \frac{\gamma_\varepsilon}{d_\varepsilon^{N-1}}\int_{\Omega_\varepsilon} (F_\varepsilon v_\varepsilon + H_\varepsilon : e(v_\varepsilon))\,\mathrm{d}x \\
&= \gamma_\varepsilon \sum_{i=1}^N \int_{\mathbb{R}^N\times Y^N} (F^i \hat v^i_\varepsilon + H^i : e^i_\varepsilon(\hat v^i_\varepsilon))\,\mathrm{d}y\,\mathrm{d}x + O_\varepsilon.
\end{aligned}$$

Passing to the limit, we deduce that $(\hat u^i_0, \hat u^i_1, \hat u^i_2, \hat u^i_3)$, $i \in \{1,\dots,N\}$, satisfies

$$\begin{aligned}
\sum_{i=1}^N \int_{\mathbb{R}^N\times Y^N} & (A^i e^i_0(\hat u^i_0, \hat u^i_1, \hat u^i_2, \hat u^i_3) - \gamma H^i) : e^i_0(\hat v^i_0, \hat v^i_1, \hat v^i_2, \hat v^i_3)\,\mathrm{d}y\,\mathrm{d}x \\
&= \sum_{i=1}^N \int_{\mathbb{R}^N\times Y^N} \biggl(\gamma \sum_{j=1}^N F^i_j \hat v^j_0 + (1-\gamma)\sum_{m\neq i} F^i_m \hat v^i_{1,m}\biggr)\,\mathrm{d}y\,\mathrm{d}x \\
&\qquad\qquad\qquad \forall(\hat v^i_0, \hat v^i_1, \hat v^i_2, \hat v^i_3) \in \mathcal{E}^i.
\end{aligned}$$

By density, we conclude that $(\hat u^i_0, \hat u^i_1, \hat u^i_2, \hat u^i_3)$, $i \in \{1,\dots,N\}$, is a solution of (2.12).

To conclude the proof of theorem 2.5, thanks to theorem 4.10, it is enough to prove (2.11). For every $i \in \{1,\dots,N\}$, the monotonicity properties (2.1) and (2.8) of A_ε give

$$\alpha \sum_{m\neq i} \|(e^m_\varepsilon(\gamma_\varepsilon \hat{u}^m_\varepsilon) - e^m_0(\hat{u}^m_0, \hat{u}^m_1, \hat{u}^m_2, \hat{u}^m_3))\chi_{\{|y_m|>d_\varepsilon/2\}}\|^2_{L^2(\mathbb{R}^N\times Y^N;\mathcal{S}_N)} + \alpha\|e^i_\varepsilon(\gamma_\varepsilon \hat{u}^i_\varepsilon) - e^i_0(\hat{u}^i_0, \hat{u}^i_1, \hat{u}^i_2, \hat{u}^i_3)\|^2_{L^2(\mathbb{R}^N\times Y^N;\mathcal{S}_N)}$$
$$\leqslant \frac{1}{d_\varepsilon^{N-1}} \int_{\omega_\varepsilon} A_\varepsilon e(\gamma_\varepsilon u_\varepsilon) : e(\gamma_\varepsilon u_\varepsilon)\,\mathrm{d}x + \alpha \int_{\mathbb{R}^N} \int_{|y_i|<d_\varepsilon/2} |e^i_\varepsilon(\gamma_\varepsilon \hat{u}^i_\varepsilon) - e^i_0(\hat{u}^i_0, \hat{u}^i_1, \hat{u}^i_2, \hat{u}^i_3)|^2\,\mathrm{d}y\,\mathrm{d}x$$
$$+ \sum_{j=1}^N \int_{\mathbb{R}^N} \int_{|y_j|>d_\varepsilon/2} A^j(e^j_\varepsilon(\gamma_\varepsilon \hat{u}^j_\varepsilon) - e^j_0(\hat{u}^j_0, \hat{u}^j_1, \hat{u}^j_2, \hat{u}^j_3)) : (e^j_\varepsilon(\gamma_\varepsilon \hat{u}^j_\varepsilon) - e^j_0(\hat{u}^j_0, \hat{u}^j_1, \hat{u}^j_2, \hat{u}^j_3))\,\mathrm{d}y\,\mathrm{d}x. \quad (4.64)$$

