

SEPARATING DIAGONAL STATIONARY REFLECTION PRINCIPLES

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Abstract. We introduce three families of diagonal reflection principles for matrices of stationary sets of ordinals. We analyze both their relationships among themselves and their relationships with other known principles of simultaneous stationary reflection, the strong reflection principle, and the existence of square sequences.

§1. Introduction. The study of compactness and reflection principles has been the subject of a significant amount of set theoretic research, and the careful investigation of the tension existing between compactness principles that arise due to the presence of large cardinals and incompactness principles that tend to hold, for example, in canonical inner models, has been quite fruitful. Particularly prominent among the compactness principles that have been studied are various principles of stationary reflection. In this article, we investigate the relationships between different principles of stationary reflection, particularly focusing on diagonal stationary reflection principles. We also prove some results closely linking these diagonal reflection principles with certain square principles, which provide concrete instances of incompactness. We begin by giving some background, to motivate the questions we are going to address. The notation in the upcoming definition follows [4] and [8].

DEFINITION 1.1. Suppose that λ is a regular uncountable cardinal and $S \subseteq \lambda$ is stationary.

- (1) If $\alpha < \lambda$, then S reflects at α if $\text{cf}(\alpha) > \omega$ and $S \cap \alpha$ is stationary in α . We say that S reflects if there is $\alpha < \lambda$ such that S reflects at α .
- (2) If \mathcal{T} is a family of stationary subsets of λ and $\alpha < \lambda$, then \mathcal{T} reflects simultaneously at α if T reflects at α for every $T \in \mathcal{T}$. We say that \mathcal{T} reflects simultaneously if there is $\alpha < \lambda$ such that \mathcal{T} reflects simultaneously at α .
- (3) If $1 < \kappa \leq \lambda$, then $\text{Refl}(<\kappa, S)$ is the assertion that every family of fewer than κ -many stationary subsets of S reflects simultaneously. $\text{Refl}(<\kappa^+, S)$ is typically denoted by $\text{Refl}(\kappa, S)$.
- (4) $\text{Refl}^*(<\kappa, S)$ is the assertion that, if \mathbb{P} is a λ -directed closed forcing and $|\mathbb{P}| \leq \lambda$, then $\Vdash_{\mathbb{P}}$ “ $\text{Refl}(<\kappa, S)$ ”.

A major appeal of these principles is that they imply the failure of certain square principles.

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DEFINITION 1.2. Let λ be a limit of limit ordinals.

- (1) A sequence $\vec{C} = \langle C_\alpha \mid \alpha < \lambda, \alpha \text{ limit} \rangle$ is a *coherent sequence of length λ* if, for every limit ordinal $\alpha < \lambda$,
 - (a) C_α is a nonempty collection of clubs in α ; and
 - (b) for every $C \in C_\alpha$ and every limit point β of C , we have $C \cap \beta \in C_\beta$.

If, moreover, κ is a cardinal and $|C_\alpha| < \kappa$ for every limit $\alpha < \lambda$, then \vec{C} is said to have *width $< \kappa$* .

- (2) If \vec{C} is a coherent sequence of length λ , then a *thread* through \vec{C} is a club subset T of λ that coheres with \vec{C} , i.e., for every limit point β of T , we have $T \cap \beta \in C_\beta$.
- (3) If κ is a cardinal, then a $\square(\lambda, < \kappa)$ -*sequence* is a coherent sequence of length λ and width $< \kappa$ that does not have a thread. The principle $\square(\lambda, < \kappa)$ asserts that there is a $\square(\lambda, < \kappa)$ sequence. In place of $\square(\lambda, < \kappa^+)$, we may write $\square(\lambda, \kappa)$.

$\square(\lambda, 1)$ is known as $\square(\lambda)$, and $\square(\lambda, < \kappa)$ becomes weaker as κ increases. In [8], Hayut and the second author analyzed the effects of simultaneous stationary reflection on these kinds of square principles. In what follows and throughout the paper, if $\kappa < \beta$ where κ is an infinite cardinal and β is an ordinal, then we will write S_{κ}^β for the set of limit ordinals less than β of cofinality κ . Similarly, $S_{< \kappa}^\beta$ denotes the set of limit ordinals less than β of cofinality less than κ , and $S_{\leq \kappa}^\beta, S_{\geq \kappa}^\beta, S_{> \kappa}^\beta$ have the obvious meaning.

THEOREM 1.3 [8, Theorem 2.13]. *Suppose that $\kappa < \lambda$ are cardinals, λ is regular, and $\text{Refl}(< \kappa, S)$ holds for some stationary $S \subseteq S_{\geq \kappa}^\lambda$. Then $\square(\lambda, < \kappa)$ fails.*

The first author came to this from a different angle, looking for ways to derive the strongest possible failure of these square principles from the assumption of certain forcing axioms. Forcing axioms such as Martin’s Maximum (MM) or the subcomplete forcing axiom (SCFA) imply reflection principles of the form $\text{Refl}(\omega_1, S_\omega^\lambda)$ for sufficiently large regular λ (in the case of MM, $\lambda > \omega_1$ is enough, while in the case of SCFA, $\lambda > 2^\omega$ is needed). So the above theorem yields only the failure of $\square(\lambda, < \omega)$ as a consequence of these types of stationary reflection principles.

Looking to find improvements of Theorem 1.3 that would remove the hypothesis that $S \subseteq S_{\geq \kappa}^\lambda$, Hayut and the second author introduced the following definition.

DEFINITION 1.4 [8]. A $\square(\lambda, < \kappa)$ -sequence $\langle C_\alpha \mid \alpha < \lambda, \alpha \text{ limit} \rangle$ is *full* if for unboundedly many $\alpha < \lambda$, there is a club of $\beta < \lambda$ such that α is a limit point of some $C \in C_\beta$.

They then proved that the requirement that $S \subseteq S_{\geq \kappa}^\lambda$ can be removed if one only wants to preclude the existence of *full* $\square(\lambda, < \kappa)$ -sequences.

THEOREM 1.5 [8, Theorem 2.18]. *Suppose that $\kappa < \lambda$ are uncountable cardinals, where λ is regular, and suppose that $\text{Refl}(< \kappa, S)$ holds, for some stationary $S \subseteq \lambda$. Then there is no full $\square(\lambda, < \kappa)$ -sequence.*

So this theorem shows that there is no full $\square(\lambda, \omega_1)$ -sequence if $\text{Refl}(\omega_1, S_\omega^\lambda)$ holds and $\lambda > \omega_2$, for example. The following theorem serves to preclude the existence of a sequence that is not full:

THEOREM 1.6 [8, Theorem 2.20]. *Suppose that $\kappa < \lambda$ are regular, uncountable cardinals. If $\text{Refl}(2, \lambda)$ holds, then there is no $\square(\lambda, < \kappa)$ sequence that is not full.*

Notice that $\text{Refl}(1, \lambda)$, and hence $\text{Refl}(2, \lambda)$, fails if $\lambda = \mu^+$ where μ is regular, as witnessed by S_μ^λ . Therefore, Theorem 1.6 is only nontrivial if λ is either weakly inaccessible or the successor of a singular cardinal. In addition, since the stationary reflection principles derived from the abovementioned forcing axioms only yield stationary reflection for subsets of S_ω^λ , Theorem 1.6 is not particularly relevant for the study of their consequences. For these reasons, the first author was led to introduce the diagonal stationary reflection principle, which we formulate in the following definition, along with some natural variants.

DEFINITION 1.7. Let λ be a regular cardinal, let $S \subseteq \lambda$ be stationary, and let $\kappa < \lambda$. The *diagonal stationary reflection principle* $\text{DSR}(< \kappa, S)$ asserts that for every matrix $\langle S_{\alpha,i} \mid \alpha < \lambda, i < j_\alpha \rangle$ of stationary subsets of S such that $j_\alpha < \kappa$ for every $\alpha < \lambda$, there are a $\gamma < \lambda$ of uncountable cofinality and a club $F \subseteq \gamma$ such that for every $\alpha \in F$ and every $i < j_\alpha$, $S_{\alpha,i}$ reflects at γ . The version of the principle in which we only require $j_\alpha \leq \kappa$ is denoted $\text{DSR}(\kappa, S)$.

We will also be considering weakenings of this principle. Let $\text{uDSR}(< \kappa, S)$ and $\text{sDSR}(< \kappa, S)$ result from modifying the definition of $\text{DSR}(< \kappa, S)$ by requiring that the subset $F \subseteq \gamma$ be unbounded in γ or stationary in γ , respectively. We note that, for all three principles, Clause (1) of Definition 1.1 implies that any ordinal γ witnessing an instance of the principle must have uncountable cofinality. As with Refl , if S is a stationary subset of a regular uncountable cardinal λ , let $\text{DSR}^*(< \kappa, S)$ denote the statement that $\text{DSR}(< \kappa, S)$ holds and continues to hold in any extension by a λ -directed closed forcing notion of size at most λ (and similarly for uDSR^* and sDSR^*).

REMARK 1.8. It is easily seen that each of the above principles, as well as each of the other stationary reflection principles considered in this paper, is equivalent to the apparent strengthening formed by requiring the existence of stationarily many such γ at which reflection holds as opposed to just one.

The original point of introducing these reflection principles was the fact that, for all regular $\lambda \geq \omega_2$, the principle $\text{DSR}^*(\omega_1, S_\omega^\lambda)$ follows from either MM or SCFA + CH, and the first author proved that $\text{DSR}(\omega_1, S_\omega^\lambda)$ implies the failure of $\square(\lambda, \omega_1)$ if $\lambda > \omega_2$ and the failure of $\square(\lambda, \omega)$ if $\lambda = \omega_2$:

THEOREM 1.9 [7, Theorem 3.4]. *Let $\kappa < \lambda$ be cardinals, with λ regular, and suppose that $\text{DSR}(< \kappa, S)$ holds for some stationary set $S \subseteq \lambda$. Then $\square(\lambda, < \kappa)$ fails.*

We will reduce the hypothesis of the theorem from $\text{DSR}(< \kappa, S)$ to $\text{sDSR}(< \kappa, S)$ in Theorem 2.1.

If λ is a regular cardinal, $S \subseteq \lambda$ is stationary, and $\kappa < \lambda$, then it is clear from Definition 1.7 that

$$\text{DSR}(< \kappa, S) \Rightarrow \text{sDSR}(< \kappa, S) \Rightarrow \text{uDSR}(< \kappa, S) \Rightarrow \text{Refl}(< \kappa, S).$$

Our obvious initial question, raised in light of Theorems 1.3 and 1.9, was whether $\text{Refl}(\omega_1, S)$ implies $\text{DSR}(\omega_1, S)$ and, more generally, the extent to which the arrows in the above sequence of implications can be either reversed or strengthened.

The paper is organized as follows. In Section 2, we establish some implications and equivalences in ZFC. We will also prove our strengthening of Theorem 1.9, showing that $sDSR(\langle \kappa, S \rangle)$ for some stationary $S \subseteq \lambda$ is enough to ensure the failure of $\square(\lambda, \langle \kappa \rangle)$. Section 3 contains results that separate various stationary reflection principles by showing that certain implications do not hold in ZFC. A consequence of the results of this section will be the fact that, in general, none of the arrows in the above sequence of implications are reversible. For reference, a detailed summary of these results can be found at the beginning of Section 3.3. In Section 4, we extend a result of Larson [14, Theorem 4.6] by proving, among other things, that the strong reflection principle, SRP, does not imply $uDSR(1, S_{\omega}^{\lambda})$ for any regular $\lambda > \omega_2$. Then, in Section 5, we prove that our strengthening of Theorem 1.9 is sharp in the sense that the principles $sDSR(\langle \kappa, S \rangle)$ and $\square(\lambda, \kappa)$ are compatible with one another for infinite regular cardinals $\kappa < \lambda$ and a stationary $S \subseteq \lambda$. Finally, in Section 6, we list some open questions that are raised by our work.

§2. Implications. We begin this section with our strengthening of Theorem 1.9. Here and throughout the paper, for a set of ordinals C , we will use the notation $\lim(C)$ for the set of limit points of C below the supremum of C .

THEOREM 2.1. *Suppose that $1 < \kappa < \lambda$ are cardinals, with λ regular, and suppose that there is a stationary $S \subseteq \lambda$ for which $sDSR(\langle \kappa, S \rangle)$ holds. Then $\square(\lambda, \langle \kappa \rangle)$ fails.*

PROOF. Suppose for sake of contradiction that $\vec{C} = \langle C_{\alpha} \mid \alpha \in \lim(\lambda) \rangle$ is a $\square(\lambda, \langle \kappa \rangle)$ -sequence. For each $\alpha \in \lim(\lambda)$, let $C_{\alpha} = \{C_{\alpha,i} \mid i < j_{\alpha}\}$, where $j_{\alpha} < \kappa$. For each limit $\alpha \in \lim(\lambda)$ and each $i < j_{\alpha}$, let

$$S_{\alpha,i} = \{\beta \in S \setminus (\alpha + 1) \mid \text{for all } k < j_{\beta}, C_{\beta,k} \cap \alpha \neq C_{\alpha,i}\}.$$

CLAIM 2.2. *There is $\alpha_0 < \lambda$ such that $S_{\alpha,i}$ is stationary for all limit α with $\alpha_0 \leq \alpha < \lambda$ and all $i < j_{\alpha}$.*

PROOF. Otherwise, there would be an unbounded subset $A \subseteq \lim(\lambda)$ such that, for all $\alpha \in A$, there is $i_{\alpha} < j_{\alpha}$ such that $S_{\alpha,i_{\alpha}}$ is nonstationary. For each $\alpha \in A$, let D_{α} be a club in λ such that $D_{\alpha} \cap S_{\alpha,i_{\alpha}} = \emptyset$. Define an ordering $<_T$ on A as follows. For all $\alpha, \beta \in A$, set $\alpha <_T \beta$ if and only if $\alpha < \beta$ and $C_{\beta,i_{\beta}} \cap \alpha = C_{\alpha,i_{\alpha}}$. It is easily verified that $T = (A, <_T)$ is a tree.

We claim that T has no antichains of size κ . To this end, fix a set $B \in [A]^{\kappa}$. We will find two elements of B that are $<_T$ -comparable. Fix a $\gamma \in S \cap \bigcap_{\alpha \in B} D_{\alpha}$ with $\gamma > \sup(B)$. Then, for all $\alpha \in B$, we have $\gamma \notin S_{\alpha,i_{\alpha}}$, so there is $k_{\alpha} < j_{\gamma}$ such that $C_{\gamma,k_{\alpha}} \cap \alpha = C_{\alpha,i_{\alpha}}$. Since $j_{\gamma} < \kappa$, we can find $\alpha < \beta$ in B such that $k_{\alpha} = k_{\beta}$. But then $C_{\beta,i_{\beta}} \cap \alpha = C_{\gamma,k_{\beta}} \cap \alpha = C_{\alpha,i_{\alpha}}$, so $\alpha <_T \beta$.

The tree T therefore has height λ and all of its levels have size less than κ . Since $\kappa < \lambda$, it then follows from a result of Kurepa [11] that T has a cofinal branch. Let A^* be such a cofinal branch. Then A^* is cofinal in λ and, for all $\alpha < \beta$ in A^* , we have $C_{\alpha,i_{\alpha}} = C_{\beta,i_{\beta}} \cap \alpha$. Therefore, $\bigcup_{\alpha \in A^*} C_{\alpha,i_{\alpha}}$ is a thread through \vec{C} , contradicting the assumption that \vec{C} is a $\square(\lambda, \langle \kappa \rangle)$ -sequence. \dashv

Fix $\alpha_0 < \lambda$ as in the claim. By $sDSR(\langle \kappa, S \rangle)$, we can find a $\gamma \in S_{>\omega}^{\lambda} \setminus (\alpha_0 + 1)$ and a stationary $F \subseteq \gamma \setminus \alpha_0$ such that, for all $\alpha \in F$ and all $i < j_{\alpha}$, $S_{\alpha,i}$ reflects at γ . Fix an

arbitrary $C \in \mathcal{C}_\gamma$, and find an $\alpha \in \text{lim}(C) \cap F$. Since $\alpha \in \text{lim}(C)$, it follows that there is $i < j_\alpha$ such that $C \cap \alpha = C_{\alpha,i}$. Since $\alpha \in F$, we know that $S_{\alpha,i} \cap \gamma$ is stationary in γ , so we can find $\beta \in \text{lim}(C) \cap S_{\alpha,i}$. Then $\beta > \alpha$ and, since $\beta \in \text{lim}(C)$, there is $k < j_\beta$ such that $C_{\beta,k} = C \cap \beta$. But then $C_{\beta,k} \cap \alpha = C \cap \alpha = C_{\alpha,i}$, contradicting the fact that $\beta \in S_{\alpha,i}$ and finishing the proof of the theorem. \dashv

The diagonal stationary reflection principles we are dealing with here are closely related to Paul Larson’s principle OSR_{ω_2} from [14]. Here is a slight reformulation and generalization of the original principle.

DEFINITION 2.3. Suppose that $\lambda \geq \omega_2$ is a regular cardinal and $S \subseteq \lambda$ is stationary. $\text{OSR}(S)$ is the assertion that, for every sequence $\langle S_\alpha \mid i < \lambda \rangle$ of stationary subsets of S , there is a $\delta \in S_{>\omega}^\lambda$ such that for all $\alpha < \delta$, S_α reflects at δ .

Larson wrote OSR_{ω_2} for the principle $\text{OSR}(S^{\omega_2})$. The following lemma shows among other things that, for stationary $S \subseteq \lambda$, the principle $\text{OSR}(S)$ is equivalent to each of the principles $\text{DSR}(<\lambda, S)$, $\text{uDSR}(<\lambda, S)$, and $\text{sDSR}(<\lambda, S)$. Thus, differences between the principles $\text{DSR}(<\kappa, S)$, $\text{uDSR}(<\kappa, S)$, and $\text{sDSR}(<\kappa, S)$ can only be observed when $\text{sup}(S) > \kappa$.

LEMMA 2.4. Suppose that $\lambda \geq \aleph_2$ is a regular cardinal and $S \subseteq \lambda$ is stationary. Then the following are equivalent.

