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STOCHASTIC COMPARISONS OF SYSTEMS BASED ON SEQUENTIAL ORDER STATISTICS VIA PROPERTIES OF DISTORTED DISTRIBUTIONS

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We consider systems based on sequential order statistics (SOS) with underlying distributions possessing proportional hazard rates (PHRs). In that case, the lifetime distribution of the system can be expressed as a distorted distribution. Motivated by the distribution structure in the case of pairwise different model parameters, a particular class of distorted distributions, the generalized PHR model, is introduced and characterizations of stochastic comparisons for several stochastic orders are obtained. Moreover, results on the asymptotic behavior of some aging characteristics, for example, the hazard rate and the mean residual life function, of general distorted distributions as well as related bounds are given. The results are supplemented with limiting properties of the systems in the case of possibly equal model parameters. Some examples are presented in order to illustrate the application of the findings to systems based on SOS and also to systems with independent heterogeneous components.

1. INTRODUCTION

An important focus of reliability theory is the study of the performance of technical systems. Using the notion of coherent systems of Barlow and Proschan [5], the lifetime T of a system consisting of n components can be expressed as $T = \phi(X_1, \ldots, X_n)$, where $X_1, \ldots, X_n$ denote the lifetimes of the components and ϕ the coherent life function of the system (see Esary and Marshall [20] or Barlow and Proschan [5], p. 12). In this paper, the component lifetimes $X_1, \ldots, X_n$ are mainly modeled via random variables with a particular dependence structure induced by sequential order statistics (SOS). The model of SOS has been proposed in Kamps [24] (see also Cramer [14], Cramer and Kamps [16]) for describing increasingly ordered failure times $X^*_{1:n} \leq \cdots \leq X^*_{n:n}$ of system components in situations when failures may have an impact on the lifetimes of remaining components.

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Employing component lifetimes given by exchangeable random variables with a dependence model induced by SOS, component failures can affect the performance of intact components in the resulting system (see Aki and Hirano [1], Burkschat [7], Navarro and Burkschat [31], see also Hollander and Peña [22]). The corresponding reliability is then given by the mixture distribution (cf. Burkschat [7], Navarro and Burkschat [31])

$$P(T > t) = \sum_{i=1}^{n} s_i P(X^*_{i:n} > t), \quad t \in \mathbb{R}, \tag{1.1}$$

where the weights are determined by the signature $\mathbf{s} = (s_1, \ldots, s_n)$ of the system (see Samaniego [37,38]).

In the following, stochastic comparisons of systems based on SOS are obtained by using properties of distorted distributions. It is assumed that the underlying distributions possess proportional hazard rates (PHRs) (see, e.g., Cramer and Kamps [15]). In that case, the marginal distributions of the SOS and consequently also the distribution of the system lifetime can be expressed as distorted distributions. Therefore, we also give results on the asymptotic behavior of some aging characteristics and on bounds for general distorted distributions and then apply them to systems based on SOS. These results complement the findings in Burkschat and Navarro [11], Navarro and Burkschat [31]. Distorted distributions have been recently extensively studied in Hürlimann [23], Navarro et al. [32,33], Navarro and Gomis [34]. Aside for general results on distorted distributions, we define a particular model for a distortion function, the generalized PHR model, which is motivated by the structure of the marginal distribution in the considered setting of SOS. However, it is also useful for analyzing systems with independent heterogeneous (i.e. non-identically distributed) component lifetimes.

The paper is organized as follows. In Section 2, the survival function of the lifetime of a system based on SOS is expressed as a distorted distribution and representations of the distortion function are given. The case of pairwise different model parameters is treated explicitly. Motivated by the distribution structure in this case, in Section 3, a particular class of distorted distributions, the generalized PHR model, is introduced and characterizations of stochastic comparisons for several stochastic orders are obtained. As a consequence, we also establish stochastic orders for SOS. In Section 4, limiting properties and bounds for aging functions, like the hazard rate and the mean residual life (MRL) function, of general distorted distributions are derived. As particular case, the generalized PHR model is examined. In Section 5, two examples of systems based on SOS are studied by applying the results from the preceding sections. Moreover, it is illustrated by another example that results for the generalized PHR model can be applied to systems with independent heterogeneous component lifetimes. In the last section, results on the asymptotic behavior of systems based on SOS are proven for the case of possibly equal model parameters. These properties supplement the corresponding results for pairwise different model parameters stated in Section 4.

2. REPRESENTATIONS FOR SYSTEMS BASED ON SOS

Let F be a continuous distribution function with support $[0, \infty)$, $\overline{F} = 1 - F$, and

$$\overline{F}_i = \overline{F}^{\alpha_i}, \quad i = 1, \ldots, n, \tag{2.1}$$

for parameters $\alpha_1, \ldots, \alpha_n \in \mathbb{R}^+ = (0, \infty)$, that is, $F_1, ..., F_n$ belong to the same PHR model (cf. (3.1) below) with baseline survival function $\overline{F}$. Let $X^*_{1:n}, \ldots, X^*_{n:n}$ denote the SOS based

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on distributions functions $F_1, \ldots, F_n$. In this particular case, the joint distribution of these random variables coincides with the distribution of n generalized order statistics based on the model parameters $\gamma_i = \alpha_i(n-i+1), i = 1, \ldots, n$, and the distribution function F (see Kamps [24]). Recall that the reliability function of the lifetime T of a coherent system based on the SOS is given by (1.1). Let $U^*_{1:n}, \ldots, U^*_{n:n}$ denote the uniform generalized order statistics (see also Kamps [24]). Because

$$X^*_{r:n} \sim F^{-1}(U^*_{r:n}), \quad r = 1, \ldots, n,$$

we obtain

$$P(T > t) = \sum_{k=1}^{n} s_k P(U^*_{k:n} > F(t)) = \overline{q}(\overline{F}(t)), \quad t \geq 0,$$

with

$$\overline{q}(x) = \sum_{k=1}^{n} s_k P(U^*_{k:n} > 1-x) = 1 - \sum_{k=1}^{n} s_k P(U^*_{k:n} \leq 1-x), \quad x \in [0,1].$$

It can be shown that $\overline{q}$ is an increasing continuous function with $\overline{q}(0) = 0, \overline{q}(1) = 1$, that is, $\overline{q}$ is a distortion function (see, e.g., Hürlimann [23]). From results given in Burkschat and Lenz [17], Cramer and Kamps [8], it follows that the distribution function of the rth uniform generalized order statistic with arbitrary model parameters $\gamma_1, \ldots, \gamma_r > 0$ is given by

$$F_{U^*_{r:n}}(t) = 1 - \left(\prod_{j=1}^{r} \gamma_j\right) \int_0^{1-t} \mathbf{G}^{r,0}_{r,r}\left[x|\gamma_1, \ldots, \gamma_r\right] dx \tag{2.2}$$

$$= \left(\prod_{j=1}^{r} \gamma_j\right) \mathbf{G}^{r+1,0}_{r+1,r+1}\left[1-t|\gamma_1+1, \ldots, \gamma_r+1, 1\right], \tag{2.3}$$

where

$$\mathbf{G}^{r,0}_{r,r}\left[x|\gamma_1, \ldots, \gamma_r\right] = \mathbf{G}^{r,0}_{r,r}\left[x\left|\begin{array}{c}\gamma_1, \ \ldots, \ \gamma_r \\ \gamma_1-1, \ldots, \gamma_r-1\end{array}\right.\right], \quad x \in (0,1),$$

denotes a particular Meijer's G-function (for its definition, see, e.g., Luke [28] or Mathai [29]).

Therefore, the preceding distortion function $\overline{q}$ has the representation

$$\overline{q}(x) = 1 - \sum_{k=1}^{n} s_k \left(\prod_{j=1}^{k} \gamma_j\right) \mathbf{G}^{k+1,0}_{k+1,k+1}\left[x|\gamma_1+1, \ldots, \gamma_k+1, 1\right], \quad x \in (0,1).$$

The derivative of the distortion function is given by

$$\overline{q}'(x) = \sum_{k=1}^{n} s_k \left(\prod_{j=1}^{k} \gamma_j\right) \mathbf{G}^{k,0}_{k,k}\left[x|\gamma_1, \ldots, \gamma_k\right], \quad x \in (0,1),$$

due to the relation (see, e.g, Cramer, Kamps, and Rychlik [18])

$$\frac{d}{dx}\mathbf{G}^{r+1,0}_{r+1,r+1}\left[x|\gamma_1+1, \ldots, \gamma_r+1, 1\right] = -\mathbf{G}^{r,0}_{r,r}\left[x|\gamma_1, \ldots, \gamma_r\right].$$

Moreover, it follows from Cramer et al. [18] that $\mathbf{G}^{r,0}_{r,r}[x|\gamma_1,\ldots,\gamma_r] > 0$ for $x \in (0,1)$ and thus $\overline{q}'(x) > 0$ for $x \in (0,1)$.

Furthermore, if the parameters $\gamma_1,\ldots,\gamma_r$ are pairwise different, that is, $\gamma_i \neq \gamma_j$ for $i \neq j$, then it is known (see Kamps and Cramer [25]) that the distribution function of the rth uniform generalized order statistic can be expressed as follows:

$$F_{U^*_{r:n}}(t) = 1 - \left(\prod_{j=1}^{r}\gamma_j\right)\sum_{i=1}^{r}\frac{a_{i,r}}{\gamma_i}(1-t)^{\gamma_i}, \quad t \in [0,1], \tag{2.4}$$

with the constants

$$a_{i,r} = \prod_{\substack{j=1\\ j\neq i}}^{r}\frac{1}{\gamma_j - \gamma_i}, \quad 1 \le i \le r \le n \quad (\text{with } a_{1,1} = 1), \tag{2.5}$$

i.e. $F_{U^*_{r:n}}(1-x) = 1 - \overline{q}_{r:n}(x)$ with

$$\overline{q}_{r:n}(x) = \left(\prod_{j=1}^{r}\gamma_j\right)\sum_{i=1}^{r}\frac{a_{i,r}}{\gamma_i}x^{\gamma_i}, \quad x \in [0,1].$$

Consequently, for pairwise different $\gamma_1,\ldots,\gamma_n$, the distortion function is given by

$$\overline{q}(x) = \sum_{k=1}^{n}s_k\left(\prod_{j=1}^{k}\gamma_j\right)\sum_{i=1}^{k}\frac{a_{i,k}}{\gamma_i}x^{\gamma_i} = \sum_{i=1}^{n}a_i x^{\gamma_i}, \quad x \in [0,1], \tag{2.6}$$

with the coefficients

$$a_i = \frac{1}{\gamma_i}\sum_{k=i}^{n}s_k a_{i,k}\left(\prod_{j=1}^{k}\gamma_j\right), \quad i = 1,\ldots,n.$$

Note that in this case $\overline{q}$ is equal to the distribution function of a generalized mixture of n power function distributions with the parameters $\gamma_1,\ldots,\gamma_n$ (since some coefficients a_i can be negative; for generalized mixtures; see e.g. Navarro [30]). In particular, the derivative becomes

$$\overline{q}'(x) = \sum_{i=1}^{n}a_i\gamma_i x^{\gamma_i - 1} = \sum_{i=1}^{n}\left(\sum_{k=i}^{n}s_k a_{i,k}\left(\prod_{j=1}^{k}\gamma_j\right)\right)x^{\gamma_i-1}, \quad x \in (0,1].$$

3. STOCHASTIC COMPARISONS IN THE GENERALIZED PHR MODEL

Motivated by the preceding results for the distortion function of systems based on generalized order statistics, we study stochastic comparisons in a generalization of the classical PHR model

$$\overline{G} = \overline{F}^{\gamma} \tag{3.1}$$

with $\gamma \in \mathbb{R}^+$. Taking into account representation (2.6), the PHR model can be extended as follows.