Using equations (4.13), (4.14) and (4.15), taking $\gamma_\varepsilon^2 u_\varepsilon$ as a test function in (1.1) and $(\hat{u}^j_0, \hat{u}^j_1, \hat{u}^j_2, \hat{u}^j_3)$, $j \in \{1, \dots, N\}$, as a test function in (2.12), we get

$$\sum_{j=1}^N \int_{\mathbb{R}^N} \int_{|y_j|>d_\varepsilon/2} A^j e^j_\varepsilon(\gamma_\varepsilon \hat{u}^j_\varepsilon) : e^j_\varepsilon(\gamma_\varepsilon \hat{u}^j_\varepsilon)\,\mathrm{d}y\,\mathrm{d}x + \frac{1}{d_\varepsilon^{N-1}} \int_{\omega_\varepsilon} A_\varepsilon e(\gamma_\varepsilon u_\varepsilon) : e(\gamma_\varepsilon u_\varepsilon)\,\mathrm{d}x + O_\varepsilon$$
$$= \frac{1}{d_\varepsilon^{N-1}} \int_{\Omega_\varepsilon} A_\varepsilon e(\gamma_\varepsilon u_\varepsilon) : e(\gamma_\varepsilon u_\varepsilon)\,\mathrm{d}x$$
$$= \frac{\gamma_\varepsilon}{d_\varepsilon^{N-1}} \int_{\Omega_\varepsilon} (F_\varepsilon \gamma_\varepsilon u_\varepsilon + H_\varepsilon : e(\gamma_\varepsilon u_\varepsilon))\,\mathrm{d}y\,\mathrm{d}x$$
$$= \gamma_\varepsilon \sum_{j=1}^N \int_{\mathbb{R}^N} \int_{Y^N} (F^j \gamma_\varepsilon \hat{u}^j_\varepsilon + H^j : e^j_\varepsilon(\gamma_\varepsilon \hat{u}^j_\varepsilon))\,\mathrm{d}y\,\mathrm{d}x + O_\varepsilon$$
$$= \sum_{j=1}^N \int_{\mathbb{R}^N\times Y^N} \biggl(\gamma \sum_{l=1}^N F^j_l \hat{u}^l_0 + (1-\gamma) \sum_{m\neq j} F^j_m \hat{u}^j_{1,m} + \gamma H^j : e^j_0(\hat{u}^j_0, \hat{u}^j_1, \hat{u}^j_2, \hat{u}^j_3)\biggr)\,\mathrm{d}y\,\mathrm{d}x + O_\varepsilon$$
$$= \sum_{j=1}^N \int_{\mathbb{R}^N\times Y^N} A^j e^j_0(\hat{u}^j_0, \hat{u}^j_1, \hat{u}^j_2, \hat{u}^j_3) : e^j_0(\hat{u}^j_0, \hat{u}^j_1, \hat{u}^j_2, \hat{u}^j_3)\,\mathrm{d}y\,\mathrm{d}x + O_\varepsilon. \quad (4.65)$$

Due to (4.15), we also have

$$\liminf_{\varepsilon\to 0} \int_{\mathbb{R}^N} \int_{|y_j|>d_\varepsilon/2} A^j e^j_\varepsilon(\gamma_\varepsilon \hat{u}^j_\varepsilon) : e^j_\varepsilon(\gamma_\varepsilon \hat{u}^j_\varepsilon)\,\mathrm{d}y\,\mathrm{d}x \geqslant \int_{\mathbb{R}^N\times Y^N} A^j e^j_0(\hat{u}^j_0, \hat{u}^j_1, \hat{u}^j_2, \hat{u}^j_3) : e^j_0(\hat{u}^j_0, \hat{u}^j_1, \hat{u}^j_2, \hat{u}^j_3)\,\mathrm{d}y\,\mathrm{d}x \quad \forall j \in \{1, \dots, N\}.$$

