
Planting Colourings Silently

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Let $k \geq 3$ be a fixed integer and let $Z_k(G)$ be the number of k -colourings of the graph G . For certain values of the average degree, the random variable $Z_k(G(n, m))$ is known to be concentrated in the sense that $\frac{1}{n}(\ln Z_k(G(n, m)) - \ln \mathbb{E}[Z_k(G(n, m))])$ converges to 0 in probability (Achlioptas and Coja-Oghlan, *Proc. FOCS 2008*). In the present paper we prove a significantly stronger concentration result. Namely, we show that for a wide range of average degrees, $\frac{1}{\omega}(\ln Z_k(G(n, m)) - \ln \mathbb{E}[Z_k(G(n, m))])$ converges to 0 in probability for any diverging function $\omega = \omega(n) \rightarrow \infty$. For k exceeding a certain constant k_0 this result covers all average degrees up to the so-called *condensation phase transition* $d_{k, \text{cond}}$, and this is best possible. As an application, we show that the experiment of choosing a k -colouring of the random graph $G(n, m)$ uniformly at random is contiguous with respect to the so-called ‘planted model’.

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1. Introduction

1.1. Background and motivation

Let $G(n, m)$ denote the random graph on the vertex set $[n] = \{1, \dots, n\}$ with precisely m edges. As usual, $G(n, m)$ has a property \mathcal{A} ‘with high probability’ (w.h.p.) if $\lim_{n \rightarrow \infty} \mathbb{P}[G(n, m) \in \mathcal{A}] = 1$.

The study of the graph colouring problem on $G(n, m)$ goes back to the seminal paper of Erdős and Rényi [17]. A wealth of research has since been devoted to either estimating the typical value of the chromatic number of $G(n, m)$ [5, 9, 26, 28], its concentration [6, 27, 37], or the problem of colouring random graphs by means of efficient algorithms [3, 18, 22]; for a more complete survey see [10, 20]. Some of the methods developed in this line of work have had a wide impact on combinatorics (e.g. the use of martingale tail bounds).

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Since the 1990s substantial progress has been made in the case of *sparse* random graphs, where $m = O(n)$ as $n \rightarrow \infty$. For instance, Achlioptas and Friedgut [2] proved that for any $k \geq 3$ there exists a *sharp threshold sequence* $d_{k\text{-col}}(n)$ such that for any fixed $\varepsilon > 0$ the random graph $G(n, m)$ is k -colourable w.h.p. if $2m/n < d_{k\text{-col}}(n) - \varepsilon$, whereas $G(n, m)$ fails to be k -colourable w.h.p. if $2m/n > d_{k\text{-col}}(n) + \varepsilon$. The best current bounds [11, 15] on $d_{k\text{-col}}(n)$ show that there is a sequence $(\gamma_k)_{k \geq 3}$, $\lim_{k \rightarrow \infty} \gamma_k = 0$, such that

$$(2k - 1) \ln k - 2 \ln 2 - \gamma_k \leq \liminf_{n \rightarrow \infty} d_{k\text{-col}}(n) \leq \limsup_{n \rightarrow \infty} d_{k\text{-col}}(n) \leq (2k - 1) \ln k - 1 + \gamma_k. \tag{1.1}$$

In recent work, to a large extent inspired by predictions from statistical physics [30], it has emerged that properties of *typical* k -colourings have a very significant impact both on combinatorial and algorithmic aspects of the random graph colouring problem. To be precise, by a typical k -colouring we mean a k -colouring of the random graph $G(n, m)$ chosen uniformly at random from the set of all of its k -colourings (provided that this set is non-empty). Properties of such randomly chosen colourings have been harnessed to study the ‘geometry’ of the set of k -colourings of a random graph [1, 31] as well as the nature of correlations between the colours taken by different vertices [33]. In particular, the proofs of the bounds (1.1) on $d_{k\text{-col}}(n)$ exploit structural properties such as the ‘clustering’ of the set of k -colourings and the emergence of ‘frozen variables’.

1.2. Quiet planting

The notion of choosing a random colouring of a random graph $G(n, m)$ can be formalized as follows. Let $\Lambda_{k,n,m}$ be the set of all pairs (G, σ) such that G is a graph on $[n]$ with precisely m edges, and σ is a k -colouring of G . Further, for a graph G let $Z_k(G)$ signify the number of k -colourings of G . Now, define a probability distribution $\pi_{k,n,m}^{\text{rc}}(G, \sigma)$, called the *random colouring model*, on $\Lambda_{k,n,m}$ by letting

$$\pi_{k,n,m}^{\text{rc}}(G, \sigma) = \left[Z_k(G) \binom{\binom{n}{2}}{m} \mathbb{P}[G(n, m) \text{ is } k\text{-colourable}] \right]^{-1}.$$

Perhaps more intuitively, this is the distribution produced by the following experiment.

RC1 Generate a random graph $G = G(n, m)$ subject to the condition that $Z_k(G) > 0$.

RC2 Choose a k -colouring τ of G uniformly at random. The result of the experiment is (G, τ) .

Since we are going to be interested in values of m/n where $G(n, m)$ is k -colourable w.h.p., the conditioning in step **RC1** is harmless. But what turns the direct study of the distribution $\pi_{k,n,m}^{\text{rc}}$ into a challenge is step **RC2**. This is illustrated by the fact that the best current algorithms for sampling a k -colouring of $G(n, m)$ are known to be efficient only for average degrees $d < k$ [16], a far cry from $d_{k\text{-col}}(n)$ (see (1.1)).

Achlioptas and Coja-Oghlan [1] suggested circumventing this problem by means of an alternative probability distribution on $\Lambda_{k,n,m}$ called the *planted model*. This distribution is induced by the following experiment. For $\sigma : [n] \rightarrow [k]$ let

$$\mathcal{F}(\sigma) = \sum_{i=1}^k \binom{|\sigma^{-1}(i)|}{2}$$

denote the number of edges of the complete graph that are monochromatic under σ .

PL1 Choose a map $\sigma : [n] \rightarrow [k]$ uniformly at random, subject to the condition that $\mathcal{F}(\sigma) \leq \binom{n}{2} - m$.

PL2 Generate a graph G on $[n]$ consisting of m edges that are bichromatic under σ uniformly at random. The result of the experiment is (G, σ) .

Thus, the probability that the planted model assigns to a pair (G, σ) is

$$\pi_{k,n,m}^{\text{pl}}(G, \sigma) \sim \left[\binom{\binom{n}{2}}{m} k^n \mathbb{P}[\sigma \text{ is a } k\text{-colouring of } G(n, m)] \right]^{-1},$$

where $q(n) \sim p(n)$ if $\lim_{n \rightarrow \infty} q(n)/p(n) = 1$. In contrast to the ‘difficult’ experiment **RC1–RC2**, **PL1–PL2** is quite convenient to work with.

Of course, the two probability distributions $\pi_{k,n,m}^{\text{rc}}$ and $\pi_{k,n,m}^{\text{pl}}$ differ. For instance, under $\pi_{k,n,m}^{\text{pl}}$ a graph G arises with a probability that is proportional to its number of k -colourings, which is not the case under $\pi_{k,n,m}^{\text{rc}}$. However, the two models are related if $m = m(n)$ is such that

$$\ln Z_k(G(n, m)) = \ln \mathbb{E}[Z_k(G(n, m))] + o(n) \quad \text{w.h.p.} \tag{1.2}$$

Indeed, if (1.2) is satisfied, then the following is true [1]:

$$\begin{aligned} &\text{If } (\mathcal{E}_n) \text{ is a sequence of events in } \Lambda_{k,n,m} \text{ such that} \\ &\pi_{k,n,m}^{\text{pl}}[\mathcal{E}_n] \leq \exp(-\Omega(n)), \text{ then } \pi_{k,n,m}^{\text{rc}}[\mathcal{E}_n] = o(1). \end{aligned} \tag{1.3}$$

The statement (1.3), baptized ‘quiet planting’ by Krzalačka and Zdeborová [25], has provided the foundation for the study of the geometry of the set of colourings, freezing, *etc.* [1, 8, 31, 33]. Moreover, similar statements have proved useful in the study of other random constraint satisfaction problems [13, 32, 33]. Yet a significant complication in the use of (1.3) is that \mathcal{E}_n is required to be *exponentially* unlikely in the planted model. This has caused substantial difficulties in several applications (*e.g.* [8, 31]).

1.3. Results

The contribution of the present paper is to show that the statement (1.3) can be sharpened in the strongest possible sense. Roughly speaking, we are going to show that if (1.2) holds, then the random colouring model is contiguous with respect to the planted model, that is, in (1.3) it suffices that $\pi_{k,n,m}^{\text{pl}}[\mathcal{E}_n] = o(1)$ (see Theorem 1.2 below for a precise statement). We obtain this result by establishing that under certain conditions the number $Z_k(G(n, m))$ of k -colourings of the random graph is concentrated remarkably tightly.

To state the result, we need a bit of notation. From here on we always assume that $m = \lceil \bar{d}n \rceil$ for a number $\bar{d} > 0$ that remains fixed as $n \rightarrow \infty$. Furthermore, for $k \geq 3$ we define

$$d_{k,\text{cond}} = \sup \{ \bar{d} > 0 : \lim_{n \rightarrow \infty} \mathbb{E}[Z_k(G(n, m))^{1/n}] = k(1 - 1/k)^{\bar{d}/2} \}. \tag{1.4}$$

This definition is motivated by the well-known fact that

$$\mathbb{E}[Z_k(G(n, m))] = \Theta(k^n (1 - 1/k)^m). \tag{1.5}$$

Thus, Jensen’s inequality shows that

$$\limsup_{n \rightarrow \infty} \mathbb{E}[Z_k(G(n, m))^{1/n}] \leq k(1 - 1/k)^{\bar{d}/2} \quad \text{for all } \bar{d},$$

and $d_{k,\text{cond}}$ marks the greatest average degree up to which this upper bound is tight. Under the assumption that $k \geq k_0$ for a certain constant k_0 , it is possible to calculate the number $d_{k,\text{cond}}$ precisely [8], and an asymptotic expansion in k yields

$$d_{k,\text{cond}} = (2k - 1) \ln k - 2 \ln 2 + \gamma_k, \quad \text{where } \lim_{k \rightarrow \infty} \gamma_k = 0.$$

Theorem 1.1. *There is a constant $k_0 > 3$ such that the following is true. Assume either that $k \geq 3$ and $\bar{d} \leq 2(k - 1) \ln(k - 1)$ or that $k \geq k_0$ and $\bar{d} < d_{k,\text{cond}}$. Then*

$$\lim_{\omega \rightarrow \infty} \lim_{n \rightarrow \infty} \mathbb{P}[|\ln Z_k(G(n, m)) - \ln \mathbb{E}[Z_k(G(n, m))]| \leq \omega] = 1. \tag{1.6}$$

On the other hand, for any fixed number $\omega > 0$, any $k \geq 3$ and any $\bar{d} > 0$ we have

$$\lim_{n \rightarrow \infty} \mathbb{P}[|\ln Z_k(G(n, m)) - \ln \mathbb{E}[Z_k(G(n, m))]| \leq \omega] < 1.$$

For \bar{d}, k covered by the first part of Theorem 1.1 we have $\ln Z_k(G(n, m)) = \Theta(n)$ w.h.p. Whilst one might expect *a priori* that $\ln Z_k(G(n, m))$ has fluctuations of order, say, \sqrt{n} , the first part of Theorem 1.1 shows that actually $\ln Z_k(G(n, m))$ fluctuates by no more than $\omega(n)$ for any $\omega(n) \rightarrow \infty$ w.h.p. The second part shows that this is best possible. In addition, for $k \geq k_0$ Theorem 1.1 is best possible with respect to the range of \bar{d} . In fact, it has been shown in [8] that $\ln Z_k(G(n, m)) < \ln \mathbb{E}[Z_k(G(n, m))] - \Omega(n)$ w.h.p. for $\bar{d} > d_{k,\text{cond}}$.

Theorem 1.1 enables us to establish a very strong connection between the random colouring model and the planted model. To state this, we recall the following definition. Suppose that $\mu = (\mu_n)_{n \geq 1}, \nu = (\nu_n)_{n \geq 1}$ are two sequences of probability measures such that μ_n, ν_n are defined on the same probability space Ω_n for every n . Then $(\mu_n)_{n \geq 1}$ is *contiguous* with respect to $(\nu_n)_{n \geq 1}$, in symbols $\mu \triangleleft \nu$, if for any sequence $(\mathcal{E}_n)_{n \geq 1}$ of events such that $\lim_{n \rightarrow \infty} \nu_n(\mathcal{E}_n) = 0$, we have $\lim_{n \rightarrow \infty} \mu_n(\mathcal{E}_n) = 0$.

Theorem 1.2. *There is a constant $k_0 > 3$ such that the following is true. Assume either that $k \geq 3$ and $\bar{d} \leq 2(k - 1) \ln(k - 1)$ or that $k \geq k_0$ and $\bar{d} < d_{k,\text{cond}}$. Then*

$$(\pi_{k,n,m}^{\text{rc}})_{n \geq 1} \triangleleft (\pi_{k,n,m}^{\text{pl}})_{n \geq 1}.$$

Inspired by the term ‘quiet planting’ which has been used to describe (1.3), we are inclined to refer to the contiguity statement of Theorem 1.2 as ‘silent planting’.

1.4. Discussion and further related work

The proof of Theorem 1.1 combines the second moment arguments from Achlioptas and Naor [5] and its enhancements from [8, 15] with the ‘small subgraph conditioning’ method [19, 36]. More precisely, the key observation on which the proof of Theorem 1.1 is based is that the fluctuations of $\ln Z_k(G(n, m))$ can be attributed to the variations of the number of bounded length cycles in the random graph.

