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Twisted Brauer monoids

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We investigate the structure of the twisted Brauer monoid $\mathcal{B}_n^\tau$, comparing and contrasting it with the structure of the (untwisted) Brauer monoid $\mathcal{B}_n$. We characterize Green's relations and pre-orders on $\mathcal{B}_n^\tau$, describe the lattice of ideals and give necessary and sufficient conditions for an ideal to be idempotent generated. We obtain formulae for the rank (smallest size of a generating set) and (where applicable) the idempotent rank (smallest size of an idempotent generating set) of each principal ideal; in particular, when an ideal is idempotent generated, its rank and idempotent rank are equal. As an application of our results, we describe the idempotent generated subsemigroup of $\mathcal{B}_n^\tau$ (which is not an ideal), as well as the singular ideal of $\mathcal{B}_n^\tau$ (which is neither principal nor idempotent generated), and we deduce that the singular part of the Brauer monoid $\mathcal{B}_n$ is idempotent generated, a result previously proved by Maltcev and Mazorchuk.

Keywords: Brauer monoids; twisted Brauer monoids; idempotents; ideals; rank; idempotent rank

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1. Introduction

The *Temperley–Lieb algebras* were introduced in [41] to study lattice problems in (planar) statistical mechanics. These algebras have played important roles in many different areas of mathematics, most notably in the foundational works of Jones [30] and Kauffman [32] on knot polynomials. As noted by Kauffman in [32], the structure of the Temperley–Lieb algebra is governed by an underlying (countably infinite) monoid that has now become known as the *Kauffman monoid* [8,34]; an approach via a natural finite quotient of this monoid was described in [42]. Kauffman also noted in [32] that the Temperley–Lieb algebras are closely related to the algebras introduced by Brauer in his famous 1937 article [9] on invariant theory and representations of orthogonal groups. The Temperley–Lieb and Brauer algebras both have bases consisting of certain diagrams that are concatenated in a natural way (see below), so that the product of two basis elements is a scalar multiple of another basis element. Other such algebras, known collectively as *diagram algebras*, include partition algebras [26,31,36], partial Brauer algebras [24,37], Motzkin algebras [6], rook monoid algebras [25,40] and many more. These diagram algebras are all *twisted semigroup algebras* [42] of certain finite *diagram semigroups* (such as the partition monoid, Brauer monoid and Jones monoid), but they may also be viewed as (ordinary) semigroup algebras of the so-called *twisted diagram semigroups* (the Kauffman monoid is a canonical example).

Studies of diagram semigroups have led to important results concerning the associated algebras, including cellularity [42], presentations [15, 16] and idempotent enumeration [11, 12]; see also [19] for an alternative approach to calculating dimensions of irreducible representations. But it is also interesting to note that diagram semigroups have played a part in the development of semigroup theory itself, particularly in the context of regular $*$-semigroups [17, 18] and pseudovarieties of finite semigroups [2–4]. Although the *twisted* diagram semigroups are more closely related to diagram algebras, they have so far received less attention than their untwisted relatives, with existing studies (see [5, 7, 8, 10, 12, 14, 34]) focusing mostly on the Kauffman monoid (which we have already discussed). This paper therefore aims to further the study of twisted diagram semigroups, and here we focus on the *twisted Brauer monoid*.[1] In particular, we conduct a thorough investigation of the algebraic structure of the monoid, paying particular attention to Green's relations and pre-orders (which govern divisibility in the monoid and formalize several natural parameters associated to Brauer diagrams) and the lattice of ideals (which plays an important role in the cellular structure of the associated algebra [21]). We also consider combinatorial problems, such as determining which ideals are idempotent generated and calculating invariants such as the smallest size of (idempotent-)generating sets.

The paper is organized as follows. In § 2, we recall the definition of the Brauer monoid $\mathcal{B}_n$, and record some known results we shall need in what follows. Section 3, which concerns the twisted Brauer monoid $\mathcal{B}_n^\tau$, forms the bulk of the paper, and consists of four subsections. In § 3.1, we describe Green's relations and pre-orders on $\mathcal{B}_n^\tau$, and we also characterize the regular elements of $\mathcal{B}_n^\tau$. Section 3.2 contains a classification of the ideals of $\mathcal{B}_n^\tau$. We calculate the smallest size of a generating set for each principal ideal of $\mathcal{B}_n^\tau$ in § 3.3, where we also give necessary and sufficient conditions for an ideal to be idempotent generated; we also calculate the smallest size of an idempotent generating set for such an ideal. Finally, in § 3.4, we apply the results of the previous sections to prove results about the singular part of $\mathcal{B}_n^\tau$ and the idempotent generated subsemigroups of $\mathcal{B}_n^\tau$ and $\mathcal{B}_n$.

2. The Brauer monoid

Fix a non-negative integer n, and write $[n] = \{1, \ldots, n\}$ and $[n]' = \{1', \ldots, n'\}$. Denote by $\mathcal{B}_n$ the set of all set partitions of $[n] \cup [n]'$ into blocks of size 2. For

[1] The twisted Brauer monoid also played a role in [5], where it was called the *wire monoid*. We use the current terminology because of the above-mentioned links with twisted semigroup algebras.

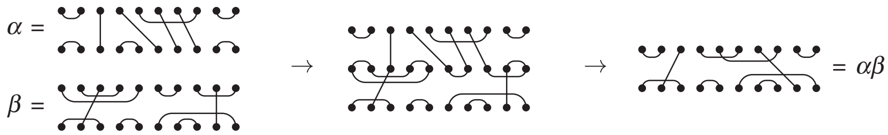


Figure 1. Two Brauer diagrams $\alpha, \beta \in \mathcal{B}_{10}$ (left), their product $\alpha\beta \in \mathcal{B}_{10}$ (right) and the graph $\Gamma(\alpha, \beta)$ (centre).

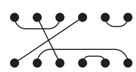
example, here is an element of $\mathcal{B}_6$:

$$\alpha = \{\{1,3\},\{2,3'\},\{4,1'\},\{5,6\},\{2',6'\},\{4',5'\}\}.$$

There is a unique element of $\mathcal{B}_0$, namely the empty partition. It is easy to see that

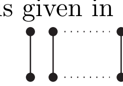
$$|\mathcal{B}_n| = (2n-1)!! = (2n-1)\cdot(2n-3)\cdots 3\cdot 1 = \frac{(2n)!}{2^n \cdot n!} = \frac{n!}{2^n}\cdot\binom{2n}{n}.$$

An element of $\mathcal{B}_n$ may be represented (uniquely) by a graph on vertex set $[n] \cup [n]'$; a single edge is included between vertices $u, v \in [n] \cup [n]'$ if and only if $\{u, v\}$ is a block of α. Such a graph is called a *Brauer n-diagram* (or just a *Brauer diagram* if n is understood from context). We typically identify $\alpha \in \mathcal{B}_n$ with its corresponding Brauer diagram. When drawing a Brauer diagram, the vertices $1, \ldots, n$ are arranged in a horizontal line, with vertices $1', \ldots, n'$ in a parallel line below; unless otherwise specified, the vertices are assumed to be increasing from left to right. For example, with $\alpha \in \mathcal{B}_6$ as above, we have

$\alpha =$ 

It will often be convenient to order the top and/or bottom vertices differently, but the ordering will always be made clear (see figure 3, for example).

The set $\mathcal{B}_n$ forms a monoid, known as the *Brauer monoid* of degree n, under an operation we now describe. Let $\alpha, \beta \in \mathcal{B}_n$. Write $[n]'' = \{1'', \ldots, n''\}$. Let $\alpha_\vee$ be the graph obtained from α by changing the label of each lower vertex i' to i''. Similarly, let $\beta^\wedge$ be the graph obtained from β by changing the label of each upper vertex i to i''. Consider now the graph $\Gamma(\alpha, \beta)$ on the vertex set $[n] \cup [n]' \cup [n]''$ obtained by joining $\alpha_\vee$ and $\beta^\wedge$ together so that each lower vertex i'' of $\alpha_\vee$ is identified with the corresponding upper vertex i'' of $\beta^\wedge$. Note that $\Gamma(\alpha, \beta)$, which we call the *product graph of* α, β, may contain parallel edges. We define $\alpha\beta \in \mathcal{B}_n$ to be the Brauer diagram that has an edge $\{x, y\}$ if and only if $x, y \in [n] \cup [n]'$ are connected by a path in $\Gamma(\alpha, \beta)$. An example calculation (with $n = 10$) is given in figure 1.

The identity element of $\mathcal{B}_n$ is the Brauer diagram $1 =$ . The set

$$\mathcal{S}_n = \{\alpha \in \mathcal{B}_n : \operatorname{dom}(\alpha) = \operatorname{codom}(\alpha) = [n]\}$$

is the group of units of $\mathcal{B}_n$, and is (isomorphic to) the symmetric group on $[n]$.

Let $\alpha \in \mathcal{B}_n$. A block of α is called a *transversal* if it has non-empty intersection with both $[n]$ and $[n]'$, and an *upper hook* (respectively, *lower hook*) if it is contained in $[n]$ (respectively, $[n]'$). The *rank* of α, denoted by $\operatorname{rank}(\alpha)$, is equal to the number

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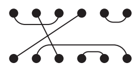
of transversals of α. For $x \in [n] \cup [n]'$, let $[x]_\alpha$ denote the block of α containing x. We define the *domain* and *codomain* of α to be the sets

$$\mathrm{dom}(\alpha) = \{x \in [n] \colon [x]_\alpha \cap [n]' \neq \emptyset\} \quad \text{and} \quad \mathrm{codom}(\alpha) = \{x \in [n] \colon [x']_\alpha \cap [n] \neq \emptyset\}.$$

Note that $\mathrm{rank}(\alpha) = |\mathrm{dom}(\alpha)| = |\,\mathrm{codom}(\alpha)|$, and that $n - \mathrm{rank}(\alpha)$ is equal to the number of hooks of α (half of which are upper hooks, and half lower). We also define the *kernel* and *cokernel* of α to be the equivalences

$$\ker(\alpha) = \{(x,y) \in [n] \times [n] \colon [x]_\alpha = [y]_\alpha\},$$
$$\mathrm{coker}(\alpha) = \{(x,y) \in [n] \times [n] \colon [x']_\alpha = [y']_\alpha\}.$$

To illustrate these ideas, with $\alpha =$  $\in \mathcal{B}_6$ as above, we have $\mathrm{rank}(\alpha) = 2$,

$$\mathrm{dom}(\alpha) = \{2,4\}, \qquad \mathrm{codom}(\alpha) = \{1,3\},$$
$$\ker(\alpha) = (1,3 \mid 2 \mid 4 \mid 5,6), \qquad \mathrm{coker}(\alpha) = (1 \mid 2,6 \mid 3 \mid 4,5),$$

using an obvious notation for equivalences.