Then, from (4.65), we deduce that

$$\lim_{\varepsilon\to 0}\frac{\gamma_\varepsilon^2}{d_\varepsilon^{N-1}}\int_{\omega_\varepsilon} A_\varepsilon e(u_\varepsilon):e(u_\varepsilon)\,\mathrm{d}x = 0 \tag{4.66}$$

and

$$\begin{aligned}\lim_{\varepsilon\to 0}\sum_{j=1}^{N}\int_{\mathbb{R}^N}\int_{|y_j|>d_\varepsilon/2} A^j e_\varepsilon^j(\gamma_\varepsilon\hat{u}_\varepsilon^j):e_\varepsilon^j(\gamma_\varepsilon\hat{u}_\varepsilon^j)\,\mathrm{d}y\,\mathrm{d}x \\ = \sum_{j=1}^{N}\int_{\mathbb{R}^N\times Y^N} A^j e_0^j(\hat{u}_0^j,\hat{u}_1^j,\hat{u}_2^j,\hat{u}_3^j):e_0^j(\hat{u}_0^j,\hat{u}_1^j,\hat{u}_2^j,\hat{u}_3^j)\,\mathrm{d}y\,\mathrm{d}x.\end{aligned} \tag{4.67}$$

By (4.66) and (2.1), we also have

$$\begin{aligned}\int_{\mathbb{R}^N}\int_{|y_i|<d_\varepsilon/2}|e_\varepsilon^i(\gamma_\varepsilon\hat{u}_\varepsilon^i)|^2\,\mathrm{d}y\,\mathrm{d}x &= \frac{\gamma_\varepsilon^2}{d_\varepsilon^{N-1}}\int_{\omega_\varepsilon}|e(u_\varepsilon)|^2\,\mathrm{d}x \\ &\leqslant \frac{\gamma_\varepsilon^2}{\alpha d_\varepsilon^{N-1}}\int_{\omega_\varepsilon}A_\varepsilon e(u_\varepsilon):e(u_\varepsilon)\,\mathrm{d}x \\ &= O_\varepsilon.\end{aligned} \tag{4.68}$$

From (4.66) and (4.68), we easily deduce that the first and second terms on the right-hand side of (4.64) tend to zero, while, using (4.67) and (4.15), we deduce that

$$\begin{aligned}\sum_{j=1}^{N}\int_{\mathbb{R}^N}\int_{|y_j|>d_\varepsilon/2} A^j(e_\varepsilon^j(\gamma_\varepsilon\hat{u}_\varepsilon^j)-e_0^j(\hat{u}_0^j,\hat{u}_1^j,\hat{u}_2^j,\hat{u}_3^j)): \\ (e_\varepsilon^j(\gamma_\varepsilon\hat{u}_\varepsilon^j)-e_0^j(\hat{u}_0^j,\hat{u}_1^j,\hat{u}_2^j,\hat{u}_3^j))\,\mathrm{d}y\,\mathrm{d}x \\ = \sum_{j=1}^{N}\int_{\mathbb{R}^N}\int_{|y_j|>d_\varepsilon/2} A^j e_\varepsilon^j(\gamma_\varepsilon\hat{u}_\varepsilon^j):e_\varepsilon^j(\gamma_\varepsilon\hat{u}_\varepsilon^j)\,\mathrm{d}y\,\mathrm{d}x \\ -\sum_{j=1}^{N}\int_{\mathbb{R}^N}\int_{|y_j|>d_\varepsilon/2} A^j e_\varepsilon^j(\gamma_\varepsilon\hat{u}_\varepsilon^j):e_0^j(\hat{u}_0^j,\hat{u}_1^j,\hat{u}_2^j,\hat{u}_3^j)\,\mathrm{d}y\,\mathrm{d}x \\ -\sum_{j=1}^{N}\int_{\mathbb{R}^N}\int_{|y_j|>d_\varepsilon/2} A^j e_0^j(\hat{u}_0^j,\hat{u}_1^j,\hat{u}_2^j,\hat{u}_3^j):e_\varepsilon^j(\gamma_\varepsilon\hat{u}_\varepsilon^j)\,\mathrm{d}y\,\mathrm{d}x \\ +\sum_{j=1}^{N}\int_{\mathbb{R}^N}\int_{|y_j|>d_\varepsilon/2} A^j e_0^j(\hat{u}_0^j,\hat{u}_1^j,\hat{u}_2^j,\hat{u}_3^j):e_0^j(\hat{u}_0^j,\hat{u}_1^j,\hat{u}_2^j,\hat{u}_3^j)\,\mathrm{d}y\,\mathrm{d}x = O_\varepsilon.\end{aligned}$$