This was already known to be the case for random regular graphs. Specifically, Kemkes, Perez-Gimenez and Wormald [21] combined the small subgraph conditioning argument with the second

moment argument from [5] to bound the chromatic number of the random d -regular graph from above. While it had been pointed out by Achlioptas and Moore [4] that the second moment argument from [5] can be used rather directly to conclude that the same upper bound holds with a probability that remains bounded away from 0 as $n \rightarrow \infty$, small subgraph conditioning was used in [21] to boost this probability to $1 - o(1)$. Improved bounds on the chromatic number of random regular graphs, also based on the second moment method and small subgraph conditioning, were recently obtained in [12]. In the case of the $G(n, m)$ model, small subgraph conditioning is not necessary to bound the chromatic number from above, because the sharp threshold result [2] can be used instead.¹

A priori it might seem reasonable to expect that the random variable $\ln Z_k$ is more tightly concentrated in random regular graphs than in the $G(n, m)$ model, and that therefore small subgraph conditioning cannot be applied in the case of $G(n, m)$. In fact, in the random regular graph for any fixed number ω , the depth- ω neighbourhood of all but a bounded number of vertices is just a d -regular tree. Thus, there are only extremely limited fluctuations in the local structure of the random regular graph. By contrast, in the $G(n, m)$ -model the depth- ω neighbourhoods can be of varying shapes and sizes (although all but a bounded number will be acyclic), and also the number of vertices/edges in the largest connected component and the k -core fluctuate. Nonetheless, perhaps somewhat surprisingly, we are going to show that even in the case of the $G(n, m)$ model, the fluctuations of $\ln Z_k$ are merely due to the appearance of short cycles. Finally, Theorem 1.2 will follow from Theorem 1.1 by means of an argument similar to that used in [1].

We expect that the present approach of combining the second moment method with small subgraph conditioning can be applied successfully to a variety of other random constraint problems. Immediate examples that spring to mind include random k -NAESAT or random k -XORSAT, random hypergraph k -colourability or, more generally, the family of problems studied in [33]. (On the other hand, we expect that in problems such as random k -SAT the logarithm of the number of satisfying assignments exhibits stronger fluctuations, due to a lack of symmetry.)

Independently of the present work, Neeman and Netrapalli [35] used small subgraph conditioning very elegantly to investigate the non-reconstructability problem in the stochastic block model.² More precisely, Neeman and Netrapalli consider a very general class of block models where the edges are chosen independently from a distribution characterized by a density matrix of bounded rank. Edges are then inserted randomly according to the density matrix such that the average degree of the resulting graph is bounded. This model encompasses a ‘binomial’ version of the planted colouring model in which edges are inserted independently. The main result shows that the block model and the ‘plain’ binomial random graph are mutually contiguous under certain assumptions. Comparing [35] with the present work, we note that for Theorem 1.1 it is important to actually fix the number of edges, that is, the result does not hold for the binomial model. Moreover, Theorem 1.2 establishes contiguity not just for the graph distributions but for graph/colouring pairs. Yet it would be interesting to see if the techniques of [35] can be

¹ While the combination of the second moment method and the sharp threshold result can be used to show that (1.2) implies (1.3), this approach does *not* yield Theorem 1.1. For instance, even the sharp threshold analysis from [1] allows for the possibility that $Z_k(G(n, m)) = (3 - o(1))\mathbb{E}[Z_k(G(n, m))]$ with probability $1/3$, while $Z_k(G(n, m)) \leq \exp(-n^{0.99})\mathbb{E}[Z_k(G(n, m))]$ with probability $2/3$.

² Although [35] was posted on arXiv in April 2014, we only learned of the paper from [7].

combined with the present arguments to obtain results like Theorem 1.1 and Theorem 1.2 for more general models. Building upon [35], Banks and Moore [7] recently proved explicit upper and lower bounds on the information-theoretic threshold in a stochastic block model where there are k classes such that each edge joining two vertices in the same class is present with a certain probability, while crossing edges are present with another probability.³

Our contiguity results imply that the condensation threshold is a lower bound on the information-theoretic detectability threshold for the planted colouring problem. This means that no algorithm can decide with high probability whether a given graph was generated from the planted or $G(n, m)$ model. This bound is tight, that is, above the condensation threshold there is an algorithm, albeit an exponential one.

1.5. Preliminaries and notation

We always assume that $n \geq n_0$ is large enough for our various estimates to hold. Moreover, if $p = (p_1, \dots, p_l)$ is a vector with entries $p_i \geq 0$, then we let

$$H(p) = - \sum_{i=1}^l p_i \ln p_i.$$

Here and throughout, we use the convention that $0 \ln 0 = 0$. Hence, if $\sum_{i=1}^l p_i = 1$, then $H(p)$ is the entropy of the probability distribution p . Further, for a number x and an integer $h > 0$ we let $(x)_h = x(x-1) \cdots (x-h+1)$ denote the h th falling factorial of x .

We use the following version of the small subgraph technique.

Theorem 1.3 ([19]). *Suppose that $(\delta_l)_{l \geq 2}, (\lambda_l)_{l \geq 2}$ are sequences of real numbers such that $\delta_l \geq -1$ and $\lambda_l > 0$ for all l . Moreover, assume that $(C_{l,n})_{l \geq 2, n \geq 1}$ and $(Z_n)_{n \geq 1}$ are random variables such that each $C_{l,n}$ takes values in the non-negative integers. Additionally, suppose that for each n the random variables $C_{2,n}, \dots, C_{n,n}$ and Z_n are defined on the same probability space. Moreover, let $(X_l)_{l \geq 2}$ be a sequence of independent random variables such that X_l has distribution $\text{Po}(\lambda_l)$, and assume that the following four conditions hold.*

SSC1 For any integer $L \geq 2$ and any integers $x_2, \dots, x_L \geq 0$,

$$\lim_{n \rightarrow \infty} \mathbb{P}[\forall 2 \leq l \leq L : C_{l,n} = x_l] = \prod_{l=2}^L \mathbb{P}[X_l = x_l].$$

SSC2 For any integer $L \geq 2$ and any integers $x_2, \dots, x_L \geq 0$,

$$\lim_{n \rightarrow \infty} \frac{\mathbb{E}[Z_n \mid \forall 2 \leq l \leq L : C_{l,n} = x_l]}{\mathbb{E}[Z_n]} = \prod_{l=2}^L (1 + \delta_l)^{x_l} \exp(-\lambda_l \delta_l).$$

SSC3 $\sum_{l=2}^{\infty} \lambda_l \delta_l^2 < \infty$.

SSC4 $\lim_{n \rightarrow \infty} \mathbb{E}[Z_n^2] / \mathbb{E}[Z_n]^2 \leq \exp[\sum_{l=2}^{\infty} \lambda_l \delta_l^2]$.

Then the sequence $(Z_n / \mathbb{E}[Z_n])_{n \geq 1}$ converges in distribution to $\prod_{l=2}^{\infty} (1 + \delta_l)^{X_l} \exp(-\lambda_l \delta_l)$. □

³ A combined version of the two papers [7, 35] is due to appear in the proceedings of COLT 2016.

Condition **SSC4** in Theorem 1.3 makes it apparent that we need very precise computation of the first and second moments of the number of k -colouring of $G(n, m)$. For this reason, we need to distinguish between the quantity \bar{d} defined as $m = \lceil \bar{d}n/2 \rceil$ and the quantity $d = 2m/n$, which arises naturally in the computations of the first and second moment. Note that the quantity $d = d(n)$ whereas \bar{d} is assumed to be fixed, that is, independent of n . However, it is elementary to show that $d \sim \bar{d}$.

2. Outline of the proof

It turns out to be convenient to prove Theorems 1.1 and 1.2 by way of another random graph model $\mathcal{G}(n, m)$. This is a random (multi-)graph on the vertex set $[n]$ obtained by choosing m edges e_1, \dots, e_m of the complete graph on n vertices uniformly and independently at random (i.e. with replacement).

To bound $Z_k(\mathcal{G}(n, m))$ from below, we will confine ourselves to k -colourings in which all the colour classes have very nearly the same size. More precisely, for a map $\sigma : [n] \rightarrow [k]$ we define

$$\rho(\sigma) = (\rho_1(\sigma), \dots, \rho_k(\sigma)), \quad \text{where } \rho_i(\sigma) = |\sigma^{-1}(i)|/n \quad (i = 1 \dots k).$$

Thus, $\rho(\sigma)$ is a probability distribution on $[k]$, which we refer to as the *colour density* of σ . Let $\mathcal{C}_k(n)$ signify the set of all possible colour densities $\rho(\sigma)$, $\sigma : [n] \rightarrow [k]$. Further, let $\bar{\mathcal{C}}_k$ be the set of all probability distributions $\rho = (\rho_1, \dots, \rho_k)$ on $[k]$, and let $\rho^* = (1/k, \dots, 1/k)$ signify the barycentre of $\bar{\mathcal{C}}_k$. We say that $\rho = (\rho_1, \dots, \rho_k) \in \bar{\mathcal{C}}_k$ is (ω, n) -balanced if

$$|\rho_i - k^{-1}| \leq \omega^{-1}n^{-1/2} \quad \text{for all } i \in [k].$$

Let $\mathcal{B}_{n,k}(\omega)$ denote the set of all (ω, n) -balanced $\rho \in \mathcal{C}_k(n)$. Now, for a graph G on $[n]$ let $Z_{k,\omega}(G)$ signify the number of (ω, n) -balanced k -colourings, that is, k -colourings σ such that $\rho(\sigma) \in \mathcal{B}_{n,k}(\omega)$. In Section 3 we will calculate the first moment of $Z_{k,\omega}$ to obtain the following.

Proposition 2.1. *Fix an integer $k \geq 3$ and a number $\bar{d} \in (0, \infty)$ and assume that $\omega = \omega(n)$ is a sequence such that $\lim_{n \rightarrow \infty} \omega(n) = \infty$, while $d = 2m/n$. Then*

$$\begin{aligned} \mathbb{E}[Z_k(\mathcal{G}(n, m))] &= \Theta(k^n(1 - 1/k)^m) \quad \text{and} \\ \frac{\mathbb{E}[Z_{k,\omega}(\mathcal{G}(n, m))]}{\mathbb{E}[Z_k(\mathcal{G}(n, m))]} &\sim \frac{|\mathcal{B}_{n,k}(\omega)|k^{k/2}}{(2\pi n)^{(k-1)/2}} \left(1 + \frac{d}{k-1}\right)^{(k-1)/2}. \end{aligned}$$

In particular,

$$\ln \mathbb{E}[Z_{k,\omega}(\mathcal{G}(n, m))] = \ln \mathbb{E}[Z_k(\mathcal{G}(n, m))] + O(\ln \omega(n)).$$

As outlined in Section 1.4, our basic strategy is to show that the fluctuations of $Z_{k,\omega}(\mathcal{G}(n, m))$ can be attributed to fluctuations in the number of cycles of bounded length. Hence, for an integer $\ell \geq 2$ we let $C_{\ell,n}$ denote the number of cycles of length (exactly) ℓ in $\mathcal{G}(n, m)$. Let

$$\lambda_\ell = \frac{\bar{d}^\ell}{2^\ell} \quad \text{and} \quad \delta_\ell = \frac{(-1)^\ell}{(k-1)^{\ell-1}}, \tag{2.1}$$

It is well known that $C_{2,n}, C_{3,n}, \dots$ are asymptotically independent Poisson variables (e.g. [10, Theorem 5.16]). More precisely, we have the following.

Fact 2.2. *If x_2, \dots, x_L are non-negative integers, then*

$$\lim_{n \rightarrow \infty} \mathbb{P}[\forall 2 \leq \ell \leq L : C_{\ell,n} = x_\ell] = \prod_{\ell=2}^L \mathbb{P}[\text{Po}(\lambda_\ell) = x_\ell].$$

In order to apply Theorem 1.3 to the random variables $C_{\ell,n}$ and $Z_{k,\omega}(\mathcal{G}(n,m))$, we need to investigate the impact of the cycle counts $C_{\ell,n}$ on the first moment of $Z_{k,\omega}(\mathcal{G}(n,m))$. This is the task that we tackle in Section 4, where we prove the following.

Proposition 2.3. *Assume that $k \geq 3$ and that $\bar{d} < (k - 1)^2$. Then*

$$\sum_{\ell=2}^{\infty} \lambda_\ell \delta_\ell^2 < \infty. \tag{2.2}$$

Moreover, let $\omega = \omega(n) > 0$ be any sequence such that $\lim_{n \rightarrow \infty} \omega(n) = \infty$. If x_2, \dots, x_L are non-negative integers, then

$$\frac{\mathbb{E}[Z_{k,\omega}(\mathcal{G}(n,m)) \mid \forall 2 \leq \ell \leq L : C_{\ell,n} = x_\ell]}{\mathbb{E}[Z_{k,\omega}(\mathcal{G}(n,m))]} \sim \prod_{\ell=2}^L [1 + \delta_\ell]^{x_\ell} \exp(-\delta_\ell \lambda_\ell). \tag{2.3}$$

Further, to invoke Theorem 1.3 we need to know the second moment of $Z_{k,\omega}(\mathcal{G}(n,m))$ very precisely. To obtain the required estimate, we consider two regimes of \bar{d}, k separately. In the simpler case, based on the second moment argument from [5], we obtain the following result.

Proposition 2.4. *Assume that $k \geq 3$ and $\bar{d} < 2(k - 1) \ln(k - 1)$. Then*

$$\frac{\mathbb{E}[Z_{k,\omega}(\mathcal{G}(n,m))^2]}{\mathbb{E}[Z_{k,\omega}(\mathcal{G}(n,m))]^2} \sim \exp\left(\sum_{\ell \geq 2} \lambda_\ell \delta_\ell^2\right).$$

The second regime of \bar{d}, k is that $k \geq k_0$ for a certain constant $k_0 \geq 3$ and $\bar{d} < d_{k,\text{cond}}$ (with $d_{k,\text{cond}}$ the number defined in (1.4)). In this case, it is necessary to replace $Z_{k,\omega}$ with the slightly tweaked random variable $\tilde{Z}_{k,\omega}$ used in the second moment arguments from [8, 15].

