

Transient dynamics at the onset of Taylor vortices

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(Received 1 October 2002 and in revised form 6 November 2002)

The effect of boundary conditions on the ‘critical dynamics’ at the onset of Taylor vortices is investigated in a combined numerical and experimental study. Numerical calculations of Navier–Stokes equations with ‘stress-free’ boundary conditions show that the Landau amplitude equation provides a good model of the transient dynamics. However, this rapidly breaks down when the ‘no-slip’ condition is approached. Apparent ‘critical’ behaviour observed in experiments is shown to have a surprising dependence on the length of the system.

1. Introduction

The interpretation of hydrodynamic instabilities in terms of low-order bifurcation events has been a fruitful area for the application of modern mathematical ideas to pattern formation. Examples of these are in Rayleigh–Bénard convection as discussed by Cross & Hohenberg (1993), in Taylor–Couette flows discussed by diPrima & Swinney (1981) and in electrohydrodynamic convection in nematic liquid crystals as reported by Kramer & Pesch (1996). Bifurcations play an important rôle in the organization of dynamics found at supercritical values of the control parameters and an in-depth knowledge of the bifurcation set is necessary to understand complex behaviour such as global bifurcations and chaos, as outlined by Mullin (1993).

In many of these applications the effects of physical imperfections are not taken into account since close approximations to the mathematical idealization of perfect symmetry can be achieved in experiments. Alternatively, improved models can be constructed using imperfect bifurcation theory and these can be used to make closer comparison with observation and test the robustness of the predictions of the perfect models. It is clear that the symmetries of the models must be relevant to the physical system for these procedures to have predictive power.

The hydrodynamic system we have chosen to study is Taylor–Couette flow between rotating cylinders where the inner rotates and the outer is stationary. A cellular pattern arises suddenly on a basic featureless flow as the Reynolds number Re is increased. This event has formed a cornerstone of hydrodynamic stability theory since the pioneering experimental and theoretical investigations of Taylor (1923) where he found remarkable quantitative agreement between observation and prediction. In his model the assumptions of cylinders of infinite length and a narrow gap between them

were applied. In this case, the cellular Taylor vortex flow arises at a supercritical pitchfork bifurcation from the trivial state of Couette flow at a critical Reynolds number Re_c . It may therefore be supposed that a good model of this event would be provided by the Landau amplitude equation and it is this notion that we have investigated. Evidence for a supercritical pitchfork bifurcation is provided by the following:

(a) The onset of Taylor cells is sharp for almost all values of the aspect ratio (non-dimensionalized cylinder length) in experiments. The exceptions are very short cylinders and aspect ratios where there is an exchange of stability between neighbouring modes as discussed by Mullin (1982).

(b) The observed value of Re for the onset of cells is in agreement with the theoretical Re_c to within experimental uncertainty for a wide range of aspect and radius ratios of the cylinders as reviewed by diPrima & Swinney (1981).

(c) Detailed measurements of the onset of cells using laser Doppler velocimetry to monitor the flow at mid-length of the cylinders shows close agreement with square-root behaviour (See the review by diPrima & Swinney 1981 and references therein).

(d) The finite extent of cylinders in experimental studies gives rise to 'end effects' which take the form of Ekman vortices adjacent to the stationary ends of the apparatus. These naturally produce a smoothed onset but it has been shown that the resulting spatial variation can be modelled by the steady Ginzburg–Landau equation as shown by Pfister & Rehberg (1981).

(e) Transient dynamics close to the onset of Taylor cells contain slowing-down effects in 'large' systems as demonstrated in the experiments of Donnelly & Schwarz (1963, 1965) and the numerical studies of Neitzel (1984). It has been conjectured by Gollub & Freilich (1976) and Hirshfield & Rapaport (1998, 2000) that this is the result of critical slowing down close to a pitchfork bifurcation. Since all of the above evidence points to an imperfect supercritical pitchfork bifurcation it is natural to enquire about the properties of the disconnected branch since the presence of an unstable fixed point may have important consequences for the dynamics at higher Re . Surprisingly, this issue was not addressed until the work of Benjamin, who postulated and observed the so-called 'anomalous modes' in Benjamin (1978), and it was discussed in detail in Benjamin & Mullin (1981).