So, passing to the limit in (4.64), we deduce, in particular, that

$$\lim_{\varepsilon\to 0}\|e_\varepsilon^i(\gamma_\varepsilon\hat{u}_\varepsilon^i)-e_0^i(\hat{u}_0^i,\hat{u}_1^i,\hat{u}_2^i,\hat{u}_3^i)\|_{L^2(\mathbb{R}^N\times Y^N;\mathcal{S}_N)}=0 \quad \forall i\in\{1,\dots,N\}.$$

This proves (2.11). □

https://doi.org/10.1017/S0308210500003620 Published online by Cambridge University Press

4.4. Reinforced structure

We now turn our attention to the proof of theorem 3.1, which gives the asymptotic behaviour of the elasticity problem (1.1) posed on the reinforced structure defined in § 3. The proof follows along the same lines of the proof of theorem 2.5 (*a priori* estimates, compactness result and passage to the limit in (1.1)). We briefly sketch these steps.

First, we obtain a Korn inequality for functions in $H^1_{\Gamma_\varepsilon}(\Omega_\varepsilon)^2$.

Theorem 4.11. *There exists $C > 0$ such that, for every $u \in H^1_{\Gamma_\varepsilon}(\Omega_\varepsilon)^2$ and every $\varepsilon > 0$, we have*

$$\int_{\Omega^3_\varepsilon} |\nabla(u\zeta)\tau|^2 \,\mathrm{d}x \leqslant C\left(\frac{1}{\varepsilon^2} + \frac{1}{d^2_\varepsilon}\right) \int_{\Omega_\varepsilon} |e(u)|^2 \,\mathrm{d}x, \tag{4.69}$$

$$\int_{\Omega_\varepsilon} |u|^2 \,\mathrm{d}x \leqslant C\left(1 + \frac{\varepsilon^2}{d^2_\varepsilon}\right) \int_{\Omega_\varepsilon} |e(u)|^2 \,\mathrm{d}x. \tag{4.70}$$

Proof. Taking into account the fact that the constant in (4.3) is invariant by translations and rotations and using Poincaré's inequality, the same reasoning that gives (4.9) provides

$$\int_{\Omega^3_\varepsilon} |\nabla(u\zeta)\tau|^2 \,\mathrm{d}x \leqslant C\left(\frac{1}{d^2_\varepsilon}\int_{\Omega^3_\varepsilon} |e(u)|^2 \,\mathrm{d}x + \frac{1}{\varepsilon^2 d_\varepsilon}\int_{\omega_\varepsilon} |u\zeta|^2 \,\mathrm{d}x\right).$$

Since $\omega_\varepsilon \subset \Omega^1_\varepsilon \cap \Omega^2_\varepsilon$ and $\Omega^1_\varepsilon \cup \Omega^2_\varepsilon \subset \Omega_\varepsilon$ (we remark that $\Omega^1_\varepsilon \cup \Omega^2_\varepsilon$ is the same structure that was considered in the previous subsections with $N = 2$), from (4.5), we deduce that

$$\int_{\omega_\varepsilon} |u\zeta|^2 \,\mathrm{d}x \leqslant \int_{\Omega^1_\varepsilon \cap \Omega^2_\varepsilon} |u\zeta|^2 \,\mathrm{d}x \leqslant d_\varepsilon \int_{\Omega^1_\varepsilon \cup \Omega^2_\varepsilon} |e(u)|^2 \,\mathrm{d}x \leqslant d_\varepsilon \int_{\Omega_\varepsilon} |e(u)|^2 \,\mathrm{d}x.$$

Reasoning as in theorem 4.3, these inequalities (which are similar to (4.9) and (4.5)) easily give (4.69) and (4.70). □

The next compactness result is analogous to theorem 4.7.