It is immediate from the definitions that

$$\mathrm{dom}(\alpha\beta) \subseteq \mathrm{dom}(\alpha), \qquad \ker(\alpha\beta) \supseteq \ker(\alpha),$$
$$\mathrm{codom}(\alpha\beta) \subseteq \mathrm{codom}(\beta), \qquad \mathrm{coker}(\alpha\beta) \supseteq \mathrm{coker}(\beta),$$

for all $\alpha, \beta \in \mathcal{B}_n$. For example, the identity $\ker(\alpha\beta) \supseteq \ker(\alpha)$ says that any upper hook of α is an upper hook of $\alpha\beta$.

We now recall from [18] another way to specify an element of $\mathcal{B}_n$. With this in mind, let $\alpha \in \mathcal{B}_n$. We write

$$\alpha = \left(\begin{array}{c|c|c|c|c|c} i_1 & \cdots & i_r & a_1, b_1 & \cdots & a_s, b_s \\ \cline{4-6} j_1 & \cdots & j_r & c_1, d_1 & \cdots & c_s, d_s \end{array} \right) \tag{$\dagger$}$$

to indicate that α has transversals $\{i_1, j_1'\}, \ldots, \{i_r, j_r'\}$, upper hooks $\{a_1, b_1\}, \ldots,$ $\{a_s, b_s\}$ and lower hooks $\{c_1', d_1'\}, \ldots, \{c_s', d_s'\}$. Note that it is possible for either r or s to be 0, but we always have $n = r + 2s$. In particular, we always have $\mathrm{rank}(\alpha) = r \equiv n \pmod 2$.

For $\alpha \in \mathcal{B}_n$, we write α^* for the Brauer diagram obtained from α by interchanging dashed and undashed vertices (i.e. by reflecting α in the horizontal axis). The '$*$' operation gives $\mathcal{B}_n$ the structure of a *regular $*$-semigroup* [39]; that is, for all $\alpha, \beta \in \mathcal{B}_n$,

$$\alpha^{**} = \alpha, \qquad (\alpha\beta)^* = \beta^*\alpha^*, \qquad \alpha\alpha^*\alpha = \alpha, \qquad \alpha^*\alpha\alpha^* = \alpha^*.$$

(The fourth identity follows easily from the first three.) This symmetry allows us to shorten many proofs.

Recall that *Green's relations* $\mathscr{R}, \mathscr{L}, \mathscr{J}, \mathscr{H}, \mathscr{D}$ are defined on a semigroup S, for $x, y \in S$, by

$$x \mathrel{\mathscr{R}} y \iff xS^1 = yS^1, \qquad x \mathrel{\mathscr{L}} y \iff S^1x = S^1y,$$
$$x \mathrel{\mathscr{J}} y \iff S^1xS^1 = S^1yS^1,$$
$$\mathscr{H} = \mathscr{R} \cap \mathscr{L}, \qquad \mathscr{D} = \mathscr{R} \vee \mathscr{L} = \mathscr{R} \circ \mathscr{L} = \mathscr{L} \circ \mathscr{R}.$$

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Here, S^1 denotes the monoid obtained by adjoining an identity 1 to S (if necessary). If S is finite, then $\mathscr{J} = \mathscr{D}$. If $x \in S$, and if $\mathscr{K}$ is one of Green's relations, we denote by K_x the $\mathscr{K}$-class of x in S. An $\mathscr{H}$-class contains an idempotent if and only if it is a group, in which case it is a maximal subgroup of S. If e and f are $\mathscr{D}$-related idempotents of S, then the subgroups H_e and H_f are isomorphic. If S is a monoid, then the $\mathscr{H}$-class of the identity element of S is the group of units of S. An element $x \in S$ is *regular* if $x = xyx$ and $y = yxy$ for some $y \in S$ or, equivalently, if D_x contains an idempotent, in which case R_x and L_x do too. In a $\mathscr{D}$-class of S, either every element is regular or every element is non-regular. We say S is regular if every element of S is regular. For more background on semigroups, see, for example, [27, 29]. The Brauer monoid $\mathcal{B}_n$ is regular since, as noted above, it is a regular $*$-semigroup.

The next result, which describes Green's relations on $\mathcal{B}_n$, was originally proved in [38, theorem 7]; see also [20, 33, 42].

Proposition 2.1 (Marorchuk [38]). *Let $\alpha, \beta \in \mathcal{B}_n$. Then*

(i) $\alpha \mathscr{R} \beta \iff \ker(\alpha) = \ker(\beta) \iff \alpha\mathcal{S}_n = \beta\mathcal{S}_n$,

(ii) $\alpha \mathscr{L} \beta \iff \mathrm{coker}(\alpha) = \mathrm{coker}(\beta) \iff \mathcal{S}_n\alpha = \mathcal{S}_n\beta$,

(iii) $\alpha \mathscr{J} \beta \iff \alpha \mathscr{D} \beta \iff \mathrm{rank}(\alpha) = \mathrm{rank}(\beta) \iff \mathcal{S}_n\alpha\mathcal{S}_n = \mathcal{S}_n\beta\mathcal{S}_n$.

In particular,

$$R_\alpha = \alpha\mathcal{S}_n, \quad L_\alpha = \mathcal{S}_n\alpha, \quad H_\alpha = \alpha\mathcal{S}_n \cap \mathcal{S}_n\alpha, \quad D_\alpha = J_\alpha = \mathcal{S}_n\alpha\mathcal{S}_n \quad \text{for all } \alpha \in \mathcal{B}_n.$$

For the remainder of the paper, it will be convenient to define $z \in \{0, 1\}$ with $z \equiv n \pmod 2$. We shall also define the indexing set $I(n) = \{z, z+2, \ldots, n-2, n\}$. So $\mathrm{rank}(\alpha) \in I(n)$ for all $\alpha \in \mathcal{B}_n$, and the $\mathscr{D}$-classes of $\mathcal{B}_n$ are precisely the sets

$$D_r = \{\alpha \in \mathcal{B}_n : \mathrm{rank}(\alpha) = r\} \quad \text{for all } r \in I(n).$$

The following two results were proved in [19, theorem 8.4].

Proposition 2.2 (East and Gray [19]). *Let $r = n - 2s \in I(n)$, and set*

$$\rho_{nr} = \binom{n}{r} \cdot (n - r - 1)!! = \frac{n!}{2^s s! r!} \quad \text{and} \quad \delta_{nr} = \rho_{nr}^2 \cdot r! = \frac{n!^2}{2^{2s} s!^2 r!}.$$

Then

(i) *D_r contains ρ_{nr} $\mathscr{R}$-classes and ρ_{nr} $\mathscr{L}$-classes,*

(ii) *each $\mathscr{H}$-class contained in D_r has size $r!$ (and group $\mathscr{H}$-classes contained in D_r are isomorphic to $\mathcal{S}_r$),*

(iii) $|D_r| = \delta_{nr}$.

Theorem 2.3 (East and Gray [19]). *The ideals of $\mathcal{B}_n$ are precisely the sets*

$$I_r = D_z \cup D_{z+2} \cup \cdots \cup D_r = \{\alpha \in \mathcal{B}_n : \mathrm{rank}(\alpha) \leqslant r\} \quad \text{for all } r \in I(n).$$

If $r \in I(n) \setminus \{n\}$, *then*

$$I_r = \langle D_r \rangle = \langle E(D_r) \rangle \quad \textit{and} \quad \operatorname{rank}(I_r) = \operatorname{idrank}(I_r) = \rho_{nr},$$

where the numbers ρ_{nr} *are defined in proposition 2.2.*

3. The twisted Brauer monoid

When forming the product $\alpha\beta$, where $\alpha, \beta \in \mathcal{B}_n$, the product graph $\Gamma(\alpha, \beta)$ may contain components that lie completely in $[n]''$; such components are called *floating components*. We write $\tau(\alpha, \beta)$ for the number of such floating components in $\Gamma(\alpha, \beta)$. In the example from figure 1, $\Gamma(\alpha, \beta)$ has a unique floating component, namely $\{1'', 2'', 4'', 5''\}$, so $\tau(\alpha, \beta) = 1$. There are two main ways to modify the product in $\mathcal{B}_n$ to take these floating components into account. One leads to the *Brauer algebra* [9], an associative algebra with $\mathcal{B}_n$ as its basis, and the other leads to the *twisted Brauer monoid*, which we now describe. Specifically, we define

$$\mathcal{B}_n^\tau = \mathbb{N} \times \mathcal{B}_n = \{(i, \alpha) \colon i \in \mathbb{N},\ \alpha \in \mathcal{B}_n\}$$

with the product $\star$ defined, for $i, j \in \mathbb{N}$ and $\alpha, \beta \in \mathcal{B}_n$, by

$$(i, \alpha) \star (j, \beta) = (i + j + \tau(\alpha, \beta), \alpha\beta).$$

One can easily check that

$$\tau(\alpha, \beta) + \tau(\alpha\beta, \gamma) = \tau(\alpha, \beta\gamma) + \tau(\beta, \gamma) \quad \text{for all } \alpha, \beta, \gamma \in \mathcal{B}_n. \tag{3.1}$$