Proposition 2.5. *There is a constant $k_0 \geq 3$ such that the following is true. Assume that $k \geq k_0$ and $2(k - 1) \ln(k - 1) \leq \bar{d} < d_{k,\text{cond}}$. There exists an integer-valued random variable $0 \leq \tilde{Z}_{k,\omega} \leq Z_{k,\omega}$ such that*

$$\begin{aligned} \mathbb{E}[\tilde{Z}_{k,\omega}(\mathcal{G}(n,m))] &\sim \mathbb{E}[Z_{k,\omega}(\mathcal{G}(n,m))] \quad \text{and} \\ \frac{\mathbb{E}[\tilde{Z}_{k,\omega}(\mathcal{G}(n,m))^2]}{\mathbb{E}[\tilde{Z}_{k,\omega}(\mathcal{G}(n,m))]^2} &\leq (1 + o(1)) \exp\left(\sum_{\ell \geq 2} \lambda_\ell \delta_\ell^2\right). \end{aligned} \tag{2.4}$$

The proofs of Propositions 2.4 and 2.5 appear at the end of Section 5.

Of course, to apply Theorem 1.3 to the random variable $\tilde{Z}_{k,\omega}$ we need to investigate the impact of the cycle counts $C_{\ell,n}$ on the first moment of $\tilde{Z}_{k,\omega}$ as well. That is, we need a result similar to Proposition 2.3 for $\tilde{Z}_{k,\omega}$. Fortunately, this does not require reiteration of the proof of Proposition 2.3. Instead, what we need follows readily from Proposition 2.3 and (2.4). More precisely, we have the following result.

Corollary 2.6. *Let x_2, \dots, x_L be non-negative integers. With the assumptions and notation of Proposition 2.5,*

$$\frac{\mathbb{E}[\tilde{Z}_{k,\omega}(\mathcal{G}(n, m)) \mid \forall \ell \leq L : C_{\ell,n} = x_\ell]}{\mathbb{E}[\tilde{Z}_{k,\omega}(\mathcal{G}(n, m))]} \sim \prod_{\ell=2}^L [1 + \delta_\ell]^{x_\ell} \exp(-\delta_\ell \lambda_\ell). \tag{2.5}$$

Proof. Let S denote the event $\{\forall \ell \leq L : C_{\ell,n} = x_\ell\}$ and let $\mathcal{Z}_n = \tilde{Z}_{k,\omega}(\mathcal{G}(n, m))$ for the sake of brevity. Since $\mathcal{Z}_n \leq Z_{k,\omega}$, (2.4) implies the upper bound

$$\frac{\mathbb{E}[\mathcal{Z}_n \mid S]}{\mathbb{E}[\mathcal{Z}_n]} \leq \frac{\mathbb{E}[Z_{k,\omega}(\mathcal{G}(n, m)) \mid S]}{(1 + o(1))\mathbb{E}[Z_{k,\omega}(\mathcal{G}(n, m))]} \sim \prod_{\ell=2}^L [1 + \delta_\ell]^{x_\ell} \exp(-\delta_\ell \lambda_\ell). \tag{2.6}$$

To obtain a matching lower bound, we claim that

$$\mathbb{E}[\mathcal{Z}_n \mid S] \geq (1 - o(1))\mathbb{E}[Z_{k,\omega}(\mathcal{G}(n, m)) \mid S]. \tag{2.7}$$

Indeed, assume for contradiction that (2.7) is false. Then there is an n -independent $\varepsilon > 0$ such that for infinitely many n ,

$$\mathbb{E}[\mathcal{Z}_n \mid S] < (1 - \varepsilon)\mathbb{E}[Z_{k,\omega}(\mathcal{G}(n, m)) \mid S]. \tag{2.8}$$

By Fact 2.2 there exists an n -independent $\xi = \xi(x_2, \dots, x_L) > 0$ such that $\mathbb{P}[S] \geq \xi$. Hence, (2.8) and Bayes' rule imply that

$$\begin{aligned} \mathbb{E}[\mathcal{Z}_n] &= \mathbb{P}[S] \cdot \mathbb{E}[\mathcal{Z}_n \mid S] + \mathbb{P}[\neg S] \mathbb{E}[\mathcal{Z}_n \mid \neg S] \\ &\leq \mathbb{P}[S] \cdot \mathbb{E}[\mathcal{Z}_n \mid S] + \mathbb{P}[\neg S] \mathbb{E}[Z_{k,\omega}(\mathcal{G}(n, m)) \mid \neg S] \quad (\text{as } \mathcal{Z}_n \leq Z_{k,\omega}(\mathcal{G}(n, m))) \\ &\leq (1 - \varepsilon) \mathbb{P}[S] \cdot \mathbb{E}[Z_{k,\omega}(\mathcal{G}(n, m)) \mid S] + \mathbb{P}[\neg S] \cdot \mathbb{E}[Z_{k,\omega}(\mathcal{G}(n, m)) \mid \neg S] \\ &\leq \mathbb{E}[Z_{k,\omega}(\mathcal{G}(n, m))] - \varepsilon \xi \cdot \mathbb{E}[Z_{k,\omega}(\mathcal{G}(n, m)) \mid S] \\ &= \mathbb{E}[Z_{k,\omega}(\mathcal{G}(n, m))] \cdot \left(1 + o(1) - \varepsilon \xi \prod_{\ell=2}^L [1 + \delta_\ell]^{x_\ell} \exp(-\delta_\ell \lambda_\ell)\right) \\ &= (1 - \Omega(1))\mathbb{E}[Z_{k,\omega}(\mathcal{G}(n, m))] \quad (\text{as } \delta_\ell, \lambda_\ell, x_\ell \text{ remain fixed as } n \rightarrow \infty). \end{aligned} \tag{2.9}$$

But (2.9) contradicts (2.4). Thus, we have established (2.7). Finally, combining (2.7) with (2.3) and (2.4), we get

$$\frac{\mathbb{E}[\mathcal{Z}_n \mid S]}{\mathbb{E}[\mathcal{Z}_n]} \geq \frac{(1 - o(1))\mathbb{E}[Z_{k,\omega}(\mathcal{G}(n, m)) \mid S]}{(1 + o(1))\mathbb{E}[Z_{k,\omega}(\mathcal{G}(n, m))]} \sim \prod_{\ell=2}^L [1 + \delta_\ell]^{x_\ell} \exp(-\delta_\ell \lambda_\ell), \tag{2.10}$$

and the assertion follows from (2.6) and (2.10). □

We now have all the pieces in place to apply Theorem 1.3.

Corollary 2.7. *Assume that either $k \geq 3$ and $\bar{d} \leq 2(k - 1)\ln(k - 1)$ or $k \geq k_0$ for a certain constant k_0 and $\bar{d} \leq d_{k,\text{cond}}$. Then*

$$\lim_{\varepsilon \rightarrow 0} \lim_{n \rightarrow \infty} \mathbb{P} \left[\frac{Z_k(\mathcal{G}(n, m))}{\mathbb{E}[Z_k(\mathcal{G}(n, m))]} \geq \varepsilon \right] = 1. \tag{2.11}$$

Proof. Let $\omega = \omega(n) > 0$ be any sequence such that $\lim_{n \rightarrow \infty} \omega(n) = \infty$. Moreover, define a sequence $(\mathcal{Z}_n)_{n \geq 1}$ of random variables as follows.

Case 1: $\bar{d} \leq 2(k - 1)\ln(k - 1)$. Let $\mathcal{Z}_n = Z_{k,\omega}(\mathcal{G}(n, m))$.

Case 2: $k \geq k_0$ and $2(k - 1)\ln(k - 1) < \bar{d} < d_{k,\text{cond}}$. Let \mathcal{Z}_n be equal to the random variable $\tilde{Z}_{k,\omega}(\mathcal{G}(n, m))$ from Proposition 2.5.

Then in either case Proposition 2.1 and 2.5 imply that

$$\mathbb{E}[\mathcal{Z}_n] \sim \mathbb{E}[Z_{k,\omega}(\mathcal{G}(n, m))]. \tag{2.12}$$

We are going to apply Theorem 1.3 to the random variables \mathcal{Z}_n and $(C_{\ell,n})_{\ell \geq 2}$. Fact 2.2 readily implies that $C_{2,n}, \dots$ satisfy **SSC1**. Furthermore, Proposition 2.3 and Corollary 2.6 imply that for any integers $x_2, \dots, x_L \geq 0$,

$$\frac{\mathbb{E}[\mathcal{Z}_n \mid \forall 2 \leq \ell \leq L : C_{\ell,n} = x_\ell]}{\mathbb{E}[\mathcal{Z}_n]} \sim \prod_{\ell=2}^L [1 + \delta_\ell]^{x_\ell} \exp(-\delta_\ell \lambda_\ell).$$

Thus, condition **SSC2** is satisfied as well. Further, (2.2) establishes **SSC3**. Finally, **SSC4** is verified by Propositions 2.4 and 2.5. Hence, Theorem 1.3 applies and shows that $\mathcal{Z}_n/\mathbb{E}[\mathcal{Z}_n]$ converges in distribution to

$$W = \prod_{\ell=2}^{\infty} (1 + \delta_\ell)^{X_\ell} \exp(-\lambda_\ell \delta_\ell),$$

where $(X_\ell)_{\ell \geq 2}$ is a family of independent random variables such that X_ℓ has distribution $\text{Po}(\lambda_\ell)$. In particular, since W takes a positive (and finite) value with probability one, we conclude that for any sequence $\omega = \omega(n)$ such that $\lim_{n \rightarrow \infty} \omega(n) = \infty$ we have

$$\lim_{\delta \rightarrow 0} \lim_{n \rightarrow \infty} \mathbb{P} \left[\frac{\mathcal{Z}_n}{\mathbb{E}[\mathcal{Z}_n]} \geq \delta \right] = 1. \tag{2.13}$$

To complete the proof, let $(\varepsilon(n))_{n \geq 1}$ be a sequence in $(0, 1)$ such that $\lim_{n \rightarrow \infty} \varepsilon(n) = 0$. Set $\omega(n) = -\ln \varepsilon(n)$. Then by Proposition 2.1 and (2.12) there exists an n -independent number $c > 0$ such that

$$\mathbb{E}[Z_k(\mathcal{G}(n, m))] \leq \omega^c \cdot \mathbb{E}[\mathcal{Z}_n], \tag{2.14}$$

for sufficiently large n . Thus, combining (2.13) and (2.14) and recalling that $Z_k(\mathcal{G}(n, m)) \geq Z_n$, we see that

$$\begin{aligned} \lim_{n \rightarrow \infty} \mathbb{P} \left[\frac{Z_k(\mathcal{G}(n, m))}{\mathbb{E}[Z_k(\mathcal{G}(n, m))]} \geq \varepsilon(n) \right] &\geq \lim_{n \rightarrow \infty} \mathbb{P} \left[\frac{Z_n}{\mathbb{E}[Z_n]} \geq \omega^c \varepsilon(n) \right] \\ &\geq \lim_{n \rightarrow \infty} \mathbb{P} \left[\frac{Z_n}{\mathbb{E}[Z_n]} \geq \sqrt{\varepsilon(n)} \right] = 1. \end{aligned}$$

Since this holds for any sequence $\varepsilon(n) \rightarrow 0$, the assertion follows. □

Proof of Theorem 1.1. Corollary 2.7 and Markov’s inequality imply that

$$\lim_{\omega \rightarrow \infty} \lim_{n \rightarrow \infty} \mathbb{P}[|\ln Z_k(\mathcal{G}(n, m)) - \ln \mathbb{E}[Z_k(\mathcal{G}(n, m))]| < \omega] = 1. \tag{2.15}$$

To derive Theorem 1.1 from (2.15), let S be the event that $\mathcal{G}(n, m)$ consists of m distinct edges. Given that S occurs, $\mathcal{G}(n, m)$ is identical to $G(n, m)$. Furthermore, Fact 2.2 implies that $\mathbb{P}[S] = \Omega(1)$. Consequently, (2.15) yields

$$\begin{aligned} 1 &= \lim_{\omega \rightarrow \infty} \lim_{n \rightarrow \infty} \mathbb{P}[|\ln Z_k(\mathcal{G}(n, m)) - \ln \mathbb{E}[Z_k(\mathcal{G}(n, m))]| < \omega \mid S] \\ &= \lim_{\omega \rightarrow \infty} \lim_{n \rightarrow \infty} \mathbb{P}[|\ln Z_k(G(n, m)) - \ln \mathbb{E}[Z_k(G(n, m))]| < \omega]. \end{aligned} \tag{2.16}$$

Furthermore, (1.5) and Proposition 2.1 imply that

$$\mathbb{E}[Z_k(G(n, m))], \mathbb{E}[Z_k(\mathcal{G}(n, m))] = \Theta(k^n(1 - 1/k)^m).$$

Hence, $\mathbb{E}[Z_k(\mathcal{G}(n, m))] = \Theta(\mathbb{E}[Z_k(G(n, m))])$ and (2.16) implies that

$$\lim_{\omega \rightarrow \infty} \lim_{n \rightarrow \infty} \mathbb{P}[|\ln Z_k(G(n, m)) - \ln \mathbb{E}[Z_k(G(n, m))]| < \omega] = 1,$$

which is the first part of Theorem 1.1.