Definition 3.1: *We say that a reliability function $\overline{G}$ satisfies the generalized proportional hazard rate (GPHR) model based on the reliability function $\overline{F}$ and on the coefficients $a_1,\ldots,a_k \in \mathbb{R}$ (not all of them are zero) and $\gamma_1,\ldots,\gamma_k \in \mathbb{R}^+$ if*

$$\overline{G}(t) = \sum_{i=1}^{k} a_i(\overline{F}(t))^{\gamma_i} \tag{3.2}$$

for all t.

Clearly, the PHR model is obtained when $k = a_1 = 1$. The preceding definition is a particular case of the distorted distributions defined in the context of risk theory (see, e.g., Hürlimann [23]), because if $\overline{G}$ satisfies GPHR model then it can be written as

$$\overline{G}(t) = \overline{q}(\overline{F}(t)) \tag{3.3}$$

where $\overline{q}(x) = \sum_{i=1}^{k} a_i x^{\gamma_i}$ is called the distortion function. This function is a continuous increasing function over $[0,1]$ which satisfies $\overline{q}(0) = 0$ and $\overline{q}(1) = 1$. Note that the right-hand side of (3.2) defines a proper reliability function for any reliability function $\overline{F}$ if and only if $\sum_{i=1}^{k} a_i = 1$ and

$$\sum_{i=1}^{k} a_i\gamma_i x^{\gamma_i} \geq 0$$

for all $x \in [0,1]$. In that case $\overline{q}$ is the distribution of a generalized mixture of power distributions with parameters $\gamma_1,\ldots,\gamma_k \in \mathbb{R}^+$. In this and the two following sections, we will assume $\gamma_i \neq \gamma_j$ for all $i \neq j$. Then, the results can be directly applied to coherent systems based on SOS using the corresponding distribution theory (see (2.4) and (2.6)).

If $\overline{G}$ satisfies (3.2) and is absolutely continuous, then the associated probability density function (pdf) is

$$g(t) = \sum_{i=1}^{k} a_i\gamma_i(\overline{F}(t))^{\gamma_i-1} f(t)$$

and its hazard rate is

$$h_G(t) = \frac{\sum_{i=1}^{k} a_i\gamma_i(\overline{F}(t))^{\gamma_i}}{\sum_{i=1}^{k} a_i(\overline{F}(t))^{\gamma_i}} h_F(t),$$

where f is the pdf of $\overline{F}$ and $h_F = f/\overline{F}$ is its hazard rate function.

Ordering properties for distorted distributions and generalized mixtures were obtained from Navarro [32], Navarro et al. [30], Navarro and Gomis [34]. Analogously, we can obtain the following ordering properties for the GPHR model.

Proposition 3.2: *Let $\overline{G}$ and $\overline{G}^*$ be two reliability functions satisfying the GPHR model with a common baseline reliability $\overline{F}$ and with respective coefficients $a_1,\ldots,a_k \in \mathbb{R}$, $\gamma_1,\ldots,\gamma_k \in \mathbb{R}^+$, $a_1^*,\ldots,a_{k^*}^* \in \mathbb{R}$ and $\gamma_1^*,\ldots,\gamma_{k^*}^* \in \mathbb{R}^+$. Then:*

(i) $\overline{G} \leq_{\text{st}} \overline{G}^*$ for all $\overline{F}$ if and only if

$$\sum_{i=1}^{k^*} a_i^* x^{\gamma_i^*} - \sum_{j=1}^{k} a_j x^{\gamma_j} \geq 0$$

for all $x \in [0,1]$.

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(ii) $\overline{G} \leq_{\mathrm{hr}} \overline{G}^*$ *for all* $\overline{F}$ *if and only if*

$$\sum_{i=1}^{k^*}\sum_{j=1}^{k} a_i^* a_j(\gamma_j - \gamma_i^*)x^{\gamma_i^*+\gamma_j} \geq 0 \tag{3.4}$$

for all $x \in [0,1]$.

(iii) $\overline{G} \leq_{\mathrm{rh}} \overline{G}^*$ *for all* $\overline{F}$ *if and only if*

$$\sum_{i=1}^{k^*}\sum_{j=1}^{k} a_i^* a_j(\gamma_i^* - \gamma_j)x^{\gamma_i^*+\gamma_j} \leq \sum_{i=1}^{k^*} a_i^*\gamma_i^* x^{\gamma_i^*} - \sum_{j=1}^{k} a_j\gamma_j x^{\gamma_j} \tag{3.5}$$

for all $x \in [0,1]$.

(iv) $\overline{G} \leq_{\mathrm{lr}} \overline{G}^*$ *for all absolutely continuous* $\overline{F}$ *if and only if*

$$\sum_{i=1}^{k^*}\sum_{j=1}^{k} a_i^* a_j\gamma_i^*\gamma_j(\gamma_j - \gamma_i^*)x^{\gamma_i^*+\gamma_j} \geq 0 \tag{3.6}$$

for all $x \in [0,1]$.

(v) $\overline{G} \leq_{\mathrm{mrl}} \overline{G}^*$ *for all* $\overline{F}$ *with* $\mu_G \leq \mu_{G^*}$ *if*

$$\frac{\sum_{i=1}^{k^*} a_i^* x^{\gamma_i^*}}{\sum_{i=1}^{k} a_i x^{\gamma_i}}$$

is bathtub in $(0,1]$.

Proof: The proof of (i) is immediate.

To prove (ii) we note that $\overline{G} \leq_{\mathrm{hr}} \overline{G}^*$ holds if and only if

$$\frac{\overline{G}^*(t)}{\overline{G}(t)} = \frac{\sum_{i=1}^{k^*} a_i^*(\overline{F}(t))^{\gamma_i^*}}{\sum_{j=1}^{k} a_j(\overline{F}(t))^{\gamma_j}} \tag{3.7}$$

is increasing in t. This property holds for any $\overline{F}$ if and only if

$$R(x) = \frac{\sum_{i=1}^{k^*} a_i^* x^{\gamma_i^*}}{\sum_{j=1}^{k} a_j x^{\gamma_j}} \tag{3.8}$$

decreases for all $x \in (0,1]$. By differentiating we obtain

$$\begin{aligned} R'(x) &=_{sign} \left[\sum_{i=1}^{k^*} a_i^*\gamma_i^* x^{\gamma_i^*}\right]\left[\sum_{j=1}^{k} a_j x^{\gamma_j}\right] - \left[\sum_{i=1}^{k^*} a_i^* x^{\gamma_i^*}\right]\left[\sum_{j=1}^{k} a_j\gamma_j x^{\gamma_j}\right] \\ &= \sum_{i=1}^{k^*}\sum_{j=1}^{k} a_i^* a_j(\gamma_i^* - \gamma_j)x^{\gamma_i^*+\gamma_j} \end{aligned}$$

for all $x \in (0,1]$. Here $=_{sign}$ means that both sides of the equation have the same sign. Therefore R decreases in $(0,1]$ if and only if (3.4) holds for all $x \in [0,1]$.

Analogously, to prove (iii) we note that $\overline{G} \leq_{\text{rh}} \overline{G}^*$ holds if

$$\frac{G^*(t)}{G(t)} = \frac{1-\overline{G}^*(t)}{1-\overline{G}(t)} = \frac{1-\sum_{i=1}^{k^*} a_i^*(\overline{F}(t))^{\gamma_i^*}}{1-\sum_{j=1}^{k} a_j(\overline{F}(t))^{\gamma_j}}$$

is increasing in t. This property holds for all $\overline{F}$ if and only if

$$\overline{R}(x) = \frac{1-\sum_{i=1}^{k^*} a_i^* x^{\gamma_i^*}}{1-\sum_{i=1}^{k} a_i x^{\gamma_i}} \tag{3.9}$$

decreases for all $x \in [0,1)$. By differentiating, we obtain

$$\begin{aligned}\overline{R}'(x) &=_{sign} -\left[\sum_{i=1}^{k^*} a_i^*\gamma_i^* x^{\gamma_i^*}\right]\left[1-\sum_{j=1}^{k} a_j x^{\gamma_j}\right] + \left[1-\sum_{i=1}^{k^*} a_i^* x^{\gamma_i^*}\right]\left[\sum_{j=1}^{k} a_j\gamma_j x^{\gamma_j}\right] \\ &= \sum_{i=1}^{k^*}\sum_{j=1}^{k} a_i^* a_j(\gamma_i^*-\gamma_j)x^{\gamma_i^*+\gamma_j} - \sum_{i=1}^{k^*} a_i^*\gamma_i^* x^{\gamma_i^*} + \sum_{j=1}^{k} a_j\gamma_j x^{\gamma_j}\end{aligned}$$

for all $x \in [0,1)$. Therefore $\overline{R}$ decreases in $[0,1)$ if and only if (3.5) holds for all $x \in [0,1]$.

To prove (iv) we note that $\overline{G} \leq_{\text{lr}} \overline{G}^*$ holds if and only if the ratio of the respective density functions

$$\frac{g^*(t)}{g(t)} = \frac{f(x)\sum_{i=1}^{k^*} a_i^*\gamma_i^*(\overline{F}(t))^{\gamma_i^*-1}}{f(x)\sum_{j=1}^{k} a_j\gamma_j(\overline{F}(t))^{\gamma_j-1}}$$

is increasing in t. This property holds for all absolutely continuous $\overline{F}$ if and only if

$$r(x) = \frac{\sum_{i=1}^{k^*} a_i^*\gamma_i^* x^{\gamma_i^*}}{\sum_{i=1}^{k} a_i\gamma_i x^{\gamma_i}} \tag{3.10}$$

decreases for all $x \in (0,1)$. By differentiating we obtain

$$\begin{aligned}r'(x) &=_{\text{sign}} \left[\sum_{i=1}^{k^*} a_i^*(\gamma_i^*)^2 x^{\gamma_i^*}\right]\left[\sum_{j=1}^{k} a_j\gamma_j x^{\gamma_j}\right] - \left[\sum_{i=1}^{k^*} a_i^*\gamma_i^* x^{\gamma_i^*}\right]\left[\sum_{j=1}^{k} a_j(\gamma_j)^2 x^{\gamma_j}\right] \\ &= \sum_{i=1}^{k^*}\sum_{j=1}^{k} a_i^* a_j\gamma_i^*\gamma_j(\gamma_i^*-\gamma_j)x^{\gamma_i^*+\gamma_j}\end{aligned}$$

for all $x \in (0,1)$. Therefore r decreases in $(0,1)$ if and only if (3.6) holds for all $x \in [0,1]$.

Finally, the proof of (v) is obtained from (3.7) and Theorem 2.3 in Navarro and Gomis [34]. ■

Remark 3.3: Note that, from the preceding proof, an alternative condition to check the HR ordering is to study whether the function R defined in (3.8) is decreasing in $[0,1]$. Analogously, to check the RHR or the LR orders we can study the functions $\overline{R}$ or r, respectively, defined in the preceding proof. This is equivalent to use (3.2) and the general results for distorted distributions obtained in Navarro et al. [32].

Clearly we can apply the general results obtained in the preceding proposition to coherent systems based on SOS from a PHR model (2.1) by using the representation given in (2.6). In particular, for the SOS (k-out-of-n systems) we obtain the following results.

