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EXISTENCE OF EXPONENTIALLY AND SUPEREXPONENTIALLY SPATIALLY LOCALIZED BREATHER SOLUTIONS FOR NONLINEAR KLEIN–GORDON LATTICES IN $\mathbb{Z}^d$, $d \geq 1$

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Abstract We prove the existence of exponentially and superexponentially localized breather solutions for discrete nonlinear Klein–Gordon systems. Our approach considers d-dimensional infinite lattice models with general on-site potentials and interaction potentials being bounded by an arbitrary power law, as well as, systems with purely anharmonic forces, cases which are much less studied particularly in a higher-dimensional set-up. The existence problem is formulated in terms of a fixed-point equation considered in weighted sequence spaces, which is solved by means of Schauder's Fixed-Point Theorem. The proofs provide energy bounds for the solutions depending on the lattice parameters and its dimension under physically relevant non-resonance conditions.

Keywords: discrete Klein–Gordon; higher-dimensional lattices; breathers; exponential and superexponential localization; Schauder fixed-point theorem

2020 Mathematics subject classification primary 37K40;
secondary 37K60, 47H10

1. Introduction

Intrinsic localized modes (ILMs) or discrete breathers in nonlinear lattices have attracted significant interest, not least due to the important role they play in many physical realms where features of localization in systems of coupled oscillators are involved (for a review see [28] and references therein) [1, 2, 8, 9, 17–24, 35–40, 42, 43, 45, 46, 53, 54, 62]. Representative results for the existence and non-existence of breathers, as spatially localized and time-periodically varying solutions, are provided in [6, 7, 12, 31, 34, 44], and for the stability of small-amplitude breathers and the notion of exponential stability in [10, 11, 49]. Analytical and numerical methods have been developed to continue breather solutions in conservative and dissipative systems starting from the anti-integrable limit [47, 48, 60].

During recent years, the existence of breathers has been verified in a number of experiments in various contexts including micromechanical cantilever arrays [57], arrays of

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coupled Josephson junctions [13], antiferromagnetic chains [56, 59], coupled optical wave guides [26, 29], Bose–Einstein condensates in optical lattices [25, 61], in coupled torsion pendula [22], electrical transmission lines [27, 50], and granular crystals [14]. Regarding their creation mechanism in conservative systems, modulational instability (MI) provides the route to the formation of breathers originating from an initially spatially homogeneous state imposed to (weak) perturbations. To be precise, the MI of band edge plane waves triggers an inherent instability leading to the formation of a spatially localized state [55].

In this work, we extend the results of [32] (where one-dimensional systems of nonlinearly interacting particles without on-site potentials were treated) and [33] (for a general class of one-dimensional systems with a nonlinear interaction between its constituents and with general potentials), *to higher-dimensional systems*, where localization effects are expected to be more intricate [8, 9, 17, 28, 45, 46].

In particular, we present a comparatively concise proof of the existence of breathers for general d-dimensional infinite nonlinear Klein–Gordon (KG) lattices based on Schauder's fixed-point theorem [58]. In detail, we discuss systems with on-site potentials $V(x)$ whose second derivative is bounded by $|V''(x)| \leq K|x|^{\alpha}$ for some constants $K > 0$ and arbitrary power $\alpha > 0$ [33], in their higher-dimensional set-up. Furthermore, as in the one-dimensional case, for the aforementioned d-dimensional infinite KG lattices, we prove not only existence but establish directly a requested degree of spatial localization [51, 52, 65]. Other important differences with [33] are the following: First, we provide, to our knowledge, a novel and general argument (compared to the one applied for the system of [33] which combined the conditions the nonlinearity and phonon-spectrum properties), in order to justify that solutions are non-trivial. Second, we not only consider breathers which are exponentially localized, but we also cover the case of superexponential localization [10] (i.e. the so-called single-site breathers). To this aim, suitable weighted function spaces will be introduced, as in [33].

We also demonstrate that the above approach may cover the problem of the existence of breathers for other examples of physically significant systems: systems of N coupled, d-dimensional KG lattices and systems with long-range harmonic and/or anharmonic interaction potentials. It may also apply to the existence of multi-site breathers arising in the limit for unity weight function, that is, in the limit of the standard l^2 sequence spaces.

The presentation of the paper has as follows. In § 2, we discuss the nonlinear discrete KG systems and the main assumptions on their interaction potentials. Section 3 contains the description and properties of the functional setting which essentially involves weighted sequence spaces and the proof of the main result. The proof combines Schauder's fixed-point theorem, with a contradiction argument based on the non-invertibility of the involved compact operator (defined by the fixed-point map), which establishes the existence of at least one non-trivial solution.