To obtain the second assertion, let \mathcal{E}_t be the event that the random graph $G(n, m)$ contains t isolated triangles (*i.e.* t connected components that are isomorphic to the complete graph on three vertices). It is well known that for $t \geq 0$ and $\lambda = (\bar{d}e^{-\bar{d}})^3/6$ we have

$$\liminf_{n \rightarrow \infty} \mathbb{P}[\mathcal{E}_t] = \frac{e^{-\lambda} \lambda^t}{t!}. \tag{2.17}$$

For $S \subset [n]$, of cardinality $3t$, let \mathcal{A}_S be the event that the vertices in S form t many isolated triangles. As the number of k -colourings of a triangle is $k(k - 1)(k - 2)$, (1.5) yields

$$\begin{aligned} \mathbb{E}[Z_k(G(n, m)) \mid \mathcal{A}_S] &= \mathbb{E}[Z_k(G(n - 3t, m - 3t))](k(k - 1)(k - 2))^t \\ &\leq C(d, k) \cdot k^n(1 - 1/k)^{m-3t}(1 - 1/k)^t(1 - 2/k)^t \\ &\leq C(d, k) \cdot k^n(1 - 1/k)^m \cdot (1 - 1/(k - 1)^2)^t \\ &\leq O(\mathbb{E}[Z_k(\mathcal{G}(n, m))]) \cdot (1 - 1/(k - 1)^2)^t. \end{aligned} \tag{2.18}$$

Furthermore, letting T be the family of cardinality $3t$ subsets of $[n]$, it holds that

$$\begin{aligned} \mathbb{E}[Z_k(G(n, m)) \mid \mathcal{E}_t] &\leq \frac{\mathbb{E}[Z_k(G(n, m))1_{\mathcal{E}_t}]}{\mathbb{P}[\mathcal{E}_t]} \\ &= \frac{1}{\mathbb{P}[\mathcal{E}_t]} \sum_{S \in \mathcal{S}} \mathbb{E}[Z_k(G(n, m))1_{\mathcal{E}_t} \mid \mathcal{A}_S] \cdot \mathbb{P}[\mathcal{A}_S] \end{aligned}$$

$$\begin{aligned} &\leq O(\mathbb{E}[Z_k(\mathcal{G}(n, m))]) \cdot (1 - 1/(k - 1)^2)^t \sum_{S \in T} \frac{\mathbb{P}[A_S]}{\mathbb{P}[\mathcal{E}_t]} \quad (\text{from (2.18)}) \\ &\leq O(\mathbb{E}[Z_k(\mathcal{G}(n, m))]) \cdot (1 - 1/(k - 1)^2)^t e^{(d^3 e^{-3d}/6)}, \end{aligned} \tag{2.19}$$

where the last equality follows by noting that $\sum_{S \in T} \mathbb{P}[A_S] \sim (\lambda)^t / t!$ and (2.17).

Hence, for any $\omega > 0$ we can choose t sufficiently large that

$$\mathbb{P}[Z_k(G(n, m)) \mid \mathcal{E}_t] \leq \mathbb{E}[Z_k(G(n, m))] / (2\omega).$$

In combination with Markov’s inequality, this implies that

$$\mathbb{P}[\ln Z_k(G(n, m)) \geq \ln \mathbb{E}[Z_k(G(n, m))] - \ln(\omega) \mid \mathcal{E}_t] \leq 1/2. \tag{2.20}$$

Finally, combining (2.17) and (2.20), we conclude that for any finite ω there is $\varepsilon > 0$ such that for large enough n ,

$$\begin{aligned} &\mathbb{P}[\ln Z_k(G(n, m)) \leq \ln \mathbb{E}[Z_k(G(n, m))] - \omega] \\ &\geq \mathbb{P}[\ln Z_k(G(n, m)) \leq \ln \mathbb{E}[Z_k(G(n, m))] - \omega \mid \mathcal{E}_t] \mathbb{P}[\mathcal{E}_t] > \varepsilon/2. \end{aligned}$$

This completes the proof of the second claim. □

Proof of Theorem 1.2. Assume for contradiction that $(\mathcal{A}_n)_{n \geq 1}$ is a sequence of events such that for some fixed number $0 < \varepsilon < 1/2$ we have

$$\lim_{n \rightarrow \infty} \pi_{k,n,m}^{\text{pl}}[\mathcal{A}_n] = 0 \quad \text{while} \quad \limsup_{n \rightarrow \infty} \pi_{k,n,m}^{\text{rc}}[\mathcal{A}_n] > \varepsilon. \tag{2.21}$$

For some $\sigma : [n] \rightarrow [k]$, let $G(n, m, \sigma)$ denote a graph on $[n]$ with precisely m edges, such that all of these edges are bichromatic under σ , chosen uniformly at random.

Given some G , we need to consider the number of k -colourings σ of G such that $(G, \sigma) \in \mathcal{A}_n$. We express this number by writing it as $Z_k(G) \langle 1\{(G, \sigma) \in \mathcal{A}_n\} \rangle_G$, where $\langle \cdot \rangle_G$ denotes the expectation operator with respect to the uniform distribution over the k -colourings of G . We have that

$$\begin{aligned} &\mathbb{E}[Z_k(G(n, m)) \langle 1\{(G(n, m), \sigma) \in \mathcal{A}_n\} \rangle_{G(n, m)}] \\ &= \sum_{\sigma: [n] \rightarrow [k]} \mathbb{P}[\sigma \text{ is a } k\text{-colouring of } G(n, m) \text{ and } (G(n, m), \sigma) \in \mathcal{A}_n] \\ &= \sum_{\sigma: [n] \rightarrow [k]} \mathbb{P}[(G(n, m), \sigma) \in \mathcal{A}_n \mid \sigma \text{ is a } k\text{-colouring of } G(n, m)] \\ &\quad \times \mathbb{P}[\sigma \text{ is a } k\text{-colouring of } G(n, m)] \\ &= \sum_{\sigma: [n] \rightarrow [k]} \mathbb{P}[(G(n, m), \sigma) \in \mathcal{A}_n] \mathbb{P}[\sigma \text{ is a } k\text{-colouring of } G(n, m)] \\ &\leq O((1 - 1/k)^m) \sum_{\sigma: [n] \rightarrow [k]} \mathbb{P}[(G(n, m), \sigma) \in \mathcal{A}_n] \\ &= O(k^n (1 - 1/k)^m) \pi_{k,n,m}^{\text{pl}}[\mathcal{A}_n] = o(k^n (1 - 1/k)^m). \end{aligned} \tag{2.22}$$

By Corollary 2.7, for any $\varepsilon > 0$ there is $\delta > 0$ such that for all large enough n we have

$$\mathbb{P}[Z_k(G(n, m)) < \delta \mathbb{E}[Z_k(G(n, m))]] < \varepsilon/2. \tag{2.23}$$

For what follows we need to note that

$$\mathbb{E}[\langle 1\{(G(n, m), \sigma) \in \mathcal{A}_n\} \rangle_{G(n, m)}] = \pi_{k, n, m}^{\text{rc}}[\mathcal{A}_n].$$

Letting \mathcal{E} be the event that $Z_k(G(n, m)) \geq \delta \mathbb{E}[Z_k(G(n, m))]$, we have

$$\begin{aligned} &\mathbb{E}[Z_k(G(n, m)) \langle 1\{(G(n, m), \sigma) \in \mathcal{A}_n\} \rangle_{G(n, m)}] \\ &\geq \mathbb{E}[Z_k(G(n, m)) \langle 1\{(G(n, m), \sigma) \in \mathcal{A}_n\} \rangle_{G(n, m)} \mid \mathcal{E}] \mathbb{P}[\mathcal{E}] \\ &\geq \delta \mathbb{E}[Z_k(G(n, m))] \mathbb{E}[\langle 1\{(G(n, m), \sigma) \in \mathcal{A}_n\} \rangle_{G(n, m)} \mid \mathcal{E}] \mathbb{P}[\mathcal{E}] \\ &\geq \delta \mathbb{E}[Z_k(G(n, m))] \pi_{k, n, m}^{\text{rc}}[\mathcal{A}_n \mid \mathcal{E}] \mathbb{P}[\mathcal{E}] \\ &\geq \frac{1}{2} \delta \pi_{k, n, m}^{\text{rc}}[\mathcal{A}_n \mid \mathcal{E}] \mathbb{E}[Z_k(G(n, m))] \quad (\text{as } \mathbb{P}[\mathcal{E}] \geq 1/2) \\ &= \frac{1}{2} \delta \pi_{k, n, m}^{\text{rc}}[\mathcal{A}_n \mid \mathcal{E}] \Omega(k^n (1 - 1/k)^m). \end{aligned} \tag{2.24}$$

Combining (2.22) and (2.24), we obtain that $\pi_{k, n, m}^{\text{rc}}[\mathcal{A}_n \mid \mathcal{E}] = o(1)$. Hence, (2.23) implies that

$$\begin{aligned} \pi_{k, n, m}^{\text{rc}}[\mathcal{A}_n] &= \pi_{k, n, m}^{\text{rc}}[\mathcal{A}_n \mid \neg \mathcal{E}] \cdot \mathbb{P}[\neg \mathcal{E}] + \pi_{k, n, m}^{\text{rc}}[\mathcal{A}_n \mid \mathcal{E}] \cdot \mathbb{P}[\mathcal{E}] \\ &\leq \mathbb{P}[\neg \mathcal{E}] + \pi_{k, n, m}^{\text{rc}}[\mathcal{A}_n \mid \mathcal{E}] \leq \varepsilon/2 + o(1), \end{aligned}$$

in contradiction to (2.21). □

3. The first moment

The aim in this section is to prove Proposition 2.1. The calculations that we perform follow the path beaten in [5, 15, 21]. Let $Z_{k, \rho}(G)$ be the number of k -colourings of the graph G with colour density ρ .

Lemma 3.1. *Let $k \geq 3$, $\bar{d} \in (0, \infty)$ and $d = 2m/n$. Set*

$$\begin{aligned} g : \rho \in \bar{\mathcal{C}}_k &\mapsto H(\rho) + \frac{d}{2} \ln \left(1 - \sum_{i=1}^k \rho_i^2 \right), \\ \alpha(d, k) &= \ln k + \frac{d}{2} \ln \left(1 - \frac{1}{k} \right), \\ c_n(d, k) &= (2\pi n)^{(1-k)/2} k^{k/2}. \end{aligned} \tag{3.1}$$

(1) *There exist numbers $C_1 = C_1(k, d), C_2 = C_2(k, d) > 0$ such that*

$$C_1 n^{(1-k)/2} \exp[ng(\rho)] \leq \mathbb{E}[Z_{k, \rho}(\mathcal{G}(n, m))] \leq C_2 \exp[ng(\rho)] \quad \text{for any } \rho \in \mathcal{C}_k(n). \tag{3.2}$$

Moreover, if $\|\rho - \rho^\|_2 = o(1)$, then*

$$\mathbb{E}[Z_{k, \rho}(\mathcal{G}(n, m))] \sim c_n(d, k) \exp[d/2 + ng(\rho)]. \tag{3.3}$$

(2) *Assume that $\omega = \omega(n) \rightarrow \infty$. Then*

$$\mathbb{E}[Z_{k, \omega}(\mathcal{G}(n, m))] \sim |\mathcal{B}_{n, k}(\omega)| c_n(d, k) \exp[d/2 + n\alpha(d, k)]. \tag{3.4}$$

Proof. By Stirling’s formula and the independence of the edges in the random graph $\mathcal{G}(n, m)$,

$$\mathbb{E}[Z_{k,\rho}(\mathcal{G}(n, m))] = \binom{n}{\rho_1 n, \dots, \rho_k n} \left(1 - \frac{1}{N} \sum_{i=1}^k \binom{\rho_i n}{2}\right)^m, \quad \text{where } N = \binom{n}{2}. \tag{3.5}$$

Further,

$$\sum_{i=1}^k \binom{\rho_i n}{2} = N \left(\sum_{i=1}^k \rho_i^2\right) + \frac{n}{2} \left(\sum_{i=1}^k \rho_i^2 - 1\right) + O(1).$$

Consequently,

$$\begin{aligned} m \ln \left(1 - \frac{1}{N} \sum_{i=1}^k \binom{\rho_i n}{2}\right) &= m \ln \left[\left(1 + \frac{n}{2N}\right) \left(1 - \sum_{i=1}^k \rho_i^2\right)\right] + o(1) \\ &= n \frac{d}{2} \ln \left(1 - \sum_{i=1}^k \rho_i^2\right) + \frac{d}{2} + o(1). \end{aligned} \tag{3.6}$$

Equation (3.2) follows from (3.5), (3.6) and Stirling’s formula. Moreover, (3.3) follows from (3.5) and (3.6) because $\|\rho - \rho^*\|_2 = o(1)$ implies that $\sum_{i=1}^k \rho_i^2 \sim 1/k$ and

$$\binom{n}{\rho_1 n, \dots, \rho_k n} \sim (2\pi n)^{(1-k)/2} k^{k/2} \exp[nH(\rho)].$$

To obtain (3.4), we observe that if $\rho \in \mathcal{B}_{n,k}(\omega)$, then from the definition of the set we have that $\|\rho - \rho^*\|_2 = \sqrt{\sum_{i \in [k]} |\rho_i - \rho_i^*|^2} = o(n^{-1/2})$. Further, by Taylor expansion we obtain

$$H(\rho) = \ln k + O\left(\sum_{i=1}^k \left(\rho_i - \frac{1}{k}\right)^2\right) = \ln k + o(n^{-1}), \tag{3.7}$$

$$\ln \left(1 - \sum_{i=1}^k \rho_i^2\right) = \ln \left(1 - \frac{1}{k}\right) + O\left(\sum_{i=1}^k \left(\rho_i - \frac{1}{k}\right)^2\right) = \ln \left(1 - \frac{1}{k}\right) + o(n^{-1}). \tag{3.8}$$

Thus, (3.4) follows from (3.3), (3.7) and (3.8). □

Corollary 3.2. *With the expressions from (3.1), for any $k \geq 3$, $\bar{d} \in (0, \infty)$*

$$\mathbb{E}[Z_k(\mathcal{G}(n, m))] \sim \exp[d/2 + n\alpha(d, k)] \left(1 + \frac{d}{k-1}\right)^{-(k-1)/2},$$

where $d = 2m/n$.