It quickly follows that $\star$ is associative. We call $\mathcal{B}_n^\tau$ (with the '$\star$' operation) the *twisted Brauer monoid* of degree n. We note that there is a natural inclusion $\iota \colon \mathcal{B}_n \to \mathcal{B}_n^\tau \colon \alpha \mapsto (0, \alpha)$, and we typically identify $\mathcal{B}_n$ with its image under ι. But it is important to note that ι is not a homomorphism, since $\alpha \star \beta = (\tau(\alpha, \beta), \alpha\beta) \neq \alpha\beta$ if $\tau(\alpha, \beta) \neq 0$. It follows from the associativity of $\star$ that, for any $\alpha_1, \ldots, \alpha_k \in \mathcal{B}_n$,

$$\alpha_1 \star \cdots \star \alpha_k = (\tau(\alpha_1, \ldots, \alpha_k), \alpha_1 \cdots \alpha_k)$$

for some $\tau(\alpha_1, \ldots, \alpha_k) \in \mathbb{N}$. Note that, for any $\alpha, \beta, \gamma \in \mathcal{B}_n$, $\tau(\alpha, \beta, \gamma)$ is equal to the common value in (3.1). It is of special importance (and easily seen) that $\tau(\alpha, \beta) = 0$ if either α or β belongs to $\mathcal{S}_n$. It is also immediate that $\tau(\alpha, \beta) = \tau(\beta^*, \alpha^*)$, so if we define $(i, \alpha)^* = (i, \alpha^*)$, then

$$(i, \alpha)^{**} = (i, \alpha) \quad \text{and} \quad ((i, \alpha) \star (j, \beta))^* = (j, \beta)^* \star (i, \alpha)^*$$

for all $(i, \alpha), (j, \beta) \in \mathcal{B}_n^\tau$. In other words, $\mathcal{B}_n^\tau$ is a $*$-semigroup (a semigroup with involution). But this '$*$' operation does not give $\mathcal{B}_n^\tau$ the structure of a regular $*$-semigroup [39], since it is not necessarily the case that $(i, \alpha) \star (i, \alpha)^* \star (i, \alpha) = (i, \alpha)$; for example, the latter does not hold if $i \geqslant 1$ or if $\tau(\alpha, \alpha^*) \geqslant 1$. In fact, $\mathcal{B}_n^\tau$ is not a regular $*$-semigroup at all, as it is not even regular, as we shall see in the next section.

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3.1. Green's relations and pre-orders

Our next goal is to describe Green's relations on the twisted Brauer monoid $\mathcal{B}_n^\tau$. In order to do this, it will be convenient to first describe Green's pre-orders on $\mathcal{B}_n^\tau$. Recall that Green's pre-orders $\leqslant_{\mathscr{R}}$, $\leqslant_{\mathscr{L}}$, $\leqslant_{\mathscr{J}}$ are defined on a semigroup S, for $x, y \in S$, by

$$\begin{aligned} x \leqslant_{\mathscr{R}} y &\iff xS^1 \subseteq yS^1, \\ x \leqslant_{\mathscr{L}} y &\iff S^1x \subseteq S^1y, \\ x \leqslant_{\mathscr{J}} y &\iff S^1xS^1 \subseteq S^1yS^1. \end{aligned}$$

So, for example, $\mathscr{R} = \leqslant_{\mathscr{R}} \cap \geqslant_{\mathscr{R}}$. In order to avoid confusion, we shall use the symbols $\mathscr{R}$, $\leqslant_{\mathscr{R}}$, etc., for Green's relations and pre-orders on $\mathcal{B}_n$, and write $\mathscr{R}^\tau$, $\leqslant_{\mathscr{R}}^\tau$, etc., for the corresponding relations and pre-orders on $\mathcal{B}_n^\tau$. We first need to prove a result concerning Green's pre-orders on $\mathcal{B}_n$, which involves the twisting map τ.

Proposition 3.1. *Let $\alpha, \beta \in \mathcal{B}_n$. Then*

(i) $\alpha \leqslant_{\mathscr{R}} \beta \iff \ker(\alpha) \supseteq \ker(\beta) \iff \alpha = \beta\delta$ *for some* $\delta \in \mathcal{B}_n$ *with* $\tau(\beta, \delta) = 0$,

(ii) $\alpha \leqslant_{\mathscr{L}} \beta \iff \operatorname{coker}(\alpha) \supseteq \operatorname{coker}(\beta) \iff \alpha = \gamma\beta$ *for some* $\gamma \in \mathcal{B}_n$ *with* $\tau(\gamma, \beta) = 0$,

(iii) $\alpha \leqslant_{\mathscr{J}} \beta \iff \operatorname{rank}(\alpha) \leqslant \operatorname{rank}(\beta) \iff \alpha = \gamma\beta\delta$ *for some* $\gamma, \delta \in \mathcal{B}_n$ *with* $\tau(\gamma, \beta, \delta) = 0$.

Proof. We begin with (i). Again, it is well-known that $\alpha \leqslant_{\mathscr{R}} \beta \iff \ker(\alpha) \supseteq \ker(\beta)$. Next, suppose $\ker(\alpha) \supseteq \ker(\beta)$. Then we may write

$$\alpha = \left(\begin{array}{c|c|c|c|c|c|c|c|c} i_1 & \cdots & i_r & a_1, b_1 & \cdots & a_s, b_s & a_{s+1}, b_{s+1} & \cdots & a_{s+t}, b_{s+t} \\ j_1 & \cdots & j_r & c_1, d_1 & \cdots & c_s, d_s & c_{s+1}, d_{s+1} & \cdots & c_{s+t}, d_{s+t} \end{array} \right)$$

and

$$\beta = \left(\begin{array}{c|c|c|c|c|c|c|c|c|c|c} i_1 & \cdots & i_r & a_1 & b_1 & \cdots & a_s & b_s & a_{s+1}, b_{s+1} & \cdots & a_{s+t}, b_{s+t} \\ k_1 & \cdots & k_r & e_1 & f_1 & \cdots & e_s & f_s & e_{s+1}, f_{s+1} & \cdots & e_{s+t}, f_{s+t} \end{array} \right).$$

It is easy to check that $\alpha = \beta\delta$ with $\tau(\beta, \delta) = 0$, where

$$\delta = \left(\begin{array}{c|c|c|c|c|c|c|c|c|c|c} k_1 & \cdots & k_r & e_{s+1} & f_{s+1} & \cdots & e_{s+t} & f_{s+t} & e_1, f_1 & \cdots & e_s, f_s \\ j_1 & \cdots & j_r & c_{s+1} & d_{s+1} & \cdots & c_{s+t} & d_{s+t} & c_1, d_1 & \cdots & c_s, d_s \end{array} \right).$$

This completes the proof of (i). Part (ii) follows by duality.

For (iii), suppose $\operatorname{rank}(\alpha) \leqslant \operatorname{rank}(\beta)$. As above, it suffices to demonstrate the existence of γ, δ with the desired properties. We may write

$$\alpha = \left(\begin{array}{c|c|c|c|c|c|c|c|c} i_1 & \cdots & i_r & a_1, b_1 & \cdots & a_s, b_s & a_{s+1}, b_{s+1} & \cdots & a_{s+t}, b_{s+t} \\ j_1 & \cdots & j_r & c_1, d_1 & \cdots & c_s, d_s & c_{s+1}, d_{s+1} & \cdots & c_{s+t}, d_{s+t} \end{array} \right)$$

and

$$\beta = \left(\begin{array}{c|c|c|c|c|c|c|c|c|c|c} l_1 & \cdots & l_r & g_1 & h_1 & \cdots & g_s & h_s & g_{s+1}, h_{s+1} & \cdots & g_{s+t}, h_{s+t} \\ k_1 & \cdots & k_r & e_1 & f_1 & \cdots & e_s & f_s & e_{s+1}, f_{s+1} & \cdots & e_{s+t}, f_{s+t} \end{array} \right).$$

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Now, set

$$\varepsilon = \left(\begin{array}{c|c|c|c|c|c|c|c|c} l_1 & \cdots & l_r & g_1, h_1 & \cdots & g_s, h_s & g_{s+1}, h_{s+1} & \cdots & g_{s+t}, h_{s+t} \\ \cline{4-9} j_1 & \cdots & j_r & c_1, d_1 & \cdots & c_s, d_s & c_{s+1}, d_{s+1} & \cdots & c_{s+t}, d_{s+t} \end{array} \right).$$

Then $\ker(\varepsilon) \supseteq \ker(\beta)$. By (i), it follows that there exists $\delta \in \mathcal{B}_n$ with $\varepsilon = \beta\delta$ and $\tau(\beta, \delta) = 0$. But also $\varepsilon \mathrel{\mathscr{L}} \alpha$, so proposition 2.1(ii) gives $\alpha = \gamma\varepsilon$ for some $\gamma \in \mathcal{S}_n$. In particular, $\alpha = \gamma\varepsilon = \gamma\beta\delta$, and $\tau(\gamma, \beta, \delta) = \tau(\gamma, \beta\delta) + \tau(\beta, \delta) = 0$. □

Proposition 3.2. *Let $i, j \in \mathbb{N}$ and $\alpha, \beta \in \mathcal{B}_n$. If $\mathscr{K}$ is any of $\mathscr{R}$, $\mathscr{L}$, $\mathscr{J}$, then*

$$(i, \alpha) \leqslant^{\tau}_{\mathscr{K}} (j, \beta) \iff i \geqslant j \text{ and } \alpha \leqslant_{\mathscr{K}} \beta.$$

Proof. We just treat the $\leqslant^{\tau}_{\mathscr{J}}$ pre-order, since the other cases are similar. Suppose first that $(i, \alpha) \leqslant^{\tau}_{\mathscr{J}} (j, \beta)$. Then

$$(i, \alpha) = (h, \gamma) \star (j, \beta) \star (k, \delta) = (h + j + k + \tau(\gamma, \beta, \delta), \gamma\beta\delta)$$

for some $h, k \in \mathbb{N}$ and $\gamma, \delta \in \mathcal{B}_n$. But then $i = h + j + k + \tau(\gamma, \beta, \delta) \geqslant j$ and $\alpha = \gamma\beta\delta \leqslant_{\mathscr{J}} \beta$. Conversely, suppose $i \geqslant j$ and $\alpha \leqslant_{\mathscr{J}} \beta$. By proposition 3.1(iii), there exist $\gamma, \delta \in \mathcal{B}_n$ such that $\alpha = \gamma\beta\delta$ and $\tau(\gamma, \beta, \delta) = 0$. But then one can easily check that $(i, \alpha) = (i - j, \gamma) \star (j, \beta) \star (0, \delta)$, completing the proof. □

Let $i \in \mathbb{N}$ and $\alpha \in \mathcal{B}_n$. If $\mathscr{K}$ is one of Green's relations, we write K_α and $K^{\tau}_{(i,\alpha)}$ for the $\mathscr{K}$-class of α in $\mathcal{B}_n$ and the $\mathscr{K}^{\tau}$-class of (i, α) in $\mathcal{B}^{\tau}_n$.