The proof also demonstrates that Schauder's fixed-point theorem approach gives rise to physically relevant and consistent restrictions on the frequency of the time-periodic solutions. For example, in the case of hard on-site potential, the amplitude of the breathers tends to zero when their frequency approaches the upper edge of the frequency band of linear oscillations. In addition, the approach, if combined with an investigation of the

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contraction regime for the nonlinear map, may provide physically meaningful upper and lower bounds for the norms of the existing breather solutions. An extension of the above results is discussed in § 4, for the existence of breathers for systems of coupled KG lattices. Section 5 deals with the existence of superexponentially localized breathers for the important example of higher-dimensional KG lattices with purely anharmonic interaction forces. In this type of system, the linearized system has no continuous spectrum. All these examples suggest the potential wide applicability of the aforementioned combined method. Section 6 summarizes the main findings and highlights further extensions for future studies.

2. Description of the system

The nonlinear Klein–Gordon systems on d-dimensional infinite lattices are given by the following set of coupled oscillator equations

$$\frac{d^2u_n}{\mathrm{d}t^2} = \kappa\,(\Delta_d u)_n - (U'(u))_n, \quad n \in \mathbb{Z}^d, \tag{2.1}$$

where u_n is the displacement of an oscillator from its equilibrium position. The operator $(\Delta_d u)_n$ is the d-dimensional discrete Laplacian

$$(\Delta_d u)_{n\in\mathbb{Z}^d} = \sum_j (u_{n+j} - 2u_n + u_{n-j}),$$

where j are the d unit vectors belonging to the d axes of $\mathbb{Z}^d$. The function $U(x)$ is the on-site potential and prime $'$ stands for the derivative with respect to the argument x. Each oscillator interacts with all of its next neighbours and the strength of the interaction is determined by the value of the parameter κ.

This system has a Hamiltonian structure related to the energy

$$H = \sum_{n\in\mathbb{Z}^d}\left(\frac{1}{2}p_n^2 + (U(u))_n\right) + \frac{\kappa}{2}\sum_{n\in\mathbb{Z}^d}\sum_j (u_{n+j} - u_n)^2,$$

and it is time-reversible with respect to the involution $p \mapsto -p$.

Discrete breathers can be characterized as follows:

$$u_n(t+T) = u_n(t), \quad p_n(t+T) = p_n(t),\ n \in \mathbb{Z}^d, \tag{2.2}$$

$$\lim_{|n|\to\infty} u_n = 0, \quad \lim_{|n|\to\infty} p_n = 0, \tag{2.3}$$

Furthermore, $u_n(t)$ has zero time average, i.e.

$$\int_0^T u_n(t)\,\mathrm{d}t = 0, \quad n \in \mathbb{Z}^d. \tag{2.4}$$

The existence theorem will be proved under the following assumption:

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- **A:** The anharmonic on-site potential $U : \mathbb{R} \to \mathbb{R}$ possesses a minimum at $x = 0$ and is at least twice continuously differentiable with

$$U(0) = U'(0) = 0, \quad U''(0) = \omega_0^2 \geq 0. \tag{2.5}$$

The solutions of the system obtained when linearizing equations (2.1) around the equilibrium $u_n = 0$ are superpositions of plane wave solutions (phonons)

$$u_n(t) = \exp(i(kn - \omega t)), \quad k \in [-\pi, \pi]^d,$$

with frequencies

$$\omega^2(k) = \omega_0^2 + 4\kappa \sum_{j=1}^{d} \sin^2\left(\frac{k_j}{2}\right), \quad k_j \in [-\pi, \pi],\ j = 1, \ldots, d.$$

Note that $U''(0) = \omega_0$ can be zero. An example is $U(x) = (1/\beta)x^\beta,\ \beta > 2$. The (extended) plane wave solutions disperse. Therefore, the frequency Ω of a localized time-periodic solution must satisfy the non-resonance condition $\Omega \neq |\omega(k)|/m$ for any integer $m \geq 1$. This requires $\Omega^2 > \omega_0^2 + 4\kappa d$ as a necessary condition for the existence of localized time-periodic solutions of system (2.1).