Theorem 4.12. *We assume that there exists $\lim_{\varepsilon\to 0}(\varepsilon/d_\varepsilon) = \vartheta \in [0, +\infty]$ (this always holds for a subsequence). Let u_ε be a sequence in $H^1_{\Gamma_\varepsilon}(\Omega_\varepsilon)^2$ that satisfies (4.10), and define $\hat{u}^i_\varepsilon$, $i \in \{1, 2, 3\}$, by (2.3) and (3.1). Then there exist a subsequence of ε, still denoted by ε, $(\hat{u}^i_0, \hat{u}^i_1, \hat{u}^i_2, \hat{u}^i_3) \in E^i$, $i \in \{1, 2, 3\}$, such that (4.13)–(4.15) hold for $i \in \{1, 2\}$, and for $i = 3$, defining $\hat{u}_0 = (\hat{u}^1_0, \hat{u}^2_0)$, we have*

$$\begin{aligned} &\hat{u}^3_0 = \hat{u}^0\tau, \\ &\hat{u}^3_\varepsilon\tau \rightharpoonup \hat{u}_0\tau \quad \text{in } L^2(\Omega \times \mathfrak{D}^3), \end{aligned} \tag{4.71}$$

$$\left.\begin{aligned} &\hat{u}^3_\varepsilon\zeta \rightharpoonup \hat{u}_0\zeta && \text{if } \vartheta = 0, \\ &\hat{u}^3_\varepsilon\zeta \rightharpoonup \hat{u}_0\zeta + \vartheta\hat{u}^3_1\zeta && \text{if } \vartheta \in (0, +\infty) \quad \text{in } L^2(\Omega \times \mathfrak{D}^3), \\ &\frac{d_\varepsilon}{\varepsilon}\hat{u}^3_\varepsilon\zeta \rightharpoonup \hat{u}^3_1\zeta && \text{if } \vartheta = +\infty, \end{aligned}\right\} \tag{4.72}$$

$$e^3_\varepsilon(\hat{u}^3_\varepsilon) \rightharpoonup e^3_0(\hat{u}^3_0, \hat{u}^3_1, \hat{u}^3_2, \hat{u}^3_3) \quad \text{in } L^2(\Omega \times \mathfrak{D}^3; \mathcal{S}_2). \tag{4.73}$$

https://doi.org/10.1017/S0308210500003620 Published online by Cambridge University Press

Proof. From (4.10), using the change of variables y^i_ε, $i \in \{1,2,3\}$, we deduce that $\hat{u}^i_\varepsilon$ and $e^i_\varepsilon(\hat{u}^i_\varepsilon)$ are bounded in $L^2(\mathbb{R}^2 \times \mathfrak{D}^i)^2$ and $L^2(\mathbb{R}^2 \times \mathfrak{D}^i; \mathcal{S}_2)$, respectively. On the one hand, applying theorem 4.7 to

$$u_{\varepsilon|_{\Omega^1_\varepsilon \cup \Omega^2_\varepsilon}}$$

(which satisfies the hypothesis of that theorem), we deduce that there exists

$$(\hat{u}^i_0, \hat{u}^i_1, \hat{u}^i_2, \hat{u}^i_3) \in E^i, \quad i \in \{1,2\},$$

such that (4.13)–(4.15) hold. On the other hand, taking into account the equality

$$|e^3_\varepsilon(\hat{u}^3_\varepsilon)| = \left| \begin{pmatrix} \nabla_y(\hat{u}^3_\varepsilon\tau)\tau & \frac{1}{2}(\nabla_y(\hat{u}^3_\varepsilon\tau)\zeta + \nabla_y(\hat{u}^3_\varepsilon\zeta)\tau) \\ \frac{1}{2}(\nabla_y(\hat{u}^3_\varepsilon\tau)\zeta + \nabla_y(\hat{u}^3_\varepsilon\zeta)\tau) & \nabla_y(\hat{u}^3_\varepsilon\zeta)\zeta \end{pmatrix} \right|,$$