- (1) $\text{uDSR}(<\lambda, S)$.
- (2) $\text{DSR}(<\lambda, S)$.
- (3) $\text{OSR}(S)$.
- (4) For every matrix $\langle S_{\alpha,i} \mid \alpha < \lambda, i < j_\alpha \rangle$ of stationary subsets of S such that $j_\alpha < \lambda$ for all $\alpha < \lambda$, there is $\gamma \in S_{>\omega}^\lambda$ such that $S_{\alpha,i}$ reflects at γ for all $\alpha < \gamma$ and $i < j_\alpha$.

PROOF. It is immediate that clause (4) implies clauses (1), (2), and (3), and that clause (2) implies clause (1). To see that clause (3) implies clause (4), suppose that $\text{OSR}(S)$ holds and that we are given a matrix $\langle S_{\alpha,i} \mid \alpha < \lambda, i < j_\alpha \rangle$ as in the statement of clause (4). Assume without loss of generality that $j_\alpha > 0$ for all $\alpha < \lambda$. Let $\pi : \lambda \rightarrow \bigcup_{\alpha < \lambda} (\{\alpha\} \times j_\alpha)$ be a bijection. Form a sequence $\langle T_\alpha \mid \alpha < \lambda \rangle$ of stationary subsets of S by letting $T_\alpha = S_{\pi(\alpha)}$ for all $\alpha < \lambda$. Let C be the set of $\gamma < \lambda$ such that $\pi''\gamma = \bigcup_{\alpha < \gamma} (\{\alpha\} \times j_\alpha)$. Then C is a club in λ . Using $\text{OSR}(S)$, we can find $\gamma \in C \cap S_{>\omega}^\lambda$ such that T_α reflects at γ for all $\alpha < \gamma$. By our definition of T_α and our choice of γ , it follows that $S_{\beta,i}$ reflects at γ for all $\beta < \gamma$ and $i < j_\beta$, so γ witnesses this instance of clause (4).

It remains to argue that clause (1) implies clause (3). To this end, suppose that $\text{uDSR}(<\lambda, S)$ holds, and let $\langle S_\alpha \mid \alpha < \lambda \rangle$ be a sequence of stationary subsets of S . Define a matrix $\langle T_{\alpha,i} \mid \alpha < \lambda, i < \alpha \rangle$ of stationary subsets of S by letting $T_{\alpha,i} = S_i$ for all $\alpha < \lambda$ and $i < \alpha$. By $\text{uDSR}(<\lambda, S)$, we can find an ordinal $\gamma \in S_{>\omega}^\lambda$ and an unbounded subset $F \subseteq \gamma$ such that, for all $\alpha \in F$ and all $i < \alpha$, we have that $T_{\alpha,i}$ reflects at γ .

We claim that S_β reflects at γ for all $\beta < \gamma$. To see this, fix $\beta < \gamma$, and let $\alpha = \min(F \setminus (\beta + 1))$. Then $T_{\alpha,\beta} = S_\beta$. But, since $\alpha \in F$, we know that $T_{\alpha,\beta}$ reflects at γ . Therefore, γ witnesses this instance of $\text{OSR}(S)$. \dashv

We will now explore the relationships between diagonal and simultaneous reflection. Let λ be a regular uncountable cardinal. As noted in the introduction, $\text{uDSR}(\langle \kappa, \lambda \rangle)$ implies $\text{Refl}(\langle \kappa, \lambda \rangle)$, because given any sequence $\langle S_i \mid i < \bar{\kappa} \rangle$, where $\bar{\kappa} < \kappa$, one can consider the matrix defined by setting $S_{\alpha,i} = S_i$ for all $\alpha < \lambda$ and $i < \bar{\kappa}$. It is a more interesting question whether diagonal stationary reflection implies any amount of simultaneous reflection when this is not explicitly built into the principle at hand. The following lemma provides one instance in which this is the case.

LEMMA 2.5. *Let λ be a regular uncountable cardinal, and let $S \subseteq \lambda$ be stationary. Then $\text{DSR}(1, S)$ implies $\text{Refl}(\omega, S)$.*

PROOF. Assume that $\text{DSR}(1, S)$ holds, and let $\langle T_n \mid n < \omega \rangle$ be a sequence of stationary subsets of S . We will find an ordinal $\delta \in S_{>\omega}^\lambda$ such that, for all $n < \omega$, T_n reflects at δ . By shrinking the sets if necessary, we may assume that $\langle T_n \mid n < \omega \rangle$ is a sequence of pairwise disjoint sets. For each $n < \omega$, let $T_n = \bigcup_{l < \omega} T_{n,l}$ be a partition of T_n into stationary sets. Define a function $s : \omega \times \omega \rightarrow \omega \times \omega$ by setting

$$s(n, l) = \begin{cases} \langle 0, l + 1 \rangle & \text{if } n \geq l, \\ \langle n + 1, l \rangle & \text{if } n < l. \end{cases}$$

Notice that, denoting the p -fold application of s by s^p (for $p < \omega$), s is defined in such a way that the following holds.

CLAIM 2.6. *For any $\langle n, l \rangle \in \omega \times \omega$ and any $m < \omega$, there is a $p \in \omega$ such that the first component of $s^p(n, l)$ is m .*

Now define a function $f : \lambda \rightarrow \omega \times \omega$ by

$$f(\alpha) = \begin{cases} s(n, l) & \text{if } \alpha \in T_{n,l}, \\ \langle 0, 0 \rangle & \text{if } \alpha \notin \bigcup_{n,l < \omega} T_{n,l}. \end{cases}$$

Note that $\text{DSR}(1, S)$ is equivalent to the assertion that, given any sequence $\langle S_\alpha \mid \alpha < \lambda \rangle$ of stationary subsets of S , there are γ, F such that

(*) $\gamma \in S_{>\omega}^\lambda$, $F \subseteq \gamma$ is club, and for all $\alpha \in F$, S_α reflects at γ .

Define

$$S_\alpha = T_{f(\alpha)}$$

for $\alpha < \lambda$, and let γ and F satisfy (*) with respect to $\langle S_\alpha \mid \alpha < \lambda \rangle$.

CLAIM 2.7. *Let $\alpha \in F$. Then there is a $\beta \in F$ with $f(\beta) = s(f(\alpha))$.*

PROOF. Since $\alpha \in F$, $S_\alpha \cap \gamma$ is stationary in γ , by (*). By definition, $S_\alpha = T_{f(\alpha)}$, so $T_{f(\alpha)} \cap \gamma$ is stationary in γ . Since F is club in γ , there is a $\beta \in T_{f(\alpha)} \cap F$. By definition, then, $f(\beta) = s(f(\alpha))$, so β is as wished. \dashv

Now let $m < \omega$. We will show that T_m reflects at γ . To see this, pick $\alpha \in F$. Using Claim 2.6, let $p \in \omega$ be such that the first component of $s^p(f(\alpha))$ is m , say $s^p(f(\alpha)) = \langle m, l \rangle$. By applying Claim 2.7 p times, we see that there is a $\beta \in F$ such that $f(\beta) = s^p(f(\alpha))$. By (*), $S_\beta \cap \gamma$ is stationary in γ . But $S_\beta = T_{m,l} \subseteq T_m$, so T_m reflects at γ , as wished. \dashv

The previous lemma is optimal, in the sense that DSR cannot be replaced with sDSR, by Theorem 3.19, and in the sense that $\text{DSR}(1, S)/\text{Refl}(\omega, S)$ cannot be replaced by $\text{DSR}(\mu, S)/\text{Refl}(\mu^+, S)$ for any uncountable μ , by Theorem 3.28. We doubt that other instances of this phenomenon are possible, but at present have no model in which, for example, $\text{DSR}(1, S_\omega^\lambda)$ holds but $\text{Refl}(\omega_1, S_\omega^\lambda)$ fails.

§3. Separations. In this section, we prove results separating a number of stationary reflection principles. Our standard strategy is one that dates back to Kunen's result indicating that the existence of a κ -saturated, κ -complete ideal on an inaccessible cardinal κ does not imply that κ is measurable (or even weakly compact) [10] and has since been extensively deployed in the context of the investigation of square principles and stationary reflection principles (cf. [4]).

The basic idea is as follows. To show that a reflection principle P does not imply a reflection principle P^* , we start in a model V in which P (or some strengthening thereof) holds and, moreover, is indestructible under forcing with forcing notions coming from some class Γ . (In a typical scenario, Γ will consist of all forcings that are sufficiently directed closed.) We will then design a two-step forcing iteration $\mathbb{S} * \dot{\mathbb{T}}$ such that:

- Forcing with \mathbb{S} introduces a counterexample to P^* .
- The iteration $\mathbb{S} * \dot{\mathbb{T}}$ is in Γ , and therefore P holds in $V^{\mathbb{S} * \dot{\mathbb{T}}}$.
- In $V^{\mathbb{S}}$, forcing with $\dot{\mathbb{T}}$ cannot destroy a potential counterexample to P in $V^{\mathbb{S} * \dot{\mathbb{T}}}$.
Therefore, the fact that P holds in $V^{\mathbb{S} * \dot{\mathbb{T}}}$ can be pulled back to $V^{\mathbb{S}}$.

The model $V^{\mathbb{S}}$ will then be a model in which P holds but P^* fails.

The properties of $\mathbb{S} * \dot{\mathbb{T}}$ isolated above are in tension with one another. In particular, in practice, in order for $\mathbb{S} * \dot{\mathbb{T}}$ to be in Γ , it is typically the case that, in $V^{\mathbb{S}}$, forcing with $\dot{\mathbb{T}}$ necessarily destroys the counterexample to P^* introduced by forcing with \mathbb{S} . At the same time, we must ensure that forcing with $\dot{\mathbb{T}}$ *cannot* destroy any potential counterexamples to P . A delicate balance must be achieved, and the choice of \mathbb{S} and $\dot{\mathbb{T}}$ is very dependent on the precise nature of the reflection principles P and P^* under consideration.

For simplicity and concreteness, we will primarily be considering reflection principles of the form $\text{OSR}(S)$, $\text{DSR}(<\kappa, S)$, etc. in which $S = S_\omega^\lambda$ for some regular cardinal $\lambda \geq \omega_2$. Our reasons for doing this are twofold. First, these are the reflection principles that are implied by forcing axioms such as MM and SCFA, and therefore we feel they are of the most interest. Second, doing so will simplify matters at various points due to the fact that the stationarity of subsets of S_ω^λ is always preserved by countably closed forcing. It is not in general true that the stationarity of subsets of S_κ^λ for uncountable κ is preserved by κ^+ -closed forcing; to arrange for this, one would have to consider issues of approachability that we would for the most part rather avoid here. Nonetheless, with a bit more care and attention to approachability, the reader can adapt our proofs to apply to situations in which S is of the form $S_{\leq \kappa}^\lambda$ for certain uncountable κ .

Before we turn to the actual separation results, we recall some facts about indestructible reflection principles and forcing that will be used in their proofs.

3.1. Indestructible reflection principles. In this subsection, we indicate how to arrange for the indestructible reflection principles that will appear as hypotheses in later theorems. We first deal with the reflection principles at inaccessible cardinals. Note that, if λ is weakly compact, then $\text{OSR}(\lambda)$ holds. By standard arguments (see, e.g., [3, Example 16.2]), if λ is a weakly compact cardinal, then there is a forcing extension in which λ remains weakly compact and its weak compactness is indestructible under λ -directed closed forcings of size at most λ . In particular, in this forcing extension, λ is inaccessible and $\text{OSR}^*(\lambda)$ holds (and hence, by Lemma 2.4, $\text{DSR}^*(<\lambda, \lambda)$ holds as well).

We now consider indestructible reflection principles at successors of regular cardinals.

THEOREM 3.1. *Suppose that μ is a regular uncountable cardinal and $\lambda > \mu$ is a weakly compact cardinal. Then there is a μ -closed, λ -cc forcing extension in which $\lambda = \mu^+$ and $\text{OSR}^*(S_{<\mu}^{\mu^+})$ holds.*

PROOF. This follows from a straightforward modification of the proof of [8, Theorem 3.22]. A key component of the proof is the fact that the stationarity of subsets of $S_{<\mu}^{\mu^+}$ is preserved by μ -closed forcing, which holds because, for regular μ , the set $S_{<\mu}^{\mu^+}$ is in the approachability ideal $I[\mu^+]$. For this and other facts on the approachability ideal, we direct the reader to [5, Section 3]. ⊖

REMARK 3.2. Note that $\text{OSR}^*(S_{<\mu}^{\mu^+})$ is equivalent to each of $\text{DSR}^*(\mu, S_{<\mu}^{\mu^+})$ and $\text{uDSR}^*(\mu, S_{<\mu}^{\mu^+})$ by Lemma 2.4. Also, by Theorem 3.1 and a result of Magidor [17], the existence of a regular uncountable cardinal μ for which $\text{OSR}^*(S_{<\mu}^{\mu^+})$ holds is equiconsistent over ZFC with the existence of a weakly compact cardinal.

We will now turn to the consistency of very indestructible versions of diagonal reflection that will include indestructible diagonal reflection at successors of singular cardinals. We will use the concept of generically supercompact cardinals, due to Cummings and Foreman.

DEFINITION 3.3. A cardinal κ is *generically supercompact*¹ if κ is the successor of a regular cardinal μ and, for every regular cardinal $\lambda > \kappa$, there is a μ -closed forcing notion \mathbb{R} such that whenever H is \mathbb{R} -generic, there is, in $V[H]$, an elementary embedding

$$j : V \prec M \subseteq V[H]$$

with:

- (1) $\text{crit}(j) = \kappa$,
- (2) $j(\kappa) > \lambda$,
- (3) $j^{\text{“}}\lambda \in M$,

¹Our definition of generic supercompactness differs slightly from what is called generically supercompact by Foreman [6, Definition 11.2]. The version of indestructible generic supercompactness of Foreman [6, Definition 11.4] implies ours. It should be pointed out that clause (4) in our definition follows from (1)–(3), since λ is assumed to be regular in V . We kept this clause to highlight the similarity to Foreman’s versions of these concepts.

- (4) $\text{sup}(j^{“}\lambda) < j(\lambda)$,
- (5) $M \models \text{cf}(\lambda) = \mu$.

A cardinal κ is *indestructibly* generically supercompact if whenever \mathbb{R} is a κ -directed closed notion of forcing and G is \mathbb{R} -generic, then κ is generically supercompact in $V[G]$.

It is well-known by [16] that if κ is supercompact, then there is a forcing extension in which κ is indestructibly supercompact, meaning that κ is supercompact and remains so in any further forcing extension obtained by κ -directed closed forcing. If κ is indestructibly supercompact and $\mu < \kappa$ is a regular uncountable cardinal, then in the forcing extension obtained by collapsing κ to become the cardinal successor of μ , κ is indestructibly generically supercompact in the sense of [6, Definition 11.4], which implies our version of indestructible generic supercompactness, as stated in the footnote to Definition 3.3; see [6, Remark after Definition 11.4].

LEMMA 3.4. *Suppose that μ is an uncountable regular cardinal and $\kappa = \mu^+$ is generically supercompact. Then for every regular cardinal $\lambda > \kappa$, the principle $\text{DSR}(<\kappa, S_\omega^\lambda)$ holds.*

PROOF. To show that $\text{DSR}(<\kappa, S_\omega^\lambda)$ holds, note that since $\kappa = \mu^+$, the principle is equivalent to $\text{DSR}(\mu, S_\omega^\lambda)$. Thus, let a matrix $\langle S_{\alpha,i} \mid \alpha < \lambda, i < \mu \rangle$ of stationary subsets of S_ω^λ be given.

By the generic supercompactness of κ , let H be generic for a μ -closed forcing notion \mathbb{R} such that, in $V[H]$, there is an elementary embedding $j : V \prec M$ satisfying the clauses listed in Definition 3.3.

Note that λ has cofinality μ in $V[H]$, since this is true in M , by clause (5) of Definition 3.3, and since μ is still regular in $V[H]$, by the μ -closure of \mathbb{R} . Thus, $\nu = \text{sup } j^{“}\lambda$ has cofinality μ as well. Temporarily fixing $\alpha < \lambda$ and $i < \mu$, let us verify that in $V[H]$, $j^{“}S_{\alpha,i}$ is stationary in ν . First, it follows that $S_{\alpha,i}$ is still stationary in $V[H]$, because $S_{\alpha,i} \subseteq S_\omega^\lambda$ and \mathbb{R} is at least ω_1 -closed—it is well-known that countably closed forcing preserves stationary subsets of S_ω^λ . Now, to see that $j^{“}S_{\alpha,i}$ is stationary in ν , let $C \subseteq \nu$ be club, with $C \in V[H]$. We have that $j^{“}\lambda$ is closed under limits of cofinality less than μ in $V[H]$, since V and $V[H]$ have the same sequences of ordinals of length less than μ . Thus, $\bar{C} = j^{-1}{}^{“}C$ is closed under limits of cofinality less than μ . Letting \bar{C}' be the closure of \bar{C} in λ , \bar{C}' is club in λ , and hence, as $S_{\alpha,i}$ is stationary in $V[H]$ and $\bar{C}' \in V[H]$, there is a $\beta \in \bar{C}' \cap S_{\alpha,i} \subseteq S_\omega^\lambda$, so $\text{cf}(\beta) < \mu$. But this means that $\beta \in \bar{C} \cap S_{\alpha,i}$, and hence $j(\beta) \in C \cap j^{“}S_{\alpha,i}$.