We write for the anharmonic part of the on-site potential:

$$V(x) = U(x) - \frac{\omega_0^2}{2}x^2, \tag{2.6}$$

and assume

$$|V'(x)| \leq \overline{K}|x|^{1+\alpha}, \quad |V''(x)| \leq \overline{K}_0|x|^\alpha,\ \forall x \in \mathbb{R}, \tag{2.7}$$

and that the function $V'(x)$ is one-to-one on $\mathbb{R}$.

3. Existence of exponentially localized breathers

a. Functional setting and proof of the main result.

In the following, we prove the existence of localized periodic solutions of system (2.1) on the infinite d-dimensional lattice. To this end, some appropriate function spaces are introduced, on which, the original problem is presented as a fixed-point problem for a corresponding operator. Utilizing Schauder's Fixed-Point Theorem, we establish the existence of exponentially localized solutions, extending the approach followed in [33] to the higher-dimensional systems.

In order to obtain the required spatial localization of the solutions, we introduce suitable weighted function spaces. First, we consider the exponentially weighted Hilbert space

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of square-summable sequences, $l^2_w(\mathbb{Z}^d)$, defined as

$$l^2_w = \left\{ u_n \in \mathbb{R} : ||u||^2_{l^2_w} := \sum_n w_n |u_n|^2 \right\}, \tag{3.1}$$

with exponential weight $w_n = \exp(\lambda |n|)$ and $\lambda \geq 0$. Then, we denote by

$$X_0 = \left\{ u \in L^2_{per}([0,T]; l^2_w) \ : \ \int_0^T u_n(t)\,\mathrm{d}t = 0,\ n \in \mathbb{Z}^d \right\},$$

the space of $T-$periodic square-integrable functions in time with zero time average, with values in l^2_w. Evidently, X_0 is a closed convex subspace of $L^2_{per}([0,\,T]; l^2_w)$. We also consider the Sobolev space

$$X_2 = \left\{ u \in H^2_{per}([0,T]; l^2_w) : \int_0^T u_n(t)\,\mathrm{d}t = 0, \quad n \in \mathbb{Z}^d \right\},$$

containing $T-$periodic functions of time assuming values in l^2_w which, together with their weak derivatives up to second order are in X_0. The above spaces are endowed with the following norms:

$$||u||^2_{X_0} = \frac{1}{T} \int_0^T ||u(t)||^2_{l^2_w}\,\mathrm{d}t,$$

$$||u||^2_{X_2} = \frac{1}{T} \int_0^T (||u(t)||^2_{l^2_w} + ||Du(t)||^2_{{l_w}^2} + ||D^2 u(t)||^2_{l^2_w})\,\mathrm{d}t.$$

For an element $u \in X_2$, we consider the Fourier-series expansion of $u_n(t)$ with respect to time t and space variable n, determined by

$$u_n(t) = \sum_{m \in \mathbb{Z} \setminus \{0\}} \hat{u}_{n,m} \exp(i\Omega m t),\ t \in [0,T], \quad \hat{u}_{n,m} = \frac{1}{T} \int_0^T u_n(t) \exp(-i\Omega m t)\,\mathrm{d}t, \tag{3.2}$$

$$\hat{u}_{n,m} = \overline{\hat{u}}_{n,-m}, \tag{3.3}$$

and

$$u_n(t) = \frac{1}{(2\pi)^d} \int_0^{2\pi} \cdots \int_0^{2\pi} \tilde{u}_k(t) \exp(ikn)\,\mathrm{d}k_1 \cdots \mathrm{d}k_d, \quad \tilde{u}_k(t) = \sum_{n \in \mathbb{Z}^d} u_n(t) \exp(-i\,k\,n), \tag{3.4}$$

$$\tilde{u}_k = \overline{\tilde{u}}_{-k}. \tag{3.5}$$

Then, using (3.2)–(3.5), we have the following representation of u:

$$u_n(t) = \sum_{m \in \mathbb{Z} \setminus \{0\}} \frac{1}{(2\pi)^d} \int_0^{2\pi} \cdots \int_0^{2\pi} \hat{\tilde{u}}_{k,m} \exp(ikn)\,\mathrm{d}k_1 \cdots \mathrm{d}k_d \exp(i\Omega m t). \tag{3.6}$$

In the following, we facilitate two versions of Schauder's fixed-point Theorem [58] given by

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Theorem III.1. *1. (First version): Let G be a closed bounded convex subset of a Banach space X. Assume that $f : G \mapsto G$ is compact. Then, f has at least one fixed-point in G.*

2. (Second version): Let G be a closed convex subset of a Banach space X and f a continuous map of G into a compact subset of G. Then, f has at least one fixed-point.