Proof. The functions $\rho \in \bar{\mathcal{C}}_k \mapsto H(\rho)$ and $\rho \in \bar{\mathcal{C}}_k \mapsto (d/2) \ln(1 - \sum_{i=1}^k \rho_i^2)$ are both concave and attain their maximum at $\rho = \rho^*$. Consequently, setting $B(d, k) = k(1 + d/(k - 1))$ and expanding around $\rho = \rho^*$, we obtain

$$\alpha(d, k) - \frac{B(d, k)}{2} \|\rho - \rho^*\|_2^2 - O(\|\rho - \rho^*\|_2^3) \leq g(\rho) \leq \alpha(d, k) - \frac{B(d, k)}{2} \|\rho - \rho^*\|_2^2. \tag{3.9}$$

Plugging the upper bound from (3.9) into (3.2) and observing that $|C_k(n)| \leq n^k = \exp(o(n))$, we find

$$S_1 = \sum_{\substack{\rho \in C_k(n) \\ \|\rho - \rho^*\|_2 > n^{-5/12}}} \mathbb{E}[Z_{k,\rho}(\mathcal{G}(n, m))] \leq C_2 \exp[n\alpha(d, k)] \exp\left[-\frac{B(d, k)}{2} n^{1/6}\right]. \tag{3.10}$$

On the other hand, (3.3) implies that

$$\begin{aligned} S_2 &= \sum_{\substack{\rho \in C_k(n) \\ \|\rho - \rho^*\|_2 \leq n^{-5/12}}} \mathbb{E}[Z_{k,\rho}(\mathcal{G}(n, m))] \sim \sum_{\substack{\rho \in C_k(n) \\ \|\rho - \rho^*\|_2 \leq n^{-5/12}}} c_n(d, k) \exp(d/2) \exp[ng(\rho)] \\ &\sim c_n(d, k) \exp[d/2 + n\alpha(d, k)] \sum_{\rho \in C_k(n)} \exp\left[-n\frac{B(d, k)}{2} \|\rho - \rho^*\|_2^2\right]. \end{aligned} \tag{3.11}$$

The last (asymptotic) equality follows by observing that the contribution of the summands for ρ such that $\|\rho - \rho^*\|_2 > n^{-5/12}$ is of smaller order of magnitude than the whole sum.

Moreover, the last sum in (3.11) is almost in the standard form of a Gaussian summation, except that the vectors $\rho \in C_k(n)$ that we sum over are subject to the linear constraint $\rho_1 + \dots + \rho_k = 1$. We rid ourselves of this constraint by substituting $\rho_k = 1 - \rho_1 - \dots - \rho_{k-1}$. Formally, let J be the $(k-1) \times (k-1)$ -matrix whose diagonal entries are equal to 2 and whose remaining entries are 1. Then

$$\begin{aligned} \sum_{\rho \in C_k(n)} \exp\left[-n\frac{B(d, k)}{2} \|\rho - \rho^*\|_2^2\right] &\sim \sum_{y \in \frac{1}{n}\mathbb{Z}^{k-1}} \exp\left[-n\frac{B(d, k)}{2} \langle Jy, y \rangle\right] \\ &\sim (2\pi n)^{(k-1)/2} k^{-k/2} \left(1 + \frac{d}{k-1}\right)^{-(k-1)/2} \quad (\text{as } \det J = k). \end{aligned} \tag{3.12}$$

Plugging (3.12) into (3.11), we obtain

$$\begin{aligned} S_2 &\sim c_n(d, k) \exp[d/2 + n\alpha(d, k)] (2\pi n)^{(k-1)/2} k^{-k/2} \left(1 + \frac{d}{k-1}\right)^{-(k-1)/2} \\ &= \exp[d/2 + n\alpha(d, k)] \left(1 + \frac{d}{k-1}\right)^{-(k-1)/2} \quad (\text{using (3.1)}). \end{aligned} \tag{3.13}$$

Finally, comparing (3.10) and (3.13), we see that $S_1 = o(S_2)$. Thus,

$$\mathbb{E}[Z_k(\mathcal{G}(n, m))] = S_1 + S_2 \sim S_2,$$

and the assertion follows from (3.13). □

Proof of Proposition 2.1. The first assertion is immediate from Corollary 3.2. Moreover, the second assertion follows from Corollary 3.2 and the second part of Lemma 3.1. □

4. Counting short cycles

Throughout this section, we let x_2, \dots, x_L denote a sequence of non-negative integers. Moreover, let S be the event that in $\mathcal{G}(n, m)$ we have $C_{\ell, n} = x_\ell$ for $\ell = 2, \dots, L$. Further, let $\mathcal{V}(\sigma)$ be the event that σ is a k -colouring of the random graph $\mathcal{G}(n, m)$. We also recall $\lambda_\ell, \delta_\ell$ from (2.1).

4.1. Proof of Proposition 2.3

The key ingredient in the proof is the following lemma concerning the distribution of the random variables $C_{\ell,n}$ given $\mathcal{V}(\sigma)$.

Lemma 4.1. *Let*

$$\mu_\ell = \frac{d^\ell}{2\ell} \left[1 + \frac{(-1)^\ell}{(k-1)^{\ell-1}} \right].$$

Then

$$\mathbb{P}[S \mid \mathcal{V}(\sigma)] \sim \prod_{\ell=2}^L \frac{\exp(-\mu_\ell)}{x_\ell!} \mu_\ell^{x_\ell} \quad \text{for any } \sigma \in \mathcal{B}_{n,k}(\omega),$$

where $\omega = \omega(n)$ grows with n arbitrarily slow.

Before we establish Lemma 4.1, let us point out how it implies Proposition 2.3. By Bayes' rule,

$$\begin{aligned} \mathbb{E}[Z_{k,\omega}(\mathcal{G}(n,m)) \mid S] &= \frac{1}{\mathbb{P}[S]} \sum_{\tau \in \mathcal{B}_{n,k}(\omega)} \mathbb{P}[\mathcal{V}(\tau)] \mathbb{P}[S \mid \mathcal{V}(\tau)] \\ &\sim \frac{\prod_{l=2}^L \frac{\exp(-\mu_l)}{x_l!} \mu_l^{x_l}}{\mathbb{P}[S]} \sum_{\tau \in [k]^n: \tau \in \mathcal{B}_{n,k}(\omega)} \mathbb{P}[\mathcal{V}(\tau)] \quad (\text{from Lemma 4.1}) \\ &\sim \frac{\prod_{l=2}^L \frac{\exp(-\mu_l)}{x_l!} \mu_l^{x_l}}{\mathbb{P}[S]} \mathbb{E}[Z_{k,\omega}(\mathcal{G}(n,m))]. \end{aligned}$$

From Lemma 4.1 and Fact 2.2 we get that

$$\frac{\prod_{\ell=2}^L \frac{\exp(-\mu_\ell)}{x_\ell!} \mu_\ell^{x_\ell}}{\mathbb{P}[S]} \sim \prod_{\ell=2}^L \frac{\exp(-\mu_\ell)}{\exp(-\lambda_\ell)} \frac{\mu_\ell^{x_\ell}}{\lambda_\ell^{x_\ell}} \sim \prod_{\ell=2}^L [1 + \delta_\ell]^{x_\ell} \exp(-\delta_\ell \lambda_\ell),$$

whence Proposition 2.3 follows. □

4.2. Proof of Lemma 4.1

We are going to show that for any fixed sequence of integers $m_2, \dots, m_L \geq 0$, the joint factorial moments satisfy

$$\mathbb{E}[(C_{2,n})_{m_2} \cdots (C_{L,n})_{m_L} \mid \mathcal{V}(\sigma)] \sim \prod_{\ell=2}^L \mu_\ell^{m_\ell}. \tag{4.1}$$

Then Lemma 4.1 follows from [10, Theorem 1.23].

We consider the number of sequences of $m_2 + \dots + m_L$ distinct cycles such that m_2 corresponds to the number of cycles of length 2, and so on. Clearly this is equal to $(C_{2,n})_{m_2} \cdots (C_{L,n})_{m_L}$. Let Y be the number of those sequences of cycles such that any two cycles are vertex-disjoint. Also, let

Y' denote the number of sequences which have intersecting cycles. Clearly it holds that

$$\mathbb{E}[(C_{2,n})_{m_2} \cdots (C_{L,n})_{m_L} \mid \mathcal{V}(\sigma)] = \mathbb{E}[Y \mid \mathcal{V}(\sigma)] + \mathbb{E}[Y' \mid \mathcal{V}(\sigma)]. \tag{4.2}$$

For $\mathbb{E}[Y' \mid \mathcal{V}(\sigma)]$ we use the following claim, whose proof follows below.

Claim 4.2. *It holds that $\mathbb{E}[Y' \mid \mathcal{V}(\sigma)] = O(n^{-1})$.*

Hence, we need to count vertex-disjoint cycles given $\mathcal{V}(\sigma)$. For this, we adapt the argument for random regular graphs from [21, Section 2]. We count rooted directed cycles, first. This introduces a factor of 2ℓ for the number of cycles of length ℓ . That is, if D_ℓ is the number of rooted, directed cycles of length ℓ , then $D_\ell = 2\ell C_\ell$.

For a rooted directed cycle (v_1, \dots, v_ℓ) of length ℓ , we call $(\sigma(v_1), \dots, \sigma(v_\ell))$ the *type* of the cycle under σ . For $t = (a_1, \dots, a_\ell)$ let $D_{\ell,t}$ denote the number of rooted, directed cycles (of length ℓ and) type t . We claim that

$$\mathbb{E}[D_{\ell,t} \mid \mathcal{V}(\sigma)] \sim \left(\frac{n}{k}\right)^\ell \frac{\binom{m}{\ell}}{N^\ell (1 - \mathcal{F}(\sigma)/N)^\ell} \sim \left(\frac{d}{k-1}\right)^\ell \quad \text{with } N = \binom{n}{2}. \tag{4.3}$$

Indeed, since σ is (ω, n) -balanced, the number of ways of choosing a vertex of colour t_i is $(1 + o(1))n/k$, and we have got to choose ℓ vertices in total. Thus, the total number of ways of choosing ℓ vertices (v_1, \dots, v_ℓ) such that $\sigma(v_i) = t_i$ for all i is $(1 + o(1))(n/k)^\ell$. In addition, each edge $\{v_i, v_{i+1}\}$ of the cycle is present in the graph with a probability asymptotically equal to $m/(N - \mathcal{F}(\sigma))$. This explains the first asymptotic equality in (4.3). The second one follows because $m = dn/2$ and $\mathcal{F}(\sigma) \sim 1/kN$ (as $\sigma \in \mathcal{B}_{n,k}(\omega)$).

Note that, the right-hand side of (4.3) is independent of the type t . For a given ℓ , let T_ℓ signify the number of all possible types of cycles of length ℓ . T_ℓ is equal to the number of all sequences (t_1, \dots, t_ℓ) such that $t_{i+1} \neq t_i$ for all $1 \leq i < \ell$ and $t_\ell \neq t_1$. Let $T_1 = 0$. Then T_ℓ satisfies the recurrence $T_\ell + T_{\ell-1} = k(k-1)^{\ell-1}$ (see [21, Section 2]).⁴ Hence, $T_\ell = (k-1)^\ell + (-1)^\ell(k-1)$. Combining this formula with (4.3) and the fact that $\bar{d} \sim d$, we obtain

$$\mathbb{E}[D_\ell \mid \mathcal{V}(\sigma)] \sim T_\ell \cdot \mathbb{E}[D_{\ell,t} \mid \mathcal{V}(\sigma)] \sim \left(1 + \frac{(-1)^\ell}{(k-1)^{\ell-1}}\right) \bar{d}^\ell.$$

Hence, recalling that $C_\ell = \frac{1}{2\ell}D_\ell$, we get

$$\mathbb{E}[C_\ell \mid \mathcal{V}(\sigma)] \sim \frac{\bar{d}^\ell}{2\ell} \left[1 + \frac{(-1)^\ell}{(k-1)^{\ell-1}}\right]. \tag{4.4}$$

In fact, since Y considers only vertex-disjoint cycles and L, m_2, \dots, m_L remain fixed as $n \rightarrow \infty$, (4.4) yields

$$\mathbb{E}[Y \mid \mathcal{V}(\sigma)] \sim \prod_{\ell=2}^L \left(\frac{\bar{d}^\ell}{2\ell} \left[1 + \frac{(-1)^\ell}{(k-1)^{\ell-1}}\right]\right)^{m_\ell}.$$

Plugging the above relation and Claim 4.2 into (4.2) we get (4.1). The proposition follows. □

⁴ To see this, observe that $k(k-1)^\ell$ is the number of all sequences (t_1, \dots, t_ℓ) such that $t_{i+1} \neq t_i$ for all $1 \leq i < \ell$. Any such sequence either satisfies $t_\ell \neq t_1$, which is accounted for by T_ℓ , or $t_\ell = t_1$ and $t_{\ell-1} \neq t_1$, in which case it is contained in $T_{\ell-1}$.

Proof of Claim 4.2. For m_2, m_3, \dots, m_L , let $t = \sum_{j=1}^L m_j$. Note that $(C_{2,n})_{m_2} \cdots (C_{L,n})_{m_L}$ is the number of t -tuples of cycles where the first m_2 entries consist of distinct cycles of length 2, the next m_3 entries consist of distinct cycles of length 3 and so on. Y' counts the number of these tuples under the condition that there is at least one pair of cycles which intersect.

For every set of vertices Λ , let 1_Λ be equal to 1 if the number of edges with both ends in Λ is at least $|\Lambda| + 1$. Since Y' counts tuples of cycles which intersect with each other, it is easy to see that the following holds:

$$\mathbb{E}[Y' \mid \mathcal{V}(\sigma)] \leq \mathbb{E} \left[\sum_{j=1}^R \sum_{\Lambda: |\Lambda|=j} 1_\Lambda \mid \mathcal{V}(\sigma) \right], \quad \text{where } R = \left(\sum_{i=2}^L i \cdot m_i \right) - 1.$$

Note that R is independent of n .