working with the components $\hat{u}^3_\varepsilon\tau$, $\hat{u}^3_\varepsilon\zeta$ of $\hat{u}^3_\varepsilon$ with respect to the basis $\{\tau,\zeta\}$ and following the same reasoning as that in the proof of theorem 4.7 (in fact, because we are in dimension two, the reasoning is simpler (see [10])), we obtain there exists $(\hat{u}^3_0, \hat{u}^3_1, \hat{u}^3_2, \hat{u}^3_3) \in E^3$ that satisfies, up to a subsequence, equations (4.71)–(4.73) with $\hat{u}_0 = (\hat{u}^1_0, \hat{u}^2_0)$. In order to obtain this result, it is necessary to prove that $\hat{u}^3_0 = \hat{u}_0\tau$. Defining $C_\varepsilon(x)$ as in the previous section and $P_\varepsilon(x) = C_\varepsilon(x) \cap \omega_\varepsilon$, a.e. $x \in \Omega$, this follows from the following simple application of the Poincaré–Wirtinger inequality:

$$\begin{aligned}
\hat{u}^3_0(x) &= \lim_{\varepsilon\to 0} \frac{1}{|y^3_\varepsilon(P_\varepsilon(x))|} \int_{y^3_\varepsilon(P_\varepsilon(x))} \hat{u}^3_\varepsilon(x,y)\tau \,\mathrm{d}y \\
&= \lim_{\varepsilon\to 0} \frac{1}{|P_\varepsilon(x)|} \int_{P_\varepsilon(x)} u_\varepsilon(z)\tau \,\mathrm{d}z \\
&= \lim_{\varepsilon\to 0} \frac{1}{|P_\varepsilon(x)|} \int_{P_\varepsilon(x)} \left(\frac{1}{\sqrt{2}} u_{\varepsilon,1}(z) + \frac{1}{\sqrt{2}} u_{\varepsilon,2}(z) \right) \mathrm{d}z \\
&= \lim_{\varepsilon\to 0} \left[\frac{1}{\sqrt{2}|y^1_\varepsilon(P_\varepsilon(x))|} \int_{y^1_\varepsilon(P_\varepsilon(x))} \hat{u}^1_{\varepsilon,1}(x,y)\,\mathrm{d}y \right. \\
&\qquad\qquad \left. + \frac{1}{\sqrt{2}|y^2_\varepsilon(P_\varepsilon(x))|} \int_{y^2_\varepsilon(P_\varepsilon(x))} \hat{u}^2_{\varepsilon,2}(x,y)\,\mathrm{d}y \right] \\
&= \frac{1}{\sqrt{2}} \lim_{\varepsilon\to 0} \frac{1}{\varepsilon^2 d^2_\varepsilon} \int_{\varepsilon d_\varepsilon Y^2} \hat{u}^1_{\varepsilon,1}(x,y)\,\mathrm{d}y + \frac{1}{\sqrt{2}} \lim_{\varepsilon\to 0} \frac{1}{\varepsilon^2 d^2_\varepsilon} \int_{\varepsilon d_\varepsilon Y^2} \hat{u}^2_{\varepsilon,2}(x,y)\,\mathrm{d}y \\
&= \hat{u}^0(x)\tau \quad \text{a.e. } x \in \Omega.
\end{aligned}$$

□

Proof of theorem 3.1. Taking u_ε as a test function in (1.1) and using (4.70), we prove that

$$\frac{1}{|\Omega_\varepsilon|} \int_{\Omega_\varepsilon} |e(\gamma_\varepsilon u_\varepsilon)|^2 \,\mathrm{d}x \leqslant C \quad \forall \varepsilon > 0.$$

So we can apply theorem 4.12 to the sequence $\gamma_\varepsilon u_\varepsilon$ and then deduce that, up to a subsequence, there exist $(\hat{u}^i_0, \hat{u}^i_1, \hat{u}^i_2, \hat{u}^i_3) \in E^i$, $i \in \{1,2,3\}$, which satisfy (4.13)–(4.15), for $i \in \{1,2\}$, and (4.71)–(4.73), with $\hat{u}^i_\varepsilon$ replaced by $\gamma_\varepsilon \hat{u}^i_\varepsilon$.