PROPOSITION 3.4: *Let $X^*_{r:n}$ and $Y^*_{s:m}$ be two SOS obtained from PHR models with a common reliability function $\overline{F}$ and with respective coefficients $\alpha_1, \ldots, \alpha_n \in \mathbb{R}^+$ and $\beta_1, \ldots, \beta_m \in \mathbb{R}^+$. If $\gamma_i = (n-i+1)\alpha_i$ for $i = 1, \ldots, n$ and $\delta_j = (m-j+1)\beta_j$ for $j = 1, \ldots, m$, we assume that $\gamma_i \neq \gamma_j$ and $\delta_i \neq \delta_j$ for all $i \neq j$. Let*

$$a_{i,r} = \prod_{\substack{j=1\\ j\neq i}}^{r} \frac{1}{\gamma_j - \gamma_i}, \quad 1 \leq i \leq r \leq n$$

and

$$b_{i,s} = \prod_{\substack{j=1\\ j\neq i}}^{s} \frac{1}{\delta_j - \delta_i}, \quad 1 \leq i \leq s \leq m.$$

Then:

(i) $X^*_{r:n} \leq_{\mathrm{st}} Y^*_{s:m}$ for all $\overline{F}$ if and only if

$$\left[\prod_{j=1}^{s} \delta_j\right] \sum_{i=1}^{s} \frac{b_{i,s}}{\delta_i} x^{\delta_i} - \left[\prod_{j=1}^{r} \gamma_j\right] \sum_{i=1}^{r} \frac{a_{i,r}}{\gamma_i} x^{\gamma_i} \geq 0$$

for all $x \in [0,1]$.

(ii) $X^*_{r:n} \leq_{\mathrm{hr}} Y^*_{s:m}$ for all $\overline{F}$ if and only if

$$\sum_{i=1}^{s}\sum_{j=1}^{r} a_{j,r} b_{i,s} \left(\frac{1}{\delta_i} - \frac{1}{\gamma_j}\right) x^{\delta_i+\gamma_j} \geq 0$$

for all $x \in [0,1]$.

(iii) $X^*_{r:n} \leq_{\mathrm{rh}} Y^*_{s:m}$ for all $\overline{F}$ if and only if

$$\sum_{i=1}^{s}\sum_{j=1}^{r} a_{j,r} b_{i,s} \left(\frac{1}{\gamma_j} - \frac{1}{\delta_i}\right) x^{\delta_i+\gamma_j} \leq \left[\prod_{j=1}^{r} \gamma_j\right]^{-1} \sum_{i=1}^{s} b_{i,s} x^{\delta_i} - \left[\prod_{j=1}^{s} \delta_j\right]^{-1} \sum_{i=1}^{r} a_{i,r} x^{\gamma_i}$$

for all $x \in [0,1]$.

(iv) $X^*_{r:n} \leq_{\mathrm{lr}} Y^*_{s:m}$ for all $\overline{F}$ if and only if

$$\sum_{i=1}^{s}\sum_{j=1}^{r} a_{j,r} b_{i,s} \left(\gamma_j - \delta_i\right) x^{\delta_i+\gamma_j} \geq 0$$

for all $x \in [0,1]$.

(v) $X^*_{r:n} \leq_{\mathrm{mrl}} Y^*_{s:m}$ for all $\overline{F}$ with $E(X^*_{r:n}) \leq E(Y^*_{s:m})$ if

$$\frac{\sum_{i=1}^{s} \frac{b_{i,s}}{\delta_i} x^{\delta_i}}{\sum_{j=1}^{r} \frac{a_{j,r}}{\gamma_j} x^{\gamma_j}}$$

is bathtub in $(0,1]$.

The proof is immediate from (2.6) and Proposition 3.2. For example, note that from Proposition 3.4(iv) we obtain the following trivial result for series systems from two different PHR models: $X^*_{1:n} \le_{\rm lr} Y^*_{1:m}$ holds for all $\overline{F}$ if and only if $\gamma_1 \ge \delta_1$, that is, $n\alpha_1 \ge m\beta_1$.

Remark 3.5: The results obtained here for the GPHR model can also be applied to coherent systems with independent components satisfying the PHR model. Stochastic comparisons for k-out-of-n systems in this setting have been recently reviewed in Balakrishnan and Zhao [4].

It is well known (see, e.g., Barlow and Proschan [5]) that the reliability functions of the systems with independent components can be written as

$$\overline{F}_T(t) = \overline{Q}(\overline{F}_1(t), \ldots, \overline{F}_n(t)),$$

where $\overline{F}_1, \ldots, \overline{F}_n$ are the reliability functions of the component lifetimes and $\overline{Q}$ is a multinomial, which only depends on the structure of the system. This function is given by

$$\overline{Q}(x_1, \ldots, x_n) = \sum_{i=1}^{r} \prod_{k \in P_i} x_k - \sum_{i=1}^{r-1} \sum_{j=i}^{r} \prod_{k \in P_i \cap P_j} x_k + \cdots + (-1)^{r+1} \prod_{k \in P_1 \cap \cdots \cap P_r} x_k,$$

where $P_1, \ldots, P_r$ are the minimal path sets of the system. A path set is a set P such that if all the components in P work, then the system works. A minimal path set is a path set which does not contain other path sets. Therefore, under the PHR model, the system reliability function can be written as in (3.2) and we can apply the results obtained here. An example is given in Section 5.

4. LIMITING PROPERTIES AND BOUNDS OF AGING FUNCTIONS

In this section, we study limiting properties and bounds of aging functions of distorted distributions. Specific results are obtained for the GPHR model defined in the preceding section. Therefore, these results can be applied to SOS as well as coherent systems based on them (see Section 2).

Firstly, we study the hazard rate function. In this section we assume that G is a distorted distribution from F and that both are absolutely continuous with support $[0, \infty)$. For the respective density functions g and f of G and F, it is assumed that $g(x) > 0$ and $f(x) > 0$ for $x > 0$. The respective reliability functions satisfy

$$\overline{G}(t) = \overline{q}(\overline{F}(t))$$

where q is a distortion function. The respective pdfs satisfy

$$g(t) = f(t)\overline{q}'(\overline{F}(t))$$

and the hazard rate functions are related via

$$h_G(t) = \frac{g(t)}{\overline{G}(t)} = f(t)\frac{\overline{q}'(\overline{F}(t))}{\overline{q}(\overline{F}(t))} = h_F(t)\alpha(\overline{F}(t)),$$

where $h_F = f/\overline{F}$ and $\alpha(u) = u\overline{q}'(u)/\overline{q}(u)$. Therefore, we obtain the following immediate results.

PROPOSITION 4.1: *If G is a distorted distribution from F, then*

$$\lim_{t\to\infty}\frac{h_G(t)}{h_F(t)}=\lim_{u\to 0+}\alpha(u)$$

and

$$h_F(t)\inf_{u\in(0,1]}\alpha(u)\le h_G(t)\le h_F(t)\sup_{u\in(0,1]}\alpha(u),$$

where $\alpha(u)=u\overline{q}'(u)/\overline{q}(u)$.

We should mention here that the function α was also used in Navarro et al. [33] to determine whether the IFR and DFR classes are preserved under the formation of distorted distributions. Specifically, there it is proved that the IFR (resp. DFR) class is preserved if and only if α is decreasing (resp. increasing). Results on aging properties, like the IFR and DFR classes, of SOS can be found in Cramer and Kamps [16], Cramer [13], Burkschat and Navarro [9], Torrado, Lillo, and Wiper [39], Burkschat and Torrado [12].

If G satisfies the GPHR model (3.2), then

$$\alpha(u)=\frac{\sum_{i=1}^k a_i\gamma_i u^{\gamma_i}}{\sum_{i=1}^k a_i u^{\gamma_i}}$$

and

$$\lim_{t\to\infty}\frac{h_G(t)}{h_F(t)}=\lim_{u\to 0+}\alpha(u)=\gamma_{1:k},$$

where $\gamma_{1:k}=\min(\gamma_1,\ldots,\gamma_k)$, provided that $a_1,\ldots,a_k\in\mathbb{R}\setminus\{0\}$.

To obtain a similar result for MRL functions, we need to study at first the limiting behavior of the reliability functions. The result can be stated as follows.

PROPOSITION 4.2: *If G is a distorted distribution from F and there exists a distortion function $\overline{p}$ with $\overline{p}(u)>0$ for $u\in(0,1)$ such that*

$$\lim_{u\to 0+}\frac{\overline{q}(u)}{\overline{p}(u)}=c\in\mathbb{R}^+,$$

then

$$\lim_{t\to\infty}\frac{\overline{G}(t)}{\overline{p}(\overline{F}(t))}=c$$

and

$$\lim_{t\to\infty}\frac{m_G(t)}{m_p(t)}=1,$$

where m_G is the MRL of G and m_p is the MRL of $\overline{p}(\overline{F})$.

Proof: The limit of the reliability functions is given by

$$\lim_{t\to\infty}\frac{\overline{G}(t)}{\overline{p}(\overline{F}(t))}=\lim_{u\to 0+}\frac{\overline{q}(u)}{\overline{p}(u)}=c>0.$$

From the definition of the MRL, we have

$$\lim_{t\to\infty}\frac{m_G(t)}{m_p(t)}=\lim_{t\to\infty}\frac{\overline{p}(\overline{F}(t))}{\overline{G}(t)}\lim_{t\to\infty}\frac{\int_t^\infty \overline{G}(x)dx}{\int_t^\infty \overline{p}(\overline{F}(x))dx}.$$

The first limit is

$$\lim_{t\to\infty}\frac{\overline{p}(\overline{F}(t))}{\overline{G}(t)}=\lim_{u\to 0+}\frac{\overline{p}(u)}{\overline{q}(u)}=1/c>0.$$

By applying the L'Hôpital's rule to the second limit, we get

$$\lim_{t\to\infty}\frac{\int_t^\infty \overline{G}(x)dx}{\int_t^\infty \overline{p}(\overline{F}(x))dx}=\lim_{t\to\infty}\frac{\overline{G}(t)}{\overline{p}(\overline{F}(t))}=c>0$$

and so the stated result holds. ■

Note that the function $\overline{p}$ can also be used to obtain bounds for the reliability function as

$$\overline{p}(\overline{F}(t))\inf_{u\in(0,1]}\frac{\overline{q}(u)}{\overline{p}(u)}\le \overline{G}(t)\le \overline{p}(\overline{F}(t))\sup_{u\in(0,1]}\frac{\overline{q}(u)}{\overline{p}(u)}.$$

Analogously, for the expected values we have

$$\mu_p\inf_{u\in(0,1]}\frac{\overline{q}(u)}{\overline{p}(u)}\le \mu_G\le \mu_p\sup_{u\in(0,1]}\frac{\overline{q}(u)}{\overline{p}(u)}$$

where μ_G is the mean of G and

$$\mu_p=\int_0^\infty \overline{p}(\overline{F}(x))dx$$

is the mean of $\overline{p}(\overline{F})$.

In particular, if G satisfies the GPHR model (3.2) with $a_1,\ldots,a_k\in\mathbb{R}\setminus\{0\}$, then

$$\lim_{u\to 0+}\frac{\overline{q}(u)}{u^{\gamma_{1:k}}}=a_j>0,$$

where $\gamma_j=\gamma_{1:k}<\gamma_i$ for all $i\neq j$ and

$$\lim_{t\to\infty}\frac{m_G(t)}{m_p(t)}=1$$

for $\overline{p}(u)=u^{\gamma_{1:k}}$.

Next, we present an alternative way of computing the limit of the hazard rate function of a distorted distribution. The proof is similar to that of the preceding proposition.

PROPOSITION 4.3: *If G is a distorted distribution from F and there exists a distortion function $\overline{p}$ with derivative $\overline{p}'(u) > 0$ for $0 < u < \varepsilon$ and some $\varepsilon \in (0,1)$ such that*

$$\lim_{u\to 0+} \frac{\overline{q}'(u)}{\overline{p}'(u)} = c \in \mathbb{R}^+,$$

then

$$\lim_{t\to\infty} \frac{\overline{G}(t)}{\overline{p}(\overline{F}(t))} = \lim_{t\to\infty} \frac{g(t)}{f(t)\overline{p}'(\overline{F}(t))} = c$$

and

$$\lim_{t\to\infty} \frac{h_G(t)}{h_p(t)} = 1,$$

where h_G and h_p are the hazard rate function of G and $\overline{p}(\overline{F})$, respectively.