Corollary 3.3. *Let $i, j \in \mathbb{N}$ and $\alpha, \beta \in \mathcal{B}_n$. If $\mathscr{K}$ is any of $\mathscr{R}$, $\mathscr{L}$, $\mathscr{H}$, $\mathscr{J}$, $\mathscr{D}$, then*

$$(i, \alpha) \mathrel{\mathscr{K}^{\tau}} (j, \beta) \iff i = j \text{ and } \alpha \mathrel{\mathscr{K}} \beta.$$

Consequently, $K^{\tau}_{(i,\alpha)} = \{i\} \times K_\alpha$ for any $(i, \alpha) \in \mathcal{B}^{\tau}_n$.

Proof. The descriptions of the $\mathscr{R}^{\tau}$, $\mathscr{L}^{\tau}$, $\mathscr{H}^{\tau}$, $\mathscr{J}^{\tau}$ relations follow immediately from proposition 3.2. It remains only to show that $\mathscr{J}^{\tau} \subseteq \mathscr{D}^{\tau}$. But this is true because

$$\begin{aligned} (i, \alpha) \mathrel{\mathscr{J}} (j, \beta) &\implies [i = j \text{ and } \alpha \mathrel{\mathscr{J}} \beta] \\ &\implies [i = j \text{ and } \alpha \mathrel{\mathscr{D}} \beta] \\ &\implies [i = j \text{ and } \alpha \mathrel{\mathscr{R}} \gamma \mathrel{\mathscr{L}} \beta \text{ for some } \gamma \in \mathcal{B}_n] \\ &\implies (i, \alpha) \mathrel{\mathscr{R}^{\tau}} (i, \gamma) \mathrel{\mathscr{L}^{\tau}} (j, \beta) \\ &\implies (i, \alpha) \mathrel{\mathscr{D}^{\tau}} (j, \beta). \end{aligned}$$

□

So the $\mathscr{D}^{\tau}$-classes of $\mathcal{B}^{\tau}_n$ are precisely the sets

$$D_{r;k} = \{k\} \times D_r = \{(k, \alpha) \colon \operatorname{rank}(\alpha) = r\} \quad \text{for all } r \in I(n) \text{ and } k \in \mathbb{N}.$$

Note that, under the identification of $\alpha \in \mathcal{B}_n$ with $(0, \alpha) \in \mathcal{B}^{\tau}_n$, we have $D_{r;0} = D_r$ for all $r \in I(n)$.

Recall that the set $S/\mathscr{J}$ of all $\mathscr{J}$-classes of a semigroup S is a partially ordered set under the order $\leqslant$ defined, for $x, y \in S$, by $J_x \leqslant J_y \iff x \leqslant_{\mathscr{J}} y$. We shall write $\leqslant$ and $\leqslant^{\tau}$ for the partial orders on $\mathcal{B}_n/\mathscr{D}$ and $\mathcal{B}^{\tau}_n/\mathscr{D}^{\tau}$, respectively (recall

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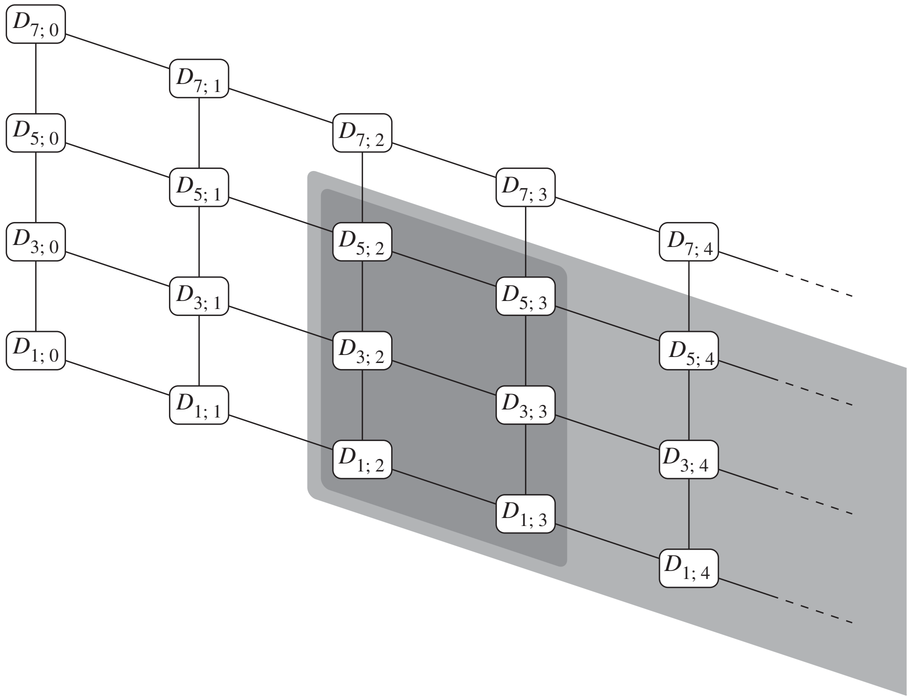

Figure 2. The structure of the partially ordered set $(\mathcal{B}_7^\tau/\mathscr{D}^\tau, \leqslant^\tau)$. The principal ideal $I_{5;2}$ is shaded light grey, and its generating set $M_{5;2}$ is shaded dark grey.

that $\mathscr{J} = \mathscr{D}$ and $\mathscr{J}^\tau = \mathscr{D}^\tau$ in $\mathcal{B}_n$ and $\mathcal{B}_n^\tau$). So, by propositions 3.1 and 3.2, we have

$$D_r \leqslant D_s \iff r \leqslant s \quad \text{and} \quad D_{r;k} \leqslant D_{s;l} \iff [r \leqslant s \text{ and } k \geqslant l].$$

So the partially ordered set $(\mathcal{B}_n^\tau/\mathscr{D}^\tau, \leqslant^\tau)$ is a lattice, and is order-isomorphic to the direct product of the chains $(I(n), \leqslant)$ and $(\mathbb{N}, \geqslant)$; this is analogous to the case of the Kauffman monoid [34]. Figure 2 gives an illustration for $n = 7$ (the reader may ignore the shading in the diagram for now).

We conclude this section with a description of the regular elements of $\mathcal{B}_n^\tau$.

Proposition 3.4. *An element* $(i, \alpha) \in \mathcal{B}_n^\tau$ *is regular if and only if* $i = 0$ *and* $\operatorname{rank}(\alpha) > 0$. *In particular,* $\mathcal{B}_n^\tau$ *is not regular.*

Proof. From $(i, \alpha) \star (j, \beta) \star (i, \alpha) = (2i + j + \tau(\alpha, \beta, \alpha), \alpha\beta\alpha)$, we deduce that (i, α) cannot be regular if

(i) $i \geqslant 1$, since then $2i + j + \tau(\alpha, \beta, \alpha) \geqslant 2i > i$, or

(ii) $\operatorname{rank}(\alpha) = 0$, since then $2i + j + \tau(\alpha, \beta, \alpha) \geqslant 2i + j + 1 > i$.

Conversely, if $i = 0$ and $r > 0$, then one may easily check that $\varepsilon =$  $\in D_{r;0}$ is an idempotent of $\mathcal{B}_n^\tau$ (i.e. $\varepsilon = \varepsilon \star \varepsilon$). It follows that the $\mathscr{D}^\tau$-classes $D_{r;0}$ with $r > 0$ are all regular. □

3.2. Ideals

We may now describe the ideals of $\mathcal{B}_n^\tau$. Recall that a *principal ideal* of a semigroup S is of the form

$$S^1aS^1 = \{xay\colon x, y \in S^1\} = \{x \in S\colon x \leqslant_{\mathscr{J}} a\} \quad \text{for } a \in S.$$

By proposition 3.2, we may immediately describe the principal ideals of $\mathcal{B}_n^\tau$. These are precisely the sets

$$I_{r;k} = \{(i, \alpha)\colon \operatorname{rank}(\alpha) \leqslant r,\ i \geqslant k\} \quad \text{for } r \in I(n) \text{ and } k \in \mathbb{N}.$$

Note that $I_{r;k} \subseteq I_{s;l} \iff D_{r;k} \leqslant^\tau D_{s;l} \iff [r \leqslant s \text{ and } k \geqslant l]$. The principal ideal $I_{5;2}$ of $\mathcal{B}_7^\tau$ is illustrated in figure 2. We now show that every ideal of $\mathcal{B}_n^\tau$ is the union of finitely many principal ideals (not every infinite semigroup shares this property).

Proposition 3.5.

(i) *Let* $r_1, \ldots, r_s \in I(n)$ *and* $k_1, \ldots, k_s \in \mathbb{N}$, *with* $r_1 > \cdots > r_s$ *and* $k_1 > \cdots > k_s$. *Then* $I_{r_1;k_1} \cup \cdots \cup I_{r_s;k_s}$ *is an ideal of* $\mathcal{B}_n^\tau$.