Let $(\hat{v}_0^i, \hat{v}_1^i, \hat{v}_2^i, \hat{v}_3^i)$ be in E^i, $i \in \{1,2,3\}$, with $\hat{v}_0^3 = \hat{v}_0\tau$, $\hat{v}_0 = (\hat{v}_0^1, \hat{v}_0^2)$ (indeed, we can assume that $(\hat{v}_0^i, \hat{v}_1^i, \hat{v}_2^i, \hat{v}_3^i)$ is sufficiently smooth and that it is in a dense set of E^i). Reasoning analogously as in proposition 4.8 (see also remark 4.9), we can prove that there exists $v_\varepsilon \in H^1_{\Gamma_\varepsilon}(\Omega_\varepsilon)^2$ such that the corresponding sequences $\hat{v}_\varepsilon^i$, $i \in \{1,2,3\}$, satisfy

$$\begin{aligned} \hat{v}_{\varepsilon,i}^i &\to \hat{v}_0^i && \text{in } L^2(\mathbb{R}^2 \times \mathfrak{D}^i), \\ \hat{v}_{\varepsilon,m}^i - \hat{v}_0^m - \frac{\varepsilon}{d_\varepsilon}\hat{v}_{1,m}^i &\to 0 && \text{in } L^2(\mathbb{R}^2 \times \mathfrak{D}^i), \quad m \in \{1,2\} \setminus \{i\}, \\ e_\varepsilon^i(\hat{v}_\varepsilon^i) &\to e_0^i(\hat{v}_0^i, \hat{v}_1^i, \hat{v}_2^i, \hat{v}_3^i) && \text{in } L^2(\mathbb{R}^2 \times \mathfrak{D}^i; \mathcal{S}_2), \end{aligned}$$

for $i \in \{1,2\}$, and

$$\begin{aligned} \hat{v}_\varepsilon^3\tau &\to \hat{v}_0\tau && \text{in } L^2(\mathbb{R}^2 \times \mathfrak{D}^3), \\ \hat{v}_\varepsilon^3\zeta - \hat{v}_0\zeta - \frac{\varepsilon}{d_\varepsilon}\hat{v}_1^3\zeta &\to 0 && \text{in } L^2(\mathbb{R}^2 \times \mathfrak{D}^3), \\ e_\varepsilon^3(\hat{v}_\varepsilon^3) &\to e_0^3(\hat{v}_0^3, \hat{v}_1^3, \hat{v}_2^3, \hat{v}_3^3) && \text{in } L^2(\mathbb{R}^2 \times \mathfrak{D}^3; \mathcal{S}_2). \end{aligned}$$

Using $\gamma_\varepsilon v_\varepsilon$ as test functions in (1.1), we deduce that $(\hat{u}_0^i, \hat{u}_1^i, \hat{u}_2^i, \hat{u}_3^i)$, $i \in \{1,2,3\}$, is a solution of (3.2). As in the proof of theorem 2.5, we can also prove that the convergence of $e(\gamma_\varepsilon \hat{u}_\varepsilon^i)$ is strong in $L^2(\Omega \times \mathfrak{D}^i; \mathcal{S}_2)$, and this allows us to obtain (2.15) and (3.4). □

Acknowledgments

The authors thank F. Murat for stimulating discussions on the topic. This work has been partly supported by the project BFM2002-00672 of the Ministerio de Ciencia y Tecnología of Spain.

References

1 G. Allaire. Homogenization and two-scale convergence. *SIAM J. Math. Analysis* **23** (1992), 1482–1518.

2 G. Allaire and M. Briane. Multiscale convergence and reiterated homogenization. *Proc. R. Soc. Edinb.* A **126** (1996), 297–342.

3 G. Allaire and C. Conca. Bloch wave homogenization for a spectral problem in fluid–solid structures. *Arch. Ration. Mech. Analysis* **135** (1996), 197–257.

4 T. Arbogast, J. Douglas and U. Hornung. Derivation of the double porosity model of single phase flow via homogenization theory. *SIAM J. Math. Analysis* **21** (1990), 823–836.

5 N. S. Bakhvalov and G. P. Panasenko. *Homogenization: averaging processes in periodic media*. Mathematics and its Applications, vol. 36 (Dordrecht: Kluwer, 1989).

6 G. Bouchitté and I. Fragalà. Homogenization of thin structures by two-scale method with respect to measures. *SIAM J. Math. Analysis* **32** (2001), 1198–1226.