In particular, if G satisfies the GPHR model (3.2) with $a_1, \ldots, a_k \in \mathbb{R} \setminus \{0\}$ and $\overline{p}(u) = u^{\gamma_{1:k}}$ with $\gamma_{1:k} = \min(\gamma_1, \ldots, \gamma_k)$, then

$$\lim_{u\to 0+} \frac{\overline{q}'(u)}{\overline{p}'(u)} = \lim_{u\to 0+} \frac{\sum_{i=1}^{k} a_i\gamma_i u^{\gamma_i - 1}}{\gamma_{1:k} u^{\gamma_{1:k}-1}} = a_j > 0,$$

where $\gamma_j = \gamma_{1:k} < \gamma_i$ for all $i \neq j$. Hence

$$\lim_{t\to\infty} \frac{h_G(t)}{h_p(t)} = 1,$$

where $h_p(t) = \gamma_{1:k} h_F(t)$. Therefore, as above, we obtain

$$\lim_{t\to\infty} \frac{h_G(t)}{h_F(t)} = \gamma_{1:k}.$$

Remark 4.4: Analogously to the preceding results, we can conclude from (2.6) for a system based on SOS with signature vector $\mathbf{s} = (s_1, \ldots, s_r, 0, \ldots, 0)$ with $s_r > 0$ and decreasingly ordered model parameters $\gamma_1 > \cdots > \gamma_r$ that (cf. Burkschat and Navarro [11], Theorems 4.7 and 4.1)

$$\lim_{t\to\infty} \frac{\overline{F}_T(t)}{\overline{F}^{\gamma_r}(t)} = s_r a_{r,r} \prod_{j=1}^{r-1} \gamma_j = s_r \prod_{j=1}^{r-1} \frac{\gamma_j}{\gamma_j - \gamma_r}, \quad \lim_{t\to\infty} \frac{h_T(t)}{h_F(t)} = \gamma_r.$$

For the corresponding results for systems with iid components, see Block, Dugas, and Samaniego [6] and Samaniego [38], Section 5.3; see also Liu, Mao, and Hu [27].

Finally, we study the dispersion of the distorted distribution by using the Gini mean difference defined as

$$\Delta_F = E(X_{2:2} - X_{1:2}) = 2\int_0^\infty \overline{F}(t)(1 - \overline{F}(t))dt,$$

where $X_{1:2}, X_{2:2}$ are the order statistics obtained from two IID random variables with the same distribution F. This dispersion measure was used in Kozyra and Rychlik [26] to obtain bounds for L-statistics and coherent systems with IID components. Let us see how these results can be extended to distorted distributions.

Proposition 4.5: *If* $\overline{G} = \overline{q}(\overline{F})$ *is the reliability function of* Y *and*

$$\mu_F = \int_0^\infty \overline{F}(t)dt < \infty,$$

then

$$\inf_{u\in(0,1)} \beta(u) \leq E\left(\frac{Y-\mu_F}{\Delta_F}\right) \leq \sup_{u\in(0,1)} \beta(u),$$

where

$$\beta(u) = \frac{\overline{q}(u)-u}{2u(1-u)}, \quad u \in (0,1). \tag{4.1}$$

Proof: The lower bound can be obtained as follows:

$$\begin{aligned} E\left(\frac{Y-\mu_F}{\Delta_F}\right) &= \int_0^\infty \frac{\overline{q}(\overline{F}(t))-\overline{F}(t)}{\Delta_F}dt \\ &= \int_0^\infty \frac{\overline{q}(\overline{F}(t))-\overline{F}(t)}{2\overline{F}(t)(1-\overline{F}(t))}\frac{2\overline{F}(t)(1-\overline{F}(t))}{\Delta_F}dt \\ &\geq \inf_{u\in(0,1)} \beta(u) \int_0^\infty \frac{2\overline{F}(t)(1-\overline{F}(t))}{\Delta_F}dt \\ &= \inf_{u\in(0,1)} \beta(u) \end{aligned}$$

with β given in (4.1). The upper bound can be obtained in a similar manner. ■

5. EXAMPLES

In this section, we apply the results from the first sections in several examples. In the first example, we compare in the HR order two differently structured systems based on SOS from the same PHR model. Note that the considered system structures cannot be ordered in the HR order by using signatures.

Example 5.1: We want to compare in the HR order the systems with lifetimes

$$T_1 = \min(X_1, \max(X_2, X_3), \max(X_3, X_4))$$

and

$$T_2 = \max(\min(X_1, X_2), \min(X_1, X_3, X_4), \min(X_2, X_3, X_4))$$

(i.e. systems numbers 12 and 15 of Table 1 in Navarro et al. [36]). These systems are ST ordered for any exchangeable model (see Figure 1 in Navarro et al. [36]) which includes the dependence model defined by the SOS (see Navarro and Burkschat [31]). In particular, $T_1 \leq_{\text{st}} T_2$ for SOS based on any PHR model (2.1) and any F. However, they are not necessarily HR ordered for any exchangeable model (see Figure 2 in Navarro et al. [36]). So we want to study whether they are HR ordered for a particular (known) SOS–PHR model, that is, for fixed parameter values $\alpha_1, \dots, \alpha_4$. Their respective signatures are

$\boldsymbol{s}_1 = (1/4, 7/12, 1/6, 0)$ and $\boldsymbol{s}_2 = (0, 5/6, 1/6, 0)$ (see, e.g., Table 1 in Navarro et al. [36]). Hence, from (2.6), the respective distortion functions are

$$\overline{q}_1(x) = \frac{1}{4}\overline{q}_{1:4}(x) + \frac{7}{12}\overline{q}_{2:4}(x) + \frac{1}{6}\overline{q}_{3:4}(x)$$

and

$$\overline{q}_2(x) = \frac{5}{6}\overline{q}_{2:4}(x) + \frac{1}{6}\overline{q}_{3:4}(x),$$

where $\overline{q}_{1:4}(x) = x^{\gamma_1}$,

$$\overline{q}_{2:4}(x) = \frac{\gamma_2}{\gamma_2-\gamma_1}x^{\gamma_1} + \frac{\gamma_1}{\gamma_1-\gamma_2}x^{\gamma_2},$$

$$\overline{q}_{3:4}(x) = \frac{\gamma_2}{\gamma_2-\gamma_1}\frac{\gamma_3}{\gamma_3-\gamma_1}x^{\gamma_1} + \frac{\gamma_1}{\gamma_1-\gamma_2}\frac{\gamma_3}{\gamma_3-\gamma_2}x^{\gamma_2} + \frac{\gamma_1}{\gamma_1-\gamma_3}\frac{\gamma_2}{\gamma_2-\gamma_3}x^{\gamma_3},$$

$\gamma_1 = 4\alpha_1$, $\gamma_2 = 3\alpha_2$ and $\gamma_3 = 2\alpha_3$. Therefore

$$\overline{q}_1(x) = \left(\frac{1}{4} + \frac{7}{12}\frac{\gamma_2}{\gamma_2-\gamma_1} + \frac{1}{6}\frac{\gamma_2}{\gamma_2-\gamma_1}\frac{\gamma_3}{\gamma_3-\gamma_1}\right)x^{\gamma_1} + \left(\frac{7}{12}\frac{\gamma_1}{\gamma_1-\gamma_2} + \frac{1}{6}\frac{\gamma_1}{\gamma_1-\gamma_2}\frac{\gamma_3}{\gamma_3-\gamma_2}\right)x^{\gamma_2} + \frac{1}{6}\frac{\gamma_1}{\gamma_1-\gamma_3}\frac{\gamma_2}{\gamma_2-\gamma_3}x^{\gamma_3}$$

and

$$\overline{q}_2(x) = \left(\frac{5}{6}\frac{\gamma_2}{\gamma_2-\gamma_1} + \frac{1}{6}\frac{\gamma_2}{\gamma_2-\gamma_1}\frac{\gamma_3}{\gamma_3-\gamma_1}\right)x^{\gamma_1} + \left(\frac{5}{6}\frac{\gamma_1}{\gamma_1-\gamma_2} + \frac{1}{6}\frac{\gamma_1}{\gamma_1-\gamma_2}\frac{\gamma_3}{\gamma_3-\gamma_2}\right)x^{\gamma_2} + \frac{1}{6}\frac{\gamma_1}{\gamma_1-\gamma_3}\frac{\gamma_2}{\gamma_2-\gamma_3}x^{\gamma_3}.$$

Then we can use Proposition 3.2(*ii*), to study if these two systems are HR ordered (or not) for any F. From Remark 3.3, we just need to study whether the function R defined in (3.8) is decreasing in $(0,1]$. For example, if $\alpha_1 = 1$, $\alpha_2 = 2$ and $\alpha_3 = 1$, we obtain the R function plotted in Figure 1 (solid). As this function is not decreasing, these systems are not HR ordered for all F for these parameter values (note that even for $\alpha_1 = \alpha_2 = \alpha_3 = 1$, that is, the situation of iid component lifetimes, we also get a function R which is not decreasing). However, if $\alpha_1 = 1$, $\alpha_2 = 2$ and $\alpha_3 = 4$, we obtain the R function plotted in Figure 1 (dotted). It can be shown that $R'(x) =_{sign} x^4 - 2x^2 - 1 < 0$ for $x \in (0,1)$, that is, R is decreasing. Consequently, $T_1 \leq_{\rm hr} T_2$ holds for all F. Note that in both cases we have $R \geq 1$ and so $T_1 \leq_{\rm st} T_2$ for all F. This last result can be obtained by using signatures (for any exchangeable model); see Navarro et al. [36]. Moreover, from the results given in the preceding section the system hazard rate functions satisfy $\lim_{t\to\infty} h_{T_i}(t)/h_F(t) = 2$ in the first case and $\lim_{t\to\infty} h_{T_i}(t)/h_F(t) = 4$ in the second, for $i = 1, 2$.

The following example shows that some coherent systems can be HR ordered in the iid case but not HR ordered in the case of components coming from the SOS model.

Example 5.2: Let us consider the coherent systems with lifetimes $T_1 = \min(X_1, \max(X_2, X_3, X_4))$ and $T_2 = \max(X_1, \min(X_2, X_3, X_4))$. Example 3.1 in Navarro [30] proves that if the components are iid with a common distribution F, then $T_1 \leq_{\rm lr} T_2$ for any F. However, their respective signatures $(1/4, 1/4, 1/2, 0)$ and $(0, 1/2, 1/4, 1/4)$ are not HR-ordered. As

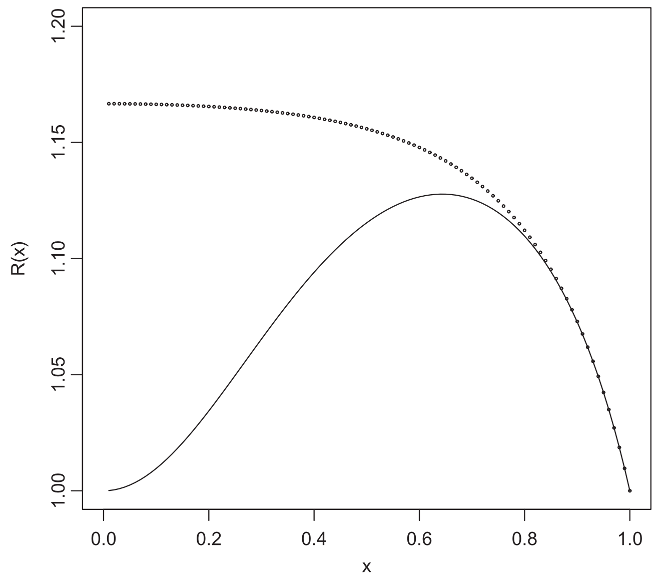

FIGURE 1. Plots of the function R defined in (3.8) for the systems studied in Example 5.1 when $\alpha_1 = 1$, $\alpha_2 = 2$ and $\alpha_3 = 1$ (solid) and $\alpha_1 = 1$, $\alpha_2 = 2$ and $\alpha_3 = 4$ (dotted). In the first case, the systems are not HR ordered, but in the second they are HR ordered, since R is decreasing in $(0, 1]$.

the signatures are ST ordered, we have $T_1 \leq_{\text{st}} T_2$ for any exchangeable random vector (X_1, X_2, X_3, X_4).