(ii) *Each ideal of* $\mathcal{B}_n^\tau$ *is of the form described in (i).*

(iii) *Each ideal of* $\mathcal{B}_n^\tau$ *is uniquely determined by (and uniquely determines) the parameters* $r_1, \ldots, r_s$, $k_1, \ldots, k_s$, *as described in (i).*

Proof. Part (i) is clear. Next, suppose I is an arbitrary non-empty ideal of $\mathcal{B}_n^\tau$. Set

$$r_1 = \max\{\operatorname{rank}(\alpha)\colon (k, \alpha) \in I\ (\exists k \in \mathbb{N})\},$$
$$k_1 = \min\{k \in \mathbb{N}\colon (k, \alpha) \in I\ (\exists \alpha \in D_{r_1})\}.$$

(Note that k_1 is defined in terms of r_1.) Then $I_{r_1;k_1} \subseteq I$. If $I = I_{r_1;k_1}$, then we are done. Otherwise, set

$$r_2 = \max\{\operatorname{rank}(\alpha)\colon (k, \alpha) \in I \setminus I_{r_1;k_1}\ (\exists k \in \mathbb{N})\},$$
$$k_2 = \min\{k \in \mathbb{N}\colon (k, \alpha) \in I \setminus I_{r_1;k_1}\ (\exists \alpha \in D_{r_2})\}.$$

(Note that $r_1 > r_2$ is obvious, while $k_1 > k_2$ follows from the fact that $I_{r_1;k_1}$ already contains $I_{r_2;k_1}$.) Then $I_{r_2;k_2} \subseteq I$. If $I = I_{r_1;k_1} \cup I_{r_2;k_2}$, then we are done. Otherwise, we define r_3 and k_3 similarly. Continuing in this fashion, since $I(n)$ is a finite chain, we eventually obtain $I = I_{r_1;k_1} \cup \cdots \cup I_{r_s;k_s}$ for some $r_1, \ldots, r_s \in I(n)$ and $k_1, \ldots, k_s \in \mathbb{N}$ with $r_1 > \cdots > r_s$ and $k_1 > \cdots > k_s$, giving (ii). For (iii), it is clear that $I_{r_1;k_1} \cup \cdots \cup I_{r_s;k_s} = I_{q_1;l_1} \cup \cdots \cup I_{q_t;l_t}$ if and only if $(r_1, \ldots, r_s) = (q_1, \ldots, q_t)$ and $(k_1, \ldots, k_s) = (l_1, \ldots, l_t)$. □

Remark 3.6. Note that

$$\begin{aligned} I_{r_1;k_1} \cup \cdots \cup I_{r_s;k_s} \subset I_{q_1;l_1} \cup \cdots \cup I_{q_t;l_t} & \\ \iff (\forall i \in [s])(\exists j \in [t])\ I_{r_i;k_i} \subset I_{q_j;l_j} & \\ \iff (\forall i \in [s])(\exists j \in [t])\ [r_i \leqslant q_j \text{ and } k_i \geqslant l_j]. & \end{aligned}$$

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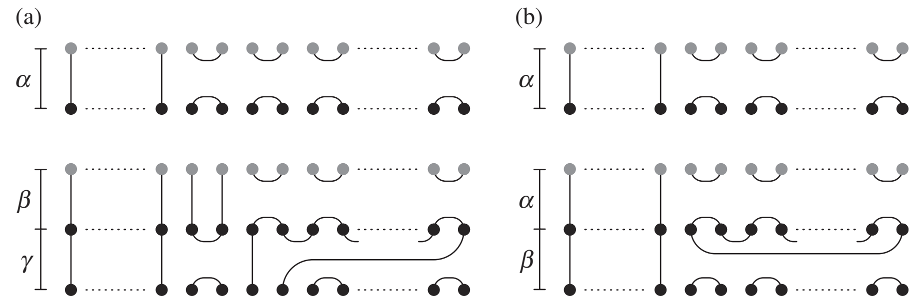

Figure 3. Diagrammatic verification that (a) $\alpha = \beta\gamma$ with $\tau(\beta,\gamma) = 0$ from the proof of lemma 3.7, and (b) $\alpha = \alpha\beta$ with $\tau(\alpha,\beta) = 1$ from the proof of lemma 3.9; see the text for more details. In both cases, grey vertices are ordered $i_1, \ldots, i_r, a_1, b_1, \ldots, a_s, b_s$, and black vertices are ordered $j_1, \ldots, j_r, c_1, d_1, \ldots, c_s, d_s$.

3.3. Small generating sets

We now turn to the question of minimal generation of the principal ideals. Recall that if S is a semigroup, then the *rank* of S, denoted $\operatorname{rank}(S)$, is the minimum cardinality of a subset $A \subseteq S$ such that $S = \langle A \rangle$. If S is idempotent generated, then the *idempotent rank* of S, denoted $\operatorname{idrank}(S)$, is defined analogously with respect to generating sets consisting of idempotents. In this section, we give necessary and sufficient conditions for a principal ideal $I_{r;k}$ to be idempotent generated. We also calculate the rank and idempotent rank (if appropriate) for an arbitrary principal ideal $I_{r;k}$; in particular, we show that $\operatorname{rank}(I_{r;k}) = \operatorname{idrank}(I_{r;k})$ if $I_{r;k}$ is idempotent generated.

If $\Sigma \subseteq \mathcal{B}_n$ (respectively, $\Gamma \subseteq \mathcal{B}_n^\tau$), we write $\langle \Sigma \rangle$ (respectively, $\langle\langle \Gamma \rangle\rangle$) for the subsemigroup of $\mathcal{B}_n$ (respectively, $\mathcal{B}_n^\tau$) generated by Σ (respectively, Γ). Since we identify $\mathcal{B}_n$ with a subset of $\mathcal{B}_n^\tau$, via the mapping $\alpha \mapsto (0, \alpha)$, it is possible to consider both $\langle \Sigma \rangle$ and $\langle\langle \Sigma \rangle\rangle$ for a subset $\Sigma \subseteq \mathcal{B}_n$; these are obviously not equal in general.

It will be necessary to consider the ideals $I_{r;k}$ in a number of separate cases, depending on the values of the parameters r, k (see theorem 3.20). We begin with the ideals $I_{r;0}$ with $r < n$. For this, we shall need the following two lemmas, the second of which will also be used later.

Lemma 3.7. *If $r \leqslant n-4$, then $D_r \subseteq D_{r+2} \star D_{r+2}$.*

Proof. Write α as in equation (†) on p. 734, where $r \leqslant n-4$. We show in figure 3(a) that $\alpha = \beta\gamma$ for some $\beta, \gamma \in D_{r+2}$ with $\tau(\beta,\gamma) = 0$. □

Remark 3.8. A weaker version of lemma 3.7 was proved in [19, lemma 8.3], where it was shown that $D_r \subseteq \langle D_{r+2} \rangle$; the proof of that result was much simpler, as no conditions were imposed on the twisting map τ, and the $*$-regular structure of $\mathcal{B}_n$ played a role.

Lemma 3.9. *If $\alpha \in \mathcal{B}_n \setminus \mathcal{S}_n$, then $\alpha = \alpha\beta$ for some $\beta \in \mathcal{B}_n$ with* $\operatorname{rank}(\beta) = \operatorname{rank}(\alpha)$ *and $\tau(\alpha,\beta) = 1$.*

Proof. Write α as in equation (†) on p. 734. We demonstrate the existence of β in figure 3(b). ☐

PROPOSITION 3.10. *If $r \in I(n) \setminus \{n\}$, then $I_{r;0} = \langle\!\langle D_r \rangle\!\rangle$.*

Proof. We first show, by descending induction, that $D_s \subseteq \langle\!\langle D_r \rangle\!\rangle$ for all $s \in I(r)$. Indeed, this is obvious if $s = r$, while if $s < r$, then lemma 3.7 and an induction hypothesis gives $D_s \subseteq D_{s+2} \star D_{s+2} \subseteq \langle\!\langle D_r \rangle\!\rangle$. It follows that $I_r \subseteq \langle\!\langle D_r \rangle\!\rangle$. Now suppose that $i \geqslant 1$ and $\alpha \in I_r$. We have seen that $\alpha \in \langle\!\langle D_r \rangle\!\rangle$. By lemma 3.9, we may choose some $\beta \in D_r$ such that $\alpha = \alpha\beta$ and $\tau(\alpha, \beta) = 1$. But then it quickly follows that

$$(i, \alpha) = \alpha \star \underbrace{\beta \star \cdots \star \beta}_{i} \in \langle\!\langle D_r \rangle\!\rangle.$$

We have shown that $I_{r;0} \subseteq \langle\!\langle D_r \rangle\!\rangle$. The reverse inclusion is clear. ☐

Proposition 3.10 does not hold for the top ideal $I_{n;0} = \mathcal{B}_n^\tau$, but we may use it as a stepping stone to calculate $\operatorname{rank}(\mathcal{B}_n^\tau)$. Recall that $\operatorname{rank}(\mathcal{S}_n) = 2$ if $n \geqslant 3$.

PROPOSITION 3.11. *Suppose $n \geqslant 3$. Let $\alpha, \beta \in \mathcal{S}_n$ be such that $\mathcal{S}_n = \langle \alpha, \beta \rangle$, and let $\gamma \in D_{n-2;0}$ and $(1, \delta) \in D_{n;1}$ be arbitrary. Then $\mathcal{B}_n^\tau = \langle\!\langle \alpha, \beta, \gamma, (1, \delta) \rangle\!\rangle$. Further, $\operatorname{rank}(\mathcal{B}_n^\tau) = 4$.*

Proof. Write $S = \langle\!\langle \alpha, \beta, \gamma, (1, \delta) \rangle\!\rangle$. First note that $\mathcal{S}_n = \langle \alpha, \beta \rangle = \langle\!\langle \alpha, \beta \rangle\!\rangle \subseteq S$. Together with proposition 2.1, it then follows that S contains $D_{n-2} = D_\gamma = \mathcal{S}_n \gamma \mathcal{S}_n = \mathcal{S}_n \star \gamma \star \mathcal{S}_n$. Proposition 3.10 then gives $I_{n-2;0} = \langle\!\langle D_{n-2} \rangle\!\rangle \subseteq S$. Finally, let $i \geqslant 1$ and $\sigma \in \mathcal{S}_n$ be arbitrary. Then

$$(i, \sigma) = (0, \sigma\delta^{-i}) \star \underbrace{(1, \delta) \star \cdots \star (1, \delta)}_{i} \in S,$$

which completes the proof that $\mathcal{B}_n^\tau = S = \langle\!\langle \alpha, \beta, \gamma, (1, \delta) \rangle\!\rangle$. It also follows that $\operatorname{rank}(\mathcal{B}_n^\tau) \leqslant 4$.