Since $j^{“}S_{\alpha,i} \subseteq j(S_{\alpha,i}) \cap \nu$, this means that $j(S_{\alpha,i})$ reflects at ν in $V[H]$, and hence in M . Let $\bar{T} = j(\bar{S}) = \langle T_{\alpha,i} \mid \alpha < j(\lambda), i < \mu \rangle$. Since in M , the cofinality of ν is μ and $j^{“}\lambda$ is closed under limits of cofinality less than μ , there is in M a club $F \subseteq \nu$ of order type μ with $F \subseteq j^{“}\lambda$ —recall that $j^{“}\lambda \in M$, by clause (3) of Definition 3.3. Thus, ν and F witness that

$$M \models \exists \nu' < j(\lambda) \exists F' (F' \subseteq \nu' \text{ is club, } \text{cf}(\nu') = \mu \\ \text{and } \forall \alpha \in F' \forall i < \mu \quad T_{\alpha,i} \text{ reflects at } \nu').$$

This uses the fact that $F \subseteq j^{“}\lambda$, so that if $\alpha \in F$, then $\alpha = j(\bar{\alpha})$ for some $\bar{\alpha} < \lambda$, and hence $T_{\alpha,i} = j(S_{\bar{\alpha},i})$ reflects at ν . The same statement is then true in V about \bar{S} , by the elementarity of j . But this means that, in V there are $\nu' < \lambda$ and F' such

that $\text{cf}(v') = \mu$, $F' \subseteq v'$ is club, $\text{otp}(F') = \mu$, and, for all $\alpha \in F'$ and all $i < \mu$, $S_{\alpha,i}$ reflects at v' , as wished. \dashv

COROLLARY 3.5. *Suppose that μ is an uncountable regular cardinal and $\kappa = \mu^+$ is indestructibly generically supercompact. Then if G is generic for a κ -directed closed notion of forcing and if $\lambda > \kappa$ is regular in $V[G]$, then the principle $\text{DSR}(<\kappa, S_\omega^\lambda)$ holds in $V[G]$.*

REMARK 3.6. The only way in which we used the assumption that we were only working with stationary subsets of S_ω^λ in the above proofs is the fact that the stationarity of subsets of S_ω^λ is preserved by μ -closed forcing. This property is also satisfied by all stationary subsets of $S_{<\mu}^\lambda$ that lie in the approachability ideal $I[\lambda]$. If v is a regular cardinal and $v^+ < \lambda$, then there is a stationary set $\Sigma \subseteq S_v^\lambda$ with $\Sigma \in I[\lambda]$ (cf. [5, Section 3.3]). Therefore, if $v < \mu$ is a fixed regular cardinal, then we can find a stationary $\Sigma \subseteq S_v^\lambda$ such that, in the conclusions of Lemma 3.4 and Corollary 3.5, $\text{DSR}(<\kappa, S_\omega^\lambda)$ can be strengthened to $\text{DSR}(<\kappa, S_\omega^\lambda \cup \Sigma)$. This will be relevant below, in particular in Theorem 3.21.

3.2. Forcing preliminaries. In this subsection, we introduce some basic forcing notions and facts that will be used in our separation results. An essential tool will be the following canonical forcing notion to destroy the stationarity of a set by “shooting a club through its complement.”

DEFINITION 3.7. Let λ be an uncountable regular cardinal, and let $S \subseteq \lambda$ be unbounded. The forcing notion \mathbb{T}_S consists of all closed, bounded subsets t of λ such that $t \cap S = \emptyset$. The ordering is defined by setting, for $t_0, t_1 \in \mathbb{T}_S$, $t_1 \leq_{\mathbb{T}_S} t_0$ if t_1 end-extends t_0 , that is, $t_1 \cap \sup\{\xi + 1 \mid \xi \in t_0\} = t_0$.

In many natural cases, the forcing \mathbb{T}_S is λ -distributive, i.e., forcing with it does not add any new sequences of ordinals of length less than λ . For example, if $\lambda \setminus S$ is fat, meaning that for every club $C \subseteq \lambda$ and every $\alpha < \lambda$, there is a closed set $D \subseteq C \cap (\lambda \setminus S)$ of order type α , and if λ is inaccessible, or $\lambda = \mu^+$, where μ is regular and $\mu^{<\mu} = \mu$, then \mathbb{T}_S is λ -distributive and adds a club subset of λ that is disjoint from S ; see [2, Theorem 1]. In particular, this is the case if $\lambda = \omega_1$ and $\omega_1 \setminus S$ is stationary. So in these situations, the assumptions of the following lemma are satisfied. The lemma is entirely standard, but as we could not find this precise formulation in the literature, we provide a proof for completeness.

LEMMA 3.8. *Let λ be an uncountable regular cardinal, and let $S \subseteq \lambda$ be such that both S and $\lambda \setminus S$ are unbounded in λ .*

- (1) *Let G be \mathbb{T}_S -generic over V . Then $\bigcup G$ is a club subset of λ that is disjoint from S .*
- (2) *Suppose that \mathbb{T}_S preserves the fact that λ has uncountable cofinality (in our applications, \mathbb{T}_S will be λ -distributive, so this will certainly be the case). Then, for a set $T \subseteq \lambda$ such that $T \setminus S \subseteq S_\omega^\lambda$, the following are equivalent:*
 - (a) $\Vdash_{\mathbb{T}_S}$ “ \check{T} is stationary in $\check{\lambda}$.”
 - (b) *There is a $t \in \mathbb{T}_S$ such that $t \Vdash_{\mathbb{T}_S}$ “ \check{T} is stationary in $\check{\lambda}$.”*
 - (c) $T \setminus S$ is stationary.

PROOF. Claim (1) is immediate, since the fact that $\lambda \setminus S$ is unbounded implies that, for every $\alpha < \lambda$, the set $\{p \in \mathbb{T}_S \mid \max(p) > \alpha\}$ is dense in \mathbb{T}_S .

Let us now prove claim (2).

The implication (a) \implies (b) is obvious.

For the implication (b) \implies (c), let $t \in \mathbb{T}_S$ be as in (b), suppose T is stationary in $V[G]$, where G is generic for \mathbb{T}_S with $t \in G$, and assume that $T \setminus S$ is not stationary in V . Let C be a club subset of λ disjoint from $T \setminus S$. Let $D = \bigcup G$, so D is a club subset of λ disjoint from S , by (1). Since \mathbb{T}_S preserves the fact that λ has uncountable cofinality, $C \cap D$ is a club subset of λ . But $T = (T \cap S) \dot{\cup} (T \setminus S)$, $(T \cap S) \cap D = \emptyset$, and $(T \setminus S) \cap C = \emptyset$, so $T \cap (C \cap D) = \emptyset$, contradicting the fact that T is stationary in $V[G]$.

For the implication (c) \implies (a), assume that $T \setminus S \subseteq S_\omega^\lambda$ is stationary, but suppose that there are a $\dot{C} \in V^{\mathbb{T}_S}$ and a condition $t \in \mathbb{T}$ forcing that \dot{C} is a club subset of $\check{\lambda}$ disjoint from \check{T} . Since $T \setminus S$ is stationary, we can find an elementary submodel $M \prec H_\theta$, where θ is some sufficiently large regular cardinal, such that the cardinality of M is less than λ , $\delta = M \cap \lambda \in T \setminus S$, and M contains all relevant objects, including t, S, T , and \dot{C} . Since $\delta \in (T \setminus S) \subseteq S_\omega^\lambda$, we can fix a strictly increasing sequence of ordinals $\langle \xi_n \mid n < \omega \rangle$ cofinal in δ . We can then simultaneously define sequences $\langle t_n \mid n < \omega \rangle$ and $\langle \alpha_n \mid n < \omega \rangle$ by recursion on n so that the following conditions are satisfied for all $n < \omega$:

- (1) $\xi_n < \alpha_n < \alpha_{n+1} < \delta$,
- (2) $t_n \in \mathbb{T}_S \cap M$,
- (3) $t_{n+1} \leq t_n, t_0 \leq t, \max(t_n) > \xi_n$,
- (4) $t_n \Vdash_{\mathbb{T}_S} \check{\alpha}_n \in \dot{C}$.

For the construction, in the case $n = 0$, working in M , since t forces that \dot{C} is unbounded in δ , there are a $t_0 \leq t$ and an $\alpha_0 > \xi_0$ such that t_0 forces that $\check{\alpha}_0 \in \dot{C}$. t_0 may be chosen so that $\max(t_0) > \xi_0$, since $\delta \setminus S$ is unbounded in δ , by elementarity. Similarly, if t_n, α_n have already been defined, then, again working in M , there are a $t_{n+1} \leq t_n$ with $\max(t_{n+1}) > \xi_{n+1}$ and an $\alpha_{n+1} > \xi_{n+1}$ such that t_{n+1} forces that $\check{\alpha}_{n+1} \in \dot{C}$. This completes the construction, and we can set $t^* = (\bigcup_{n < \omega} t_n) \cup \{\delta\}$. Since $\delta \notin S$, it follows that $t^* \in \mathbb{T}_S$, and since $t^* \leq t_n$, it follows that t^* forces that $\alpha_n \in \dot{C}$, for every $n < \omega$. Since for all $n < \omega, \delta > \alpha_n > \xi_n$, it follows that $\sup_{n < \omega} \alpha_n = \delta$. Thus, since t^* forces that \dot{C} is closed below $\check{\lambda}$, we have that t^* forces $\check{\delta} \in \dot{C} \cap \check{T}$, a contradiction. ⊥

We briefly recall the concept of strategic closure.

DEFINITION 3.9. Let \mathbb{P} be a notion of forcing, and let β be an ordinal. In the game $\ominus_\beta(\mathbb{P})$, two players collaborate to play a $\leq_{\mathbb{P}}$ -descending sequence $\langle p_\alpha \mid \alpha < \beta \rangle$ as follows. Player II plays at even stages of the game (including limit stages) and in round 0 must play $p_0 = 1_{\mathbb{P}}$. If, during the course of play, a limit ordinal $\alpha < \beta$ is reached such that $\langle p_\xi \mid \xi < \alpha \rangle$ has no lower bound in \mathbb{P} (and hence Player II cannot continue playing), then Player I wins. Otherwise, Player II wins. \mathbb{P} is β -strategically closed if Player II has a winning strategy in the game $\ominus_\beta(\mathbb{P})$.

Observe that if \mathbb{P} is λ -strategically closed for some regular cardinal λ , then \mathbb{P} is λ -distributive.

3.3. Separation results. We are now ready to turn to our various separation results. Before jumping into their proofs, we collect a summary of the results here for the convenience of the reader. Recall the following sequence of implications from the introduction, where S is a stationary subset of some regular cardinal λ and $\kappa < \lambda$ is a cardinal:

$$\text{DSR}(\langle \kappa, S \rangle) \Rightarrow \text{sDSR}(\langle \kappa, S \rangle) \Rightarrow \text{uDSR}(\langle \kappa, S \rangle) \Rightarrow \text{Refl}(\langle \kappa, S \rangle).$$

We prove here that, in general, none of the arrows in this diagram are reversible (in fact, our results will be stronger than this). We also prove that the arrows in the diagram are optimal in the sense that $\text{sDSR}(\langle \kappa, S \rangle)$ does not imply $\text{Refl}(\kappa, S)$ (and, with one known exception given by Lemma 2.5 and one possible exception discussed in Question 6.1, neither does $\text{DSR}(\langle \kappa, S \rangle)$). In particular, we will in turn construct models showing that, modulo large cardinal hypotheses, each of the following conjunctions is consistent (see the respective theorems for more precise statements and the hypotheses that must be satisfied by κ and λ):

- (Theorem 3.10) $\text{Refl}(\langle \kappa, S_\omega^\lambda \rangle) \wedge \neg \text{uDSR}(1, S_\omega^\lambda)$;
- (Theorem 3.15) $\text{uDSR}(\langle \kappa, S_\omega^\lambda \rangle) \wedge \neg \text{sDSR}(1, S_\omega^\lambda)$;
- (Theorem 3.19) $\text{sDSR}(\langle \kappa, S_\omega^\lambda \rangle) \wedge \neg \text{Refl}(\kappa, S_\omega^\lambda)$;
- (Theorem 3.21) $\text{sDSR}(\langle \kappa, S_\omega^\lambda \rangle) \wedge \neg \text{DSR}(1, S_\omega^\lambda)$;
- (Theorem 3.28) $\text{DSR}(\langle \kappa, S_\omega^\lambda \rangle) \wedge \neg \text{Refl}(\kappa, S_\omega^\lambda)$ (assuming $\omega_1 < \kappa$).

Without further ado, let us jump into the proofs. Our first result shows that even the strongest simultaneous reflection principles do not imply unbounded diagonal reflection.

THEOREM 3.10. *Suppose that $\lambda = \lambda^{<\lambda} \geq \aleph_2$ is a regular cardinal, $1 < \kappa \leq \lambda$, and $\text{Refl}^*(\langle \kappa, S_\omega^\lambda \rangle)$ holds. Then there is a cofinality-preserving forcing extension in which*

- (1) $\text{Refl}(\langle \kappa, S_\omega^\lambda \rangle)$ holds;
- (2) $\text{uDSR}(1, S_\omega^\lambda)$ fails.

PROOF. Let $S = S_\omega^\lambda$, and let \mathbb{P} be the forcing poset whose conditions are all functions of the form $p : \gamma^p \times \gamma^p \rightarrow 2$ such that

- $\gamma^p < \lambda$;
- for each $\alpha < \gamma^p$, letting $S_\alpha^p = \{\eta < \gamma^p \mid p(\alpha, \eta) = 1\}$, we have
 - $S_\alpha^p \subseteq S \setminus (\alpha + 1)$;
 - for all $\beta \in S_{>\omega}^{\gamma^p+1}$, the set

$$\{\alpha < \beta \mid S_\alpha^p \cap \beta \text{ is stationary in } \beta\}$$

is bounded below β ;

- for all $\alpha < \beta < \gamma^p$, we have $S_\alpha^p \cap S_\beta^p = \emptyset$.

For all $p, q \in \mathbb{P}$, we say that $q \leq_{\mathbb{P}} p$ if $\gamma^q \geq \gamma^p$ and $q \upharpoonright (\gamma^p \times \gamma^p) = p$.

CLAIM 3.11. \mathbb{P} is λ -strategically closed.

PROOF. We describe a winning strategy for Player II in $\mathcal{D}_\lambda(\mathbb{P})$. In a run of $\mathcal{D}_\lambda(\mathbb{P})$, the players will produce a $\leq_{\mathbb{P}}$ -descending sequence $\bar{p} = \langle p_\eta \mid \eta < \lambda \rangle$. For each $\eta \in \text{lim}(\lambda)$, we will let $\delta_\eta = \sup\{\gamma^{p_\xi} \mid \xi < \eta\}$ and then let $D = \{\delta_\eta \mid \eta \in \text{lim}(\lambda)\}$. D will be a club in λ . Also, for each $\alpha < \lambda$, we will let $S_\alpha = \bigcup_{\eta < \lambda} S_\alpha^{p_\eta}$. Player II will play in a

way that ensures that, for all $\alpha < \lambda$, $S_\alpha \cap D = \emptyset$. Since $S_\alpha \cap (\alpha + 1) = \emptyset$ for all $\alpha < \lambda$, this amounts to ensuring that, for all $\eta \in \text{lim}(\lambda)$ and all $\alpha < \delta_\eta$, we have $\delta_\eta \notin S_\alpha$.

To start, Player II must play the empty condition as p_0 . Next, suppose that $\eta < \lambda$ is an odd ordinal and $\langle p_\xi \mid \xi \leq \eta \rangle$ has been played. Player II then plays any condition $p_{\eta+1}$ with $\gamma^{p_{\eta+1}} > \gamma^{p_\eta}$. Finally, suppose that $\eta < \lambda$ is a limit ordinal and $\langle p_\xi \mid \xi < \eta \rangle$ has been played. Recall that $\delta_\eta = \sup\{\gamma^{p_\xi} \mid \xi < \eta\}$. Player II will play a condition p_η with $\gamma^{p_\eta} = \delta_\eta + 1$, defined by letting $p_\eta \upharpoonright (\delta_\eta \times \delta_\eta) = \bigcup_{\xi < \eta} p_\xi$ and $p_\eta(\alpha, \eta) = 0$ for all $(\alpha, \eta) \in (\gamma^{p_\eta} \times \gamma^{p_\eta}) \setminus (\delta_\eta \times \delta_\eta)$. To see that p_η is indeed a condition, it remains to show that, if $\text{cf}(\delta_\eta) > \omega$, then the set

$$\{\alpha < \delta_\eta \mid S_\alpha^{p_\eta} \cap \delta_\eta \text{ is stationary in } \delta_\eta\}$$

is bounded below δ_η . But, in fact, this set is empty: if $\text{cf}(\delta_\eta) > \omega$, then $D \cap \delta_\eta = \{\delta_\xi \mid \xi \in \text{lim}(\eta)\}$ is a club in δ_η , and Player II's previous plays at limit stages have ensured that, for every $\alpha < \delta_\eta$, we have $S_\alpha^{p_\eta} \cap D \cap \delta_\eta = \emptyset$, so $S_\alpha^{p_\eta} \cap \delta_\eta$ is nonstationary in δ_η . It follows that p_η is indeed a condition in \mathbb{P} and that this describes a winning strategy for Player II in $\mathcal{D}_\lambda(\mathbb{P})$. \dashv

Note that it is implicit in this proof that \mathbb{P} is countably closed. It is now straightforward to see that, for every $\alpha < \lambda$, the set $E_\alpha = \{p \in \mathbb{P} \mid \alpha < \gamma^p\}$ is a dense open subset of \mathbb{P} . Suppose that G is a \mathbb{P} -generic filter over V . It follows that $f = \bigcup G$ is a function from $\lambda \times \lambda$ to 2. For $\alpha < \lambda$, let $S_\alpha = \{\eta < \lambda \mid f(\alpha, \eta) = 1\}$.

$V[G]$ is our desired model. Since \mathbb{P} is λ -strategically closed and of size $\lambda = \lambda^{<\lambda}$, we know that $V[G]$ is a cofinality-preserving forcing extension of V . The following claim will allow us to conclude that the sequence $\langle S_\alpha \mid \alpha < \lambda \rangle$ witnesses the failure of $\text{uDSR}(1, S_\omega^\lambda)$.

CLAIM 3.12. *In $V[G]$, for all $\alpha < \lambda$, S_α is a stationary subset of S .*

PROOF. Fix $\alpha < \lambda$. The fact that S_α is a subset of S follows immediately from the definition of \mathbb{P} . To see that S_α is stationary in $V[G]$, work in V , let \dot{S}_α be the canonical \mathbb{P} -name for S_α , let $p \in \mathbb{P}$, and let \dot{C} be a \mathbb{P} -name such that $p \Vdash_{\mathbb{P}} \text{“}\dot{C} \text{ is a club in } \dot{\lambda}\text{”}$. We will find a condition $q \leq p$ and an ordinal $\eta \in S$ such that $q \Vdash_{\mathbb{P}} \text{“}\check{\eta} \in \dot{S}_\alpha \cap \dot{C}\text{”}$.