Let us assume that the components are dependent with the dependence model defined by the SOS obtained from a PHR model with parameters $\alpha_1, \ldots, \alpha_4$ and baseline reliability $\overline{F}$. Note that it is crucial that the signatures of the considered systems are not HR ordered. For systems based on the SOS–PHR model with signatures which are HR ordered, it follows $T_1 \leq_{\text{hr}} T_2$ (see Navarro and Burkschat [31, Sections 2 and 3]). Now, if we assume that $\gamma_i = (4 - i + 1)\alpha_i$, $i = 1, \ldots, 4$ are pairwise different, then the reliability functions of these systems can be written as a distortion of $\overline{F}$ with respective distortion functions p_1 and p_2 given by (2.6). For example, for $\alpha_1 = \alpha_2 = \alpha_3 = 1$ and $\alpha_4 = 10$, we get $\gamma_1 = 4$, $\gamma_2 = 3$, $\gamma_3 = 2$ and $\gamma_4 = 10$ and the respective distortion functions are

$$\overline{q}_1(x) = x^4 - 3x^3 + 3x^2 + 0x^{10}$$

and

$$\overline{q}_2(x) = \frac{1}{2}x^4 - \frac{20}{7}x^3 + \frac{27}{8}x^2 - \frac{1}{56}x^{10}.$$

By plotting the ratio $R(x) = \overline{q}_2(x)/\overline{q}_1(x)$ (see Figure 2, left) we see that R is not monotonic and so these systems are not HR ordered for any F. The hazard rate functions in the case of a baseline exponential distribution with mean 1 can be seen in Figure 2, right (solid, T_1, dotted, T_2). In this case they are not HR ordered. However, for other parameters, we might obtain different results. For example, for $\alpha_1 = \alpha_2 = \alpha_3 = 1$ and $\alpha_4 = 5$, we obtain the ratio $R(x) = \overline{q}_2(x)/\overline{q}_1(x)$ plotted in Figure 3 (left). Therefore, $T_1 \leq_{\text{hr}} T_2$ for any F since R is decreasing. Hence we can study whether they are LR ordered. To this end we plot the function r defined in (3.10) in Figure 3 (right). As r is decreasing, $T_1 \leq_{\text{lr}} T_2$ holds for any F.

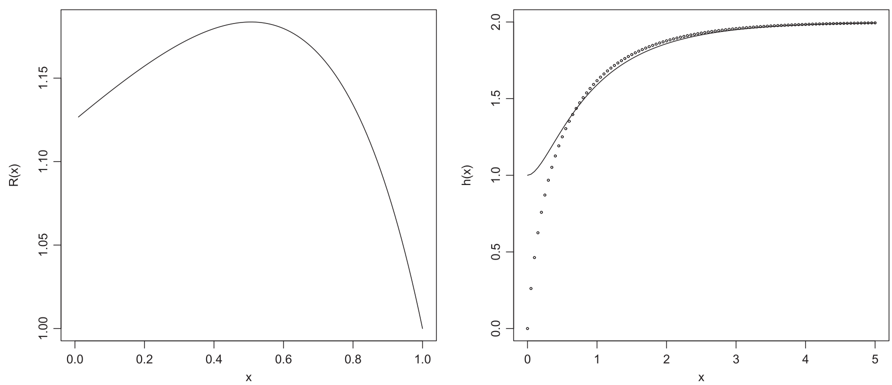

FIGURE 2. Plot (left) of the function R defined in (3.8) for the systems studied in Example 5.2 when $\alpha_1 = \alpha_2 = \alpha_3 = 1$ and $\alpha_4 = 10$. The systems are not HR ordered for any F since R is not monotonic. Hazard rate plots (right) in the case of a baseline exponential distribution with mean 1 (solid, T_1, dotted, T_2). In this case, they are not HR ordered.

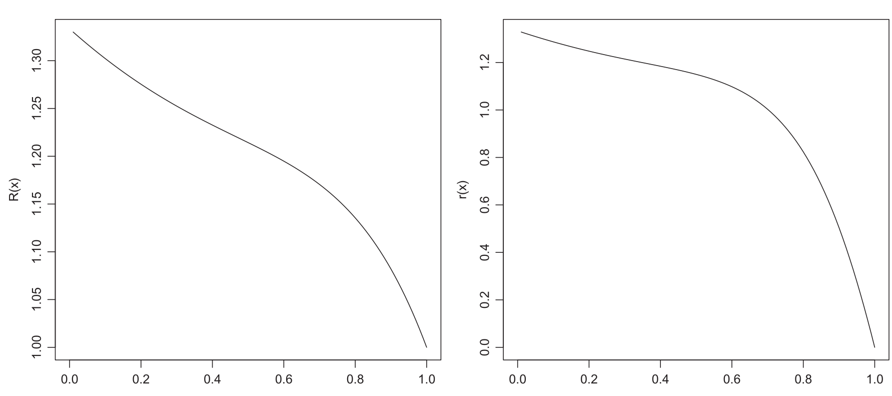

FIGURE 3. Plot (left) of the function R defined in (3.8) for the systems studied in Example 5.2 when $\alpha_1 = \alpha_2 = \alpha_3 = 1$ and $\alpha_4 = 5$. The systems are HR ordered for any F since R is decreasing. Plot (right) of the function r defined in (3.10). The systems are LR ordered for any F since r is decreasing.

We can also use the results given above to study the limiting behavior of the aging functions of these systems. For example, for the first system we obtain

$$\alpha_1(x) = \frac{x\overline{q}_1'(x)}{\overline{q}_1(x)} = \frac{4x^2 - 9x + 6}{x^2 - 3x + 3}.$$

Therefore, from Proposition 4.1, we have

$$\lim_{t\to\infty} \frac{h_{T_1}(t)}{h_F(t)} = \lim_{u\to 0+} \alpha_1(u) = 2 = \gamma_{1:3}$$

for any F. Of course, this is what we have in Figure 2 (right, solid line) since for this exponential distribution $h_F(t) = 1$ for $t \geq 0$. Moreover, we have

$$\inf_{u\in(0,1]} \alpha_1(u) = 1, \text{ and } \sup_{u\in(0,1]} \alpha_1(u) = 2,$$

and so $h_F(t) \leq h_{T_1}(t) \leq 2h_F(t)$. Analogously, for the other system we obtain

$$\lim_{t\to\infty} \frac{h_{T_2}(t)}{h_F(t)} = 2$$

and $0 \leq h_{T_2}(t) \leq 2h_F(t)$ (i.e., in this case we do not have a lower bound). Of course, this is what we have in Figure 2 (right, dotted line). It is easy to see that the functions $\alpha_1(u)$ and $\alpha_2(u)$ are strictly decreasing in $(0, 1]$ and so, from the results given in Navarro et al. [33], the IFR class is preserved (i.e., if F is IFR, then T_1 and T_2 are IFR). For that reason we obtain strictly increasing hazard rate functions in Figure 2 (right). However, the DFR class is not preserved for all F as can be seen in Figure 2 (right) where a DFR model (the exponential distribution is both IFR and DFR since it has a constant hazard rate) gives systems with strictly increasing hazard rate functions.

To study the limits of the MRL functions we note that

$$\lim_{x\to 0+} \frac{\overline{q}_1(x)}{x^2} = \lim_{x\to 0+} \frac{x^4 - 3x^3 + 3x^2}{x^2} = 3$$

and so, from Proposition 4.2, we obtain

$$\lim_{t\to\infty} \frac{\overline{F}_{T_1}(t)}{\overline{F}^2(t)} = 3$$

and

$$\lim_{t\to\infty} \frac{m_{T_1}(t)}{m(t)} = 1,$$

where m_{T_1} is the MRL of T_1 and m is the MRL of $\overline{F}^2$. Note that 3 is the coefficient of $x^2 = x^{\gamma_{1:3}}$ in $\overline{q}_1$. To obtain bounds for the system reliability we study the function $\overline{q}_1(x)/x^2$, obtaining

$$\inf_{u\in(0,1]} \frac{\overline{q}_1(u)}{u^2} = 1, \text{ and } \sup_{u\in(0,1]} \frac{\overline{q}_1(u)}{u^2} = 3,$$

and so $\overline{F}^2(t) \leq \overline{F}_{T_1}(t) \leq 3\overline{F}^2(t)$ and $\mu \leq E(T_1) \leq 3\mu$ where $\mu = \int_0^\infty \overline{F}^2(t)dt$ is the mean of $\overline{F}^2$. For example, for an exponential distribution with mean 1, we obtain $1/2 \leq E(T_1) \leq 3/2$. Analogously, from Proposition 4.3, as

$$\lim_{x\to 0+} \frac{\overline{q}_1'(x)}{2x} = \lim_{x\to 0} \frac{4x^2 - 9x + 6}{2} = 3,$$

we obtain (again)

$$\lim_{t\to\infty} \frac{h_{T_1}(t)}{2h_F(t)} = 1,$$

where $2h_F$ is the hazard rate of $\overline{F}^2$. Similar results can be obtained for T_2.

Finally, we use the results given in Proposition 4.5 to study bounds expressed in the Gini mean difference units. To this purpose we compute the function β for T_1 obtaining

$$\beta(x) = \frac{x^4 - 3x^3 + 3x^2 - x}{2x(1-x)} = -\frac{(x-1)^2}{2}.$$

Hence, $\inf_{x\in[0,1]} \beta(x) = -1/2$ and $\sup_{x\in[0,1]} \beta(x) = 0$ and we have the following bounds

$$\mu_F - \frac{1}{2}\Delta_F \leq E(T_1) \leq \mu_F$$

for all F. Analogously, for the second system we obtain

$$\mu_F - \frac{1}{2}\Delta_F \leq E(T_2) \leq \mu_F + \frac{1}{2}\Delta_F.$$

The following example shows how the results obtained here for the GPHR model can also be applied to systems with independent heterogeneous (non i.d.) components satisfying the PHR model (see Remark 3.5).