The even 'anomalous modes' arise as a consequence of the disconnection of the pitchfork of the 'periodic' model as pointed out by Schaeffer (1980). In this case, the modes contain even numbers of cells all rotating in the opposite sense to the normal one which evolves with a smooth increase of Re as described above. Anomalous modes may also contain odd numbers of cells which fully disconnect in a systematic way as the no-slip boundary condition is applied at the ends in numerical calculations, as explored by Cliffe & Mullin (1985). Both sets of anomalous modes have now been the subject of extensive experimental and numerical research over wide parameter ranges and excellent quantitative agreement has been found between observations and calculations, as reviewed by Cliffe, Kobine & Mullin (1992). Hence their existence as essential features of the complete steady solution set is assured.

Perhaps the most surprising aspect of anomalous modes is that their lower limit of stability is between a factor of 2 and 10 different from the onset of cells as shown by Cliffe *et al.* (1992). Hence the appearance of cells in Taylor–Couette flow cannot be modelled as the simple softening of a pitchfork bifurcation. Intriguingly, the onset is distinctly sharp and yet the saddle-node of the disconnected branch may be an order of magnitude different in Re . Moreover the onset of cells appears to show critical behaviour of the type associated with the Landau amplitude equation. The Landau

equation for the amplitude A of a critical mode is

$$\tau_0 \frac{dA}{dt} = \varepsilon A - \frac{A^3}{A_0^2}, \quad (1.1)$$

with $\varepsilon = (Re - Re_c)/Re_c$ representing the relative distance from the critical point and τ_0 and A_0 scaling factors for time and amplitude, respectively. A time constant may be defined as $T := \tau_0/\varepsilon$, a final amplitude $A_f := A_0\sqrt{\varepsilon}$, and an initial amplitude $A_i = A(t = 0)$. The solution of equation (1.1) is given by

$$A(t) = \frac{A_f e^{t/T}}{\sqrt{e^{2t/T} + (A_f/A_i)^2 - 1}}. \quad (1.2)$$

This equation has been successfully applied to determine critical slowing down at a Hopf bifurcation in Taylor–Couette flow by Pfister & Gerdtts (1981) and in magnetoconvection by Hof & Mullin (2001). It was also used by Gollub & Freilich (1974) to model the onset of Taylor cells. Here we report on an experimental and numerical investigation into the nonlinear response close to the onset of Taylor cells.

2. Experimental apparatus

The experiments were performed in a Taylor–Couette flow where viscous fluid confined between two concentric cylinders was driven by rotation of the inner cylinder. The dynamical parameter which was used as the primary control was the Reynolds number which is defined by $Re = (\Omega_i r_i d)/\nu$, where the rotation rate of the inner cylinder is Ω_i , $d = r_o - r_i$ is the gap between the cylinders and r_i and r_o denote the radius of the inner and outer cylinders, respectively. The radius ratio $\eta = r_i/r_o$ was 0.5. The second control parameter was the aspect ratio $\Gamma = h/d$, where h denotes the height of the system. Γ was varied by moving the stationary collars which defined the end boundary conditions. The inner cylinder was machined from stainless steel and had a radius of $r_i = (12.50 \pm 0.01)$ mm, while the outer cylinder was made from optically polished glass with a radius of $r_o = (25.00 \pm 0.01)$ mm. The eccentricity of the cylinders was measured to $\epsilon \leq 0.005$ mm. The axial location of the bottom plate was fixed while the top plate could be positioned with an accuracy of 0.01 mm. A phase-locked loop (PLL) circuit controlled the speed of the inner cylinder to an accuracy of better than one part in 10^{-4} in the short term and 10^{-7} in the long-term average.

The working fluid was silicone oil and its kinematic viscosity was determined to be $\nu = 35.43$ cSt by comparison with square-root behaviour at $\Gamma = 20$ obtained from experimental data and from numerical calculations of Navier–Stokes equations. The temperature of the fluid was held constant to within 0.01 K by circulating thermostatically controlled oil through a jacket which surrounded the apparatus. Further control was provided by an air-temperature controlled cabinet and the laboratory was air conditioned.

The local vertical velocity was measured by a real-fringe laser-Doppler-velocimeter (LDV) and processed using a phase-locked loop analogue tracker which produced a voltage linearly proportional to the velocity. The time series were recorded via a 14-bit AD-converter and the data processing was performed off line on a computer.