We will define a decreasing sequence of conditions $\langle p_k \mid k \leq \omega \rangle$ and an increasing sequence of ordinals $\langle \eta_k \mid k < \omega \rangle$. To begin, let p_0 be any condition extending p such that $\gamma^{p_0} > \alpha$. If $k < \omega$ and p_k has been specified, let p_{k+1} be any extension of p_k such that there is an ordinal η_k for which

- $\gamma^{p_k} < \eta_k < \gamma^{p_{k+1}}$ and
- $p_{k+1} \Vdash_{\mathbb{P}} \text{“}\check{\eta}_k \in \dot{C}\text{”}$.

This is straightforward given the discussion in the paragraph preceding the claim and the fact that p forces \dot{C} to be unbounded in λ .

After $\langle p_k \mid k < \omega \rangle$ has been defined, let $\eta = \sup\{\gamma^{p_k} \mid k < \omega\} = \sup\{\eta_k \mid k < \omega\}$, and define a condition p_ω with $\gamma^{p_\omega} = \eta + 1$ by letting

- $p_\omega \upharpoonright (\eta \times \eta) = \bigcup_{k < \omega} p_k$;
- $p_\omega(\alpha, \eta) = 1$;
- $p_\omega(\beta, \xi) = 0$ for all $(\beta, \xi) \in (\gamma^{p_\omega} \times \gamma^{p_\omega}) \setminus ((\eta \times \eta) \cup \{(\alpha, \eta)\})$.

It is clear that p_ω is a lower bound for $\langle p_k \mid k < \omega \rangle$. Also, since $p_\omega(\alpha, \eta) = 1$, we have $p_\omega \Vdash_{\mathbb{P}} \text{“}\check{\eta} \in \dot{S}_\alpha\text{”}$. Moreover, for all $k < \omega$, $p_\omega \Vdash_{\mathbb{P}} \text{“}\check{\eta}_k \in \dot{C}\text{”}$, so, since p_ω extends p and p forces \dot{C} to be a club, we also have $p_\omega \Vdash_{\mathbb{P}} \text{“}\check{\eta} \in \dot{C}\text{”}$, as desired. \dashv

The fact that $\langle S_\alpha \mid \alpha < \lambda \rangle$ witnesses the failure of $\text{uDSR}(1, S_\omega^\lambda)$ in $V[G]$ now follows immediately from the previous claim and the definition of \mathbb{P} . It remains to verify that $\text{Refl}(\langle \kappa, S \rangle)$ holds. To do this, we need to introduce some auxiliary forcing notions. In $V[G]$, for all $\beta < \lambda$, let $S_{\geq \beta} = \bigcup_{\beta \leq \gamma < \lambda} S_\gamma$, and let $\mathbb{T}_\beta = \mathbb{T}_{S_{\geq \beta}}$ be the forcing to shoot a club through λ disjoint from $S_{\geq \beta}$, introduced in Definition 3.7.

CLAIM 3.13. *In V , for all $\beta < \lambda$, $\mathbb{P} * \dot{\mathbb{T}}_\beta$ has a dense λ -directed closed subset of size λ .*

PROOF. Fix $\beta < \lambda$. For notational simplicity, let us assume that the name $\dot{\mathbb{T}}_\beta$ is closed under equivalent names, in the sense that it has the following property: whenever $p \in \mathbb{P}$, i and \dot{u} are such that $(p, i) \in \mathbb{P} * \dot{\mathbb{T}}_\beta$, $\dot{u} \in V^\mathbb{P} \cap H_\lambda$, and $p \Vdash_{\mathbb{P}} i = \dot{u}$, it follows that $(p, \dot{u}) \in \mathbb{P} * \dot{\mathbb{T}}_\beta$. (In the future, we will assume without comment that names for forcing posets are closed under equivalent names.) Let \mathbb{U}_β be the set of conditions in $\mathbb{P} * \dot{\mathbb{T}}_\beta$ of the form (p, \dot{t}) with $\gamma^p = \max(t) + 1 > \beta$. We will show that \mathbb{U}_β has the desired properties.

First, since $\lambda^{<\lambda} = \lambda$, it follows that \mathbb{U}_β has size λ .

To see that \mathbb{U}_β is dense, fix $(p_0, i_0) \in \mathbb{P} * \dot{\mathbb{T}}_\beta$. Since \mathbb{P} is λ -strategically closed and hence does not add new bounded subsets of λ , by strengthening p_0 if necessary, we can assume that p_0 decides the value of i_0 , and hence we can assume that i_0 is of the form \check{t}_0 , where $t_0 \in V$. We can also assume that $\gamma^{p_0} > \max\{\max(t_0), \beta\}$. Now define a condition $p \leq p_0$ with $\gamma^p = \gamma^{p_0} + 1$ and $p(\alpha, \gamma^{p_0}) = 0$ for all $\alpha < \gamma^p$. Next, let $t = t_0 \cup \{\gamma^{p_0}\}$. Then $(p, \dot{t}) \in \mathbb{U}_\beta$ extends (p_0, i_0) , showing that \mathbb{U}_β is dense.

We next show that \mathbb{U}_β is λ -directed closed. Note that \mathbb{U}_β is *tree-like*, i.e., if $u, v, w \in \mathbb{U}_\beta$ and w extends both u and v , then u and v are $\leq_{\mathbb{U}_\beta}$ -comparable. It thus suffices to show that \mathbb{U}_β is λ -closed. To this end, let $\eta < \lambda$ be a limit ordinal, and suppose that $\langle (p_\xi, \check{t}_\xi) \mid \xi < \eta \rangle$ is a strictly decreasing sequence of conditions from \mathbb{U}_β .

Let $\delta = \sup\{\gamma^{p_\xi} \mid \xi < \eta\}$. We begin by defining a condition $p \in \mathbb{P}$ extending $\langle p_\xi \mid \xi < \eta \rangle$ with $\gamma^p = \delta + 1$. We do this by letting $p \upharpoonright (\delta \times \delta) = \bigcup_{\xi < \eta} p_\xi$ and $p(\alpha, \varepsilon) = 0$ for all $(\alpha, \varepsilon) \in (\gamma^p \times \gamma^p) \setminus (\delta \times \delta)$. To verify that p is a condition, it suffices to show that, if $\text{cf}(\delta) > \omega$, then $\{\alpha < \delta \mid S_\alpha^p \cap \delta \text{ is stationary in } \delta\}$ is bounded below δ . In fact, this set does not contain any ordinals greater than β , which will yield the desired conclusion since $\delta > \gamma^{p_0} > \beta$. To see this, note that, if $\text{cf}(\delta) > \omega$, then $t' = \bigcup_{\xi < \eta} t_\xi$ is a club in δ and, by the definition of \mathbb{T}_β , it follows that, for all α with $\beta \leq \alpha < \delta$, we have $S_\alpha^p \cap t' = \emptyset$, so $S_\alpha^p \cap \delta$ is nonstationary in δ .

Finally, let $t = t' \cup \{\delta\}$. Then $(p, \dot{t}) \in \mathbb{U}_\beta$ is a lower bound for the sequence given at the outset, thus showing that \mathbb{U}_β is λ -directed closed, which completes the proof of the claim. ⊖

Since \mathbb{P} is λ -strategically closed in V and, for all $\alpha < \lambda$, $\mathbb{P} * \mathbb{T}_\alpha$ has a λ -directed closed dense subset, it follows that \mathbb{T}_α is λ -distributive in $V[G]$. In particular, Lemma 3.8 applies, showing that the following are equivalent in $V[G]$, for a stationary subset T of S :

- $\Vdash_{\mathbb{T}_\alpha}$ “ \check{T} is stationary in $\check{\lambda}$ ”.
- $T \setminus S_{\geq \alpha}$ is stationary.

Also note that, if $\alpha \leq \beta < \lambda$ and $T \setminus S_{\geq \alpha}$ is stationary, then trivially $T \setminus S_{\geq \beta}$ is also stationary. With this in mind, the following claim will be instrumental in proving that $\text{Refl}(\langle \kappa, S \rangle)$ holds in $V[G]$.

CLAIM 3.14. *In $V[G]$, for every stationary $T \subseteq S$, there is $\alpha < \lambda$ such that $T \setminus S_{\geq \alpha}$ is stationary.*

PROOF. Fix a stationary $T \subseteq S$, and suppose to the contrary that, for every $\alpha < \lambda$, $T \setminus S_{\geq \alpha}$ is nonstationary. Then, for each $\alpha < \lambda$, there is a club C_α in λ such that $C_\alpha \cap T \subseteq S_{\geq \alpha}$. Let $C = \Delta_{\alpha < \lambda} C_\alpha$. Then C is a club in λ , so we can fix some $\beta \in C \cap T$. Then $\beta \in T \cap \bigcap_{\alpha < \beta} S_{\geq \alpha} = T \cap S_{\geq \beta}$. But, for all γ with $\beta \leq \gamma < \lambda$, we have $S_\gamma \cap (\gamma + 1) = \emptyset$, so $\beta \notin S_{\geq \beta}$. This is a contradiction. \dashv

To see that $\text{Refl}(<\kappa, S)$ holds in $V[G]$, fix $\mu < \kappa$ and a sequence $\langle T_\zeta \mid \zeta < \mu \rangle$ of stationary subsets of S . By Claim 3.14, for each $\zeta < \mu$, there is $\alpha_\zeta < \lambda$ such that $T_\zeta \setminus S_{\geq \alpha_\zeta}$ is stationary. Let $\beta = \sup\{\alpha_\zeta \mid \zeta < \mu\}$. Since $\mu < \kappa \leq \lambda$, we have $\beta < \lambda$, and $T_\zeta \setminus S_{\geq \beta}$ is stationary for all $\zeta < \mu$. Let H be \mathbb{T}_β -generic over $V[G]$.

By Claim 3.13, in V , $\mathbb{P} * \mathbb{T}_\beta$ has a dense λ -directed closed subset of size λ . Since $\text{Refl}^*(<\kappa, S)$ holds in V , this implies that $\text{Refl}(<\kappa, S)$ holds in $V[G * H]$. For all $\zeta < \mu$, since $T_\zeta \setminus S_{\geq \beta}$ is stationary in $V[G]$, we know that T_ζ is stationary in $V[G * H]$. It follows that, in $V[G * H]$, there is $\delta \in S_{> \omega}^\lambda$ such that $T_\zeta \cap \delta$ is stationary in δ for all $\zeta < \mu$. Since stationarity is downward absolute, the same holds in $V[G]$, so $\langle T_\zeta \mid \zeta < \mu \rangle$ reflects simultaneously in $V[G]$. Since this sequence was chosen arbitrarily, it follows that $\text{Refl}(<\kappa, S)$ holds in $V[G]$. \dashv

We now show that even simultaneous unbounded diagonal reflection does not imply stationary diagonal reflection. Note that, by Lemma 2.4, the assumption that $\kappa < \lambda$ in the following theorem is necessary.

THEOREM 3.15. *Suppose that $\lambda = \lambda^{<\lambda} \geq \aleph_2$ is a regular cardinal, κ is a cardinal with $1 < \kappa < \lambda$, and $\text{uDSR}^*(<\kappa, S_\omega^\lambda)$ holds. Then there is a cofinality-preserving forcing extension in which*

- (1) $\text{uDSR}(<\kappa, S_\omega^\lambda)$ holds;
- (2) $\text{sDSR}(1, S_\omega^\lambda)$ fails.

PROOF. Let $S = S_\omega^\lambda$, and let \mathbb{P} be the forcing poset whose conditions are all functions of the form $p : \gamma^p \times \gamma^p \rightarrow 2$ such that

- $\gamma^p < \lambda$;
- for each $\alpha < \gamma^p$, letting $S_\alpha^p = \{\eta < \gamma^p \mid p(\alpha, \eta) = 1\}$, we have
 - $S_\alpha^p \subseteq S \setminus (\alpha + 1)$;
 - for all $\beta \in S_{> \omega}^{\gamma^p + 1}$, the set

$$\{\alpha < \beta \mid S_\alpha^p \cap \beta \text{ is stationary in } \beta\}$$

- is nonstationary in β ;
- for all $\alpha < \beta < \gamma^p$, we have $S_\alpha^p \cap S_\beta^p = \emptyset$.

For $p, q \in \mathbb{P}$, we say that $q \leq_{\mathbb{P}} p$ if and only if $\gamma^q \geq \gamma^p$ and $q \upharpoonright (\gamma^p \times \gamma^p) = p$.

Let G be \mathbb{P} -generic over V and, for $\alpha < \lambda$, let $S_\alpha = \bigcup_{p \in G} S_\alpha^p$. Arguments exactly as in the proofs of Claims 3.11 and 3.12 yield the truth of the following statements:

- In V , \mathbb{P} is λ -strategically closed.
- In $V[G]$, for every $\alpha < \lambda$, S_α is stationary in λ . Moreover, if we let $S_{-1} = S \setminus (\bigcup_{\alpha < \lambda} S_\alpha)$, then S_{-1} is stationary as well.

In $V[G]$, let \mathbb{T} be the forcing notion whose conditions are all closed, bounded subsets t of λ such that, for all $\alpha \in t$, we have $t \cap S_\alpha = \emptyset$. As before, these conditions are ordered by end-extension. Let $\check{\mathbb{T}} \in V$ be a canonical \mathbb{P} -name for \mathbb{T} .

CLAIM 3.16. *In V , $\mathbb{P} * \check{\mathbb{T}}$ has a dense λ -directed closed subset of size λ .*

PROOF. Define \mathbb{U} to consist of all conditions in $\mathbb{P} * \check{\mathbb{T}}$ that are of the form (p, \check{t}) with the property that $\gamma^p = \max(t) + 1$. Since $\lambda^{<\lambda} = \lambda$, the cardinality of \mathbb{U} is λ .

To see that \mathbb{U} is dense, fix $(p_0, \check{t}_0) \in \mathbb{P} * \check{\mathbb{T}}$. Since \mathbb{P} is λ -strategically closed, we can assume, by extending p_0 if necessary, that p_0 decides \check{t}_0 , and hence we may assume that \check{t}_0 is of the form \check{t}_0 . We may also assume that $\gamma^{p_0} > \max(t_0)$. Now define a condition $p \leq_{\mathbb{P}} p_0$ by letting $\gamma^p = \gamma^{p_0} + 1$, $p \upharpoonright (\gamma^{p_0} \times \gamma^{p_0}) = p_0$, and $p(\alpha, \gamma^{p_0}) = p(\gamma^{p_0}, \alpha) = 0$ for all $\alpha \leq \gamma^p$. Next, let $t = t_0 \cup \{\gamma^{p_0}\}$. Then (p, \check{t}) is a condition in \mathbb{U} extending (p_0, \check{t}_0) , showing that \mathbb{U} is dense.

We next show that \mathbb{U} is λ -directed closed. Since \mathbb{U} is tree-like, it suffices to show λ -closure. To this end, let $\eta < \lambda$ be a limit ordinal, and suppose that $\langle (p_\xi, \check{t}_\xi) \mid \xi < \eta \rangle$ is a strictly decreasing sequence of conditions from \mathbb{U} . Let $\delta = \sup\{\gamma^{p_\xi} \mid \xi < \eta\} = \sup\{\max(\check{t}_\xi) \mid \xi < \eta\}$.

We first construct a condition $p \in \mathbb{P}$ such that p is a lower bound for $\langle p_\xi \mid \xi < \eta \rangle$ and $\gamma^p = \delta + 1$. To do this, simply let $p \upharpoonright (\delta \times \delta) = \bigcup_{\xi < \eta} p_\xi$ and $p(\alpha, \zeta) = 0$ for all $(\alpha, \zeta) \in (\gamma^p \times \gamma^p) \setminus (\delta \times \delta)$. The only nontrivial statement to check to verify that p is indeed a condition is that, if $\text{cf}(\delta) > \omega$, then the set

$$X = \{\alpha < \delta \mid S_\alpha^p \cap \delta \text{ is stationary in } \delta\}$$

is nonstationary in δ . But notice that, if $\text{cf}(\delta) > \omega$, then $t_\infty = \bigcup_{\xi < \eta} t_\xi$ is a club in δ and, for all $\alpha \in t_\infty$, we have $t_\infty \cap S_\alpha^p = \emptyset$. Therefore, X is disjoint from t_∞ and hence nonstationary in δ . Letting $t = t_\infty \cup \{\delta\}$, it now follows that (p, \check{t}) is a condition in \mathbb{U} that is a lower bound for the given sequence. \dashv

$V[G]$ is our desired model. It is clear from what has been written that, in $V[G]$, $\langle S_\alpha \mid \alpha < \lambda \rangle$ is a witness to the failure of $\text{sDSR}(1, S)$. To verify that $\text{uDSR}(<\kappa, S)$ holds, the following claim will be useful.

CLAIM 3.17. *In $V[G]$, suppose that $-1 \leq \varepsilon < \lambda$, R is a stationary subset of S_ε , and $t_0 \in \mathbb{T}$ is a condition such that $\varepsilon \notin t_0$ and $\max(t_0) > \varepsilon$. Then $t_0 \Vdash_{\mathbb{T}} \check{R}$ is stationary in $\check{\lambda}$.*

PROOF. Work in $V[G]$. Let $t \leq_{\mathbb{T}} t_0$ be arbitrary, and let \check{C} be a \mathbb{T} -name forced by t to be a club in λ . It will suffice to find $r \leq_{\mathbb{T}} t$ and $\delta \in R$ such that $r \Vdash_{\mathbb{T}} \check{\delta} \in \check{C}$.

We will proceed much as in the proof of Lemma 3.8. Let θ be a sufficiently large regular cardinal, let \triangleleft be a fixed well-ordering of H_θ , and let M be an elementary submodel of $(H_\theta, \in, \triangleleft)$ of size less than λ such that

- $\{\mathbb{T}, t, \check{C}, R\} \subseteq M$;
- $\delta := \sup(M \cap \lambda) \in R$.