Suppose now that $\mathcal{B}_n^\tau = \langle\!\langle \Sigma \rangle\!\rangle$. The proof will be complete if we can show that $|\Sigma| \geqslant 4$. Since $\mathcal{B}_n^\tau \setminus \mathcal{S}_n = I_{n-2;0} \cup I_{n;1}$ is an ideal of $\mathcal{B}_n^\tau$, it follows that Σ contains a generating set for $\mathcal{S}_n$, so that $|\Sigma \cap \mathcal{S}_n| \geqslant 2$. Now let $\sigma \in \mathcal{S}_n$ be arbitrary, and consider an expression

$$(1, \sigma) = (i_1, \alpha_1) \star \cdots \star (i_k, \alpha_k) = (i_1 + \cdots + i_k + \tau(\alpha_1, \ldots, \alpha_k), \alpha_1 \cdots \alpha_k),$$

where $(i_1, \alpha_1), \ldots, (i_k, \alpha_j) \in \Sigma$. Since $\alpha_1 \cdots \alpha_k = \sigma \in \mathcal{S}_n$, and since $\mathcal{B}_n \setminus \mathcal{S}_n$ is an ideal of $\mathcal{B}_n$, it follows that $\alpha_1, \ldots, \alpha_k \in \mathcal{S}_n$. Then $\tau(\alpha_1, \ldots, \alpha_k) = 0$, so $1 = i_1 + \cdots + i_k$, which gives $i_s = 1$ for some (unique) $s \in [k]$. Thus, Σ contains an element of $D_{n;1}$: namely, $(1, \alpha_s)$. Similarly, consideration of an element of $D_{n-2;0}$ as a product of elements from Σ shows that Σ contains an element of $D_{n-2;0}$. As noted above, this completes the proof. ☐

Next, we calculate $\operatorname{rank}(I_{r;0})$ in the case that $0 < r < n$. In fact, since the ideal $I_{r;0}$ is idempotent generated for such a value of r (as we shall soon show), we shall also calculate $\operatorname{idrank}(I_{r;0})$. Since

$$(i, \alpha) \star (i, \alpha) = (2i + \tau(\alpha, \alpha), \alpha^2),$$

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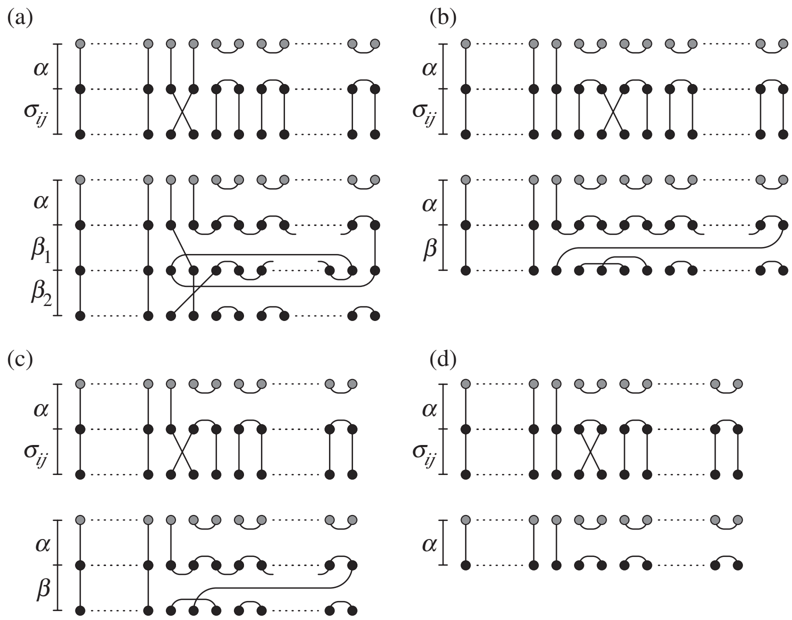
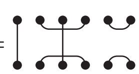
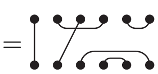


Figure 4. Diagrammatic verification that $\alpha\sigma_{ij} = \alpha \star \beta$, where $\beta \in \langle\!\langle E^\tau(D_r)\rangle\!\rangle$, as in the proof of proposition 3.12; see the text for more details. In all cases, grey vertices are ordered $i_1, \ldots, i_r, a_1, b_1, \ldots, a_s, b_s$, and black vertices are ordered $j_1, \ldots, j_r, c_1, d_1, \ldots, c_s, d_s$.

it follows that all idempotents of $\mathcal{B}_n^\tau$ are contained in $\mathcal{B}_n$. However, not every idempotent of $\mathcal{B}_n$ is an idempotent of $\mathcal{B}_n^\tau$; that is, $\alpha = \alpha^2$ in $\mathcal{B}_n$ does not necessarily imply $\alpha = \alpha \star \alpha$ in $\mathcal{B}_n^\tau$. In order to avoid confusion when discussing idempotents from $\mathcal{B}_n$ and $\mathcal{B}_n^\tau$, if $\Sigma \subseteq \mathcal{B}_n$, we shall write

$$E(\Sigma) = \{\alpha \in \Sigma \colon \alpha = \alpha^2\} \quad \text{and} \quad E^\tau(\Sigma) = \{\alpha \in \Sigma \colon \alpha = \alpha \star \alpha\}.$$

For example, one may easily check that

$\alpha =$  $\in E(\mathcal{B}_6) \setminus E^\tau(\mathcal{B}_6^\tau)$ but $\beta =$  $\in E^\tau(\mathcal{B}_6^\tau)$.

Indeed, $\alpha \star \alpha = (2, \alpha) \neq \alpha$ in $\mathcal{B}_6^\tau$. The idempotents of $\mathcal{B}_n$ and $\mathcal{B}_n^\tau$ (and a number of other diagram semigroups) were characterized and enumerated in [11], but we shall not need to use these descriptions here.

Proposition 3.12. *Suppose $r \in I(n) \setminus \{0, n\}$. Then $I_{r;0} = \langle\!\langle E^\tau(D_r)\rangle\!\rangle$.*

Proof. By proposition 3.10, it suffices to show that $D_r \subseteq \langle\!\langle E^\tau(D_r)\rangle\!\rangle$. By proposition 3.4, $D_r = D_{r;0}$ is a regular $\mathscr{D}^\tau$-class of $\mathcal{B}_n^\tau$, so we may choose an idempotent $\varepsilon \in E^\tau(D_r)$. Since $D_r = D_\varepsilon = \mathcal{S}_n \varepsilon \mathcal{S}_n$, by proposition 2.1, it suffices to show that $\lambda\varepsilon\rho \in \langle\!\langle E^\tau(D_r)\rangle\!\rangle$ for all $\lambda, \rho \in \mathcal{S}_n$. In fact, by a simple induction on the length of λ and ρ as products of transpositions, it suffices to show that

(I) for all $\alpha \in D_r$ and all $1 \leqslant i < j \leqslant n$, $\alpha\sigma_{ij} = \alpha \star \beta$ for some $\beta \in \langle\!\langle E^\tau(D_r)\rangle\!\rangle$, and

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(II) for all $\alpha \in D_r$ and all $1 \leqslant i < j \leqslant n$, $\sigma_{ij}\alpha = \beta \star \alpha$ for some $\beta \in \langle\langle E^\tau(D_r)\rangle\rangle$,

where we denote by $\sigma_{ij} \in \mathcal{S}_n$ the transposition that interchanges i and j. By symmetry, it suffices just to prove (I). So, let $\alpha \in D_r$ and $1 \leqslant i < j \leqslant n$ be arbitrary, and write α as in equation (†) on p. 734. Recall that $r, s \geqslant 1$. We now consider four separate cases:

(i) $i, j \in \operatorname{codom}(\alpha)$;

(ii) $i \in \operatorname{codom}(\alpha)$ but $j \in [n] \setminus \operatorname{codom}(\alpha)$;

(iii) $i, j \in [n] \setminus \operatorname{codom}(\alpha)$ but $(i, j) \notin \operatorname{coker}(\alpha)$;

(iv) $(i, j) \in \operatorname{coker}(\alpha)$.

We show that, in all cases, $\alpha\sigma_{ij} = \alpha \star \beta$ for some $\beta \in \langle\langle E^\tau(D_r)\rangle\rangle$. First, we consider case (i). Relabelling the vertices, if necessary, we may assume that $(i, j) = (j_{r-1}, j_r)$. In figure 4(a), we show that $\alpha\sigma_{ij} = \alpha\beta_1\beta_2$ for some $\beta_1, \beta_2 \in E^\tau(D_r)$ with $\tau(\alpha, \beta_1, \beta_2) = 0$, giving $\alpha\sigma_{ij} = \alpha \star (\beta_1 \star \beta_2)$, as required (we leave it to the reader to verify that $\beta_1, \beta_2 \in E^\tau(\mathcal{B}_n^\tau)$). Similarly, for cases (ii)–(iv), we may assume that $(i, j) = (j_r, a_1)$, $(i, j) = (b_1, a_2)$ and $(i, j) = (a_1, b_1)$, respectively. In figure 4, we show that $\alpha\sigma_{ij} = \alpha \star \beta$ for some $\beta \in E^\tau(D_r)$ in cases (ii) and (iii), and that $\alpha\sigma_{ij} = \alpha$ in case (iv). As noted above, this completes the proof. □

Remark 3.13. The trick in the above proof, of considering expressions of the form $\alpha\sigma_{ij}$ and $\sigma_{ij}\alpha$, bears some resemblance to the proof of [1, lemma 1.2].

The proof of the next result uses several ideas and results from [22]; see also [23].

Proposition 3.14. *Suppose $r \in I(n) \setminus \{0, n\}$. Then $I_{r;0}$ is idempotent generated, and*

$$\operatorname{rank}(I_{r;0}) = \operatorname{idrank}(I_{r;0}) = \rho_{nr},$$

where the numbers ρ_{nr} are defined in proposition 2.2.