3. Numerical method

A Galerkin finite-element discretization of the governing Navier–Stokes equations in toroidal geometry was produced on a mesh of 20×160 elements with suitable

corner refinements where appropriate. This was used to produce a set of nonlinear differential-algebraic equations for the nodal values of the velocities and the pressure. Continuation methods were used to trace out steady flows in the same way that has been successful in previous studies of the Taylor problem by Cliffe *et al.* (1992). The new feature here is that for calculations of transient behaviour, time-dependent terms are included. The equations take the form of a differential-algebraic set because of the incompressibility constraint, as discussed by Gresho & Sani (1998).

The differential-algebraic equations were solved using a variable-order variable-step-length backward-difference algorithm of Byrne & Hindmarsh (1975). The method uses a predictor-corrector scheme to estimate the errors and chooses the order of the method to obtain the largest possible time-step size while keeping the error in the solution less than a prescribed tolerance. For the results reported here, the relative error tolerance was set to 10^{-5} , which gives very accurate solutions to the time-dependent problem. Grid refinement tests were carried out to ensure that the spatial discretization errors were acceptably small.

No-slip boundary conditions were used at the inner and outer cylinders. Calculations were performed on a half-domain so that flows were assumed to be symmetric about the mid-plane. At one end of the annulus, boundary conditions were applied in the form of the model proposed by Schaeffer (1980), which involved the introduction of a homotopy parameter $\tau \in [0, 1]$ which relates a solid stationary end ($\tau = 1$) to the case of impermeable, stress-free boundary conditions ($\tau = 0$). A discussion of the relationship between the latter boundary condition and full periodic boundary conditions is given by Cliffe *et al.* (1992). It should be noted that increasing τ from zero to one does not move the apparent critical value observed in experiments from the bifurcation point of the periodic model.

If (u, v, w) are the velocity components with respect to cylindrical polar coordinates in the region $\{(r, z) : r_1 \leq r \leq r_2, 0 \leq z \leq \frac{1}{2}h\}$, then the boundary conditions are

$$(u, v, w) = (0, 0, 0) \quad \text{for } r = r_2, \quad 0 \leq z \leq \frac{1}{2}h, \quad (3.1)$$

$$(u, v, w) = (0, 1, 0) \quad \text{for } r = r_1, \quad 0 \leq z \leq \frac{1}{2}h, \quad (3.2)$$

$$\left. \begin{aligned} w &= 0 \\ (1 - \tau)u_z \pm \tau u &= 0 \\ (1 - \tau)v_z \pm \tau\{v - F(r)\} &= 0 \end{aligned} \right\} \quad \text{for } z = \frac{1}{2}h, \quad r_1 \leq r \leq r_2, \quad (3.3)$$

$$w = 0, \quad u_z = 0, \quad v_z = 0 \quad \text{for } z = 0, \quad r_1 \leq r \leq r_2. \quad (3.4)$$

The function $F(r)$ is smooth and satisfies the criteria that $F(r_1) = 1$ and $F(r) = 0$ for $r \geq r_1 + \xi$, with $0 < \xi \ll r_2 - r_1$, and is introduced to avoid the discontinuity in the boundary conditions at the inner cylinder. In practice, the distance ξ is taken to be approximately 1% of the gap between the two cylinders.

4. Results

We first show in figure 1 a direct comparison between numerical and experimental time series for the onset and decay of cells for a 16-cell flow at $\Gamma = 16$ and $Re_f = 69.02$, where Re_f is the set Reynolds number of the final state of the system. The measure used to represent the flow is the radial velocity component at mid-height and mid-gap. Steady-state solutions at $Re = 63.74$ and $Re = 74.81$ were used as the initial conditions for both the measurements and calculations. For the accelerating flow Re was impulsively changed from 63.74 to 69.02, and from 74.81 to 69.02 for the