We need the following results for compact operators on infinite-dimensional Banach spaces which can be either linear or nonlinear (see [64]). The proofs generalize the corresponding results for linear compact operators (see [16, 63]), to the nonlinear ones, as it is evident that the linearity or nonlinearity of the operator is not involved in the arguments.

Lemma III.2. *Let $f : X \mapsto X$ be a compact operator on an infinite-dimensional normed linear space X. Suppose $g : X \mapsto X$ is bounded and continuous. Then fg and gf are compact too.*

Proof. First, consider the operator fg. Let $\{x_n\}_{n\in\mathbb{N}}$ be a bounded sequence in X. Then by assumption $\{gx_n\}_{n\in\mathbb{N}}$ is bounded. Since f is compact, there exists a subsequence $\{fgx_{n_k}\}_{k\in\mathbb{N}}$ that converges in X. Hence, fg is compact. In order to show that gf is compact, take again a bounded sequence $\{x_n\}_{n\in\mathbb{N}} \in X$. Then by the compactness of f there exists a subsequence $\{fx_{n_k}\}_{k\in\mathbb{N}}$ of $\{fx_n\}_{n\in\mathbb{N}}$ that converges in X: $fx_{n_k} \to y$ as $k \to \infty$. Continuity of g implies $gfx_{n_k} \to g(y)$ which means that gfx_{n_k} converges in X and thus, gf is compact. □

Lemma III.3. *A compact operator on an infinite-dimensional normed linear space does not possess a bounded and continuous inverse.*

Proof. Suppose $f : X \mapsto X$ possesses an inverse f^{-1} that is bounded and continuous. Then, by Lemma III.2, $I = ff^{-1} = f^{-1}f$ is also compact implying that the closed unit ball in X is compact. However, then by the Riesz theorem X must be finite dimensional, contradicting the hypothesis that X is infinite dimensional. Therefore, f is not invertible. □

We now present the statement and proof of the main result.

Theorem III.4. *Let condition* ***A*** *hold and suppose*

$$|\Omega| > \sqrt{\omega_0^2 + 4d\kappa}. \tag{3.7}$$

Then, there exists at least one non-zero sequence $x \equiv \{x_n\}_{n\in\mathbb{Z}^d} \in X_2$ and $||x||_{X_0} \leq R$, where

$$R \leq \left[\frac{\Omega^2 - (\omega_0^2 + 4\mathrm{d}\kappa)}{\overline{K}}\right]^{1/\alpha} := R_{\max}. \tag{3.8}$$

The sequence x is an exponentially localized, time-periodic solution of system (2.1) with period $T = \frac{2\pi}{\Omega}$.

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Proof. We shall provide two alternatives of the proof by applying the two versions of the Schauder's fixed- point Theorem III.1. For this purpose, it is convenient to rewrite Equation (2.1) using (2.6) as:

$$\ddot{u}_n + \omega_0^2 u_n - \kappa\,(\Delta_d u)_n = -(V'(u))_n. \tag{3.9}$$

Thus, only the right-hand side of (3.9) features terms nonlinear in u. Ultimately, we shall express (3.9) as a fixed-point equation in u.

I. First version of the proof:

We relate the left-hand side of (3.9) to the linear mapping: $M : X_2 \to X_0$:

$$M(u_n) = \ddot{u}_n + \omega_0^2 u_n - \kappa\,(\Delta_d u)_n.$$

Then, applying the operator M to the Fourier elements $\exp(ikn)\exp(i\,\Omega mt)$ in the representation (3.6), we get that

$$M\exp(ikn)\exp(i\,\Omega mt) = \nu_m(k)\exp(ikn)\exp(i\,\Omega mt),$$

where

$$\nu_m(k) = -\Omega^2\,m^2 + \omega_0^2 + 4\kappa\sum_{j=1}^{d}\sin^2\left(\frac{k_j}{2}\right), \quad m\in\mathbb{Z}\backslash\{0\}.$$