For any set Λ such that $|\Lambda| = \ell$, we can put $\ell + 1$ edges inside the set in at most $\binom{\ell}{\ell+1}$ ways. Clearly conditioning on $\mathcal{V}(\sigma)$ can only reduce the number of different placings of the edges. Using inclusion/exclusion, for a fixed set Λ of cardinality ℓ we get that

$$\begin{aligned} \mathbb{E}[1_\Lambda \mid \mathcal{V}(\sigma)] &\leq \binom{\ell}{\ell+1} \sum_{i=0}^{\ell+1} \binom{\ell+1}{i} (-1)^i \left(1 - \frac{i}{N - \mathcal{F}(\sigma)} \right)^m \\ &\leq \binom{\ell}{\ell+1} \left(\frac{m}{N - \mathcal{F}(\sigma)} \right)^{\ell+1} \quad (\text{from the Binomial theorem}) \\ &\sim \binom{\ell}{\ell+1} \left(\frac{d}{n(1-1/k)} \right)^{\ell+1} \quad (\text{since } m = dn/2, \text{ and } \mathcal{F}(\sigma) \sim \frac{1}{k}N). \end{aligned}$$

Therefore,

$$\begin{aligned} &\mathbb{E} \left[\sum_{j=1}^R \sum_{\Lambda: |\Lambda|=j} 1_\Lambda \mid \mathcal{V}(\sigma) \right] \\ &\leq (1 + o(1)) \sum_{\ell=1}^R \binom{n}{\ell} \binom{\ell}{\ell+1} \left(\frac{d}{n(1-1/k)} \right)^{\ell+1} \\ &\leq (1 + o(1)) \sum_{\ell=1}^R \left(\frac{ne}{\ell} \right)^\ell \left(\frac{\ell e}{2} \right)^{\ell+1} \left(\frac{d}{n(1-1/k)} \right)^{\ell+1} \quad (\text{since } \binom{i}{j} \leq (ie/j)^j) \\ &\leq \frac{1 + o(1)}{n} \sum_{\ell=1}^R \frac{\ell e d}{2(1-1/k)} \left(\frac{e^2 d}{2(1-1/k)} \right)^\ell = O(n^{-1}), \end{aligned}$$

where the last equality holds since R is a fixed number. The claim follows. □

5. The second moment computation

In this section we prove the second moment bounds claimed in Propositions 2.4 and 2.5, which constitute the main technical contribution of this work. While here we need an asymptotically tight expression for the second moment, in prior work on colouring $G(n, m)$ the second moment was merely computed up to a constant factor [5, 8, 15]. Only in the case of random regular graphs was the second moment computed up to a factor of $1 + o(1)$ [21]. In addition, all of these papers confine themselves to the case of colourings whose colour densities are $(O(1), n)$ -

balanced, whereas here we need to deal with (ω, n) -balanced colour densities for a diverging function $\omega = \omega(n) \rightarrow \infty$.

Thus, the plan is to extend the arguments from [5, 8, 15] to get a precise asymptotic result, and to cover the (ω, n) -balanced case. Unsurprisingly, in the course of this we will frequently encounter formulas that resemble those of [5, 8, 15], and occasionally we will be able to re-use some of the calculations done in those papers. Furthermore, to determine the precise constant we can harness a bit of linear algebra from [21]. Throughout this section $\omega = \omega(n)$ stands for a function that tends to ∞ (slowly).

5.1. The overlap

Following [5], for $\sigma, \tau : [n] \rightarrow [k]$ we define the *overlap matrix* $\rho(\sigma, \tau) = (\rho_{ij}(\sigma, \tau))_{i,j \in [k]}$ as the $(k \times k)$ -matrix with entries

$$\rho_{ij}(\sigma, \tau) = \frac{1}{n} \cdot |\sigma^{-1}(i) \cap \tau^{-1}(j)|.$$

Moreover, for a $(k \times k)$ -matrix $\rho = (\rho_{ij})$ we introduce the shorthand notation

$$\rho_{i*} = \sum_{j=1}^k \rho_{ij}, \quad \rho_{*i} = (\rho_{i*})_{i \in [k]}, \quad \rho_{*j} = \sum_{i=1}^k \rho_{ij}, \quad \rho_{*i} = (\rho_{*i})_{i \in [k]}.$$

Thus, for any $\sigma, \tau : [n] \rightarrow [k]$ we have $\rho_{*i}, \rho_{*i} \in \mathcal{C}_k(n)$.

Let $\bar{\mathcal{R}}_k$ denote the set of all probability measures $\rho = (\rho_{ij})_{i,j \in [k]}$ on $[k] \times [k]$ and let $\bar{\rho}$ signify the $(k \times k)$ -matrix with all entries equal to k^{-2} , the barycentre of $\bar{\mathcal{R}}_k$. Further, we introduce

$$\begin{aligned} \mathcal{R}_{n,k} &= \{\rho(\sigma, \tau) : \sigma, \tau : [n] \rightarrow [k]\}, \\ \mathcal{R}_{n,k}^{\text{int}} &= \{\rho \in \mathcal{R}_{n,k} : \rho_{ij} > 1/k^3 \text{ for all } i, j \in [k]\}, \\ \mathcal{R}_{n,k}^{\text{bal}}(\omega) &= \{\rho \in \mathcal{R}_{n,k}^{\text{int}} : |\rho_{i*} - k^{-1}| \leq \omega^{-1} n^{-1/2}, |\rho_{*i} - k^{-1}| \leq \omega^{-1} n^{-1/2} \text{ for all } i \in [k]\}, \\ \mathcal{R}_{n,k}^{\text{bal}}(\omega, \eta) &= \{\rho \in \mathcal{R}_{n,k}^{\text{bal}}(\omega) : \|\rho - \bar{\rho}\|_2 \leq \eta\} \quad (\eta > 0). \end{aligned}$$

For a given graph G on $[n]$, let $Z_{k,\rho}^{(2)}(G)$ be the number of pairs (σ, τ) of k -colourings of G whose overlap is ρ . Then, by the linearity of expectation,

$$\mathbb{E}[Z_{k,\omega}(\mathcal{G}(n, m))^2] = \sum_{\rho \in \mathcal{R}_{n,k}^{\text{bal}}(\omega)} \mathbb{E}[Z_{k,\rho}^{(2)}(\mathcal{G}(n, m))]. \tag{5.1}$$

We are going to show that the right-hand side of (5.1) is dominated by the contributions with ρ ‘close to’ $\bar{\rho}$. More precisely, let

$$Z_{k,\omega,\eta}^{(2)}(G) = \sum_{\rho \in \mathcal{R}_{n,k}^{\text{bal}}(\omega,\eta)} Z_{k,\rho}^{(2)}(G) \quad \text{for any } \eta > 0.$$

Then the second moment argument performed in [5] fairly directly yields the following statement.

Proposition 5.1. *Assume that $k \geq 3$ and that $\bar{d} < 2(k - 1) \ln(k - 1)$. Then for any fixed $\eta > 0$ it holds that*

$$\mathbb{E}[Z_{k,\omega}(\mathcal{G}(n, m))^2] \sim \mathbb{E}[Z_{k,\omega,\eta}^{(2)}(\mathcal{G}(n, m))].$$

In addition, the second moment argument from [15] implies the following result.

Proposition 5.2. *There is a constant $k_0 > 3$ such that for $k \geq k_0$ and $2(k - 1) \ln(k - 1) \leq \bar{d} < d_{k,\text{cond}}$ the following is true. There exists an integer-valued random variable $0 \leq \tilde{Z}_{k,\omega} \leq Z_{k,\omega}$ that satisfies*

$$\mathbb{E}[\tilde{Z}_{k,\omega}(\mathcal{G}(n, m))] \sim \mathbb{E}[Z_{k,\omega}(\mathcal{G}(n, m))]$$

and such that for any fixed $\eta > 0$ we have

$$\mathbb{E}[\tilde{Z}_{k,\omega}(\mathcal{G}(n, m))^2] \leq (1 + o(1))\mathbb{E}[Z_{k,\omega,\eta}^{(2)}(\mathcal{G}(n, m))].$$

Since the above statements do not quite appear in this form in [5, 15], we will prove them in Sections 5.4 and 5.5, respectively.

5.2. Homing in on $\bar{\rho}$

Having reduced our task to studying overlaps ρ such that $\|\rho - \bar{\rho}\|_2 \leq \eta$ for a small but fixed $\eta > 0$, in this section we are going to argue that, in fact, it suffices to consider ρ such that $\|\rho - \bar{\rho}\|_2 \leq n^{-5/12}$ (where the constant 5/12 is somewhat arbitrary; any number smaller than 1/2 would do). More precisely, we have the following result.

Proposition 5.3. *Assume that $k \geq 3$ and that $\bar{d} < d_{k,\text{cond}}$. There exists a number $\eta_0 = \eta_0(\bar{d}, k)$ such that for any $0 < \eta < \eta_0$ we have*

$$\mathbb{E}[Z_{k,\omega,\eta}^{(2)}(\mathcal{G}(n, m))] \sim \mathbb{E}[Z_{k,\omega,n^{-5/12}}^{(2)}(\mathcal{G}(n, m))].$$

In order to prove Proposition 5.3, we first need the following elementary estimates.

Fact 5.4. *For any $k \geq 3$, $\bar{d} \in (0, \infty)$ and $d = 2m/n$, the following estimates are true.*

(1) *Let $\rho \in \mathcal{R}_{n,k}^{\text{int}}$. Then*

$$\begin{aligned} \mathbb{E}[Z_{k,\rho}^{(2)}(\mathcal{G}(n, m))] &\sim \frac{\sqrt{2\pi}n^{(1-k^2)/2}}{\prod_{i,j=1}^k \sqrt{2\pi\rho_{ij}}} \exp[d/2 + nH(\rho)] \\ &\quad + m \ln(1 - \|\rho_{\cdot\star}\|_2^2 - \|\rho_{\star\cdot}\|_2^2 + \|\rho\|_2^2) \end{aligned} \tag{5.2}$$

(2) *For any $\rho \in \mathcal{R}_{n,k}^{\text{bal}}(\omega)$ we have*

$$\mathbb{E}[Z_{k,\rho}^{(2)}(\mathcal{G}(n, m))] \sim \frac{\sqrt{2\pi}n^{(1-k^2)/2}}{\prod_{i,j=1}^k \sqrt{2\pi\rho_{ij}}} \exp[d/2 + nH(\rho) + m \ln(1 - 2/k + \|\rho\|_2^2)]. \tag{5.3}$$

Proof. By Stirling’s formula, the total number of σ, τ with overlap $\rho \in \mathcal{R}_{n,k}^{\text{int}}$ is given by

$$\binom{n}{\rho_{11}n, \dots, \rho_{kk}n} \sim \sqrt{2\pi}n^{-(k^2-1)/2} \left(\prod_{i,j} \frac{1}{\sqrt{2\pi\rho_{ij}}} \right) \exp[nH(\rho)]. \tag{5.4}$$

To obtain $\mathbb{E}[Z_{k,\rho}^{(2)}(\mathcal{G}(n,m))]$, we need to multiply this number by the probability that two maps σ, τ with overlap ρ are both colourings of a randomly chosen graph. The number of ‘forbidden’ edges joining two vertices with the same colour under either σ or τ is given by

$$\begin{aligned} \mathcal{F}(\sigma, \tau) &= \sum_{i=1}^k \binom{\rho_{i*}n}{2} + \sum_{j=1}^k \binom{\rho_{*j}n}{2} - \sum_{i,j=1}^k \binom{\rho_{ij}n}{2} \\ &= N \left(\sum_{i=1}^k \rho_{i*}^2 + \sum_{j=1}^k \rho_{*j}^2 - \sum_{i,j=1}^k \rho_{ij}^2 \right) + \frac{n}{2} \left(\sum_{i=1}^k \rho_{i*}^2 + \sum_{j=1}^k \rho_{*j}^2 - \sum_{i,j=1}^k \rho_{ij}^2 - 1 \right) + O(1). \end{aligned}$$

Therefore, the probability that σ and τ are both colourings of $\mathcal{G}(n,m)$ depends only on their overlap ρ , and is

$$\begin{aligned} \mathbb{P}[\sigma, \tau \text{ are } k\text{-colourings of } \mathcal{G}(n,m)] &= \frac{(N - \mathcal{F}(\sigma, \tau))^m}{N^m} \tag{5.5} \\ &\sim \exp \left[m \ln \left(1 - \sum_{i=1}^k \rho_{i*}^2 - \sum_{j=1}^k \rho_{*j}^2 + \sum_{i,j=1}^k \rho_{ij}^2 \right) + \frac{d}{2} \right]. \end{aligned}$$

Equation (5.2) is obtained by multiplying (5.5) with (5.4).

To prove the second claim, let $\varepsilon_i = \rho_{i*} - 1/k$ for $i \in [k]$. Because $\sum_{i,j=1}^k \rho_{ij} = 1$ we have $\sum_{i=1}^k \varepsilon_i = 0$. Consequently,

$$\|\rho_{* \cdot}\|_2^2 = \frac{1}{k} + \sum_{i=1}^k \varepsilon_i^2. \tag{5.6}$$

Further, if ρ is (ω, n) -balanced, then $\varepsilon_i = o(n^{-1/2})$ for all $i \in [k]$. Hence, (5.6) yields $\|\rho_{* \cdot}\|_2^2 = 1/k + o(n^{-1})$. Similarly, $\|\rho_{\cdot *}\|_2^2 = 1/k + o(n^{-1})$. Therefore, for any (ω, n) -balanced ρ ,

$$\exp(m \cdot \ln(1 - \|\rho_{* \cdot}\|_2^2 - \|\rho_{\cdot *}\|_2^2 + \|\rho\|_2^2)) \sim \exp \left(m \cdot \ln \left(1 - \frac{2}{k} + \|\rho\|_2^2 \right) \right).$$

Plugging the above into (5.2) completes the proof. □

To evaluate the exponential part in equation (5.3), we require the following lemma.