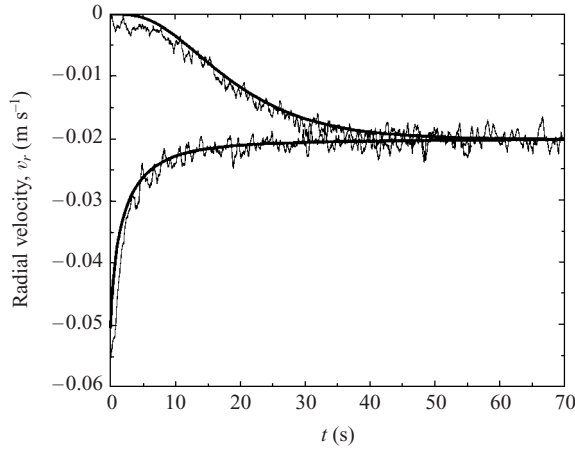


FIGURE 1. Comparison of experimental and numerical time series for the onset and decay in a 16-cell flow at $\Gamma = 16$ with $Re_f = 69.02$. The upper trace was obtained using an initial value of $Re = 63.74$ and the lower $Re = 74.81$.

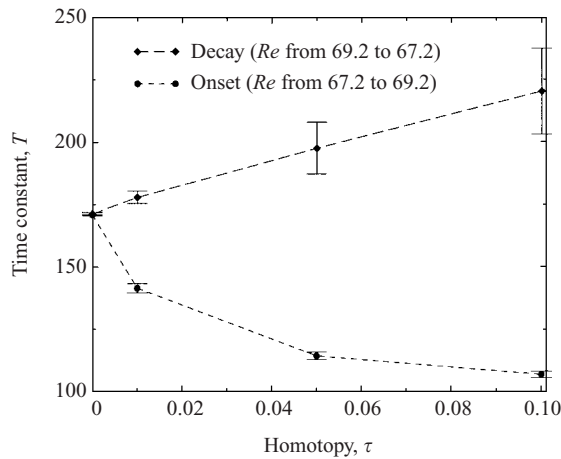


FIGURE 2. Time constants T obtained from fits of the Landau model to time series obtained from numerical solutions of the Navier–Stokes equations for various values of the Schaeffer homotopy parameter. The aspect ratio of the periodic domain is 16.

decelerating case with sufficient time being allowed for equilibrium to be reached each time. It may be seen that there is excellent agreement between experiments and numerics, giving confidence in both our numerical calculations and experimental techniques. The obvious noise which is present on the experimental time traces is purely instrumental and arises from the random arrival of particles in the measuring volume of the LDV system.

The results shown in figure 2 are the time constants T obtained from least-squares fits of equation (1.2) to time series such as those shown in figure 1. In this case they were obtained from numerical solutions of the Navier–Stokes equations for various values of the Schaeffer homotopy parameter, τ . The numerical results were calculated using the method described above using equal steps in Re either side of $Re_c \approx 68.2$. (Note that a more precise estimate of $Re_c = 68.189$ was obtained in the latter stages of

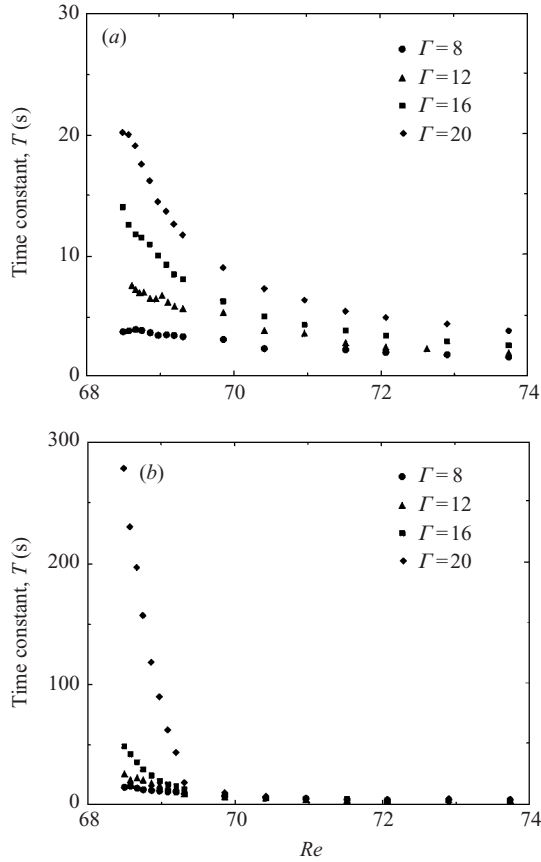


FIGURE 3. Time constants obtained from fits of the Landau theory to experimental time series for (a) onset and (b) decay for 8 to 20 cells at the respective aspect ratios $\Gamma = 8$ –20.

the investigation so the steps were not precisely equal on either side of Re_c . However, the error introduced is insignificant and is $\approx 0.04\%$.) When $\tau = 0$, ‘periodic’ boundary conditions (Schaeffer 1980) apply and both values of T are equal to within the fitting error. However, when τ is increased the value of T associated with the appearance of cells rapidly becomes much shorter than for the disappearance. We have curtailed the range of τ displayed since there is very little further variation in T as the no-slip boundary condition ($\tau = 1$) is reached.