Let $\langle \delta_n \mid n < \omega \rangle$ enumerate a cofinal subset of $M \cap \lambda$. Now, working at each step inside M , recursively construct a decreasing sequence of conditions $\langle r_n \mid n < \omega \rangle$ from $\mathbb{T} \cap M$ together with an increasing sequence $\langle \xi_n \mid n < \omega \rangle$ of ordinals from $M \cap \lambda$ such that

- $r_0 \leq_{\mathbb{T}} t$;

- for each $n < \omega$, we have $\delta_n < \min\{\max(r_n), \xi_n\}$;
- for each $n < \omega$, $r_n \Vdash_{\mathbb{T}} \check{\xi}_n \in \dot{C}$.

The construction is straightforward, using the fact that t forces \dot{C} to be unbounded in λ . Notice that $\delta = \sup\{\max(r_n) \mid n < \omega\} = \sup\{\xi_n \mid n < \omega\}$. Also recall that $\delta \in R \subseteq S_\varepsilon$. Since $\varepsilon \notin t_0$ and $\max(t_0) > \varepsilon$, it follows that $r := \{\delta\} \cup \bigcup_{n < \omega} r_n$ is a condition in \mathbb{T} that is a lower bound for $\langle r_n \mid n < \omega \rangle$ and hence extends t . For each $n < \omega$, $r \Vdash_{\mathbb{T}} \check{\xi}_n \in \dot{C}$, so, since r forces \dot{C} to be a club in λ , we see that $r \Vdash_{\mathbb{T}} \check{\delta} \in \dot{C} \cap \check{R}$, as desired. \dashv

Note also that, letting $S_{\geq \alpha} = \bigcup_{\alpha \leq \beta < \lambda} S_\beta$ for each $\alpha < \lambda$, the proof of Claim 3.14 applies in this case as well to yield the fact that, in $V[G]$, for every stationary $R \subseteq S$, there is $\alpha < \lambda$ such that $R \setminus S_{\geq \alpha}$ is stationary. But for such an α , we then have $R \setminus S_{\geq \alpha} \subseteq \bigcup_{-1 \leq \beta < \alpha} S_\beta$, and hence, by the completeness of the nonstationary ideal, there is in fact some $-1 \leq \beta < \alpha$ such that $R \cap S_\beta$ is stationary.

To see that $\text{uDSR}(\langle \kappa, S \rangle)$ holds in $V[G]$, fix a matrix $\langle R_{\alpha,i} \mid \alpha < \lambda, i < j_\alpha \rangle$ of stationary subsets of S such that, for each $\alpha < \lambda$, we have $j_\alpha < \kappa$. We will find $\gamma \in S_{> \omega}^\lambda$ and an unbounded $F \subseteq \gamma$ such that, for all $\alpha \in F$ and all $i < j_\alpha$, we have that $R_{\alpha,i}$ reflects at γ .

For each $\alpha < \lambda$ and $i < j_\alpha$, let $\beta(\alpha,i) < \lambda$ be such that $R_{\alpha,i} \cap S_{\beta(\alpha,i)}$ is stationary. Set $B_\alpha = \{\beta(\alpha,i) \mid i < j_\alpha\}$, and let $\langle \ell_\xi^\alpha \mid \xi < \theta_\alpha \rangle$ enumerate B_α in increasing order. Fix a stationary set $\Sigma^* \subseteq \lambda$ on which the map $\alpha \mapsto (j_\alpha, \theta_\alpha)$ is constant, with value (j, θ) .

Let ρ be the least ordinal below θ such that the set $\{\ell_\rho^\alpha \mid \alpha \in \Sigma^*\}$ is unbounded in λ , if such an ordinal exists, and let $\rho = \theta$ otherwise. For all $\xi < \rho$, then, there is $\beta_\xi < \lambda$ such that $\ell_\xi^\alpha < \beta_\xi$ for all $\alpha \in \Sigma^*$. Let $\varepsilon = \sup_{\xi < \rho} \beta_\xi < \lambda$, and let $t = \{\varepsilon\} \in \mathbb{T}$.

CLAIM 3.18. For all $\zeta < \lambda$, the set

$$D_\zeta := \{t^* \in \mathbb{T} \mid \exists \alpha \in \Sigma^* \setminus \zeta [\forall i < j(t^* \Vdash_{\mathbb{T}} \check{R}_{\alpha,i} \text{ is stationary in } \check{\lambda})]\}$$

is dense in \mathbb{T} below t .

PROOF. Fix $\zeta < \lambda$ and $t' \leq_{\mathbb{T}} t$. Fix $\alpha \in \Sigma^*$ such that $\alpha \geq \zeta$ and, if $\rho < \theta$, then $\ell_\rho^\alpha > \max(t')$. Now find $\gamma > \max(t')$ with $\sup\{\ell_\xi^\alpha \mid \xi < \theta\} < \gamma < \lambda$ and $\gamma \in S_{-1}$, and let $t^* = t' \cup \{\gamma\}$. Then t^* is a condition in \mathbb{T} extending t' , and we claim that, for all $i < j$, $t^* \Vdash_{\mathbb{T}} \check{R}_{\alpha,i}$ is stationary in $\check{\lambda}$, and hence $t^* \in D_\zeta$.

To see this, fix $i < j$, and let $\xi < \theta$ be such that $\beta(\alpha,i) = \ell_\xi^\alpha$. If $\xi < \rho$, then $\beta(\alpha,i) < \varepsilon = \min(t^*)$. Therefore, since $R_{\alpha,i} \cap S_{\beta(\alpha,i)}$ is stationary, Claim 3.17 implies that t^* preserves the stationarity of $R_{\alpha,i} \cap S_{\beta(\alpha,i)}$ and hence, *a fortiori*, of $R_{\alpha,i}$ itself. If, on the other hand, $\rho \leq \xi < \theta$, then we have

$$\max(t') < \ell_\rho^\alpha \leq \ell_\xi^\alpha = \beta(\alpha,i) < \gamma = \min(t^* \setminus (\max(t') + 1)).$$

In particular, we have $\beta(\alpha,i) \notin t^*$ and $\beta(\alpha,i) < \max(t^*)$, so Claim 3.17 again implies that t^* preserves the stationarity of $R_{\alpha,i}$. \dashv

Let H be \mathbb{T} -generic over $V[G]$ with $t \in H$. Claim 3.18 implies that, in $V[G * H]$, the set

$$A := \{\alpha \in \Sigma^* \mid \text{for all } i < j, R_{\alpha,i} \text{ is stationary in } \lambda\}$$

is unbounded in λ . Let $f : \lambda \rightarrow A$ be the monotone enumeration of A , and let C be the set of closure points of f , that is, the set of $\eta < \lambda$ such that $f''\eta \subseteq \eta$. Note that C is a club in λ . Note also that since, in V , $\text{uDSR}^*(\langle \kappa, S \rangle)$ holds and $\mathbb{P} * \mathbb{T}$ has a dense λ -directed closed subset of cardinality λ , it follows that $\text{uDSR}(\langle \kappa, S \rangle)$ holds in $V[G * H]$. Therefore, we can apply this principle to the matrix $\langle R'_{\xi,i} \mid \xi < \lambda, i < j \rangle$ defined by $R'_{\xi,i} = R_{f(\xi),i} \cap C$, yielding an ordinal $\gamma \in S^>_{\omega}$ and an unbounded set $F' \subseteq \gamma$ such that, for all $\xi \in F'$ and all $i < j$, $R'_{\xi,i}$ reflects at γ , i.e., $R_{f(\xi),i} \cap C$ reflects at γ . In particular, $\gamma \in C$, and hence $F = f''F'$ is an unbounded subset of γ . Moreover, since $V[G]$ and $V[G * H]$ have the same bounded subsets of λ , we also know that $F \in V[G]$ and, in $V[G]$, for all $\alpha \in F$ and all $i < j$, $R_{\alpha,i} \cap \gamma$ is stationary in γ . We have thus found γ and F as desired, so we have shown that $\text{uDSR}(\langle \kappa, S \rangle)$ holds in $V[G]$. \dashv

We now show that stationary diagonal reflection implies no more simultaneous reflection than explicitly stated. Recall that this is not true of the full diagonal reflection principle, by Lemma 2.5.

THEOREM 3.19. *Suppose that $1 < \kappa < \lambda = \lambda^{<\lambda}$ are cardinals, $\lambda \geq \aleph_2$ is regular, and either $\kappa \geq \omega$ and $\text{DSR}^*(\langle \kappa, S^>_{\omega} \rangle)$ holds, or $\kappa < \omega$ and $\text{DSR}^*(\kappa, S^>_{\omega})$ holds. Then there is a cofinality-preserving forcing extension in which*

- (1) $\text{sDSR}(\langle \kappa, S^>_{\omega} \rangle)$ holds;
- (2) $\text{Refl}(\kappa, S^>_{\omega})$ fails.

PROOF. Let $S = S^>_{\omega}$, and let \mathbb{P} be the forcing poset whose conditions are all functions of the form $p : \kappa \times \gamma^p \rightarrow 2$ such that

- $\gamma^p < \lambda$;
- for each $\ell < \kappa$, letting $S^p_{\ell} = \{\eta < \gamma^p \mid p(\ell, \eta) = 1\}$, we have
 - $S^p_{\ell} \subseteq S$;
 - for all $\beta \in S^{\gamma^p+1}_{>\omega}$, there is $\ell < \kappa$ such that $S^p_{\ell} \cap \beta$ is nonstationary in β ;
- for all $\ell < \ell^* < \kappa$, we have $S^p_{\ell} \cap S^p_{\ell^*} = \emptyset$.

For $p, q \in \mathbb{P}$, we say that $q \leq_{\mathbb{P}} p$ if and only if $\gamma^q \geq \gamma^p$ and $q \upharpoonright (\kappa \times \gamma^p) = p$.

Let G be \mathbb{P} -generic over V and, for $\ell < \kappa$, let $S_{\ell} = \bigcup_{p \in G} S^p_{\ell}$. By standard arguments analogous to those in the proofs of Claims 3.11 and 3.12, it follows that

- in V , \mathbb{P} is λ -strategically closed;
- in $V[G]$, for every $\ell < \kappa$, S_{ℓ} is stationary in λ . Moreover, if we let $S_{-1} = S \setminus (\bigcup_{\ell < \kappa} S_{\ell})$, then S_{-1} is stationary as well.

In $V[G]$, for each $\ell < \kappa$, let $\mathbb{T}_{\ell} = \mathbb{T}_{S_{\ell}}$ be the forcing notion to shoot a club through λ disjoint from S_{ℓ} introduced in Definition 3.7. Let $\dot{\mathbb{T}}_{\ell} \in V$ be a canonical \mathbb{P} -name for \mathbb{T}_{ℓ} . By arguments like those in the proof of Claim 3.13, in V , $\mathbb{P} * \dot{\mathbb{T}}_{\ell}$ has a dense λ -directed closed subset of cardinality λ .

$V[G]$ is our desired model. It follows from the previous two paragraphs that the sequence $\langle S_{\ell} \mid \ell < \kappa \rangle$ witnesses the failure of $\text{Refl}(\kappa, S)$ in $V[G]$. To see that $\text{sDSR}(\langle \kappa, S \rangle)$ holds, let $\langle T_{\alpha,i} \mid \alpha < \lambda, i < j_{\alpha} \rangle$ be a matrix of stationary subsets of S , with $j_{\alpha} < \kappa$ for all $\alpha < \lambda$. Notice that, for every stationary subset $T \subseteq S$, there is at most one $\ell < \kappa$ for which $T \setminus S_{\ell}$ is nonstationary. Therefore, for all $\alpha < \lambda$, there is some $\ell(\alpha) < \kappa$ such that, for all $i < j_{\alpha}$, we have that $T_{\alpha,i} \setminus S_{\ell(\alpha)}$ is stationary. In

particular, by Lemma 3.8, each $T_{\alpha,i}$ remains stationary after forcing over $V[G]$ with $\mathbb{T}_{\ell(\alpha)}$.

Find a stationary $\Sigma \subseteq S_{-1}$ and a fixed $\ell < \kappa$ such that $\ell(\alpha) = \ell$ for all $\alpha \in \Sigma$. Let H be \mathbb{T}_ℓ -generic over $V[G]$. For all $\alpha \in \Sigma$ and $i < j_\alpha$, $T_{\alpha,i}$ remains stationary in $V[G * H]$. Moreover, since Σ is a subset of S_{-1} and hence disjoint from S_ℓ , Σ also remains stationary in $V[G * H]$. Since, in V , either $\kappa \geq \omega$ and $\text{DSR}^*(\langle \kappa, S \rangle)$ holds, or $\kappa < \omega$ and $\text{DSR}^*(\kappa, S)$ holds, and since $\mathbb{P} * \mathbb{T}_\ell$ has a dense λ -directed closed subset of cardinality λ , it follows that the corresponding diagonal reflection principle holds in $V[G * H]$.

In $V[G * H]$, form a matrix $\langle T_{\alpha,i}^* \mid \alpha < \lambda, i < j_\alpha^* \rangle$ as follows. For $\alpha \in \Sigma$, let $j_\alpha^* = j_\alpha + 1$, let $T_{\alpha,i}^* = T_{\alpha,i}$ for $i < j_\alpha$, and let $T_{\alpha,j_\alpha} = \Sigma$. For $\alpha \in \lambda \setminus \Sigma$, let $j_\alpha^* = 1$ and $T_{\alpha,0} = \Sigma$. We can apply our diagonal reflection principle to this matrix in $V[G * H]$, finding $\gamma \in S_{>\omega}^\lambda$ and a club $F^* \subseteq \gamma$ such that, for all $\alpha \in F^*$ and all $i < j_\alpha^*$, we have that $T_{\alpha,i}^*$ reflects at γ . Since $\Sigma \in \{T_{\alpha,i}^* \mid i < j_\alpha^*\}$ for every $\alpha < \lambda$, it follows that Σ reflects at γ , and hence $F := \Sigma \cap F^*$ is stationary in γ as well. Moreover, for each $\alpha \in F$ and each $i < j_\alpha$, we know that $T_{\alpha,i} = T_{\alpha,i}^*$ reflects at γ . This is downward absolute to $V[G]$ and, since $V[G]$ and $V[G * H]$ have the same bounded subsets of λ , we know that F is in $V[G]$. Therefore, in $V[G]$, γ and F witness this instance of $\text{sDSR}(\langle \kappa, S \rangle)$. \dashv

COROLLARY 3.20. *Suppose that $\lambda = \lambda^{<\lambda} \geq \aleph_2$ is a regular cardinal and that $\text{DSR}^*(\langle \omega, S_\omega^\lambda \rangle)$ holds. Then there is a cofinality-preserving extension in which*

- (1) $\text{sDSR}(\langle \omega, S_\omega^\lambda \rangle)$ holds;
- (2) $\text{DSR}(1, S_\omega^\lambda)$ fails.

PROOF. This follows immediately from Theorem 3.19 and Lemma 2.5. \dashv

We now prove a more general version of the above corollary, indicating that simultaneous stationary diagonal reflection does not imply any amount of the full diagonal stationary reflection principle. Recall that, by the discussion in Section 3.1, the hypothesis of the following theorem can be arranged from a weakly compact cardinal if λ is to be inaccessible or the successor of a regular cardinal, or from a supercompact cardinal if λ is to be the successor of a singular cardinal.

THEOREM 3.21. *Suppose that $\kappa < \lambda$ are infinite cardinals such that $\kappa^+ < \lambda$ and $\lambda^{<\lambda} = \lambda$. Suppose moreover that there is a stationary subset $\Sigma \subseteq S_{\geq \kappa}^\lambda$ such that $\text{DSR}^*(\langle \kappa, S_\omega^\lambda \cup \Sigma \rangle)$ holds. Then there is a cofinality-preserving forcing extension in which*

- (1) $\text{sDSR}(\langle \kappa, S_\omega^\lambda \rangle)$ holds;
- (2) $\text{DSR}(1, S_\omega^\lambda)$ fails.

PROOF. Let $S = S_\omega^\lambda$, and let \mathbb{P} be the forcing poset whose conditions are all functions of the form $p : \gamma^p \times \gamma^p \rightarrow 2$ such that

- $\gamma^p < \lambda$;
- for each $\alpha < \gamma^p$, letting $S_\alpha^p = \{\eta < \gamma^p \mid p(\alpha, \eta) = 1\}$, we have
 - $S_\alpha^p \subseteq S \setminus (\alpha + 1)$;
 - for all $\beta \in S_{>\omega}^{\gamma^p+1}$, the set

$$\{\alpha < \beta \mid S_\alpha^p \cap \beta \text{ is nonstationary in } \beta\}$$

is stationary in β ;

- for all $\alpha < \alpha^* < \gamma^p$, we have $S_\alpha^p \cap S_{\alpha^*}^p = \emptyset$.

Let G be \mathbb{P} -generic over V and, for $\alpha < \lambda$, let $S_\alpha = \bigcup_{p \in G} S_\alpha^p$. By arguments as in the proofs of Claims 3.11 and 3.12, we have the following facts.

- In V , \mathbb{P} is λ -strategically closed.
- In $V[G]$, for every $\alpha < \lambda$, S_α is stationary in λ . Moreover, if we let $S_{-1} = S \setminus (\bigcup_{\alpha < \lambda} S_\alpha)$, then S_{-1} is stationary as well.

As a result, by the definition of conditions in \mathbb{P} , $\langle S_\alpha \mid \alpha < \lambda \rangle$ witnesses the failure of $\text{DSR}(1, S_\omega^\lambda)$ in $V[G]$.

In $V[G]$, we define a poset \mathbb{T} designed to destroy the stationarity of S_α for a generic ω -closed unbounded set of α 's. Recall that a set X of ordinals is ω -closed if, for all ordinals α , if $\text{cf}(\alpha) = \omega$ and $\sup(X \cap \alpha) = \alpha$, then $\alpha \in X$. When defining similar forcings in previous arguments, it was sufficient to force a single club simultaneously destroying the stationarity of many stationary sets, but here it will pay dividends to more carefully add a distinct club disjoint from each S_α . To be more precise, conditions in \mathbb{T} are pairs $t = (c^t, d^t)$ satisfying the following requirements:

- c^t is an ω -closed bounded subset of λ ;
- d^t is a function with domain c^t such that, for all $\alpha \in c^t$, $d^t(\alpha)$ is a closed bounded subset of λ with $d^t(\alpha) \cap S_\alpha = \emptyset$.