Lemma 5.5. *Let $k \geq 3$, $\bar{d} < (k - 1)^2$ and $d = 2m/n$. Let $\alpha(d, k)$ be as in (3.1) and set*

$$C_n(d, k) = \exp(d/2)k^{k^2} (2\pi n)^{(1-k^2)/2}, \quad D(d, k) = k^2 \left(1 - \frac{d}{(k-1)^2} \right).$$

- If $\rho \in \mathcal{R}_{n,k}^{\text{bal}}(\omega)$ satisfies $\|\rho - \bar{\rho}\|_2 \leq n^{-5/12}$, then

$$\mathbb{E}[Z_{k,\rho}^{(2)}(\mathcal{G}(n,m))] \sim C_n(d, k) \exp \left[2n\alpha(d, k) - n \frac{D(d, k)}{2} \|\rho - \bar{\rho}\|_2^2 \right]. \tag{5.7}$$

- There exist numbers $\eta = \eta(d, k) > 0$ and $A = A(d, k) > 0$ such that if $\rho \in \mathcal{R}_{n,k}^{\text{bal}}(\omega)$ satisfies $\|\rho - \bar{\rho}\|_2 \in (n^{-5/12}, \eta)$, then

$$\mathbb{E}[Z_{k,\rho}^{(2)}(\mathcal{G}(n,m))] \leq \exp[2n\alpha(d, k) - An^{1/6}]. \tag{5.8}$$

Proof. Following [5], we consider

$$f : \overline{\mathcal{R}}_k \rightarrow \mathbb{R}, \quad \rho \mapsto H(\rho) + \frac{d}{2} \ln \left(1 - \frac{2}{k} + \sum_{i,j=1}^k \rho_{ij}^2 \right). \tag{5.9}$$

Then Fact 5.4 yields

$$\mathbb{E}[Z_{k,\rho}^{(2)}(\mathcal{G}(n, m))] \sim C_n(d, k) \exp[nf(\rho)].$$

The function f satisfies $f(\bar{\rho}) = 2\alpha(d, k)$. Further, expanding f around $\bar{\rho}$ by writing $\varepsilon = \rho - \bar{\rho}$ (so that $\sum_{i,j=1}^k \varepsilon_{ij} = 0$) gives

$$\begin{aligned} f(\rho) &= H(\bar{\rho}) - \frac{k^2}{2} \sum_{i,j=1}^k \varepsilon_{ij}^2 + O(\|\varepsilon\|_2^3) + \frac{d}{2} \ln \left(1 - \frac{2}{k} + \frac{1}{k^2} + \sum_{i,j=1}^k \varepsilon_{ij}^2 \right) \\ &= f(\bar{\rho}) - \frac{D(d, k)}{2} \|\varepsilon\|_2^2 + O(\|\varepsilon\|_2^3). \end{aligned} \tag{5.10}$$

Consequently, for $\|\rho - \bar{\rho}\|_2 \leq n^{-5/12}$,

$$\exp[nf(\rho)] = \exp \left[nf(\bar{\rho}) - n \frac{D(d, k)}{2} \|\rho - \bar{\rho}\|_2^2 + O(n^{-1/4}) \right],$$

whence (5.7) follows.

We now prove equation (5.8). Similarly to (5.10) and because f is smooth in a neighbourhood of $\bar{\rho}$, there exist $\eta > 0$ and $A > 0$ such that for $\|\rho - \bar{\rho}\|_2 \leq \eta$,

$$f(\rho) \leq f(\bar{\rho}) - A \|\rho - \bar{\rho}\|_2^2.$$

Hence, if $\|\rho - \bar{\rho}\|_2 \in (n^{-5/12}, \eta)$, then

$$\mathbb{E}[Z_{k,\rho}^{(2)}(\mathcal{G}(n, m))] = O(n^{(1-k^2)/2}) \exp[nf(\rho)] \leq \exp[2n\alpha(d, k) - An^{1/6}],$$

as claimed. □

Proof of Proposition 5.3. We fix $\eta > 0$ and $A > 0$ as given by Lemma 5.5. Fixing $\rho_0 \in \mathcal{R}_{n,k}^{\text{bal}}(\omega, \eta)$ such that $\|\rho_0 - \bar{\rho}\|_2 \leq k/n$, we obtain from the first part of Lemma 5.5 that

$$\mathbb{E}[Z_{k,\omega,n^{-5/12}}^{(2)}(\mathcal{G}(n, m))] \geq \mathbb{E}[Z_{k,\rho_0}^{(2)}(\mathcal{G}(n, m))] \sim C_n(d, k) \exp[2n\alpha(d, k)]. \tag{5.11}$$

On the other hand, because $|\mathcal{R}_{n,k}^{\text{bal}}(\omega, \eta)|$ is bounded by a polynomial in n , the second part of Lemma 5.5 yields

$$\sum_{\substack{\rho \in \mathcal{R}_{n,k}^{\text{bal}}(\omega, \eta) \\ \|\rho - \bar{\rho}\|_2 > n^{-5/12}}} \mathbb{E}[Z_{k,\rho}^{(2)}(\mathcal{G}(n, m))] \leq \exp[2n\alpha(d, k) - An^{1/6} + O(\ln n)]. \tag{5.12}$$

Combining (5.11) and (5.12), we obtain

$$\mathbb{E}[Z_{k,\omega,\eta}^{(2)}(\mathcal{G}(n, m))] \sim \sum_{\rho \in \mathcal{R}_{n,k}^{\text{bal}}(\omega, n^{-5/12})} \mathbb{E}[Z_{k,\rho}^{(2)}(\mathcal{G}(n, m))] \sim \mathbb{E}[Z_{k,\omega,n^{-5/12}}^{(2)}(\mathcal{G}(n, m))],$$

as claimed. □

5.3. The leading constant

Here we compute the contribution of overlap matrices $\rho \in \mathcal{R}_{n,k}^{\text{bal}}(\omega, n^{-5/12})$.

Proposition 5.6. *Assume that $k \geq 3$, $\bar{d} < (k - 1)^2$, while $d = 2m/n$. Then with $c_n(d, k)$ from (3.1),*

$$\mathbb{E}[Z_{k,\omega,n^{-5/12}}^{(2)}(\mathcal{G}(n, m))] \sim (|\mathcal{B}_{n,k}(\omega)|c_n(d, k) \exp[n\alpha(d, k)])^2 \exp(d/2) \left(1 - \frac{d}{(k - 1)^2}\right)^{-((k-1)^2)/2}.$$

In order to prove the proposition, we will need the following lemma regarding Gaussian summations over matrices with coefficients in $\frac{1}{n}\mathbb{Z}$ whose lines and columns sums to zero. Thus, we let

$$\mathcal{S}_n = \left\{ (\varepsilon_{i,j})_{\substack{1 \leq i \leq k, \\ 1 \leq j \leq k}}, \forall i, j \in [k], \varepsilon_{i,j} \in \frac{1}{n}\mathbb{Z}, \forall j \in [k], \sum_{i=1}^k \varepsilon_{ij} = \sum_{i=1}^k \varepsilon_{ji} = 0 \right\}. \tag{5.13}$$

Lemma 5.7. *Let $k \geq 2$, $d < (k - 1)^2$ and $D > 0$ be fixed. Then*

$$\sum_{\varepsilon \in \mathcal{S}_n} \exp \left[-n \frac{D}{2} \|\varepsilon\|_2^2 + o(n^{1/2}) \|\varepsilon\|_2 \right] \sim (\sqrt{2\pi n})^{(k-1)^2} D^{-\frac{(k-1)^2}{2}} k^{-(k-1)}. \tag{5.14}$$

Lemma 5.7 and its proof are very similar to an argument used in [21, Section 3]. In fact, Lemma 5.7 follows from the next lemma, which is a restatement of Lemmas 6(b) and 7(c) of [21].

Lemma 5.8 ([21]). *There is a $(k - 1)^2 \times (k - 1)^2$ -matrix $\mathcal{H} = (\mathcal{H}_{(i,j),(i',j')})_{i,j,i',j' \in [k-1]}$ such that for any $\varepsilon = (\varepsilon_{ij})_{i,j \in [k]} \in \mathcal{S}_n$ we have*

$$\sum_{i,j,i',j' \in [k-1]} \mathcal{H}_{(i,j),(i',j')} \varepsilon_{ij} \varepsilon_{i'j'} = \|\varepsilon\|_2^2.$$

This matrix \mathcal{H} is positive definite and $\det \mathcal{H} = k^{2(k-1)}$. □

Proof of Lemma 5.7. Together with the Euler–Maclaurin formula and Lemma 5.8, a Gaussian integration yields

$$\begin{aligned} & \sum_{\varepsilon \in \mathcal{S}_n} \exp \left[-n \frac{D}{2} \|\varepsilon\|_2^2 + o(n^{1/2}) \|\varepsilon\|_2 \right] \\ &= \sum_{\varepsilon \in (\mathbb{Z}/n)^{(k-1)^2}} \exp \left[-n \frac{D}{2} \sum_{i,j,i',j' \in [k-1]} \mathcal{H}_{(i,j),(i',j')} \varepsilon_{ij} \varepsilon_{i'j'} + o(n^{1/2}) \|\varepsilon\|_2 \right] \\ &\sim n^{(k-1)^2} \int \dots \int \exp \left[-n \frac{D}{2} \sum_{i,j,i',j' \in [k-1]} \mathcal{H}_{(i,j),(i',j')} \varepsilon_{ij} \varepsilon_{i'j'} \right] d\varepsilon_{11} \dots d\varepsilon_{(k-1)(k-1)} \\ &\sim (\sqrt{2\pi n})^{(k-1)^2} D^{-(k-1)^2/2} (\det \mathcal{H})^{-1/2} \sim (\sqrt{2\pi n})^{(k-1)^2} D^{-(k-1)^2/2} k^{-(k-1)}, \end{aligned}$$

as desired. □

Proof of Proposition 5.6. For $\rho^{(1)}, \rho^{(2)} \in \mathcal{B}_{n,k}(\omega)$, we introduce the set of overlap matrices

$$\mathcal{R}_{n,k}^{\text{bal}}(\omega, n^{-5/12}, \rho^{(1)}, \rho^{(2)}) = \{\rho \in \mathcal{R}_{n,k}^{\text{bal}}(\omega, n^{-5/12}) : \rho_{\cdot * } = \rho^{(1)}, \rho_{* \cdot } = \rho^{(2)}\}.$$

In particular, $\mathcal{R}_{k,n}^{\text{bal}}(\omega, n^{-5/12}, \rho^{(1)}, \rho^{(2)})$ contains the ‘product’ overlap $\rho^{(1)} \otimes \rho^{(2)}$ defined by $(\rho^{(1)} \otimes \rho^{(2)})_{ij} = \rho_i^{(1)} \rho_j^{(2)}$. Because $\rho^{(1)}$ and $\rho^{(2)}$ are (ω, n) -balanced, we find

$$\|\rho^{(1)} \otimes \rho^{(2)} - \bar{\rho}\|_2 = o(n^{-1/2}). \tag{5.15}$$

With these definitions we see that

$$\mathbb{E}[Z_{k,\omega,n^{-5/12}}^{(2)}(\mathcal{G}(n, m))] = \sum_{\rho^{(1)} \in \mathcal{B}_{n,k}(\omega)} \sum_{\rho^{(2)} \in \mathcal{B}_{n,k}(\omega)} \sum_{\rho \in \mathcal{R}_{n,k}^{\text{bal}}(\omega, n^{-5/12}, \rho^{(1)}, \rho^{(2)})} \mathbb{E}[Z_{k,\rho}^{(2)}(\mathcal{G}(n, m))]. \tag{5.16}$$

Let us fix from now on two (ω, n) -balanced colour densities $\rho^{(1)}, \rho^{(2)}$ and simplify the notation by writing

$$\widehat{\mathcal{R}} = \mathcal{R}_{n,k}^{\text{bal}}(\omega, n^{-5/12}, \rho^{(1)}, \rho^{(2)}), \quad \widehat{\rho} = \rho^{(1)} \otimes \rho^{(2)}.$$

Thus, we are going to evaluate

$$\Sigma_1 = \sum_{\rho \in \widehat{\mathcal{R}}} \mathbb{E}[Z_{k,\rho}^{(2)}(\mathcal{G}(n, m))].$$

Equation (5.7) of Lemma 5.5 gives

$$\Sigma_1 \sim \sum_{\rho \in \widehat{\mathcal{R}}} C_n(d, k) \exp\left[2n\alpha(d, k) - n\frac{D(d, k)}{2} \|\rho - \bar{\rho}\|_2^2\right]. \tag{5.17}$$

Further, by the triangle inequality,

$$\|\rho - \widehat{\rho}\|_2 - \|\widehat{\rho} - \bar{\rho}\|_2 \leq \|\rho - \bar{\rho}\|_2 \leq \|\rho - \widehat{\rho}\|_2 + \|\widehat{\rho} - \bar{\rho}\|_2. \tag{5.18}$$

Along with (5.15) this gives

$$\|\rho - \bar{\rho}\|_2^2 = \|\rho - \widehat{\rho}\|_2^2 + o(n^{-1/2}) \|\rho - \widehat{\rho}\|_2 + o(n^{-1}).$$

Hence by replacing in (5.17) we obtain with the notation of Lemma 5.5

$$\begin{aligned} \Sigma_1 &\sim \sum_{\rho \in \widehat{\mathcal{R}}} C_n(d, k) \exp\left[2n\alpha(d, k) - n\frac{D(d, k)}{2} \|\rho - \widehat{\rho}\|_2^2 + o(n^{1/2}) \|\rho - \widehat{\rho}\|_2 + o(1)\right] \\ &\sim C_n(d, k) \exp[2n\alpha(d, k)] \sum_{\rho \in \widehat{\mathcal{R}}} \exp\left[-n\frac{D(d, k)}{2} \|\rho - \widehat{\rho}\|_2^2 + o(n^{1/2}) \|\rho - \widehat{\rho}\|_2\right]. \end{aligned} \tag{5.19}$$

Moreover, with \mathcal{S}_n as in (5.13), it follows from (5.18) that

$$\{\widehat{\rho} + \varepsilon : \varepsilon \in \mathcal{S}_n, \|\varepsilon\|_2 \leq n^{-5/12}/2\} \subset \{\rho \in \widehat{\mathcal{R}} : \|\rho - \bar{\rho}\|_2 \leq n^{-5/12}\} \subset \{\widehat{\rho} + \varepsilon : \varepsilon \in \mathcal{S}_n\}.$$