These results are both informative and interesting. They confirm that the Landau equation provides a good model for the instability close to the bifurcation point but this rapidly breaks down as the translational symmetry of the model is lost when τ is increased. This result differs from the effects of τ on the steady bifurcation structure where it is found that the limit point of the disconnected solution remains close to Re_c ($\tau = 0$) and only approaches the experimental values when $\tau \geq 0.9$ as discussed by Cliffe & Mullin (1985). Hence the dynamics provide a more sensitive indicator of the effects of moving towards realistic end conditions.

We next show in figure 3 sets of experimental estimates of T for four different steady cellular states plotted as a function of Re . Each point on the graph is an estimate of the time constant obtained from a fit of the Landau model to measured time series with $Re_i = 63.74$ for the onset and $Re_i = 74.81$ for the decay, i.e. sets

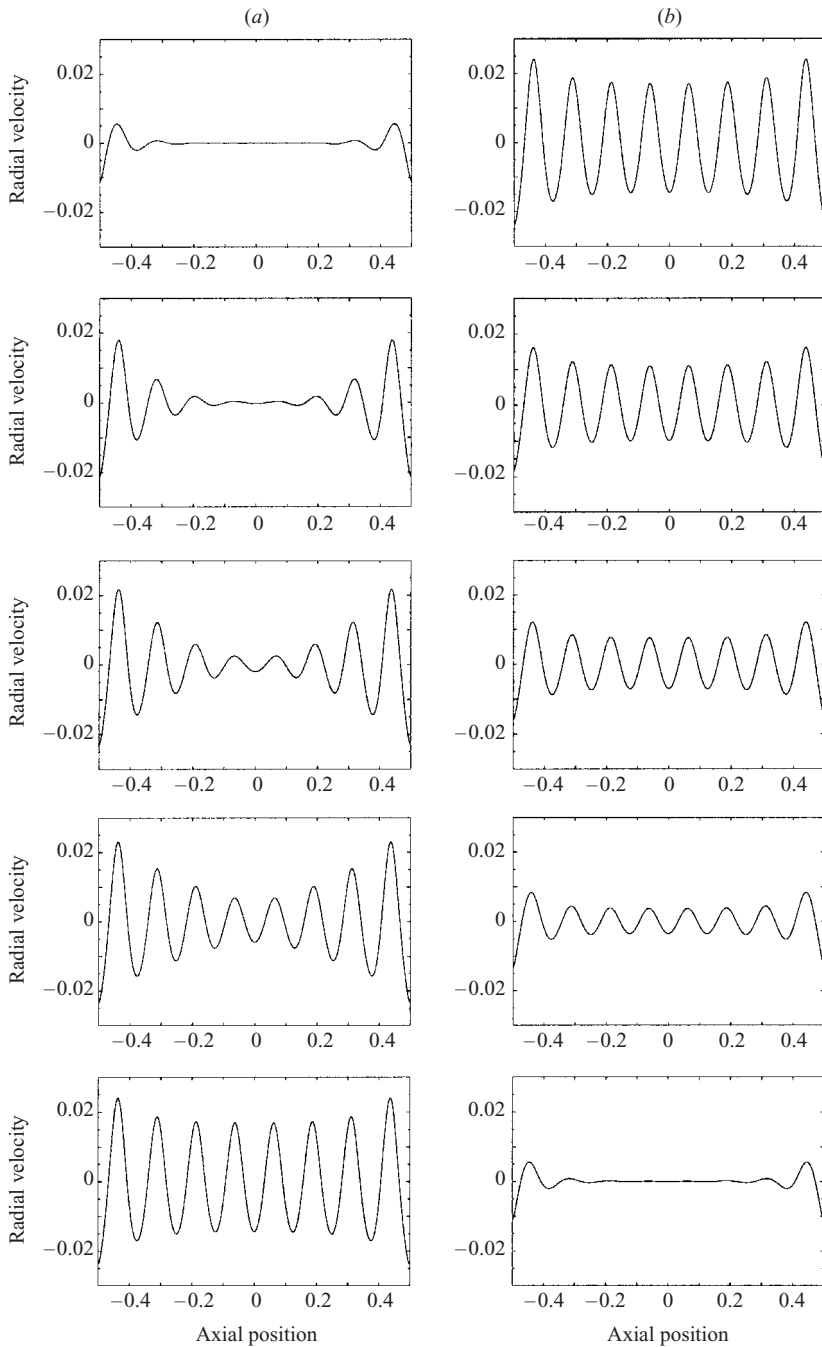


FIGURE 4. Two sequences of instantaneous radial velocity plots where Re was step changed from 63.74 to 69.02 for the onset and 69.02 to 63.74 for the decay. The times of the plots were chosen to provide optimal coverage of the most rapid changes: (a) onset 0, 100, 200, 300, 800 non-dimensional time units; (b) decay 0, 12.5, 25, 50, 200 non-dimensional time units. The Schaeffer homotopy parameter τ was 0.1.

of time series of the kind typified by the examples shown in figure 1. As expected from the Landau model there is evidence for ‘slowing down’ as the Re_c of the periodic model is approached. The effect becomes more pronounced as the aspect ratio of the system is increased. However, there is also an obvious disparity between the timescales for the onset and decay in agreement with the numerical results of figure 2. Moreover, the difference between the two timescales becomes greater as Γ is increased.

The large disparity between the time constants suggests that different mechanisms may be important in the onset and decay of cells. In order to investigate this we performed a numerical study of the onset and decay of cells by stepping either side of the critical point for the onset of cells in the periodic model. The investigation was carried out for $\tau = 0.1$ since we have shown in figure 2 that the results are qualitatively unchanged for larger values of this parameter. The results are shown in figure 4 in the form of two sets of instantaneous radial velocity profiles, one for each step. We have chosen the non-dimensional times to highlight the salient features during the periods of most rapid change.