7 G. Bouchitté and I. Fragalà. Homogenization of elastic thin structures: a measure-fattening approach. *J. Convex Analysis* **2** (2002), 339–362.

8 J. Casado-Díaz. Two-scale convergence for nonlinear Dirichlet problems in perforated domains. *Proc. R. Soc. Edinb.* A **130** (2000), 249–276.

9 J. Casado-Díaz. Asymptotic behaviour of nonlinear problems in periodically perforated domains. The case $p = N$. *Asymp. Analysis* **30** (2002), 1–26.

10 J. Casado-Díaz and M. Luna-Laynez. A multiscale method to the homogenization of elastic thin reticulated structures. In *Homogenization 2001, Proc. 1st HMS2000* (ed. L. Carbone and R. De Arcangelis). *GAKUTO Int. Ser. Math. Sci. Appl.* **18** (2003), 155–168.

11 J. Casado-Díaz, M. Luna-Laynez and J. D. Martín. An adaptation of the multi-scale methods for the analysis of very thin reticulated structures. *C. R. Acad. Sci. Paris Sér. I* **332** (2001), 223–228.

12 R. Chineb, D. Cioranescu, A. El Jatani and G. Panasenko. Structures réticulées reforcées en élasticité. *C. R. Acad. Sci. Paris Sér. I* **326** (1998), 897–902.

13 P. G. Ciarlet. *Plates and junctions in elastic multi-structures: an asymptotic analysis* (Springer, 1990).

14 P. G. Ciarlet. *Mathematical elasticity, volume II: theory of plates*. Studies in Mathematics and its Applications, vol. 27 (Amsterdam: North-Holland, 1998).

15 D. Cioranescu and J. Saint Jean Paulin. Reinforced and honeycomb structures. *J. Math. Pure Appl.* **65** (1986), 403–422.

16 D. Cioranescu and J. Saint Jean Paulin. Problèmes de Neumann et de Dirichlet dans des structures réticulées de faible épaisseur. *C. R. Acad. Sci. Paris Sér I* **303** (1986), 7–12.

17 D. Cioranescu and J. Saint Jean Paulin. *Homogenization of reticulated structures*. Applied Mathematical Sciences, vol. 136 (Springer, 1999).

18 G. Griso. Analyse asymptotique de structures réticulées. Thèse de l'Université Pierre et Marie Curie, Paris VI (1996).

19 H. Le Dret. *Problèmes variationnels dans les multi-domaines* (Paris: Masson, 1991).

20 M. Lenczner. Homogénéisation d'un circuit électrique. *C. R. Acad. Sci. Paris Sér II* **324** (1997), 537–542.

21 M. Luna-Laynez. Homogeneización de estructuras reticuladas: un método multiescala. PhD thesis, University of Seville (2003).

22 F. Murat and A. Sili. Comportement asymptotique des solutions du système de l'élasticité linéarisée anisotrope hétérogène dans des cylindres minces. *C. R. Acad. Sci. Paris Sér. I* **328** (1999), 179–184.

23 G. Nguetseng. A general convergence result for a functional related to the theory of homogenization. *SIAM J. Math. Analysis* **20** (1989), 608–623.

24 J. Něcas. *Equations aux dérivées partielles* (Presses de l'Université de Montréal, 1965).

25 G. P. Panasenko. The principle of splitting of an averaged operator for a nonlinear system of equations in periodic and random skeletal structures. *Sov. Phys. Dokl.* **25** (1982), 290–295.

26 G. P. Panasenko. Averaging processes in frame constructions with random properties. *Vych. Mat. Fiz.* **23** (1983), 1098–1109.

27 L. Trabucho and J. M. Viaño. Mathematical modelling of rods. In *Handbook of numerical analysis*, vol. IV (Amsterdam: North-Holland, 1996).

28 V. V. Zhikov. Connectedness and homogenization. Examples of fractal conductuvity. *Mat. Sb.* **187** (1996), 3–40.

29 V. V. Zhikov. On an extension of the method of two-scale convergence and its applications. *Mat. Sb.* **191** (2000), 973–1014.

30 V. V. Zhikov. Homogenization of elasticity problems on singular structures. *Dokl. Math.* **64** (2001), 258–262.

(*Issued 7 December 2004*)