Example 5.3: Let us consider a 2-out-of-3 system with independent non i.d. components satisfying the PHR model, that is, $T = X_{2:3}$ and the component reliability functions are $\overline{F}_i(t) = \overline{F}^{\alpha_i}(t)$, $\alpha_i > 0$, $i = 1, 2, 3$. The minimal path sets are $\{1, 2\}$, $\{1, 3\}$, $\{2, 3\}$. Without loss of generality, we can assume $\alpha_1 < \alpha_2 < \alpha_3$. Then the system reliability is given by

$$\begin{aligned}\overline{F}_T(t) &= \overline{F}_1(t)\overline{F}_2(t) + \overline{F}_1(t)\overline{F}_3(t) + \overline{F}_2(t)\overline{F}_3(t) - 2\overline{F}_1(t)\overline{F}_2(t)\overline{F}_3(t)\\ &= \overline{F}^{\alpha_1+\alpha_2}(t) + \overline{F}^{\alpha_1+\alpha_3}(t) + \overline{F}^{\alpha_2+\alpha_3}(t) - 2\overline{F}^{\alpha_1+\alpha_2+\alpha_3}(t).\end{aligned}$$

Hence, it is included in the GPHR model with $\gamma_1 = \alpha_1 + \alpha_2$, $\gamma_2 = \alpha_1 + \alpha_3$, $\gamma_3 = \alpha_2 + \alpha_3$ and $\gamma_4 = \alpha_1 + \alpha_2 + \alpha_3$. Therefore, we can use the results given above to study the limiting behavior of the aging functions of this system. For example, for $\alpha_i = i$ for $i = 1, 2, 3$, we obtain $\overline{F}_T = \overline{q}(\overline{F})$ where

$$\overline{q}(x) = x^3 + x^4 + x^5 - 2x^6.$$

Then

$$\alpha(x) = \frac{x\overline{q}'(x)}{\overline{q}(x)} = \frac{3 + 4x + 5x^2 - 12x^3}{1 + x + x^2 - 2x^3}.$$

Therefore, from Proposition 4.1, we have

$$\lim_{t\to\infty} \frac{h_T(t)}{h_F(t)} = \lim_{x\to 0+} \alpha(x) = 3 = \alpha_1 + \alpha_2 = \gamma_{1:4}$$

for any F. Moreover, we have

$$\inf_{u\in(0,1]} \alpha(u) = 0, \text{ and } \sup_{u\in(0,1]} \alpha(u) \cong 3.24364$$

and so $h_T(t) \leq 3.24364 h_F(t)$. It is easy to see that the function $\alpha(x)$ is first strictly increasing and then strictly decreasing in $(0, 1]$ and so, from the results given in Navarro et al. [33], the IFR and DFR classes are not preserved (e.g. for an exponential distribution we obtain a bathtub shaped hazard rate).

Analogously, to study the limit of the MRL function, we note that

$$\lim_{x\to 0+}\frac{\overline{q}(x)}{x^3}=\lim_{x\to 0+}\frac{x^3+x^4+x^5-2x^6}{x^3}=1$$

and so, from Proposition 4.2, we obtain

$$\lim_{t\to\infty}\frac{\overline{F}_T(t)}{\overline{F}^3(t)}=1$$

and

$$\lim_{t\to\infty}\frac{m_T(t)}{m(t)}=1,$$

where m_T is the MRL of T and m is the MRL of $\overline{F}^3$. Note that in the general case we have

$$\lim_{t\to\infty}\frac{\overline{F}_T(t)}{\overline{F}^{\alpha_1+\alpha_2}(t)}=1$$

and

$$\lim_{t\to\infty}\frac{m_T(t)}{m(t)}=1,$$

where m is the MRL of $\overline{F}^{\alpha_1+\alpha_2}$. To obtain bounds for the system reliability we study the function $\overline{q}(x)/x^3$, obtaining

$$\inf_{u\in(0,1]}\frac{\overline{q}(u)}{u^3}=1,\ \text{and}\ \sup_{u\in(0,1]}\frac{\overline{q}(u)}{u^3}\cong 1.5282,$$

and so $\overline{F}^3(t)\le\overline{F}_T(t)\le 1.5282\overline{F}^3(t)$ and $\mu\le E(T_1)\le 1.5282\mu$, where $\mu=\int_0^\infty\overline{F}^3(t)dt$ is the mean of $\overline{F}^3$. Analogously, from Proposition 4.3, as

$$\lim_{x\to 0+}\frac{\overline{q}'(x)}{3x^2}=\lim_{x\to 0+}\frac{3x^2+4x^3+5x^4-12x^5}{3x^2}=1,$$

we obtain (again)

$$\lim_{t\to\infty}\frac{h_T(t)}{3h_F(t)}=1,$$

where $3h_F$ is the hazard rate of $\overline{F}^3$. Moreover, in order to obtain the bounds from Proposition 4.5, we consider the following function

$$\beta(x)=\frac{x^3+x^4+x^5-2x^6-x}{2x(1-x)}=\frac{2x^4+x^3-x-1}{2}.$$

Hence, $\inf_{x\in[0,1]}\beta(x)\cong -0.6424$ and $\sup_{x\in[0,1]}\beta(x)=0.5$. The supremum is attained when $x\to 1$ and the infimum at

$$x_0=\frac{1}{8}\sqrt[3]{31+8\sqrt{15}}+\frac{1}{8\sqrt[3]{31+8\sqrt{15}}}-\frac{1}{8}\cong 0.401278.$$

Then, we have the following bounds

$$\mu_F-0.6424\Delta_F\le E(T)\le\mu_F+0.5\Delta_F$$

for all F.

6. THE CASE OF SEVERAL EQUAL MODEL PARAMETERS

In this section, we complete the analysis on the asymptotic behavior of the survival function for system lifetimes based on the dependence model induced by SOS from the PHR model (2.1) by allowing that some of the parameters γ_j are equal. Let F be an absolutely continuous distribution function with density $f(x) > 0$ for $x > 0$. For $\gamma_1, \ldots, \gamma_n > 0$ and $r \in \{1, \ldots, n\}$, let $\gamma_{1:r} \leq \cdots \leq \gamma_{r:r}$ denote the increasingly ordered values of $\gamma_1, \ldots, \gamma_r$. Remember that $\gamma_i = (n-i+1)\alpha_i, i = 1, \ldots, n$. Moreover, let $\ell^{(r)}$ denote the number of distinct values in $\gamma_1, \ldots, \gamma_r$ and let the integers $d_v^{(r)}, v = 1, \ldots, \ell^{(r)}$, be chosen such that

$$\gamma_{1:r} = \cdots = \gamma_{d_1^{(r)}:r} < \gamma_{d_1^{(r)}+1:r} = \cdots = \gamma_{d_1^{(r)}+d_2^{(r)}:r} < \cdots < \gamma_{d_1^{(r)}+\cdots+d_{\ell^{(r)}-1}^{(r)}+1:r} = \cdots = \gamma_{d_1^{(r)}+\cdots+d_{\ell^{(r)}}^{(r)}:r}.$$

Then, the (ordered) different values $\delta_1^{(r)} < \cdots < \delta_{\ell^{(r)}}^{(r)}$ among $\gamma_1, \ldots, \gamma_r$ are given by $\delta_v^{(r)} = \gamma_{d_1^{(r)}+\cdots+d_v^{(r)}:r}, v = 1, \ldots, \ell^{(r)}$. Now, the distribution function of the rth sequential order statistic is given by (cf. Cramer and Kamps [17])

$$P(U_{r:n}^* > F(t)) = \left(\prod_{j=1}^{r} \gamma_j\right) \sum_{v=1}^{\ell^{(r)}} \sum_{j=0}^{d_v^{(r)}-1} \frac{K_{v,j}^{(r)}}{(\delta_v^{(r)})^{d_v^{(r)}-j}(d_v^{(r)}-j-1)!j!} \Gamma(d_v^{(r)}-j, -\delta_v^{(r)} \ln \overline{F}(t)),$$

with $K_{v0}^{(r)} = \prod_{q=1,q\neq v}^{\ell^{(r)}} (\delta_q^{(r)} - \delta_v^{(r)})^{-d_q^{(r)}}$,

$$K_{v,j}^{(r)} = \sum_{p=0}^{j-1} \sum_{q=1,q\neq v}^{\ell^{(r)}} (-1)^{p+1} \binom{j-1}{p} \frac{p!\, d_q^{(r)}}{(\delta_q^{(r)} - \delta_v^{(r)})^{p+1}} K_{v,j-1-p}^{(r)}, \quad j \in \mathbb{N},$$

and the incomplete gamma function

$$\Gamma(r, z) = \int_z^{\infty} y^{r-1} \exp(-y)\, dy.$$

Because for $r \in \mathbb{N}, z > 0$

$$\Gamma(r, z) = (r-1)! \sum_{k=0}^{r-1} e^{-z} \frac{z^k}{k!},$$

we obtain

$$P(U_{r:n}^* > F(t)) = \left(\prod_{j=1}^{r} \gamma_j\right) \sum_{v=1}^{\ell^{(r)}} \sum_{j=0}^{d_v^{(r)}-1} \frac{K_{v,j}^{(r)}}{(\delta_v^{(r)})^{d_v^{(r)}-j} \cdot j!} (\overline{F}(t))^{\delta_v^{(r)}} \sum_{k=0}^{d_v^{(r)}-j-1} \frac{(-\ln((\overline{F}(t))^{\delta_v^{(r)}}))^k}{k!}.$$

Therefore, in this case the distribution of the rth sequential order statistic is given by a generalized mixture of the distributions of the first, second, ..., $d_v^{(r)}$-th records (see, e.g., Arnold, Balakrishnan, and Nagaraja [2]) based on the survival functions $\overline{F}^{\delta_v^{(r)}}, v = 1, \ldots, \ell^{(r)}$. Recall that the survival function of the d-th record R_d in a sequence of iid random variables with

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distribution function F is given by

$$P(R_d > t) = \overline{F}(t) \sum_{k=0}^{d-1} \frac{(-\ln(\overline{F}(t)))^k}{k!}, \quad t \in \mathbb{R}.$$

In particular, then the distribution of the lifetime of a coherent system based on SOS can be expressed as a generalized mixture of records from the same PHR model based on $\overline{F}$. Clearly, this distribution can also be interpreted as a distorted distribution based on the survival function $\overline{F}$ (see Section 2).

Similarly to the GPHR model, we can examine generalized mixtures in this situation. Let $a_1, \ldots, a_k \in \mathbb{R}$, $\delta_1, \ldots, \delta_k > 0$ and $r_1, \ldots, r_k \in \mathbb{N}$. Assume that the vectors $(r_i, \delta_i), i = 1, \ldots, k$, are different. Moreover, assume that

$$\overline{G}(t) = \sum_{i=1}^{k} a_i (\overline{F}(t))^{\delta_i} \sum_{k=0}^{r_i - 1} \frac{(-\ln((\overline{F}(t))^{\delta_i}))^k}{k!}, \quad t \in \mathbb{R},$$

defines a survival function on $\mathbb{R}$ for any survival function $\overline{F}$, that is, the associated distortion function

$$\overline{q}(x) = \sum_{i=1}^{k} a_i x^{\delta_i} \sum_{k=0}^{r_i - 1} \frac{(-\delta_i \ln(x))^k}{k!}$$

satisfies the conditions $\overline{q}(0) = 0$, $\overline{q}(1) = 1$, and $\overline{q}$ is increasing in $[0, 1]$. Let

$$\delta_* = \min\{\delta_1 \ldots, \delta_k\}, \quad r_* = \max\{r_i : i \in \{1, \ldots, k\} \text{ with } \delta_i = \delta_*\},$$

and let i_* be chosen such that $(r_*, \delta_*) = (r_{i_*}, \delta_{i_*})$. Then, if $a_1, \ldots, a_k \in \mathbb{R} \setminus \{0\}$,

$$\lim_{t \to \infty} \frac{\overline{G}(t)}{\overline{G}_*(t)} = a_{i_*} > 0, \tag{6.1}$$

where

$$\overline{G}_*(t) = (\overline{F}(t))^{\delta_*} \sum_{k=0}^{r_* - 1} \frac{(-\ln((\overline{F}(t))^{\delta_*}))^k}{k!}$$

denotes the survival function of the r_*-th record based on the survival function $\overline{F}^{\delta_*}$. With $\delta \geq 0$ and $r \in \mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$, this follows immediately from:

$$\lim_{x \to 0+} [x^\delta (-\ln(x))^r] = \begin{cases} 1, & \delta = 0, r = 0, \\ 0, & \delta = 0, r < 0, \\ 0, & \delta > 0, r \in \mathbb{Z}. \end{cases} \tag{6.2}$$

Moreover, if in the generalized mixture $a_1, \ldots, a_k \in \mathbb{R}$ (i.e. zero is included), then the limit in (6.1) holds with $a_{i_*} \geq 0$.