Since by assumption (3.7), $\Omega^2 > \omega_0^2 + 4\mathrm{d}\kappa$, it is guaranteed that $\nu_m(k)\neq 0$, for all $m\in\mathbb{Z}\backslash\{0\}$ and for all $k\in[0,\,2\pi]^d$, and that the mapping M possesses an inverse M^{-1} obeying $M^{-1}\exp(i\,(\Omega mt + kn)) = (1/\nu_m(k))\exp(i\,(\Omega mt + kn))$. First, we write the norm of the linear operator $M^{-1} : X_0 \to X_2$,

$$\begin{aligned}
||M^{-1}||^2_{X_0,X_2} &= \sup_{||u||_{X_0}=1} ||M^{-1}\,u||^2_{X_2}\\
&= \sup_{||u||_{X_0}=1} \frac{1}{T}\int_0^T [||M^{-1}u(t)||^2_{l^2_w} + ||DM^{-1}u(t)||^2_{l^2_w} + ||D^2M^{-1}u(t)||^2_{l^2_w}]\,\mathrm{d}t\\
&= \sup_{||u||_{X_0}=1} \frac{1}{T}\int_0^T \sum_{n\in\mathbb{Z}^d} w_n\Bigg(\Bigg|\sum_{m'}\frac{1}{(2\pi)^d}\int_0^{2\pi}\cdots\\
&\quad\cdot\int_0^{2\pi}\frac{\hat{\hat{u}}_{k,m}}{\nu_m(k)}\exp(ikn)\exp(i\Omega mt)\,\mathrm{d}k_1\cdots\mathrm{d}k_d\Bigg|^2\\
&\quad + \Bigg|\sum_{m'}\frac{1}{(2\pi)^d}\int_0^{2\pi}\cdots\int_0^{2\pi}\frac{i\Omega m\hat{\hat{u}}_{k,m}}{\nu_m(k)}\exp(ikn)\exp(i\Omega mt)\,\mathrm{d}k_1\cdots\mathrm{d}k_d\Bigg|^2\\
&\quad + \Bigg|\sum_{m'}\frac{1}{(2\pi)^d}\int_0^{2\pi}\cdots\int_0^{2\pi}\frac{(i\Omega m)^2\hat{\hat{u}}_{k,m}}{\nu_m(k)}\exp(ikn)\exp(i\Omega mt)\,\mathrm{d}k_1\cdots\mathrm{d}k_d\Bigg|^2\Bigg)\,\mathrm{d}t.
\end{aligned} \tag{3.10}$$

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Then, by using (3.10), $||M^{-1}||^2_{X_0,\,X_2}$ can be estimated from above, as follows:

$$
\begin{aligned}
||M^{-1}||^2_{X_0,X_2} \leq & \sup_{m\in\mathbb{Z}\setminus\{0\}} \sup_{k\in[0,2\pi]^d} \frac{1+(\Omega m)^2+(\Omega m)^4}{|\nu_m(k)|^2} \\
& \cdot \sup_{||u||_{X_0}=1} \frac{1}{T}\int_0^T \sum_{n\in\mathbb{Z}^d} w_n \left| \sum_{m'} \frac{1}{(2\pi)^d} \int_0^{2\pi} \cdots \right. \\
& \left. \int_0^{2\pi} \hat{\hat{u}}_{k,m} \exp(ikn)\exp(i\Omega mt)\,\mathrm{d}k_1 \cdots \mathrm{d}k_d \right|^2 \mathrm{d}t \\
\leq & \frac{1+\Omega^2+\Omega^4}{(\Omega^2-(\omega_0^2+4\mathrm{d}\kappa))^2} \sup_{||u||_{X_0}=1} ||u||^2_{X_0} = \frac{1+\Omega^2+\Omega^4}{(\Omega^2-(\omega_0^2+4\mathrm{d}\kappa))^2} \\
\leq & \frac{(1+\Omega^2)^2}{(\Omega^2-(\omega_0^2+4\mathrm{d}\kappa))^2} < \infty,
\end{aligned} \tag{3.11}
$$

verifying the boundedness of M^{-1}. Note that we have used the notation $\sum_{l'} = \sum_{l\in\mathbb{Z}\setminus\{0\}}$. For later use, we note that

$$
||M^{-1}||_{X_0,X_0} \leq \frac{1}{\Omega^2-(\omega_0^2+4\mathrm{d}\kappa)}. \tag{3.12}
$$