Hence,

$$\begin{aligned} \Sigma_2 &= C_n(d, k) \exp[2n\alpha(d, k)] \sum_{\substack{\varepsilon \in \mathcal{S}_n \\ \|\varepsilon\|_2 > n^{-5/12}/2}} \exp\left[-n \frac{D(d, k)}{2} \|\varepsilon\|_2^2 (1 + o(1))\right] \\ &= C_n(d, k) \exp[2n\alpha(d, k)] \sum_{\substack{l \in \mathbb{Z}/n^2 \\ l > n^{-5/6}/4}} \sum_{\substack{\varepsilon \in \mathcal{S}_n \\ \|\varepsilon\|_2^2 = l}} \exp\left[-nl \frac{D(d, k)}{2} (1 + o(1))\right] \\ &= C_n(d, k) \exp[2n\alpha(d, k)] O(n^{k^2}) \exp\left[-\frac{D(d, k)}{2} n^{1/6}\right]. \end{aligned}$$

Consequently, (5.19) yields $\Sigma_2 = o(\Sigma_1)$. Thus, we obtain from Lemma 5.7 that

$$\begin{aligned} \Sigma_1 &\sim C_n(d, k) \exp[2n\alpha(d, k)] \sum_{\varepsilon \in \mathcal{S}_n} \exp\left[-n \frac{D(d, k)}{2} \|\varepsilon\|_2^2 + o(n^{-1/2}) \|\varepsilon\|_2\right] \\ &\sim C_n(d, k) \exp[2n\alpha(d, k)] (\sqrt{2\pi n})^{(k-1)^2} k^{-k(k-1)} \left(1 - \frac{d}{(k-1)^2}\right)^{-((k-1)^2)/2}. \end{aligned} \tag{5.20}$$

In particular, the last expression is independent of the choice of the vectors ρ^1, ρ^2 that defined $\widehat{\mathcal{R}}$. Therefore, substituting (5.20) in the decomposition (5.16) completes the proof of Proposition 5.6. □

Proof of Propositions 2.4 and 2.5. First observe that

$$\exp\left(\sum_{\ell \geq 2} \lambda_\ell \delta_\ell^2\right) \sim \left(1 - \frac{d}{(k-1)^2}\right)^{-((k-1)^2)/2} \exp\left(-\frac{d}{2}\right),$$

since $d \sim \bar{d}$. Proposition 2.4 is immediately obtained by combining Lemma 3.1 with Propositions 5.1, 5.3 and 5.6. On the other hand, Proposition 2.5 is obtained by combining Lemma 3.1 with Propositions 5.2, 5.3 and 5.6. □

5.4. Proof of Proposition 5.1

Let

$$f : \rho \in \overline{\mathcal{R}}_k \rightarrow \mathbb{R}, \quad \rho \mapsto H(\rho) + \frac{d}{2} \ln\left(1 - \frac{2}{k} + \|\rho\|_2^2\right). \tag{5.21}$$

The following is a consequence of Fact 5.4.

Fact 5.9. Let $k \geq 3, \bar{d} \in (0, \infty), d = 2m/n$ and $\rho \in \mathcal{R}_{n,k}^{\text{bal}}(\omega)$. Then

$$\mathbb{E}[Z_{k,\rho}^{(2)}(\mathcal{G}(n, m))] = \exp(nf(\rho) + O(\ln n)).$$

Fact 5.9 reduces our task to studying the function $f(\rho)$. For the range of d covered by Proposition 5.1, this analysis is the main technical achievement of [5], where (essentially) the following statement is proved.

Lemma 5.10. *Assume that $k \geq 3$ and that $\bar{d} \leq 2(k-1)\ln(k-1)$ and $d = 2m/n$. For any $n > 0$ and any (ω, n) -balanced overlap matrix ρ , we have*

$$f(\rho) \leq f(\bar{\rho}) - \frac{2(k-1)\ln(k-1) - d}{4(k-1)^2} (k^2 \|\rho\|_2^2 - 1) + o(1). \tag{5.22}$$

Proof. For ρ such that $\sum_{i=1}^k \rho_{ij} = \sum_{i=1}^k \rho_{ji} = 1/k$, the bound (5.22) is proved in [5, Section 3]. This implies that (5.22) also holds for $\rho \in \mathcal{R}_{n,k}^{\text{bal}}(\omega)$, because f is uniformly continuous on the compact set $\bar{\mathcal{R}}_k$. □

Now, assume that k and d satisfy the assumptions of Proposition 5.1 and let $\eta > 0$ be any fixed number. The function $\bar{\mathcal{R}} \rightarrow \mathbb{R}, \rho \rightarrow k^2 \|\rho\|_2^2$ is smooth, strictly convex and attains its global minimum of 1 at $\rho = \bar{\rho}$. Consequently, there exist $c_k > 0$ such that if $\|\rho - \bar{\rho}\|_2 > \eta$, then $(k^2 \|\rho\|_2^2 - 1) \geq c_k$. Hence, Fact 5.9 and Lemma 5.10 yield

$$\sum_{\substack{\rho \in \mathcal{R}_{n,k}^{\text{bal}}(\omega) \\ \|\rho - \bar{\rho}\|_2 > \eta}} \mathbb{E}[Z_{k,\rho}^{(2)}(\mathcal{G}(n, m))] \leq \exp[nf(\bar{\rho}) - nc_k d_k + o(n)],$$

where $d_k = \frac{2(k-1)\ln(k-1) - d}{4(k-1)^2} > 0$. (5.23)

On the other hand, fixing any $\rho_0 \in \mathcal{R}_{n,k}^{\text{bal}}(\omega)$ such that $\|\rho_0 - \bar{\rho}\|_2 \leq k/n$, we obtain from Fact 5.9 that

$$\sum_{\substack{\rho \in \mathcal{R}_{n,k}^{\text{bal}}(\omega) \\ \|\rho - \bar{\rho}\|_2 \leq \eta}} \mathbb{E}[Z_{k,\rho}^{(2)}(\mathcal{G}(n, m))] \geq \mathbb{E}[Z_{k,\rho_0}^{(2)}(\mathcal{G}(n, m))] \geq \exp[nf(\bar{\rho}) + O(\ln n)]. \tag{5.24}$$

Combining (5.23) and (5.24), we conclude that

$$\mathbb{E}[Z_{k,\omega}^2(\mathcal{G}(n, m))] \sim \mathbb{E}[Z_{k,\omega,\eta}^{(2)}(\mathcal{G}(n, m))],$$

thereby completing the proof of Proposition 5.1.

5.5. Proof of Proposition 5.2

We continue to let f denote the function from (5.21). Let \mathcal{B} be the set of all $\rho \in \bar{\mathcal{R}}_k$ such that

$$\sum_{j=1}^k \rho_{ij} = \sum_{j=1}^k \rho_{ji} = 1/k \quad \text{for all } i \in [k].$$

Further, let us say that $\rho \in \bar{\mathcal{R}}_k$ is s -stable if ρ has precisely s entries in the interval $(0.51/k, 1]$. Then any $\rho \in \mathcal{B}$ is s -stable for some $s \in \{0, 1, \dots, k\}$. In addition, let $\kappa = \ln^{20} k/k$ and let us call $\rho \in \bar{\mathcal{R}}_k$ separable if $k\rho_{ij} \notin (0.51, 1 - \kappa)$ for all $i, j \in [k]$. The following lemma summarizes the analysis of the function f performed in [15, Section 4].

Lemma 5.11. *For any $c > 0$ there is $k_0 > 0$ such that for all $k > k_0$ and all \bar{d} such that $(2k - 1)\ln k - c \leq \bar{d} \leq (2k - 1)\ln k$ the following statements are true.*

- (i) *If $1 \leq s < k$, then for all separable s -stable $\rho \in \mathcal{B}$ we have $f(\rho) < f(\bar{\rho})$.*

- (ii) If $\rho \in \mathcal{B}$ is 0-stable and $\rho \neq \bar{\rho}$, then $f(\rho) < f(\bar{\rho})$.
- (iii) If $\bar{d} = (2k - 1) \ln k - 2$, then for all separable, k -stable $\rho \in \mathcal{B}$ we have $f(\rho) < f(\bar{\rho})$.

Further, let us call a k -colouring σ of a graph G on $[n]$ separable if for any other k -colouring τ of G the overlap matrix $\rho(\sigma, \tau)$ is separable. The following is implicit in [15, Section 3].

Lemma 5.12. *There is $k_0 > 0$ such that for all $k > k_0$ and all d such that $2(k - 1) \ln(k - 1) \leq \bar{d} \leq (2k - 1) \ln k$, the following is true. Let $\tilde{Z}_{k,\omega}(\mathcal{G}(n, m))$ denote the number of (ω, n) -balanced k -colourings of $\mathcal{G}(n, m)$ that fail to be separable. Then*

$$\mathbb{E}[\tilde{Z}_{k,\omega}(\mathcal{G}(n, m))] = o(\mathbb{E}[Z_{k,\omega}(\mathcal{G}(n, m))]). \quad \square$$

To state the final ingredient in the proof of Proposition 5.2, we need the following definition. For a graph G on $[n]$ and a k -colouring σ of G we let $\mathcal{C}(G, \sigma)$ be the set of all $\tau \in \mathcal{B}_{n,k}(\omega)$ that are k -colourings of G such that $\rho(\sigma, \tau)$ is k -stable.

Lemma 5.13 ([8, Corollary 1.1]). *There is $k_0 > 0$ such that for all $k > k_0$ and all \bar{d} such that $(2k - 1) \ln k - 2 \leq \bar{d} \leq d_{k,\text{cond}}$, the following is true. Let $\hat{Z}_{k,\omega}(\mathcal{G}(n, m))$ denote the number of (ω, n) -balanced k -colourings such that $|\mathcal{C}(\mathcal{G}(n, m), \sigma)| > \mathbb{E}[Z_{k,\omega}(\mathcal{G}(n, m))]/n$. Then*

$$\mathbb{E}[\hat{Z}_{k,\omega}(\mathcal{G}(n, m))] = o(\mathbb{E}[Z_{k,\omega}(\mathcal{G}(n, m))]). \quad \square$$

Proof of Proposition 5.2. Assume that $k \geq k_0$ for a sufficiently large number k_0 and that $\bar{d} \geq 2(k - 1) \ln(k - 1)$. We consider two different cases.

Case 1: $\bar{d} \leq (2k - 1) \ln k - 2$. Let $\tilde{Z}_{k,\omega}$ be the number of (ω, n) -balanced separable k -colourings of $\mathcal{G}(n, m)$. Then Lemma 5.12 implies that $\mathbb{E}[\tilde{Z}_{k,\omega}(\mathcal{G}(n, m))] \sim \mathbb{E}[Z_{k,\omega}(\mathcal{G}(n, m))]$. Furthermore, in the case that $\bar{d} = (2k - 1) \ln k - 2$, statements (i) and (iii) of Lemma 5.11 imply that $f(\rho) < f(\bar{\rho})$ for any separable $\rho \in \mathcal{B} \setminus \{\bar{\rho}\}$. Because $f(\rho)$ is the sum of the concave function $\rho \mapsto H(\rho)$ and the convex function $\rho \mapsto \frac{d}{2} \ln(1 - 2/k \|\rho\|_2^2)$, this implies that, in fact, for any $\bar{d} \leq (2k - 1) \ln k - 2$ we have $f(\rho) < f(\bar{\rho})$ for any separable $\rho \in \mathcal{B} \setminus \{\bar{\rho}\}$. Hence, the uniform continuity of f on $\bar{\mathcal{R}}_k$ and Fact 5.9 yield

$$\mathbb{E}[\tilde{Z}_{k,\omega}(\mathcal{G}(n, m))^2] \leq (1 + o(1)) \sum_{\substack{\rho \in \mathcal{R}_{n,k}^{\text{bal}}(\omega) \\ \rho \text{ is 0-stable}}} \mathbb{E}[Z_{k,\rho}^{(2)}(\mathcal{G}(n, m))]. \quad (5.25)$$

Finally, combining (5.25) with Fact 5.9 and Lemma 5.11(ii), we see that for any $\eta > 0$,

$$\sum_{\substack{\rho \in \mathcal{R}_{n,k}^{\text{bal}}(\omega) \\ \rho \text{ is 0-stable} \\ \|\rho - \bar{\rho}\|_2 > \eta}} \mathbb{E}[Z_{k,\rho}^{(2)}(\mathcal{G}(n, m))] \leq \sum_{\substack{\rho \in \mathcal{R}_{n,k}^{\text{bal}}(\omega) \\ \rho \text{ is 0-stable} \\ \|\rho - \bar{\rho}\|_2 > \eta}} \exp(nf(\rho) + O(\ln n)) = o(\mathbb{E}[Z_{k,\omega,\eta}^{(2)}(\mathcal{G}(n, m))]). \quad (5.26)$$

The assertion follows by combining (5.25) and (5.26).

Case 2: $(2k - 1) \ln k - 2 < \bar{d} < d_{k,\text{cond}}$. Let $\tilde{Z}_{k,\omega}$ be the number of (ω, n) -balanced separable k -colourings σ of $\mathcal{G}(n, m)$ such that $|\mathcal{C}(\mathcal{G}(n, m), \sigma)| \leq \mathbb{E}[Z_{k,\omega}(\mathcal{G}(n, m))]/n$. Then Lemmas 5.12

and 5.13 imply that $\mathbb{E}[\tilde{Z}_{k,\omega}(\mathcal{G}(n,m))] \sim \mathbb{E}[Z_{k,\omega}(\mathcal{G}(n,m))]$. Further, Lemma 5.11(i) and Fact 5.9 entail that (5.25) holds for this random variable $\tilde{Z}_{k,\omega}$. Moreover, as in the previous case (5.25), Fact 5.9 and Lemma 5.11(iii) show that (5.26) holds true for any fixed $\eta > 0$.

In either case the assertion follows by combining (5.25) and (5.26). \square

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