THEOREM 6.1: *Let T denote the lifetime of a coherent system based on SOS from the PHR model (2.1). Let $\boldsymbol{s} = (s_1, \ldots, s_r, 0, \ldots, 0)$ with $s_r > 0$ denote the signature of the system and*

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$d = \#\{j \in \{1, \ldots, r\} : \gamma_j = \gamma_{1:r}\}$. *Then,*

$$\lim_{t\to\infty} \frac{\overline{F}_T(t)}{\overline{F}_{R_d(\gamma_{1:r})}(t)} = c \in \mathbb{R}^+,$$

where

$$\overline{F}_{R_d(\gamma_{1:r})}(t) = (\overline{F}(t))^{\gamma_{1:r}} \sum_{k=0}^{d-1} \frac{(-\gamma_{1:r} \ln \overline{F}(t))^k}{k!}, \quad t \in \mathbb{R},$$

denotes the survival function of the dth record $R_d(\gamma_{1:r})$ based on an iid sequence of random variables with survival function $\overline{F}^{\gamma_{1:r}}$. Moreover, $c \geq s_r$ holds.

Proof: Rewriting $\overline{F}_T$ as a generalized mixture

$$\overline{F}_T(t) = \sum_{i=1}^{k} a_i (\overline{F}(t))^{\delta_i} \sum_{k=0}^{r_i-1} \frac{(-\ln((\overline{F}(t))^{\delta_i}))^k}{k!}, \quad t \in \mathbb{R},$$

with appropriately chosen $a_i \in \mathbb{R}$, $\delta_i > 0$, $r_i \in \mathbb{N}$ and $k \in \mathbb{N}$, it follows from the preceding results that the considered limit c exists as a real number such that $c \geq 0$. Since $s_r > 0$ by assumption, it is sufficient to prove $c \geq s_r$. At first, we obtain

$$\overline{F}_T(t) = \sum_{m=1}^{r} s_m P(X^*_{m:n} > t) \geq s_r P(X^*_{r:n} > t).$$

Using results from Cramer and Kamps [17], the mth sequential order statistic can be expressed as

$$X^*_{m:n} = H_F^{-1}\left(\sum_{j=1}^{m} \frac{Z_j}{\gamma_j}\right), \quad m = 1, \ldots, r,$$

where $Z_1, \ldots, Z_r$ are iid according to a standard exponential distribution and H_F^{-1} denotes the inverse of the cumulative hazard rate $H_F = -\ln \overline{F}$ of F. Hence, with $J = \{j \in \{1, \ldots, r\} : \gamma_j = \gamma_{1:r}\}$,

$$P(X^*_{r:n} > t) = P\left(\sum_{j=1}^{r} \frac{Z_j}{\gamma_j} > H_F(t)\right) \geq P\left(\sum_{j\in J} \frac{Z_j}{\gamma_j} > H_F(t)\right) = P\left(\sum_{j\in J} Z_j > \gamma_{1:r} H_F(t)\right).$$

Because $|J| = d$ by assumption, the random variable $\sum_{j\in J} Z_j$ is distributed according to a gamma distribution with shape parameter d and scale parameter 1. Therefore, we get

$$\begin{aligned} P\left(\sum_{j\in J} Z_j > \gamma_{1:r} H_F(t)\right) &= e^{-\gamma_{1:r} H_F(t)} \sum_{k=0}^{d-1} \frac{(\gamma_{1:r} H_F(t))^k}{k!} \\ &= (\overline{F}(t))^{\gamma_{1:r}} \sum_{k=0}^{d-1} \frac{(-\gamma_{1:r} \ln \overline{F}(t))^k}{k!} = \overline{F}_{R_d(\gamma_{1:r})}(t), \end{aligned}$$

and consequently

$$\overline{F}_T(t) \geq s_r \overline{F}_{R_d(\gamma_{1:r})}(t), \quad t > 0,$$

which yields the assertion. ■

Remark 6.2: If $\gamma_{1:r} = k \in \mathbb{N}$, then the distribution of $R_d(\gamma_{1:r})$ in Theorem 6.1 coincides with the distribution of a so-called kth record value. The kth record values describe the kth largest values in a sequence of iid random variables with distribution function F (cf. Dziubdziela and Kopociński [19]; see also Kamps [24]).

COROLLARY 6.3: *Let h_F denote the hazard rate of F. In the situation of Theorem 6.1, let h_T and $h_{R_d(\gamma_{1:r})}$ denote the hazard rates of the random variables T and $R_d(\gamma_{1:r})$, respectively. Then,*

$$\lim_{t\to\infty} \frac{h_T(t)}{h_{R_d(\gamma_{1:r})}(t)} = 1 \quad \text{and} \quad \lim_{t\to\infty} \frac{h_T(t)}{h_F(t)} = \gamma_{1:r}.$$

PROOF: The density of T can be written as

$$f_T(t) = \sum_{i=1}^{k} a_i f_{R_{r_i}(\delta_i)}(t), \quad t \in \mathbb{R},$$

with appropriately chosen $a_i \in \mathbb{R}$, $\delta_i > 0$, $r_i \in \mathbb{N}$ and $k \in \mathbb{N}$, where

$$f_{R_r(\delta)}(t) = \frac{\delta}{(r-1)!} \left(-\delta \ln \overline{F}(t)\right)^{r-1} (\overline{F}(t))^{\delta-1} f(t), \quad t \in \mathbb{R},$$

denotes the density function of the r-th record based on an iid sequence of random variables with survival function $\overline{F}^{\delta}$. Using (6.2), it follows again that the limit of $f_T(t)/f_{R_d(\gamma_{1:r})}(t)$ for $t \to \infty$ is given by a non-negative real number. By applying L'Hôpital's rule, Theorem 6.1 yields

$$\lim_{t\to\infty} \frac{f_T(t)}{f_{R_d(\gamma_{1:r})}(t)} = \lim_{t\to\infty} \frac{\overline{F}_T(t)}{\overline{F}_{R_d(\gamma_{1:r})}(t)} = c \in \mathbb{R}^+.$$

Therefore, the first limit holds and using

$$\lim_{t\to\infty} \frac{h_{R_d(\gamma_{1:r})}(t)}{\gamma_{1:r} h_F(t)} = \lim_{t\to\infty} \frac{\frac{1}{(d-1)!}\left(-\gamma_{1:r} \ln \overline{F}(t)\right)^{d-1}}{\sum_{k=0}^{d-1} \frac{(-\gamma_{1:r} \ln \overline{F}(t))^k}{k!}} = 1,$$

the second one is obtained. ■

Theorem 6.1 and Corollary 6.3 are generalizations of the results given in Remark 4.4. In the following corollary, the limiting behavior of the MRL functions is established.

COROLLARY 6.4: *In the situation of Theorem 6.1, let $E(X^*_{r:n}) < \infty$. Let m_T and $m_{R_d(\gamma_{1:r})}$ denote the MRL functions of the random variables T and $R_d(\gamma_{1:r})$, respectively. Then,*

$$\lim_{t\to\infty} \frac{m_T(t)}{m_{R_d(\gamma_{1:r})}(t)} = 1.$$

Moreover, for $d > 1$, if

$$\lim_{t\to\infty} \frac{m_{\gamma_{1:r}}(t) h_F(t)}{H_F(t)} = 0$$

holds with $H_F = -\ln \overline{F}$, then

$$\lim_{t\to\infty} \frac{m_T(t)}{m_{\gamma_{1:r}}(t)} = 1,$$

where $m_{\gamma_{1:r}}$ denotes the MRL function of $\overline{F}^{\gamma_{1:r}}$.

PROOF: In order to show the result for the MRL functions, note at first that $E(X^*_{r:n}) < \infty$ implies $E(T) < \infty$, because of the signature based representation of the survival function of T and $0 \le X^*_{1:n} \le \cdots \le X^*_{r:n}$ almost surely. Moreover, $E(R_d(\gamma_{1:r})) < \infty$ holds due to $P(X^*_{r:n} > x) \ge P(R_d(\gamma_{1:r}) > x)$ for $x \ge 0$ (cf. the proof of Theorem 6.1). In particular, $E(R_1(\gamma_{1:r})) = m_{\gamma_{1:r}}(0) < \infty$. Since $\overline{F}_T$ and $\overline{F}_{R_d(\gamma_{1:r})}$ can be interpreted as distorted distributions based on $\overline{F}$, the first limit can be concluded by utilizing Proposition 4.2 and Theorem 6.1.

For proving the second limit, it is sufficient to show for the dth record $R_d = R_d(1)$ based on the distribution function F that $E(R_d) < \infty$ and

$$\lim_{t\to\infty} \frac{m_F(t)h_F(t)}{H_F(t)} = 0 \qquad \textbf{(6.3)}$$

imply

$$\lim_{t\to\infty} \frac{m_F(t)}{m_{R_d}(t)} = 1.$$

Then the assertion follows by applying this result to records based on $\overline{F}^{\gamma_{1:r}}$, because the cumulative hazard rate and the hazard rate of $\overline{F}^{\gamma_{1:r}}$ satisfy

$$H_{\overline{F}^{\gamma_{1:r}}}(t) = \gamma_{1:r} H_F(t), \quad h_{\overline{F}^{\gamma_{1:r}}}(t) = \gamma_{1:r} h_F(t), \quad t > 0.$$

In order to prove the above statement, we will show at first that

$$\lim_{t\to\infty} \int_t^\infty \overline{F}(y)\,dy \cdot \sum_{k=0}^{d-1} \frac{(H_F(t))^k}{k!} = 0. \qquad \textbf{(6.4)}$$

Note that the cumulative hazard rate H_F is increasing with $H_F(t) \to \infty$ for $t \to \infty$. In particular,

$$\int_t^\infty \overline{F}(y) \sum_{k=0}^{d-1} \frac{(H_F(y))^k}{k!}\,dy \ge \int_t^\infty \overline{F}(y)\,dy \cdot \sum_{k=0}^{d-1} \frac{(H_F(t))^k}{k!}, \quad t \ge 0,$$

and so the limit (6.4) follows from $E(R_d) < \infty$. Now, we consider

$$\lim_{t\to\infty} \frac{m_F(t)}{m_{R_d}(t)} = \lim_{t\to\infty} \frac{\int_t^\infty \overline{F}(y)\,dy \cdot \sum_{k=0}^{d-1} \frac{(H_F(t))^k}{k!}}{\int_t^\infty \overline{F}(y) \sum_{k=0}^{d-1} \frac{(H_F(y))^k}{k!}\,dy}$$

for $d \ge 2$. Using (6.4), we can apply L'Hôpital's rule to the second limit of the type '$\frac{0}{0}$' and get

$$\begin{aligned}\lim_{t\to\infty} \frac{m_F(t)}{m_{R_d}(t)} &= \lim_{t\to\infty} \frac{\overline{F}(t)\sum_{k=0}^{d-1}\frac{(H_F(t))^k}{k!} - \int_t^\infty \overline{F}(y)\,dy \cdot h_F(t)\cdot \sum_{k=0}^{d-2}\frac{(H_F(t))^k}{k!}}{\overline{F}(t)\sum_{k=0}^{d-1}\frac{(H_F(t))^k}{k!}} \\ &= \lim_{t\to\infty}\left(1 - m_F(t)h_F(t)\frac{\sum_{k=0}^{d-2}\frac{(H_F(t))^k}{k!}}{\sum_{k=0}^{d-1}\frac{(H_F(t))^k}{k!}}\right) \\ &= \lim_{t\to\infty}\left(1 - \frac{m_F(t)h_F(t)}{H_F(t)} \frac{\sum_{k=0}^{d-2}\frac{(H_F(t))^k}{k!}}{\sum_{k=0}^{d-1}\frac{(H_F(t))^{k-1}}{k!}}\right).\end{aligned}$$