If t and s are conditions in \mathbb{T} , then $s \leq_{\mathbb{T}} t$ if and only if

- c^s end-extends c^t ; and
- for all $\alpha \in c^t$, $d^s(\alpha)$ end-extends $d^t(\alpha)$.

In V , let $\dot{\mathbb{T}}$ be a \mathbb{P} -name for \mathbb{T} .

CLAIM 3.22. *In V , $\mathbb{P} * \dot{\mathbb{T}}$ has a dense λ -directed closed subset of size λ .*

PROOF. Define \mathbb{U} to be the set of $(p, \dot{t}) \in \mathbb{P} * \dot{\mathbb{T}}$ such that, letting $t = (c^t, d^t)$, we have

- c^t has a largest element;
- $\gamma^p = \max(c^t) + 1$;
- for all $\alpha \in c^t$, $\gamma^p = \max(d^t(\alpha)) + 1$.

The verification that \mathbb{U} is a dense λ -directed closed subset of $\mathbb{P} * \dot{\mathbb{T}}$ of cardinality λ is almost the same as that in the proof of Claim 3.16, here using the fact that, if X is an ω -closed set of ordinals, β is an ordinal of uncountable cofinality, and $\sup(X \cap \beta) = \beta$, then $X \cap \beta$ is stationary in β . We therefore leave the rest of the proof to the reader. ⊖

Work for now in $V[G]$, which will be our desired model. If $t \in \mathbb{T}$, then \mathbb{T}/t denotes the set $\{s \in \mathbb{T} \mid s \leq t\}$, considered as a sub-poset of \mathbb{T} . Given a condition $t \in \mathbb{T}$ and an $\varepsilon < \lambda$, let $t \setminus \varepsilon$ denote the condition $(c^t \setminus \varepsilon, d^t \upharpoonright (c^t \setminus \varepsilon))$. Furthermore, define a function $\pi_{t,\varepsilon} : \mathbb{T}/t \rightarrow \mathbb{T}/(t \setminus \varepsilon)$ by letting $\pi_{t,\varepsilon}(s) = s \setminus \varepsilon$ for all $s \in \mathbb{T}/t$.

CLAIM 3.23. *If $t \in \mathbb{T}$, $\varepsilon < \lambda$, and $c^t \setminus \varepsilon$ is non-empty, then $\pi_{t,\varepsilon}$ is a projection² from \mathbb{T}/t to $\mathbb{T}/(t \setminus \varepsilon)$.*

²See, e.g., [1, p. 335].

PROOF. It is clear that $\pi_{t,\varepsilon}$ is order-preserving and maps the top element of \mathbb{T}/t , namely t , to the top element of $\mathbb{T}/(t \setminus \varepsilon)$, namely $t \setminus \varepsilon$. It remains to show that, for all $s \in \mathbb{T}/t$ and all $r \leq (s \setminus \varepsilon)$, there is $s^* \leq s$ such that $(s^* \setminus \varepsilon) \leq r$. To this end, fix such s and r . Since $c^t \setminus \varepsilon \neq \emptyset$ and $s \leq t$, it follows that for all $\alpha \in c^r \setminus c^s$, we have $\alpha \supseteq c^s$. Define a condition $s^* \in \mathbb{T}$ as follows:

- $c^{s^*} = c^r \cup c^s$;
- for all $\alpha \in c^r$, $d^{s^*}(\alpha) = d^r(\alpha)$;
- for all $\alpha \in c^s \setminus c^r$, $d^{s^*}(\alpha) = d^s(\alpha)$.

Note that $c^{s^*} \setminus \varepsilon = c^r$. It is now easily verified that s^* is a condition in \mathbb{T} extending s and that $s^* \setminus \varepsilon$ extends r . ⊢

CLAIM 3.24. *Suppose that, in $V[G]$, we have $t \in \mathbb{T}$, $\varepsilon < \lambda$, and a stationary $R \subseteq \lambda$ such that*

- $c^t \setminus \varepsilon \neq \emptyset$; and
- $t \Vdash_{\mathbb{T}} \text{“}\check{R} \text{ is stationary in } \check{\lambda}\text{”}$.

Then $t \setminus \varepsilon \Vdash_{\mathbb{T}} \text{“}\check{R} \text{ is stationary in } \check{\lambda}\text{”}$.

PROOF. This is an instance of a more general fact: if \mathbb{P} and \mathbb{Q} are posets with a projection $\pi : \mathbb{Q} \rightarrow \mathbb{P}$, A is some set, $\varphi(x)$ is a statement that goes down to inner models, and for some $q \in \mathbb{Q}$, $q \Vdash_{\mathbb{Q}} \varphi(A)$, then $\pi(q) \Vdash_{\mathbb{P}} \varphi(\check{A})$. If not, let $p \leq \pi(q)$ be such that $p \Vdash_{\mathbb{P}} \neg \varphi(\check{A})$. Let $q^* \leq q$ be such that $\pi(q^*) = p$. Let $I \ni q^*$ be \mathbb{Q} -generic. Then $\varphi(A)$ holds in $V[I]$, since $q^* \leq q$. But $\pi \text{“} I \text{ generates a } V\text{-generic filter } \bar{I} \text{ for } \mathbb{P}$, and $\pi(q^*) = p \in \bar{I}$. So $\varphi(A)$ fails in $V[\bar{I}]$. But $V[\bar{I}] \subseteq V[I]$, so, since $\varphi(A)$ holds in $V[I]$, it must hold in $V[\bar{I}]$, a contradiction. ⊢

CLAIM 3.25. *Suppose that $-1 \leq \varepsilon < \lambda$, R is a stationary subset of S_ε , and $t \in \mathbb{T}$ is a condition such that $\varepsilon \notin c^t$ and $\sup(c^t) > \varepsilon$. Then $t \Vdash_{\mathbb{T}} \text{“}\check{R} \text{ is stationary in } \check{\lambda}\text{”}$.*

PROOF. The proof is almost exactly the same as that of Claim 3.17, so we leave it to the reader. ⊢

It remains to verify that $\text{sDSR}(<\kappa, S)$ holds in $V[G]$. To this end, let $\langle T_{\alpha,i} \mid \alpha < \lambda, i < j_\alpha \rangle$ be a matrix of stationary subsets of S , with $j_\alpha < \kappa$ for all $\alpha < \lambda$. As in the proof of Theorem 3.15, for each $\alpha < \lambda$ and $i < j_\alpha$, we can find $\beta(\alpha, i)$ such that $-1 \leq \beta(\alpha, i) < \lambda$ and $T_{\alpha,i} \cap S_{\beta(\alpha,i)}$ is stationary.

For each $\alpha < \lambda$, set

- $X_\alpha = \{\beta(\alpha, i) \mid i < j_\alpha\}$;
- $X_\alpha^- = X_\alpha \cap \alpha$;
- $X_\alpha^+ = X_\alpha \setminus \alpha$.

Note that $|X_\alpha| < \kappa$ for all $\alpha < \lambda$. Therefore, recalling that $\Sigma \subseteq S_{\geq \kappa}^\lambda$, we have $\sup(X_\alpha^-) < \alpha$ for all $\alpha \in \Sigma$. Then the function $\alpha \mapsto \sup(X_\alpha^-)$ is regressive on the stationary set Σ and clearly continues to be in any forcing extension by \mathbb{T} . (Recall that $\mathbb{P} * \mathbb{T}$ has a dense λ -directed closed subset and hence preserves the stationarity of all stationary subsets of λ that are in V ; in particular, it preserves the stationarity of Σ). Therefore, we can find $\varepsilon < \lambda$ and $t_0 \in \mathbb{T}$ such that t_0 forces that $R_\varepsilon := \{\alpha \in \Sigma \mid \sup(X_\alpha^-) = \varepsilon\}$ is stationary in λ . By extending t_0 if necessary, we may assume that $c^{t_0} \setminus (\varepsilon + 1) \neq \emptyset$. Let $t = t_0 \setminus (\varepsilon + 1)$. Then we have the following:

- By Claim 3.24, $t \Vdash_{\mathbb{T}} \check{R}_\varepsilon$ is stationary”.
- By Claim 3.25, for all $\alpha \in R_\varepsilon$ and all $i < j_\alpha$ such that $\beta(\alpha, i) < \alpha$, $t \Vdash_{\mathbb{T}} \check{T}_{\alpha, i}$ is stationary”.

Let \dot{R}^* be a \mathbb{T} -name for the set of $\alpha \in R_\varepsilon$ such that, for all $i < j_\alpha$, $T_{\alpha, i}$ is stationary in λ (after forcing with \mathbb{T}).

CLAIM 3.26. $t \Vdash_{\mathbb{T}} \check{R}^*$ is stationary”.

PROOF. Fix $s_0 \leq t$ and a \mathbb{T} -name \dot{E} for a club in λ . We must find $s \leq s_0$ and $\alpha \in R_\varepsilon$ such that $s \Vdash_{\mathbb{T}} \check{\alpha} \in \dot{E} \cap \dot{R}^*$. Let H be \mathbb{T} -generic over $V[G]$ with $s_0 \in H$, and let E be the realization of \dot{E} . Let $c = \bigcup_{s \in H} c^s$ and define a function d on c by letting $d(\alpha) = \bigcup_{s \in H} d^s(\alpha)$. For each $\alpha \in E$, let $r_\alpha \in H$ be such that $r_\alpha \leq s_0$ and $r_\alpha \Vdash_{\mathbb{T}} \check{\alpha} \in \dot{E}$, and let $\zeta_\alpha < \lambda$ be such that $c^{r_\alpha} \subseteq \zeta_\alpha$. Then the set

$$E^* = \{\alpha \in \text{lim}(E) \mid \text{for all } \eta \in E \cap \alpha, \zeta_\eta < \alpha\}$$

is a club in λ . Since $s_0 \leq t$ and hence $t \in H$, we know that R_ε is stationary in $V[G * H]$, so we can find $\alpha \in E^* \cap R_\varepsilon$. We can also find a $\gamma < \lambda$ large enough so that, for all $\eta < \alpha$ and all $\beta \in c^{r_\eta}$, we have $d^{r_\eta}(\beta) \subseteq \gamma$. Now define $s_1 = (c^{s_1}, d^{s_1})$ by letting $c^{s_1} = c \cap \alpha$ and, for all $\beta \in c$, letting $d^{s_1}(\beta) = d(\beta) \cap (\gamma + 1)$. Note that $s_1 \in \mathbb{T}$ and $s_1 \leq r_\eta$ for all $\eta < \alpha$. Therefore, s_1 forces that α is a limit point of \dot{E} .

Now let $\delta < \lambda$ be large enough so that $\alpha < \delta$ and $X_\alpha^+ \subseteq \delta$. Define a condition $s \leq s_1$ by letting $c^s = c^{s_1} \cup \{\delta\}$, $d^s \upharpoonright c^{s_1} = d^{s_1}$, and $d^s(\delta) = \emptyset$. Notice that $c^s \cap \varepsilon = c^s \cap [\alpha, \delta) = \emptyset$, and therefore $X_\alpha \cap c^s = \emptyset$. Moreover, $\delta \in c^s$ and every element of X_α is less than δ . Therefore, by Claim 3.25,

$$s \Vdash_{\mathbb{T}} \text{“for all } i < j_\alpha, \check{T}_{\alpha, i} \text{ is stationary”}.$$

Therefore, $s \Vdash_{\mathbb{T}} \check{\alpha} \in \dot{E} \cap \dot{R}^*$, as desired. ⊣

Now let H be \mathbb{T} -generic over $V[G]$ with $t \in H$, and let R^* be the interpretation of \dot{R}^* . Define a matrix $\langle T_{\alpha, i}^* \mid \alpha < \lambda, i \leq j_\alpha \rangle$ of stationary subsets of $S_\omega^\lambda \cup \Sigma$ as follows. First, for all $\alpha < \lambda$, let $T_{\alpha, j_\alpha}^* = R^*$. Next, if $\alpha \in R^*$, then let $T_{\alpha, i}^* = T_{\alpha, i}$ for all $i < j_\alpha$. Finally, if $\alpha \in \lambda \setminus R^*$, then simply let $T_{\alpha, i}^* = S_\omega^\lambda$ for all $i < j_\alpha$.

Since $\text{DSR}(\langle \kappa, S_\omega^\lambda \cup \Sigma \rangle)$ holds in V and $\mathbb{P} * \mathbb{T}$ has a dense λ -directed closed subset of size λ , $\text{DSR}(\langle \kappa, S_\omega^\lambda \cup \Sigma \rangle)$ holds in $V[G * H]$. Therefore, we can find $\gamma \in S_{>\omega}^\lambda$ and a club F^* in γ such that, for all $\alpha \in F^*$ and all $i \leq j_\alpha$, $T_{\alpha, i}^*$ reflects at γ . In particular, R^* reflects at γ , so $F = F^* \cap R^*$ is stationary in γ . Moreover, for all $\alpha \in F$ and all $i < j_\alpha$, we know that $T_{\alpha, i}^* = T_{\alpha, i}$, so $T_{\alpha, i}$ reflects at γ . But stationarity is downward absolute and, since \mathbb{T} is λ -distributive in $V[G]$, we have $F \in V[G]$, so, in $V[G]$, γ and F witness this instance of $\text{sDSR}(\langle \kappa, S_\omega^\lambda \rangle)$. ⊣

REMARK 3.27. We do not know if the requirement that $\kappa^+ < \lambda$ is necessary in the statement of the above theorem. It was used in our proof to allow for the existence of a stationary $\Sigma \subseteq S_{\geq \kappa}^\lambda$ that reflects.

Recall that Lemma 2.5 showed that $\text{DSR}(1, S)$ implies $\text{Refl}(\omega, S)$. The next theorem provides a limitation to the extent to which this can be generalized. It also shows that Theorem 5.2, stating that $\text{sDSR}(\langle \kappa, S \rangle)$ implies the failure of $\square(\lambda, \langle \kappa \rangle)$ (where $1 < \kappa < \lambda$ and $S \subseteq \lambda$ is stationary), is in a sense optimal. The assumptions of

the theorem hold, for example, if κ is indestructibly generically supercompact and $S = S_{\omega}^{\lambda}$, or if κ is indestructibly supercompact and $S = S_{<\kappa}^{\lambda}$.

THEOREM 3.28. *Suppose that $\omega_1 < \kappa < \lambda$ are regular cardinals, $S \subseteq \lambda$ is stationary, and $\text{DSR}(<\kappa, S)$ holds. Suppose moreover that all of these statements continue to hold in any forcing extension obtained by a κ -directed closed forcing notion that preserves cofinalities up to λ . Then there is such a forcing extension in which*

- (1) $\text{DSR}(<\kappa, S)$ holds;
- (2) $\text{Refl}(\kappa, S')$ fails for all stationary $S' \subseteq \lambda$.

In fact, in this forcing extension, we have $\square^{\text{ind}}(\lambda, \kappa)$, a strengthening of $\square(\lambda, \kappa)$ (see Definition 5.1).

PROOF. Under the assumptions of the theorem, there is a forcing notion \mathbb{S} that is κ -directed closed and λ -strategically closed, such that letting G be \mathbb{S} -generic over V , a strengthening of $\square(\lambda, \kappa)$ called $\square^{\text{ind}}(\lambda, \kappa)$ holds in $V[G]$; see [8, Lemma 3.14]. We do not need the precise definition of $\square^{\text{ind}}(\lambda, \kappa)$ here. It suffices for the present purposes to know that a $\square^{\text{ind}}(\lambda, \kappa)$ -sequence is in particular a full $\square(\lambda, \kappa)$ -sequence. So by Theorem 1.5, it follows that $\text{Refl}(\kappa, S')$ fails for every stationary $S' \subseteq \lambda$ in $V[G]$. But by assumption, $\text{DSR}(<\kappa, S)$ continues to hold. \dashv

§4. Diagonal stationary reflection and the strong reflection principle. Let us now make a connection to Todorćević’s strong reflection principle. It will be useful to recall some definitions and facts.

DEFINITION 4.1. Let γ be a regular uncountable cardinal, and let $X \supseteq \gamma$ be a set. For a set $A \subseteq \gamma$, let

$$\text{lift}(A, [X]^\omega) = \{x \in [X]^\omega \mid \sup(x \cap \gamma) \in A\}.$$

Now let $S \subseteq [X]^\omega$ be stationary. Then S is *projective stationary* if for every stationary set $A \subseteq \omega_1$, the set

$$S \cap \text{lift}(A, [X]^\omega)$$

is stationary.

If $W \subseteq X \subseteq Y$, then we write

$$S \uparrow [Y]^\omega = \{y \in [Y]^\omega \mid y \cap X \in S\}$$

and

$$S \downarrow [W]^\omega = \{x \cap W \mid x \in S\}.$$

The *strong reflection principle* (SRP) is the assertion that for every regular cardinal $\theta \geq \omega_2$, if $S \subseteq [H_\theta]^\omega$ is projective stationary, then there is a continuous \in -chain $\langle M_i \mid i < \omega_1 \rangle$ of countable elementary submodels of $\langle H_\theta, \in \rangle$ such that for all $i < \omega_1$, $M_i \in S$.

FACT 4.2. *Let $W \subseteq X \subseteq Y$, and let $S \subseteq [X]^\omega$ be stationary.*

- (1) $S \downarrow [W]^\omega$ and $S \uparrow [Y]^\omega$ are stationary.
- (2) If $\omega_1 \subseteq W$ and S is projective stationary, then $S \downarrow [W]^\omega$ and $S \uparrow [Y]^\omega$ are projective stationary.

(3) *If S is projective stationary and $C \subseteq [X]^\omega$ is club, then $S \cap C$ is projective stationary.*

PROOF. Item (1) is well-known; see, for example, [15]. For item (2), we refer the reader to [9, Example 37.17]. Item (3) is immediate: if $A \subseteq \omega_1$ is stationary, then $(S \cap C) \cap \text{lift}(A, [X]^\omega) = (S \cap \text{lift}(A, [X]^\omega)) \cap C$ is stationary. \dashv

The following lemma is essentially due to Larson [14]. The original statement and proof, though, have a confusing typo and omit one implication, so we reformulate and prove it here. We use the notation of Definition 3.7; compare to Lemma 3.8.