Proof. For simplicity, write $D = D_{r;0}$ and $I = I_{r;0}$. So $I = \langle\langle E^\tau(D)\rangle\rangle$, by proposition 3.12. The *principal factor* of D, denoted D°, is the semigroup on the set $D \cup \{0\}$, with multiplication '$\circ$' defined, for $\alpha, \beta \in D$, by

$$\alpha \circ 0 = 0 \circ \alpha = 0 \circ 0 = 0 \quad \text{and} \quad \alpha \circ \beta = \begin{cases} \alpha \star \beta & \text{if } \alpha \star \beta \in D, \\ 0 & \text{otherwise.} \end{cases}$$

Suppose the $\mathscr{R}^\tau$- and $\mathscr{L}^\tau$-classes contained in D are $\{R_j : j \in J\}$ and $\{L_k : k \in K\}$, where $J \cap K = \emptyset$. The *Graham–Houghton graph* of D° is the (bipartite) graph $\varDelta = \varDelta(D^\circ)$ with vertex set $J \cup K$ and edge set $\{\{j, k\} : R_j \cap L_k \text{ contains an idempotent}\}$. We note that $\varDelta$ is *balanced*, in the sense that $|J| = |K|$; this common value is equal to ρ_{nr}, by proposition 2.2 and corollary 3.3. By [11, theorem 40], each $\mathscr{R}^\tau$- and $\mathscr{L}^\tau$-class in D contains the same number of idempotents; this number was denoted by b_{nr} in [11], and a recurrence relation was given for these numbers. It follows that $\varDelta$ is b_{nr}-*regular*, in the sense that each vertex of $\varDelta$ is adjacent to b_{nr} other vertices. Since $n \geqslant 3$ (as $I(n) \setminus \{0, n\}$ is non-empty), we have $b_{nr} \geqslant 2$. It was shown in [22, lemma 3.1] that being k-regular with $k \geqslant 2$ implies that $\varDelta$ satisfies the so-called *strong Hall condition*:

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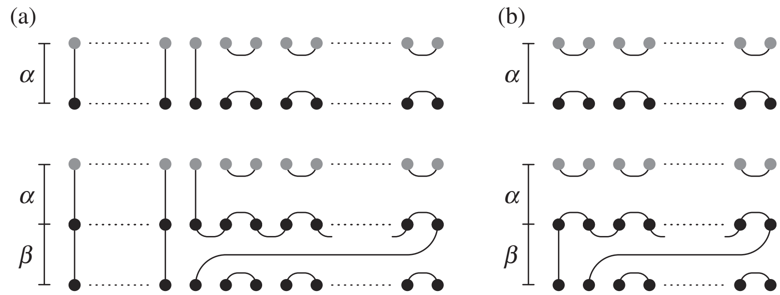

Figure 5. Diagrammatic verification that $\alpha = \alpha\beta$ from the proof of lemma 3.15 ((a) $r > 0$, (b) $r = 0$); see the text for more details. Grey vertices are ordered $i_1, \ldots, i_r, a_1, b_1, \ldots, a_s, b_s$, and black vertices are ordered $j_1, \ldots, j_r, c_1, d_1, \ldots, c_s, d_s$.

for all $\emptyset \subsetneq H \subsetneq J$, $|N(H)| > |H|$, where $N(H)$ is the set of all vertices adjacent to a vertex from H.

We also note that Δ is *connected*; indeed, this follows from the fact that D° is idempotent generated, as explained in [22, p. 61]. Since Δ is connected and balanced and satisfies the strong Hall condition, [22, lemma 2.11] gives $\text{rank}(D^\circ) = \text{idrank}(D^\circ) = |J| = |K| = \rho_{nr}$. But, since $I = \langle\!\langle D \rangle\!\rangle$, it follows that $\text{rank}(I) = \text{rank}(D^\circ)$ and $\text{idrank}(I) = \text{idrank}(D^\circ)$. □

Next we consider the ideals $I_{r;k}$, where $r, k > 0$. First we need a technical lemma.

LEMMA 3.15. *Let* $\alpha \in \mathcal{B}_n \setminus \mathcal{S}_n$.

(i) *If* $\text{rank}(\alpha) > 0$, *then* $\alpha = \alpha \star \beta$ *for some* $\beta \in D_\alpha$.

(ii) *If* $\text{rank}(\alpha) = 0$, *then* $\alpha = \alpha \star \beta$ *for some* $\beta \in D_2$.

Proof. Write α as in equation (†) on p. 734. In figure 5, we demonstrate the existence of β (of the desired rank) such that $\alpha = \alpha\beta$ with $\tau(\alpha, \beta) = 0$. □

PROPOSITION 3.16. *Let* $r \in I(n) \setminus \{0\}$, *and let* $k \geqslant 1$. *Set*

$$M_{r;k} = \bigcup_{s \in I(r),\, k \leqslant l < 2k} D_{s;l} = \{(l, \alpha) \in \mathcal{B}_n^\tau : k \leqslant l < 2k,\ \text{rank}(\alpha) \leqslant r\}.$$

Then

(i) $I_{r;k} = \langle\!\langle M_{r;k} \rangle\!\rangle$,

(ii) *any generating set for* $I_{r;k}$ *contains* $M_{r;k}$, *so* $M_{r;k}$ *is the unique minimal (with respect to size or inclusion) generating set for* $I_{r;k}$,

(iii) $\text{rank}(I_{r;k}) = |M_{r;k}| = k \cdot \sum_{s \in I(r)} \delta_{ns}$, *where the numbers* δ_{ns} *are defined in proposition 2.2.*

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Proof. We begin with (i). Let $(i,\alpha) \in I_{r;k}$ be arbitrary. If $k \leqslant i < 2k$, then $(i,\alpha) \in M_{r;k}$, so suppose $i \geqslant 2k$. Write $i = qk + s$, where $q \in \mathbb{N}$ and $k \leqslant s < 2k$. By lemma 3.15, there exists $\beta \in I_r$ such that $\alpha = \alpha\beta$ with $\tau(\alpha,\beta) = 0$. But then

$$(i,\alpha) = (s,\alpha) \star \underbrace{(k,\beta) \star \cdots \star (k,\beta)}_{q} \in \langle\!\langle M_{0;k} \rangle\!\rangle.$$

This completes the proof of (i). For (ii), suppose Γ is an arbitrary generating set for $I_{r;k}$. Let $(i,\alpha) \in M_{r;k}$ be arbitrary, and consider an expression

$$(i,\alpha) = (i_1,\alpha_1) \star \cdots \star (i_t,\alpha_t) = (i_1 + \cdots + i_t + \tau(\alpha_1,\ldots,\alpha_t), \alpha_1 \cdots \alpha_t),$$

where $(i_1,\alpha_1),\ldots,(i_t,\alpha_t) \in \Gamma$. Since $i_1,\ldots,i_t \geqslant k$ and since $i < 2k$, it follows that $t = 1$, so that $(i,\alpha) = (i_1,\alpha_1) \in \Gamma$, giving (ii). It follows immediately from (i) and (ii) that $\mathrm{rank}(I_{r;k}) = |M_{r;k}|$. The formula for $|M_{0;k}|$ follows from the fact that $|D_{s;l}| = |D_s| = \delta_{ns}$ (see corollary 3.3 and proposition 2.2). □

Remark 3.17. The generating set $M_{5;2}$ of the ideal $I_{5;2}$ of $\mathcal{B}_7^\tau$ is illustrated in figure 2.

Proposition 3.18. *Suppose n is even, and let $k \in \mathbb{N}$ be arbitrary. Set $M_{0;k} = D_{0;k} \cup \cdots \cup D_{0;2k}$. Then*

(i) $I_{0;k} = \langle\!\langle M_{0;k} \rangle\!\rangle$,

(ii) *any generating set for $I_{0;k}$ contains $M_{0;k}$, so $M_{0;k}$ is the unique minimal (with respect to size or inclusion) generating set for $I_{0;k}$,*

(iii) $\mathrm{rank}(I_{0;k}) = |M_{0;k}| = (k+1) \cdot \delta_{n0}$, *where the numbers δ_{n0} are defined in proposition 2.2.*

Proof. We omit the proof as it is very similar to that of proposition 3.16. The main difference is that we apply lemma 3.9 instead of lemma 3.15. This explains the factor of $k+1$ in the expression for $\mathrm{rank}(I_{0;k})$. □

Remark 3.19. Note that $M_{0;0} = D_{0;0} = D_0$. We saw that $I_{0;0} = \langle\!\langle D_0 \rangle\!\rangle$ in proposition 3.10.

For convenience, we gather the results on ranks of principal ideals into a single theorem.

Theorem 3.20. *Let $n \geqslant 3$, $r \in I(n)$ and $k \in \mathbb{N}$. Then*

$$\mathrm{rank}(I_{r;k}) = \begin{cases} 4 & \text{if } r = n \text{ and } k = 0, \\ \rho_{nr} & \text{if } 0 < r < n \text{ and } k = 0, \\ (k+1) \cdot \delta_{n0} & \text{if } r = 0, \\ k \cdot \sum_{s \in I(r)} \delta_{ns} & \text{if } r > 0 \text{ and } k > 0, \end{cases}$$

where the numbers ρ_{nr}, δ_{nr} are defined in proposition 2.2. Further, $I_{r;k}$ is idempotent generated if and only if $0 < r < n$ and $k = 0$, in which case $\mathrm{idrank}(I_{r;k}) = \mathrm{rank}(I_{r;k})$.

REMARK 3.21. An obvious necessary condition for an ideal I of an arbitrary semigroup S to be idempotent generated is that there must be idempotents in any maximal $\mathscr{J}$-class of I. Since idempotents of $\mathcal{B}_n^\tau$ can only exist in $\mathscr{J}^\tau = \mathscr{D}^\tau$-classes of the form $D_{r;0}$, it follows that any idempotent generated ideal of $\mathcal{B}_n^\tau$ is a principal ideal (of the form described in theorem 3.20).

3.4. Applications

A famous result of Howie [28] states that the singular ideal $\mathcal{T}_n \setminus \mathcal{S}_n$ of the full transformation semigroup $\mathcal{T}_n$ is idempotent generated. In fact, the idempotent generated subsemigroup $\langle E(\mathcal{T}_n)\rangle$ is equal to $\{1\} \cup (\mathcal{T}_n \setminus \mathcal{S}_n)$. This is true also of the Brauer monoid $\mathcal{B}_n$: specifically, $\langle E(\mathcal{B}_n)\rangle = \{1\} \cup (\mathcal{B}_n \setminus \mathcal{S}_n)$, as shown in [35], where a presentation for $\mathcal{B}_n \setminus \mathcal{S}_n$ was also given. Similar results for other diagram semigroups appear in [13, 16, 19].