Due to assumption (6.3), this limit is equal to 1. ■

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Remark 6.5: There exist distributions with a MRL function m which does not satisfy the condition (6.3). An example can be given as follows (cf. Hall and Wellner [21, p. 181]). Let $m(t) = (t + e)\ln(t + e), t \geq 0$, with $e = \exp(1)$. In particular,

$$\int_0^t \frac{1}{m(y)}\, dy = \ln(\ln(t + e)), \quad t \geq 0,$$

and therefore $\int_0^\infty \frac{1}{m(y)}\, dy = \infty$. Thus, it can be seen that m is the MRL function of a distribution function F (cf. Hall and Wellner [21, p. 172]). Moreover, $m'(t) = \ln(t + e) + 1$ with limit ∞ for $t \to \infty$. The hazard rate h, cumulative hazard rate H and survival function $\overline{F}$ are given by

$$h(t) = \frac{m'(t) + 1}{m(t)} = \frac{\ln(t + e) + 2}{(t + e)\ln(t + e)},$$

$$H(t) = \int_0^t h(y)\, dy = \ln(t + e) + 2\ln(\ln(t + e)) - 1,$$

$$\overline{F}(t) = \exp(-H(t)) = \frac{e}{(t + e)(\ln(t + e))^2}.$$

Consequently, it can be seen that

$$\lim_{t\to\infty} \frac{m(t)h(t)}{H(t)} = \lim_{t\to\infty} \frac{m'(t) + 1}{H(t)} = 1.$$

However, note that in this case

$$\int_0^t \overline{F}(y)H(y)\, dy = e\left(\ln(\ln(t + e)) + 1 - \frac{2\ln(\ln(t + e)) + 1}{\ln(t + e)}\right)$$

and therefore the MRL function of the dth record $R_d, d \geq 2$, based on F does not exist, because

$$E(R_2) = \int_0^\infty \overline{F}(y)H(y)\, dy = \infty.$$

Finally, we apply the results from this and the preceding sections to a system based on SOS with some equal model parameters.

Example 6.6: Consider a system with lifetime given by

$$T = \min(X_1, \max(X_2, \ldots, X_5)),$$

where $X_1, \ldots, X_5$ are exchangeable random variables with the dependence model defined by SOS from the PHR model (2.1). Let $\alpha_1 = 1$. After the first failure, no additional load is imposed on the remaining components, that is, $\alpha_2 = 1$. However, it is assumed that after the second failure, the current load on the system is distributed evenly among the remaining components. Therefore, now $\alpha_3 = 4/3$ is chosen (see Burkschat and Navarro [10], Remark 2.3 and Balakrishnan et al. [3], Example 1). After the third failure, the load is distributed in the same vein on the two last components, that is, $\alpha_4 = 4/2 = 2$. The system will definitely stop functioning with the next component failure. Therefore, the parameter α_5 can be chosen arbitrarily, because the system lifetime distribution does not depend on it. Recalling the definition of the model parameters, we obtain in this situation $\gamma_1 = 5, \gamma_2 = \gamma_3 = \gamma_4 = 4$.

Moreover, the signature of the system is given by $\boldsymbol{s}= (1/5, 1/5, 1/5, 2/5, 0)$ (cf. system 87 in Tables 1 and 2 of Navarro and Rubio [35]). The survival function of T can be expressed with a distorted distribution $\overline{q}$ as $\overline{F}_T = \overline{q}(\overline{F})$, where $\overline{q}$ has the representation

$$\overline{q}(x) = \frac{1}{5}\overline{q}_{1:5}(x) + \frac{1}{5}\overline{q}_{2:5}(x) + \frac{1}{5}\overline{q}_{3:5}(x) + \frac{2}{5}\overline{q}_{4:5}(x)$$

with

$$\begin{aligned}
\overline{q}_{1:5}(x) &= x^{\gamma_1} = x^5,\\
\overline{q}_{2:5}(x) &= \frac{\gamma_1}{\gamma_1-\gamma_2}x^{\gamma_2} + \frac{\gamma_2}{\gamma_2-\gamma_1}x^{\gamma_1} = -4x^5 + 5x^4,\\
\overline{q}_{3:5}(x) &= \gamma_1\gamma_2\gamma_3\left(\frac{\Gamma(2,-\gamma_2\ln(x))}{\gamma_2^2(\gamma_1-\gamma_2)} - \frac{\Gamma(1,-\gamma_2\ln(x))}{\gamma_2(\gamma_1-\gamma_2)^2} + \frac{\Gamma(1,-\gamma_1\ln(x))}{\gamma_1(\gamma_2-\gamma_1)^2}\right)\\
&= 16x^5 + 5(1-4\ln(x))x^4 - 20x^4\\
\overline{q}_{4:5}(x) &= \gamma_1\gamma_2\gamma_3\gamma_4\left(\frac{\Gamma(3,-\gamma_2\ln(x))}{2\gamma_2^3(\gamma_1-\gamma_2)} - \frac{\Gamma(2,-\gamma_2\ln(x))}{\gamma_2^2(\gamma_1-\gamma_2)^2} + \frac{\Gamma(1,-\gamma_2\ln(x))}{\gamma_2(\gamma_1-\gamma_2)^3} + \frac{\Gamma(1,-\gamma_1\ln(x))}{\gamma_1(\gamma_2-\gamma_1)^3}\right)\\
&= -64x^5 + 5(1-4\ln(x)+8\ln^2(x))x^4 - 20(1-4\ln(x))x^4 + 80x^4.
\end{aligned}$$

Therefore, we obtain

$$\overline{q}(x) = x^4(16\ln^2(x) + 20\ln(x) - 23x + 24), \quad x \in (0,1].$$

By applying Theorem 6.1, we conclude

$$\lim_{t\to\infty}\frac{\overline{F}_T(t)}{\overline{F}_{R_3(4)}(t)} = c \in \mathbb{R}^+.$$

Using the notation $\overline{F}_{R_3(4)} = \overline{p}(\overline{F})$ with

$$\overline{p}(x) = x^4\sum_{k=0}^{2}\frac{(-4\ln(x))^k}{k!},$$

we get precisely

$$c = \lim_{x\to0+}\frac{\overline{q}(x)}{\overline{p}(x)} = \lim_{x\to0+}\frac{16\ln^2(x) + 20\ln(x) - 23x + 24}{1 - 4\ln(x) + 8\ln^2(x)} = 2.$$

Moreover, bounds for the survival function of the system lifetime can be established. Utilizing

$$\inf_{u\in(0,1]}\frac{\overline{q}(u)}{\overline{p}(u)} \cong 0.80872, \qquad \sup_{u\in(0,1]}\frac{\overline{q}(u)}{\overline{p}(u)} = 2,$$

we find

$$0.80872\overline{F}_{R_3(4)}(t) \le \overline{F}_T(t) \le 2\overline{F}_{R_3(4)}(t), \quad t \ge 0.$$

In particular, bounds for the expected system lifetime are given by

$$0.80872\, E(R_3(4)) \le E(T) \le 2\, E(R_3(4)).$$

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Alternatively, the bounds based on the Gini mean difference in Proposition 4.5 can be determined by studying the function

$$\beta(x) = \frac{16\ln^2(x)x^3 + 20x^3\ln(x) - 23x^4 + 24x^3 - 1}{2(1-x)}, \quad x \in (0,1).$$

Because of

$$\inf_{x\in(0,1)} \beta(x) \cong -0.52299, \qquad \sup_{x\in(0,1)} \beta(x) = 0,$$

we obtain

$$\mu_F - 0.52299\Delta_F \le E(T) \le \mu_F.$$

Furthermore, Corollary 6.3 and Proposition 4.1 yield (cf. Figure 4 (left) for the case of an underlying standard exponential distribution)

$$\lim_{t\to\infty} \frac{h_T(t)}{h_{R_3(4)}(t)} = 1, \quad \lim_{t\to\infty} \frac{h_T(t)}{h_F(t)} = 4$$

and

$$h_F(t) \le h_T(t) \le 4h_F(t), \quad t > 0,$$

because the function

$$\alpha(u) = \frac{u\overline{q}'(u)}{\overline{q}(u)} = \frac{64\ln^2(x) + 112\ln(x) - 115x + 116}{16\ln^2(x) + 20\ln(x) - 23x + 24}$$

satisfies

$$\inf_{u\in(0,1]} \alpha(u) = 1, \text{ and } \sup_{u\in(0,1]} \alpha(u) = 4.$$

In particular, the above results yield that h_T approaches $4h_F$ from below. Additionally, since $\alpha(u)$ is strictly decreasing on $(0,1]$, it follows that the IFR class is preserved for this

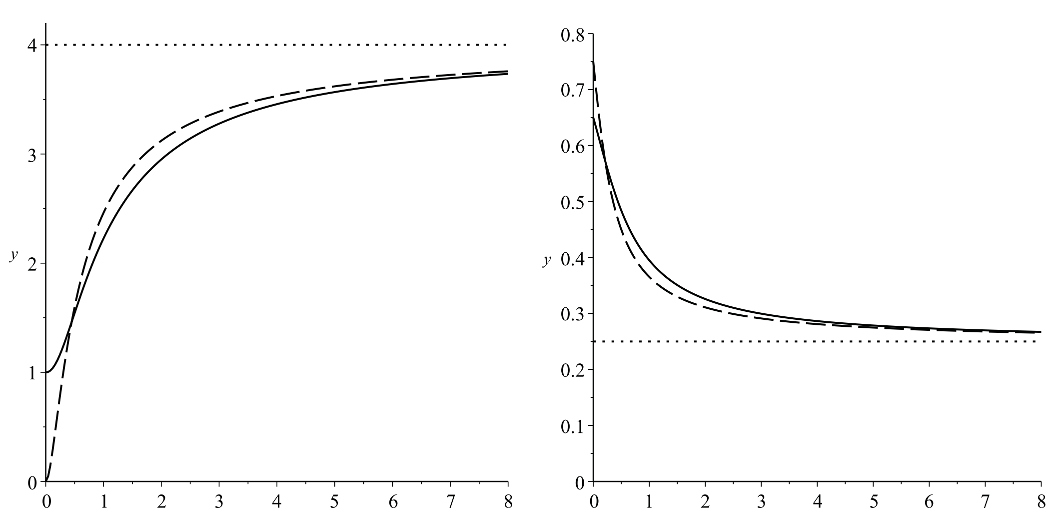

FIGURE 4. Plots of the hazard rates (left) and the MRL functions (right) of $\overline{F}_T$ (solid), $\overline{F}_{R_3(4)}$ (dashed) and $\overline{F}^4$ (dotted) in Example 6.6 with underlying reliability $\overline{F}(t) = e^{-t}$.

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system (cf. Navarro et al. [33]). Finally, using Proposition 4.2 and Corollary 6.4 we can determine the asymptotic behavior of the MRL function of the system:

$$\lim_{t\to\infty} \frac{m_T(t)}{m_{R_3(4)}(t)} = 1.$$

For example, in the case of the exponential distribution with $\overline{F}(t) = e^{-t}$, it can be shown that the conditions in Corollary 6.4 are satisfied and consequently we also get (see Figure 4, right)

$$\lim_{t\to\infty} m_T(t) = \frac{1}{4},$$

because the MRL function m_4 of $\overline{F}^4$ is given by $m_4(t) = 1/4, t \geq 0$. Furthermore, as $\overline{q}/\overline{p}$ can be seen to be bathtub, we obtain the MRL ordering $R_3(4) \leq_{\text{mrl}} T$ when the means are ordered. This is not always the case as can be seen in Figure 4 (right), where $ET < ER_3(4)$. Note that they are not ST (HR) ordered for any F, since $\overline{q}/\overline{p}$ crosses $y = 1$.

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