LEMMA 4.3. *Let $\gamma > \omega_1$ be regular, $X \supseteq \gamma$ a set, $A \subseteq \gamma$ stationary, and $S \subseteq [X]^\omega$ also stationary, such that $\gamma \setminus A$ is unbounded in γ and \mathbb{T}_A is countably distributive (this is the case, for example, if $A \subseteq S_\omega^\gamma$ and $S_\omega^\gamma \setminus A$ is stationary). Then the following are equivalent :*

- (1) $S \setminus \text{lift}(A, [X]^\omega)$ is stationary.
- (2) \mathbb{T}_A preserves the stationarity of S .
- (3) *There is a condition $p \in \mathbb{T}_A$ that forces that \check{S} is stationary.*

PROOF. (1) \implies (2): Suppose that (1) holds, and assume towards a contradiction that (2) fails. Let τ be a \mathbb{T}_A -name that is forced by some condition $p \in \mathbb{T}_A$ to be a club subset of $[X]^\omega$ disjoint from S . By (1), $S \setminus \text{lift}(A, [X]^\omega)$ is stationary. Let η be a regular cardinal large enough that H_η contains all the relevant parameters. By Fact 4.2, $(S \setminus \text{lift}(A, [X]^\omega)) \upharpoonright [H_\eta]^\omega$ is stationary, so we can pick a countable elementary submodel $N \prec \langle H_\eta, \in, p, A, S, \tau \rangle$ such that $N \cap X \in S \setminus \text{lift}(A, [X]^\omega)$. Thus, $p, A, S, \tau \in N$, $N \cap X \in S$, and $v = \sup(N \cap \gamma) \notin A$. Let $g \subseteq \mathbb{T}_A \cap N$ be N -generic for \mathbb{T}_A , with $p \in g$. Since $v = \sup(\bigcup g) = \sup(N \cap \gamma) \notin A$, it follows that $q = (\bigcup g) \cup \{v\} \in \mathbb{T}_A$. Also, there is a \subseteq -increasing sequence $\langle x_n \mid n < \omega \rangle$ of elements of τ^g such that $\bigcup_{n < \omega} x_n = N \cap X$. But if $q \in G$, where G is \mathbb{T}_A -generic over V , then $\tau^g \subseteq \tau^G$, and so $N \cap X \in \tau^G \cap S$, a contradiction.

(2) \implies (3): trivial.

(3) \implies (1): Suppose that (3) holds, and assume towards a contradiction that (1) fails. The failure of (1) means that there is a club $C \subseteq [X]^\omega$ disjoint from $S \setminus \text{lift}(A, [X]^\omega)$, i.e.,

$$(*) \quad S \cap C \subseteq \text{lift}(A, [X]^\omega).$$

Using (3), let $G \subseteq \mathbb{T}_A$ be generic so that S is stationary in $V[G]$. Hence, $S \cap C$ is stationary in $V[G]$.

Let $D = \bigcup G$, so $D \subseteq \gamma$ is club and $D \cap A = \emptyset$. Since \mathbb{T}_A preserves the fact that γ has uncountable cofinality, we can pick $x \in S \cap C$ with $\sup(x \cap \gamma) \in D$. But then $\sup(x \cap \gamma) \notin A$, which contradicts (*). \dashv

It follows from [14, Theorem 4.6] that, assuming the consistency of Martin’s Maximum, SRP does not imply $\text{OSR}(S_\omega^{\omega_2})$, which by Lemma 2.4 is equivalent to each of $\text{DSR}(\omega_1, S_\omega^{\omega_2})$, $\text{sDSR}(\omega_1, S_\omega^{\omega_2})$, and $\text{uDSR}(\omega_1, S_\omega^{\omega_2})$. We now show that one can do slightly better at regular cardinals $\lambda > \omega_2$ by adapting the proof of Theorem 3.10 to prove that SRP does not imply $\text{uDSR}(1, S_\omega^\lambda)$. The hypothesis of the following theorem follows from Martin’s Maximum.

THEOREM 4.4. *Let $\lambda > \omega_2$ be a regular cardinal, and suppose that SRP holds and continues to hold in any forcing extension obtained by a λ -directed closed forcing*

notion. Then there is a λ -strategically closed forcing notion which produces forcing extensions in which

- (1) SRP continues to hold, but
- (2) $\text{uDSR}(1, S_\omega^\lambda)$ fails.

PROOF. We use the forcing notion \mathbb{P} of Theorem 3.10. Let G be \mathbb{P} -generic over V . It was shown in the proof of that theorem that \mathbb{P} is λ -strategically closed and adds a sequence $\langle S_\alpha \mid \alpha < \lambda \rangle$ of stationary subsets of S_ω^λ which is a counterexample to $\text{uDSR}(1, S_\omega^\lambda)$. Recall that, for $\alpha < \lambda$, we set $S_{\geq \alpha} = \bigcup_{\alpha \leq \beta < \lambda} S_\beta$ and let $\dot{\mathbb{T}}_\alpha$ be a canonical name for the forcing $\mathbb{T}_{S_{\geq \alpha}}$. We have seen that $\mathbb{P} * \dot{\mathbb{T}}_\alpha$ has a dense λ -directed closed subset. Other crucial properties of the sequence \vec{S} , which we will use in the proof of the following claim, are that for $\alpha < \lambda$, $S_\alpha \cap (\alpha + 1) = \emptyset$ and that for $\alpha < \beta < \lambda$, $S_\alpha \cap S_\beta = \emptyset$. Work now in $V[G]$.

CLAIM 4.5. *Let $\theta \geq \lambda$ be a regular cardinal, and let $S \subseteq [H_\theta]^\omega$ be stationary. Then there is an $\alpha < \lambda$ such that \mathbb{T}_α preserves the stationarity of S .*

PROOF. If not, then, by Lemma 4.3, for every $\alpha < \lambda$, the set

$$B_\alpha = \{x \in S \mid \sup(x \cap \lambda) \notin S_{\geq \alpha}\}$$

is not stationary. Thus, for all $\alpha < \lambda$, let $C_\alpha \subseteq [H_\theta]^\omega$ be a club disjoint from B_α . By normality of the generalized club filter, there is a club $D \subseteq [H_\theta]^\omega$ with the property that

$$\forall x \in D \forall \xi \in x \cap \lambda \quad x \in C_\xi.$$

Now let $x \in S \cap D$, and set $\sigma = \sup(x \cap \lambda)$. For all $\xi \in x \cap \lambda$, $x \in C_\xi$. So since $x \in S$ and $C_\xi \cap B_\xi = \emptyset$, it follows that $\sigma \in S_{\geq \xi}$. So, since $x \cap \lambda$ is cofinal in σ and \vec{S} is pairwise disjoint, it follows that $\sigma \in S_{\geq \sigma}$. But for $\zeta < \lambda$ with $\zeta \geq \sigma$, we have $S_\zeta \cap (\sigma + 1) \subseteq S_\zeta \cap (\zeta + 1) = \emptyset$, so it cannot be that $\sigma \in S_{\geq \sigma}$. \dashv

Next, we are going to show that SRP holds in $V[G]$. So, still working in $V[G]$, let $S \subseteq [H_\theta]^\omega$ be projective stationary, where $\theta \geq \omega_2$ is a regular cardinal.

CLAIM 4.6. *There is an $\alpha < \lambda$ such that \mathbb{T}_α preserves the projective stationarity of S .*

PROOF. First, note that we may assume that $\theta \geq \lambda$. For otherwise, we can choose $\theta' \geq \lambda > \theta$ and set $S' = S \upharpoonright [H_{\theta'}]^\omega$. By Fact 4.2, S' is projective stationary. If we can show that the projective stationarity of S' is preserved by some \mathbb{T}_α , then by applying Fact 4.2 to $S = S' \downharpoonright [H_\theta]^\omega$, it follows that S is projective stationary in the extension.

So let us assume that $\theta \geq \lambda$. For every stationary set $A \subseteq \omega_1$, let

$$S_A = S \cap \text{lift}(A, [H_\theta]^\omega).$$

Using Claim 4.5, let $\alpha_A < \lambda$ be such that \mathbb{T}_{α_A} preserves the stationarity of S_A . Let

$$\alpha = \sup\{\alpha_A \mid A \text{ is a stationary subset of } \omega_1\}.$$

Recall that SRP implies that $2^{\omega_1} = \omega_2$ (this is due to Todorćević; cf. [18, Theorem 9.82]). Since SRP holds in V and \mathbb{P} is λ -strategically closed, this consequence still holds in $V[G]$. So we have that in $V[G]$, $2^{\omega_1} < \lambda$, which implies that $\alpha < \lambda$.

It now follows from Lemma 4.3 that \mathbb{T}_α preserves the stationarity of S_A for every stationary $A \subseteq \omega_1$ in $V[G]$. This is because by that lemma, it suffices to show that $S_A \setminus \text{lift}(S_{\geq \alpha}, [H_\theta]^\omega)$ is stationary. But we know that \mathbb{T}_{α_A} preserves the stationarity of S_A , which again by the lemma means that $S_A \setminus \text{lift}(S_{\geq \alpha_A}, [H_\theta]^\omega)$ is stationary. Since $\alpha_A \leq \alpha$, we have that $S_\alpha \subseteq S_{\alpha_A}$, and so $S_A \setminus \text{lift}(S_{\geq \alpha_A}, [H_\theta]^\omega) \subseteq S_A \setminus \text{lift}(S_{\geq \alpha}, [H_\theta]^\omega)$ is stationary, as wished.

Since, in V , $\mathbb{P} * \mathbb{T}_\alpha$ has a dense λ -directed closed subset, we know that, in $V[G]$, \mathbb{T}_α is λ -distributive and hence does not add any new subsets of ω_1 . In particular, every stationary subset of ω_1 in the extension by \mathbb{T}_α is already in $V[G]$. Therefore, forcing with \mathbb{T}_α preserves the projective stationarity of S . \dashv

We can now finish the proof by showing that, in $V[G]$, S contains a continuous \in -chain of length ω_1 . Let $\alpha < \lambda$ be such that \mathbb{T}_α preserves the projective stationarity of S , and let H be \mathbb{T}_α -generic over $V[G]$. In $V[G * H]$, S is projective stationary. Working in $V[G * H]$, let $\theta' \geq \theta$ be a regular cardinal. We can form $S' = S \uparrow ([H_{\theta'}]^\omega)^{V[G * H]}$. S' is then projective stationary in $V[G * H]$ by Fact 4.2; note that $\theta \geq \omega_2$ in $V[G * H]$. Moreover, by the same fact, $S' \cap C$ is also projective stationary, whenever $C \subseteq [H_{\theta'}]^\omega$ is a club. In particular, the set

$$T = \{x \in S' \mid x \prec \langle H_{\theta'}^{V[G * H]}, \in, H_\theta^{V[G]} \rangle\}$$

is projective stationary in $V[G * H]$.

Since $\mathbb{P} * \mathbb{T}$ is equivalent to a λ -directed closed forcing in V , it follows by our assumptions that SRP holds in $V[G * H]$. Hence, there is an ω_1 -chain $\langle N_i \mid i < \omega_1 \rangle$ of elementary submodels of $H_\theta^{V[G * H]}$ in $V[G * H]$ such that, for every $i < \omega_1$, $N_i \in T$. Set $M_i = N_i \cap H_\theta^{V[G]}$, for $i < \omega_1$. Then, since \mathbb{T}_α is λ -distributive in $V[G]$, it follows that the sequence $\langle M_i \mid i < \omega_1 \rangle$ is in $V[G]$. Moreover, it is a continuous \in -chain and, for every $i < \omega_1$, M_i is an elementary submodel of $H_\theta^{V[G]}$ (since $H_\theta^{V[G]} \cap N_i$ is available as a predicate in N_i) and $M_i \in S$. This proves this instance of SRP. \dashv

In order to use the argument of the previous proof to obtain the failure of $\text{uDSR}(1, S_\omega^{\omega_2})$ while SRP holds, we seem to need a stronger assumption. In fact, we do not know whether its consistency follows from any large cardinal assumption. Larson [14, Remark before Definition 6.5] points out that it follows from results of Woodin [18] that one can derive a model in which $\text{SRP}(\omega_2)$ holds and the nonstationary ideal on ω_1 has density ω_1 from a model of $\text{AD}_{\mathbb{R}^+}$ + “ θ is regular”. This is at least going in the direction of our assumption.

COROLLARY 4.7. *Suppose that SRP holds and continues to hold in any forcing extension obtained by an ω_2 -directed closed forcing notion. Assume furthermore that the density of the nonstationary ideal on ω_1 is ω_1 . Then there is an ω_2 -strategically closed forcing notion which produces forcing extensions where*

- (1) SRP continues to hold, but
- (2) $\text{uDSR}(1, S_\omega^{\omega_2})$ fails.

PROOF. By assumption, we may fix an ω_1 -sized collection \mathcal{A} of stationary subsets of ω_1 which is dense in the stationary subsets of ω_1 , that is, for every stationary $B \subseteq \omega_1$, there is an $A \in \mathcal{A}$ such that $A \subseteq B$.

We now argue as in the proof of Theorem 4.4, with $\lambda = \omega_2$. So we let G be generic for \mathbb{P} , adding a sequence $\langle S_\alpha \mid \alpha < \lambda \rangle$ of stationary subsets of $S_\omega^{\omega_2}$ which forms a

counterexample to $\text{uDSR}(1, S_{\omega_2}^{\omega_2})$. We define $S_{\geq \alpha}$ and \mathbb{T}_α as before, for $\alpha < \omega_2$. As before, we see that in $V[G]$, for every stationary $S \subseteq [H_\theta]^\omega$, where $\theta \geq \omega_2$, there is an $\alpha < \omega_2$ such that \mathbb{T}_α preserves the stationarity of S . We can now follow the proof of Claim 4.6 to show that in $V[G]$, if S is projective stationary in $[H_\theta]^\omega$, where $\theta \geq \omega_2$ is regular, then there is an $\alpha < \omega_2$ such that \mathbb{T}_α preserves the projective stationarity of S . To see this, let, for any stationary $B \subseteq \omega_1$, $S_B = S \cap \text{lift}(B, [H_\theta]^\omega)$ and choose, an ordinal $\alpha_B < \omega_2$ such that \mathbb{T}_{α_B} preserves the stationarity of $S_A = S \cap \text{lift}(A, [H_\theta]^\omega)$. Since the cardinality of \mathcal{A} is ω_1 , we know that $\alpha = \sup\{\alpha_A \mid A \in \mathcal{A}\} < \omega_2$. It then follows that \mathbb{T}_α preserves the stationarity of S_B , for every stationary $B \subseteq \omega_1$. Namely, given such a B , it suffices to show that $S_B \setminus \text{lift}(S_{\geq \alpha}, [H_\theta]^\omega)$ is stationary. But by density of \mathcal{A} , there is an $A \in \mathcal{A}$ with $A \subseteq B$, and we know that $S_A \setminus \text{lift}(S_{\geq \alpha_A}, [H_\theta]^\omega)$ is stationary. Clearly, $S_A \subseteq S_B$, and $S_{\geq \alpha} \subseteq S_{\geq \alpha_A}$, so that $S_A \setminus \text{lift}(S_{\geq \alpha_A}, [H_\theta]^\omega) \subseteq S_B \setminus \text{lift}(S_{\geq \alpha}, [H_\theta]^\omega)$ is stationary.

Since $\mathbb{P} * \mathbb{T}_\alpha$ has an ω_2 -directed closed dense subset, SRP holds in $V^{\mathbb{P} * \mathbb{T}_\alpha}$, where S is projective stationary, and this goes down to $V[G]$ as before. \dashv

§5. Indexed square and sDSR. In this section, we will show that Theorem 2.1 is sharp by constructing models in which both $\text{sDSR}(< \kappa, S)$ and $\square(\lambda, \kappa)$ hold, where $\kappa < \lambda$ are infinite regular cardinals and $S \subseteq \lambda$ is stationary. Notice that we have in fact already done this in the proof of Theorem 3.28 (and in fact we obtained $\text{DSR}(< \kappa, S)$ there). Our reasons for including this section are twofold. First, we can significantly reduce the large cardinals necessary. The hypotheses of Theorem 3.28 can be obtained by assuming, for instance, that κ is indestructibly generically supercompact, whereas we can achieve the hypotheses of Theorem 5.2 by starting in an inner model in which λ is weakly compact (at least, if we want λ to be either inaccessible or the successor of a regular cardinal in the final model). Second, Theorem 5.2 applies more easily to a broader class of stationary sets S . The forcing notion employed in the proof of Theorem 3.28 is only κ -directed closed and necessarily introduces a nonreflecting stationary subset of S_κ^λ (see [13, Theorem 3.4]), so the natural scenarios in which the hypotheses for Theorem 3.28 are satisfied require $S \subseteq S_{< \kappa}^\lambda$, whereas the hypotheses of our result in this section can be satisfied by $S = \lambda$, if, for instance, λ is weakly compact and remains such after any λ -directed closed forcing of size λ .

We will need the following strengthening of $\square(\lambda, \kappa)$, introduced in [12].

DEFINITION 5.1. Suppose that $\kappa < \lambda$ are infinite regular cardinals. Then a $\square^{\text{ind}}(\lambda, \kappa)$ sequence is a matrix $\vec{C} = \langle C_{\alpha, i} \mid \alpha \in \text{lim}(\lambda), i(\alpha) \leq i < \kappa \rangle$ satisfying the following conditions.

- (1) For all $\alpha \in \text{lim}(\lambda)$, we have $i(\alpha) < \kappa$.
- (2) For all $\alpha \in \text{lim}(\lambda)$ and $i(\alpha) \leq i < \kappa$, $C_{\alpha, i}$ is a club in α .
- (3) For all $\alpha \in \text{lim}(\lambda)$ and $i(\alpha) \leq i < j < \kappa$, we have $C_{\alpha, i} \subseteq C_{\alpha, j}$.
- (4) For all limit $\alpha, \beta \in \text{lim}(\lambda)$ and $i(\beta) \leq i < \kappa$, if $\alpha \in \text{lim}(C_{\beta, i})$, then $i(\alpha) \leq i$ and $C_{\alpha, i} = C_{\beta, i} \cap \alpha$.
- (5) For all $\alpha, \beta \in \text{lim}(\lambda)$ with $\alpha < \beta$, there is i with $i(\beta) \leq i < \kappa$ such that $\alpha \in \text{lim}(C_{\beta, i})$.