We now apply the results of previous sections to explore the analogous situation for the twisted Brauer monoid $\mathcal{B}_n^\tau$. This is more complicated, and it is not the case that the singular ideal $\mathcal{B}_n^\tau \setminus \mathcal{S}_n$ is idempotent generated. We may still calculate the rank of this singular ideal, and we also describe the idempotent generated subsemigroup $\langle\!\langle E^\tau(\mathcal{B}_n^\tau)\rangle\!\rangle$, and calculate its rank and idempotent rank (which are equal). We also deduce the above-mentioned result that $\mathcal{B}_n \setminus \mathcal{S}_n$ is idempotent generated.

THEOREM 3.22. *If* $n \geqslant 3$, *then* $\operatorname{rank}(\mathcal{B}_n^\tau \setminus \mathcal{S}_n) = \binom{n}{2} + n!$.

Proof. Note that $\mathcal{B}_n^\tau \setminus \mathcal{S}_n = I_{n-2;0} \cup I_{n;1}$. By (the proof of) proposition 3.14, we may choose a subset $\varSigma \subseteq D_{n-2;0}$ with $I_{n-2;0} = \langle\!\langle \varSigma \rangle\!\rangle$ and $|\varSigma| = \operatorname{rank}(I_{n-2;0}) = \rho_{n,n-2} = \binom{n}{2}$. Now set $\varGamma = \varSigma \cup D_{n;1}$. Since $\langle\!\langle \varSigma \rangle\!\rangle = I_{n-2;0} \supseteq D_{z;1} \cup \cdots \cup D_{n-2;1}$, it follows that $\langle\!\langle \varGamma \rangle\!\rangle \supseteq M_{n;1}$, so $\langle\!\langle \varGamma \rangle\!\rangle \supseteq I_{n;1}$. Thus, $\mathcal{B}_n^\tau \setminus \mathcal{S}_n = I_{n-2;0} \cup I_{n;1} = \langle\!\langle \varGamma \rangle\!\rangle$. In particular,

$$\operatorname{rank}(\mathcal{B}_n^\tau \setminus \mathcal{S}_n) \leqslant |\varGamma| = |\varSigma| + |D_{n;1}| = \binom{n}{2} + n!. \tag{3.2}$$

Conversely, suppose $\varXi$ is an arbitrary generating set for $\mathcal{B}_n^\tau \setminus \mathcal{S}_n$. Let $\alpha \in D_{n-2;0}$ be arbitrary, and consider an expression

$$\alpha = (0, \alpha) = (i_1, \alpha_1) \star \cdots \star (i_k, \alpha_k) = (i_1 + \cdots + i_k + \tau(\alpha_1, \ldots, \alpha_k), \alpha_1 \cdots \alpha_k),$$

where $(i_1, \alpha_1), \ldots, (i_k, \alpha_k) \in \varXi$. Then we must have

$$i_1 = \cdots = i_k = \tau(\alpha_1, \ldots, \alpha_k) = 0 \quad \text{and} \quad \alpha_1 \cdots \alpha_k = \alpha.$$

Then, for any $j \in [k]$, $n - 2 = \operatorname{rank}(\alpha) = \operatorname{rank}(\alpha_1 \cdots \alpha_k) \leqslant \operatorname{rank}(\alpha_j) \leqslant n - 2$. In particular, $(i_j, \alpha_j) \in D_{n-2;0}$ for each $j \in [k]$. We have shown that $D_{n-2;0} \subseteq \langle\!\langle \varXi \cap D_{n-2;0} \rangle\!\rangle$. It follows that $I_{n-2;0} = \langle\!\langle D_{n-2;0} \rangle\!\rangle \subseteq \langle\!\langle \varXi \cap D_{n-2;0} \rangle\!\rangle$. In particular,

$$|\varXi \cap D_{n-2;0}| \geqslant \operatorname{rank}(I_{n-2;0}) = \binom{n}{2}. \tag{3.3}$$

Next, let $\sigma \in \mathcal{S}_n$ be arbitrary. As in the proof of proposition 3.16, considering an expression for $(1, \sigma)$ as a product of elements from $\varXi$ shows that, in fact, $(1, \sigma) \in \varXi$. In particular, it follows that $D_{n;1} \subseteq \varXi$, so

$$|\varXi \setminus D_{n-2;0}| \geqslant |D_{n;1}| = n!. \tag{3.4}$$

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Adding (3.3) and (3.4), we obtain $|\Xi| \geqslant \binom{n}{2} + n!$. Since Ξ was an arbitrary generating set for $\mathcal{B}_n^\tau \setminus \mathcal{S}_n$, it follows that $\operatorname{rank}(\mathcal{B}_n^\tau \setminus \mathcal{S}_n) \geqslant \binom{n}{2} + n!$. Combined with (3.2), this completes the proof. □

We now describe the idempotent generated subsemigroup of $\mathcal{B}_n^\tau$ and derive a formula for its rank and idempotent rank.

Theorem 3.23. *Let $n \geqslant 3$ and let $S = \langle\langle E^\tau(\mathcal{B}_n^\tau)\rangle\rangle$ be the idempotent generated subsemigroup of $\mathcal{B}_n^\tau$. Then*

$$S = \{1\} \cup I_{n-2;0} = \{1\} \cup (\mathbb{N} \times (\mathcal{B}_n \setminus \mathcal{S}_n)) = \{1\} \cup \{(i, \alpha)\colon i \in \mathbb{N},\ \alpha \in \mathcal{B}_n \setminus \mathcal{S}_n\},$$

and $\operatorname{rank}(S) = \operatorname{idrank}(S) = \binom{n}{2} + 1$.

Proof. Since $1 \in E^\tau(\mathcal{B}_n^\tau)$, and since $I_{n-2;0}$ is idempotent generated by proposition 3.12, it is clear that $\{1\} \cup I_{n-2;0} \subseteq S$. To show the reverse containment, it suffices to show that $S \setminus I_{n-2;0} = \{1\}$. So suppose $(i, \alpha) \in S \setminus I_{n-2;0}$. In particular, $\alpha \in D_n = \mathcal{S}_n$, and we have

$$(i, \alpha) = \alpha_1 \star \cdots \star \alpha_k = (\tau(\alpha_1, \ldots, \alpha_k), \alpha_1 \cdots \alpha_k)$$

for some idempotents $\alpha_1, \ldots, \alpha_k \in E^\tau(\mathcal{B}_n^\tau)$. (Recall that $E^\tau(\mathcal{B}_n^\tau) \subseteq E(\mathcal{B}_n)$.) Since $\alpha_1 \cdots \alpha_k = \alpha \in \mathcal{S}_n$, and since $\mathcal{B}_n \setminus \mathcal{S}_n$ is an ideal of $\mathcal{B}_n$, it follows that $\alpha_1, \ldots, \alpha_k \in \mathcal{S}_n$. In particular, $\tau(\alpha_1, \ldots, \alpha_k) = 0$. But also $E(\mathcal{S}_n) = \{1\}$, as $\mathcal{S}_n$ is a group. So $\alpha_1 = \cdots = \alpha_k = 1$, and $(i, \alpha) = (\tau(\alpha_1, \ldots, \alpha_k), \alpha_1 \cdots \alpha_k) = (0, 1) = 1$, as required. The statement about the rank and idempotent rank follows immediately from proposition 3.14 and the obvious fact that $I_{n-2;0} = \langle\langle \Sigma \rangle\rangle \iff S = \{1\} \cup I_{n-2;0} = \langle\langle \{1\} \cup \Sigma \rangle\rangle$. □

As a final application, we prove the following result, which is a (slight) strengthening of a result from [35].

Theorem 3.24 (cf. Maltcev and Mazorchuk [35]). *Let $n \geqslant 3$. The singular part $\mathcal{B}_n \setminus \mathcal{S}_n$ of the Brauer monoid $\mathcal{B}_n$ is idempotent generated. In fact,*

$$\{1\} \cup (\mathcal{B}_n \setminus \mathcal{S}_n) = \langle E(\mathcal{B}_n)\rangle = \langle E^\tau(\mathcal{B}_n)\rangle.$$

Proof. First, it is clear that $\langle E^\tau(\mathcal{B}_n)\rangle \subseteq \langle E(\mathcal{B}_n)\rangle$. Next note that, since no non-identity element of $\mathcal{S}_n$ is a product of idempotents (from $\mathcal{B}_n$), we have $\langle E(\mathcal{B}_n)\rangle \subseteq \{1\} \cup (\mathcal{B}_n \setminus \mathcal{S}_n)$. Finally, suppose $\alpha \in \mathcal{B}_n \setminus \mathcal{S}_n$ is arbitrary. Then, by theorem 3.23, $(0, \alpha) = \alpha = \alpha_1 \star \cdots \star \alpha_k = (\tau(\alpha_1, \ldots, \alpha_k), \alpha_1 \cdots \alpha_k)$ for some idempotents $\alpha_1, \ldots, \alpha_k \in E^\tau(\mathcal{B}_n^\tau)$. In particular, $\alpha = \alpha_1 \cdots \alpha_k \in \langle E^\tau(\mathcal{B}_n)\rangle$ (and, in addition, $\tau(\alpha_1, \ldots, \alpha_k) = 0$). □

Remark 3.25. One may easily check that $\langle E(\mathcal{B}_2)\rangle = \{1\} \cup (\mathcal{B}_2 \setminus \mathcal{S}_2) \neq \{1\} = \langle E^\tau(\mathcal{B}_2)\rangle$. In the above proof, we showed that every element of $\mathcal{B}_n \setminus \mathcal{S}_n$ (with $n \geqslant 3$) may be written as a product of idempotents from $E^\tau(\mathcal{B}_n)$ in such a way that no floating components are created in the formation of the product. We note that this also follows from [35, proposition 2] or [19, proposition 8.7], but we omit the details.

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