The influence of the ends is clear for the onset, where the cells are seen to spread from the boundaries towards the middle. On the other hand the decay is almost uniform along the entire length of the cylinders, indicating that the ends play a minor role in this process. Of course, when periodic boundary conditions apply both the onset and decay will be spatially uniform and so, surprisingly, the decay in the finite problem appears to mimic the periodic case more closely. This qualitative difference in the two processes appears to be responsible for the large differences in the time constants and, moreover, the experimental results of figure 3 suggest that the effects of the ends on the dynamics of the onset become more important for longer cylinders.

It is interesting to note that behaviour analogous to that shown in figure 4 has been discussed by Pomeau, Zaleski & Manneville (1985) in relation to a finite-size model of Rayleigh–Bénard convection. They identified both wall and central modes, which show different scaling laws for the respective growth rates. A possible future direction of research may be to see if the dynamics uncovered here could be related to the behaviour in models of this type.

5. Conclusions

We have investigated the transient dynamics associated with the onset of Taylor cells and have confirmed by numerical investigations of the Navier–Stokes equations that the Landau amplitude equation provides an appropriate description when ‘periodic’ boundary conditions are applied. By way of contrast, the application of no-slip boundary conditions changes the dynamics radically and this finding is in agreement with our experimental observations. Specifically, the onset of cells occurs on a timescale which is an order of magnitude shorter than for their disappearance. The role of the ends is evident in the spatial velocity field for onset but the collapse of the cellular flow appears to be independent of length effects.

These results are in accord with previous observations of the static bifurcation structure, where it is well established that the bifurcation of the model is effectively destroyed in experiments despite the sharp onset of cells. Perhaps the most surprising result of all is that the disparity in timescales between appearance and disappearance of cells becomes greater for increasing aspect ratio. Long-timescale dynamics of this type have also been found in travelling wave states in a model of convection by Proctor, Tobias & Knobloch (2000).

The work of J. A. is supported by the German Science Foundation and the research of T. M. by an EPSRC ‘Senior Fellowship’.

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