(6) There is no club $D \subseteq \lambda$ such that, for every $\alpha \in \text{lim}(D)$, there is i with $i(\alpha) \leq i < \kappa$ such that $D \cap \alpha = C_{\alpha,i}$. (Such a club would be a *thread* through \vec{C} .)

$\square^{\text{ind}}(\lambda, \kappa)$ is the assertion that there is a $\square^{\text{ind}}(\lambda, \kappa)$ -sequence.

It is clear from the definition that $\square^{\text{ind}}(\lambda, \kappa)$ implies the existence of a full $\square(\lambda, \kappa)$ -sequence (see Definition 1.4.). Namely, by condition (5) if we let $C_\alpha = \{C_{\alpha,i} \mid i(\alpha) \leq i < \kappa\}$, for limit $\alpha < \lambda$, then \vec{C} is a $\square(\lambda, \kappa)$ -sequence with the property that for all $\alpha, \beta \in \text{lim}(\lambda)$ with $\alpha < \beta$, α is a limit point of some $C \in C_\beta$. This is a much stronger property than fullness.

THEOREM 5.2. *Suppose that $\kappa < \lambda$ are infinite regular cardinals, $\lambda^{<\lambda} = \lambda$, $S \subseteq \lambda$ is stationary, and $\text{DSR}^*(<\kappa, S)$ holds. Then there is a cofinality-preserving forcing extension in which S remains stationary and both $\text{sDSR}(<\kappa, S)$ and $\square^{\text{ind}}(\lambda, \kappa)$ hold.*

REMARK 5.3. Note that, since $\square^{\text{ind}}(\lambda, \kappa)$ implies the existence of a full $\square(\lambda, \kappa)$ sequence, we can conclude from Theorem 1.5 that $\text{Refl}(\kappa, T)$ fails for every stationary $T \subseteq \lambda$ in the model obtained in Theorem 5.2. Therefore, this gives an alternate proof of Theorem 3.19 in the cases in which $\kappa \geq \omega$.

PROOF OF THEOREM 5.2.. By the results of [8, Section 3.2], there is a two-step forcing iteration $\mathbb{S} * \dot{\mathbb{P}}$ with the following salient properties.

- \mathbb{S} has cardinality λ and, in $V^{\mathbb{S}}$, \mathbb{P} is a forcing iteration of length λ^+ , taken with supports of size less than λ , in which each iterand has cardinality λ . \mathbb{P} therefore has the λ^+ -cc in $V^{\mathbb{S}}$. For $\eta \leq \lambda^+$, let \mathbb{P}_η denote the initial segment of length η of this iteration.
- In $V^{\mathbb{S} * \dot{\mathbb{P}}}$, $\square^{\text{ind}}(\lambda, \kappa)$ holds, as witnessed by a sequence $\vec{C} = \langle C_{\alpha,i} \mid \alpha < \lambda, i(\alpha) \leq i < \kappa \rangle$ explicitly introduced by \mathbb{S} .
- In $V^{\mathbb{S}}$, for each $i < \kappa$, define a forcing poset \mathbb{T}_i as follows. Conditions of \mathbb{T}_i are all clubs $C_{\alpha,i}$ (from the $\square^{\text{ind}}(\lambda, \kappa)$ -sequence isolated above) such that $i(\alpha) \leq i$. If $C_{\alpha,i}$ and $C_{\beta,i}$ are in \mathbb{T}_i , then $C_{\beta,i} \leq_{\mathbb{T}_i} C_{\alpha,i}$ if and only if $C_{\alpha,i} = C_{\beta,i} \cap \alpha$. (\mathbb{T}_i is the forcing to add a thread through the i th column of \vec{C} .) Then the following hold.
 - In V , for all $i < \kappa$ and all $\eta \leq \lambda^+$, $\mathbb{S} * \dot{\mathbb{P}}_\eta * \dot{\mathbb{T}}_i$ has a dense λ -directed closed subset. Moreover, if $\eta < \lambda^+$, then this subset has cardinality λ .
 - In $V^{\mathbb{S} * \dot{\mathbb{P}}}$, for all $i < j < \kappa$, the map $C_{\alpha,i} \mapsto C_{\alpha,j}$ is a projection from \mathbb{T}_i to \mathbb{T}_j . This projection will be denoted by $\pi_{ij} : \mathbb{T}_i \rightarrow \mathbb{T}_j$.
 - In $V^{\mathbb{S} * \dot{\mathbb{P}}}$, if T is a stationary subset of λ , then there are $i < \kappa$ and $t \in \mathbb{T}_i$ such that $t \Vdash_{\mathbb{T}_i} \check{T}$ is stationary”.

Let $G * H$ be $\mathbb{S} * \dot{\mathbb{P}}$ -generic over V . For $\eta \leq \lambda^+$, let H_η be the $\dot{\mathbb{P}}_\eta$ -generic filter induced by H . $V[G * H]$ is our desired model. Notice that since, in V , $\mathbb{S} * \dot{\mathbb{P}} * \dot{\mathbb{T}}_0$ has a dense λ -directed closed subset, it preserves all stationary subsets of λ , and hence S remains stationary in $V[G * H]$. It remains to verify that $\text{sDSR}(<\kappa, S)$ holds in $V[G * H]$.

To this end, work in $V[G * H]$ and suppose that $\langle S_{\alpha,j} \mid \alpha < \lambda, j < j_\alpha \rangle$ is a matrix of stationary subsets of S , where $j_\alpha < \kappa$ for every $\alpha < \lambda$. We will find $\gamma \in S_{>\omega}^\lambda$ such that, for stationarily many $\alpha < \gamma$, for all $j < j_\alpha$, we have that $S_{\alpha,j} \cap \gamma$ is stationary in γ .

By the final property of $\mathbb{S} * \dot{\mathbb{P}}$ listed above, for each $\alpha < \lambda$ and $j < j_\alpha$, we can find an $i_{\alpha,j} < \kappa$ and a condition $t_{\alpha,j} \in \mathbb{T}_{i_{\alpha,j}}$ such that $t_{\alpha,j} \Vdash_{\mathbb{T}_{i_{\alpha,j}}} \check{S}_{\alpha,j}$ “ $\check{S}_{\alpha,j}$ is stationary”. Notice that, for each such α and j and all k with $i_{\alpha,j} < k < \kappa$, we also have $\pi_{i_{\alpha,j}k}(t_{\alpha,j}) \Vdash_{\mathbb{T}_k}$ “ $\check{S}_{\alpha,j}$ is stationary” by the arguments of Claim 3.24.

For each $\alpha < \lambda$ and $j < j_\alpha$, let $\beta_{\alpha,j}$ be such that $t_{\alpha,j} = C_{\beta_{\alpha,j}, i_{\alpha,j}}$. For each $\alpha < \lambda$, find a limit ordinal β_α with $\alpha < \beta_\alpha < \lambda$ such that $\beta_{\alpha,j} < \beta_\alpha$ for all $j < j_\alpha$. Since $j_\alpha < \kappa$, the definition of $\square^{\text{ind}}(\lambda, \kappa)$ implies that we can find an ordinal i_α with $i(\beta_\alpha) \leq i_\alpha < \kappa$ such that, for all $j < j_\alpha$, we have $\beta_{\alpha,j} \in \text{lim}(C_{\beta_\alpha, i_\alpha})$ and $i_\alpha > i_{\alpha,j}$. Letting $t_\alpha = C_{\beta_\alpha, i_\alpha}$, it follows that, for every $j < j_\alpha$, we have $t_\alpha \leq_{\mathbb{T}_{i_\alpha}} \pi_{i_{\alpha,j}i_\alpha}(t_{\alpha,j})$, and hence $t_\alpha \Vdash_{\mathbb{T}_{i_\alpha}}$ “ $\check{S}_{\alpha,j}$ is stationary”. For i with $i_\alpha \leq i < \kappa$, let $t_\alpha^i = \pi_{i_\alpha i}(t_\alpha)$.

By the chain condition of \mathbb{P} , we can find $\eta < \lambda^+$ such that

$$\langle S_{\alpha,j} \mid \alpha < \lambda, j < j_\alpha \rangle \in V[G * H_\eta].$$

Work now in $V[G * H_\eta]$. Since stationarity is downward absolute, it is still the case in $V[G * H_\eta]$ that, for all $\alpha < \lambda$, all $j < j_\alpha$, and all i with $i_\alpha \leq i < \kappa$, we have $t_\alpha^i \Vdash_{\mathbb{T}_i}$ “ $\check{S}_{\alpha,j}$ is stationary”.

For each $i < \kappa$, let \dot{J}_i be the \mathbb{T}_i -name for the generic filter, and let \dot{R}_i be the \mathbb{T}_i -name for the set

$$\{\alpha \in S \mid i_\alpha \leq i \text{ and } t_\alpha^i \in \dot{J}_i\}.$$

CLAIM 5.4. *There is $i < \kappa$ and $r \in \mathbb{T}_i$ such that*

$$r \Vdash_{\mathbb{T}_i} \check{R}_i \text{ is stationary}.$$

PROOF. Suppose not. Then, for each $i < \kappa$, there is a \mathbb{T}_i -name \dot{E}_i for a club in λ disjoint from \dot{R}_i . Since, for each $i < \kappa$, π_{0i} is a projection from \mathbb{T}_0 to \mathbb{T}_i , each \dot{E}_i can be interpreted as a \mathbb{T}_0 -name, so we can let \dot{E} be a \mathbb{T}_0 -name for $\bigcap_{i < \kappa} \dot{E}_i$.

Let J be \mathbb{T}_0 -generic over $V[G * H_\eta]$, let $D = \bigcup J$ be the generic club added by J that threads \check{C} , and let E be the interpretation of \dot{E} in $V[G * H_\eta * J]$. Notice that, since S is a stationary subset of λ in V and $\mathbb{S} * \dot{\mathbb{P}}_\eta * \dot{\mathbb{T}}_0$ has a dense λ -directed closed subset, S remains stationary in $V[G * H_\eta * J]$.

For each $\gamma \in E$, find an ordinal ξ_γ with $\gamma \leq \xi_\gamma < \lambda$ such that $C_{\xi_\gamma, 0} \in J$ and, in $V[G * H_\eta]$, $C_{\xi_\gamma, 0} \Vdash_{\mathbb{T}_0}$ “ $\check{\gamma} \in \dot{E}$ ”. Note that, by our definition of \dot{E} , it follows that, for all $i < \kappa$, we have $C_{\xi_\gamma, i} \Vdash_{\mathbb{T}_i}$ “ $\check{\gamma} \in \dot{E}_i$ ”. Let

$$E^* = \{\delta \in \text{lim}(E) \mid \text{for all } \gamma \in E \cap \delta, \text{ we have } \xi_\gamma < \delta\}.$$

Then E^* is a club in λ and $E^* \subseteq \text{lim}(D)$. We can therefore find $\delta \in E^* \cap S$. Then $C_{\delta, 0} \in J$ and $C_{\delta, 0} \leq_{\mathbb{T}_0} C_{\xi_\gamma, 0}$ for every $\gamma \in E \cap \delta$. Since $\text{sup}(E \cap \delta) = \delta$ and, in $V[G * H_\eta]$, \dot{E} is forced to be a club, we know that $C_{\delta, 0} \Vdash_{\mathbb{T}_0}$ “ $\check{\delta} \in \dot{E}$.” in $V[G * H_\eta]$. By the definition of \dot{E} , it follows that, for all $i < \kappa$, we have $C_{\delta, i} \Vdash_{\mathbb{T}_i}$ “ $\check{\delta} \in \dot{E}_i$ ”.

Recall that we previously found a limit ordinal β_δ with $\delta < \beta_\delta < \lambda$ and an ordinal $i_\delta < \kappa$ such that, for all i with $i_\delta \leq i < \kappa$, we have $t_\delta^i = C_{\beta_\delta, i}$. Let $i^* < \kappa$ be least such that $i^* \geq i_\delta$ and $\delta \in \text{lim}(C_{\beta_\delta, i^*})$. Then $t_\delta^{i^*} \leq_{\mathbb{T}_{i^*}} C_{\delta, i^*}$ and, clearly, $t_\delta^{i^*} \Vdash_{\mathbb{T}_{i^*}}$ “ $\check{\delta} \in \dot{R}_i$ ”. However, this contradicts the fact that $C_{\delta, i^*} \Vdash_{\mathbb{T}_{i^*}}$ “ $\check{\delta} \in \dot{E}_{i^*}$ ” and \dot{E}_{i^*} is forced to be disjoint from \dot{R}_i . ⊥

Choose $i < \kappa$ and $r \in \mathbb{T}_i$ as in the statement of the claim, and let J be \mathbb{T}_i -generic over $V[G * H_\eta]$ with $r \in J$. Let R be the interpretation of \hat{R}_i in $V[G * H_\eta * J]$. Note that, for all $\alpha \in R_i$ and all $j < j_\alpha$, we know that $S_{\alpha,j}$ remains stationary in $V[G * H_\eta * J]$. Since, in $V, \mathbb{S} * \mathbb{P}_\eta * \mathbb{T}_i$ has a dense λ -directed closed subset of cardinality λ and $\text{DSR}^*(\langle \kappa, S \rangle)$ holds, it follows that $\text{DSR}(\langle \kappa, S \rangle)$ holds in $V[G * H_\eta * J]$. Working in $V[G * H_\eta * J]$, define a matrix $\langle \hat{S}_{\alpha,j} \mid \alpha < \lambda, j < j_\alpha + 1 \rangle$ as follows. For all $\alpha \in R_i$, let $\hat{S}_{\alpha,j} = S_{\alpha,j}$ for all $j < j_\alpha$ and $\hat{S}_{\alpha,j_\alpha} = R$. For $\alpha \in \lambda \setminus R_i$, simply let $\hat{S}_{\alpha,j} = R$ for all $\alpha < j_\alpha + 1$.

By $\text{DSR}(\langle \kappa, S \rangle)$, we can find an ordinal $\gamma \in S_{>\omega}^\lambda$ and a club $F \subseteq \gamma$ such that $\hat{S}_{\alpha,j} \cap \gamma$ is stationary in γ for every $\alpha \in F$ and every $j < j_\alpha + 1$. Since $\hat{S}_{\alpha,j_\alpha} = R$ for every $\alpha < \lambda$, it follows that $F \cap R$ is stationary in γ and, for all $\alpha \in F \cap R$ and all $j < j_\alpha$, we have that $S_{\alpha,j} \cap \gamma$ is stationary in γ . Since stationarity is downward absolute, it follows that, in $V[G * H_\eta]$, the set of $\alpha < \gamma$ such that $S_{\alpha,j} \cap \gamma$ is stationary in γ for every $j < j_\alpha$ is itself stationary in γ . Since $V[G * H]$ has the same bounded subsets of λ as $V[G * H_\eta]$, this continues to hold in $V[G * H]$ as well. Therefore, γ witnesses this instance of $\text{sDSR}(\langle \kappa, S \rangle)$ in $V[G * H]$. \dashv

§6. Questions. We end with a few questions that remain open. First, recall Lemma 2.5, stating that if λ is a regular uncountable cardinal, $S \subseteq \lambda$ is stationary and $\text{DSR}(1, S)$ holds, then $\text{Refl}(\omega, S)$ follows. On the other hand, by Theorem 3.28, it is consistent to have regular cardinals $\omega_1 < \kappa < \lambda$ such that $\text{DSR}(\langle \kappa, S_\omega^\lambda \rangle)$ holds yet $\text{Refl}(\kappa, S)$ fails, for any set S stationary in λ . So the question that remains in this context is:

QUESTION 6.1. *Is it consistent that there is a regular cardinal $\lambda > \omega_1$ such that $\text{DSR}(1, S_\omega^\lambda)$ holds but $\text{Refl}(\omega_1, \lambda)$ fails? Or that $\text{DSR}(\omega, S_\omega^\lambda)$ holds but $\text{Refl}(\omega_1, \lambda)$ fails?*

In another direction, recall Theorem 2.1, which states that if $1 < \kappa < \lambda$, λ is regular, and $\text{sDSR}(\langle \kappa, S \rangle)$ holds for some stationary $S \subseteq \lambda$, then $\square(\lambda, \langle \kappa \rangle)$ fails. This is an improvement of the original observation, which drew the same conclusion from the assumption of $\text{DSR}(\langle \kappa, S \rangle)$. We have shown that this is optimal in some sense (see Theorems 3.28 and 5.2), but it is open whether it is optimal in another sense. Namely, the following is unknown.

QUESTION 6.2. *Suppose that $1 < \kappa < \lambda$, λ is regular, and $\text{uDSR}(\langle \kappa, S \rangle)$ holds for some stationary $S \subseteq \lambda$. Does it follow that $\square(\lambda, \langle \kappa \rangle)$ fails?*

Next, we ask whether Theorem 3.21 can be improved to cover the case in which $\lambda = \kappa^+$.

QUESTION 6.3. *Is it consistent that κ is an uncountable cardinal, $\lambda = \kappa^+$, and $\text{sDSR}(\langle \kappa, S_\omega^\lambda \rangle)$ holds but $\text{DSR}(1, S_\omega^\lambda)$ fails?*

Finally, we ask two questions around the consistency of $\text{DSR}(\langle \kappa, S \rangle)$ with $\square(\lambda, \kappa)$ addressing the optimality of Theorems 3.28 and 5.2.

QUESTION 6.4. *What is the consistency strength of the existence of infinite regular cardinals $\kappa < \lambda$, with $\lambda \geq \omega_2$, for which there exists a stationary $S \subseteq \lambda$ such that $\text{DSR}(\langle \kappa, S \rangle) + \square(\lambda, \kappa)$ holds?*

QUESTION 6.5. *Are there consistently infinite regular cardinals $\kappa < \lambda$ for which $\text{DSR}(\langle \kappa, \lambda \rangle) + \square(\lambda, \kappa)$ holds? (Here λ could be either inaccessible or the successor of a singular cardinal.)*

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