For the treatment of the nonlinear terms, we assign the nonlinear operator $N : X_0 \to X_0$ given by

$$
(N(u))_n = -(V'(u))_n,
$$

to the right-hand side of (3.9). We demonstrate that the operator N is continuous on X_0. To this end, we prove that N is Frechet differentiable at any u, with bounded derivative. We have that

$$
N'(u) : \, h \in X_0 \, \mapsto N'(u)[h] = -V''(u)h \in X_0,
$$

and by using condition (2.7), we may derive the estimate

$$
\begin{aligned}
||N'(u)[h]||^2_{X_0} &= \frac{1}{T}\int_0^T \sum_{n\in\mathbb{Z}^d} w_n \left|(V''(u))_n(t))\,h_n(t)\right|^2 \mathrm{d}t \\
&\leq \frac{1}{T}\int_0^T \sum_{n\in\mathbb{Z}^d} w_n \overline{K}_0^2 |u_n(t)|^{2\alpha}\,|h_n(t)|^2\,\mathrm{d}t \\
&\leq \overline{K}_0^2 \sup_{n\in\mathbb{Z}^d} \max_{t\in[0,T]} |u_n(t)|^{2\alpha}\,||h||^2_{X_0} = A^2||h||^2_{X_0},
\end{aligned}
$$

where $A^2 = \overline{K}_0^2 \sup_{n\in\mathbb{Z}^d} \max_{t\in[0,T]} |u_n(t)|^{2\alpha}$. Hence,

$$
||N'(u)||_{\mathcal{L}(X_0,X_0)} \leq A, \tag{3.13}
$$

implying the (uniform) boundedness of the differential. Let us now use as the closed convex subset Y_0 of X_0, its closed ball centred at 0 of radius R,

$$
Y_0 = \{u \in X_0 : ||u||_{X_0} \leq R\}\,.
$$

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Using again assumption (2.7), for the range of N on Y_0, we get the bound

$$\begin{aligned} ||N(u)||^2_{X_0} &= \frac{1}{T}\int_0^T \sum_{n\in\mathbb{Z}^d} w_n|(V'(u))_n(t))|^2\,\mathrm{d}t \\ &\leq \overline{K}^2\frac{1}{T}\int_0^T \sum_{n\in\mathbb{Z}^d} w_n|u_n(t)|^{2(\alpha+1)}\,\mathrm{d}t \\ &\leq \overline{K}^2||u||^{2(\alpha+1)}_{X_0} \leq \overline{K}^2 R^{2(\alpha+1)}, \quad \forall u\in Y_0. \end{aligned} \tag{3.14}$$

For the application of Theorem III.1, our final step is to express problem (3.9) as a fixed-point equation in terms of a mapping $S : Y_0 \to Y_0$:

$$x = M^{-1}\circ N(x) \equiv S(x). \tag{3.15}$$

Clearly, S is continuous on X_0, as relations (3.11) and (3.13) establish that its constituents M^{-1} and N are continuous. Next, we show that $S(Y_0)\subseteq Y_0$. Using (3.11) and (3.14), we have

$$\begin{aligned} ||S(x)||_{X_0} &= ||M^{-1}(\,N(x))||_{X^0} \leq ||M^{-1}||_{X_0,X_2}\cdot||\,N(x)||_{X_0} \\ &\leq \frac{\overline{K}}{\Omega^2-(\omega_0^2+4\mathrm{d}\kappa)}||x||^{\alpha+1}_{x_0}, \quad \forall x\in X_0, \end{aligned} \tag{3.16}$$

which implies that S is bounded on X_0. Then, for all $x\in Y_0$, estimate (3.16) implies that

$$||S(x)||_{X_0} \leq \frac{\overline{K}}{\Omega^2-(\omega_0^2+4\mathrm{d}\kappa)}R^{\alpha+1}\leq R, \quad \forall x\in Y_0, \tag{3.17}$$

assuring by assumption (3.8), that indeed

$$S(Y_0)\subseteq Y_0.$$

As M^{-1} maps X_0 to $X_2 \Subset X_0$ (compactly embedded), is compact, while $N : X_0 \to X_0$ is bounded and continuous. We also have that $S(Y_0)\subseteq Y_0\cap X_2$. Therefore, Lemma III.2 implies that the map $S = M^{-1}\circ N$, viewed as a map $S : Y_0\subseteq X_0 \mapsto Y_0\subseteq X_0$, is compact. The first version of Schauder's fixed point theorem implies then, that the fixed-point equation $x = S(x)$ has at least one solution.

It remains to show the existence of at least one non-trivial fixed-point solution.

Consider the operator S. Suppose that the kernel of the operator $S - I$ is trivial. Then, for every $x\in Y_0\setminus\{0\}\subseteq X_0\setminus\{0\}$, there is $y\neq 0$, $y\in X_0$ solving the inhomogeneous system

$$S(x) - x = y \neq 0. \tag{3.18}$$

This is equivalent to $S(x) = x + y$ for all $x\in Y_0\setminus\{0\}\subseteq X_0\setminus\{0\}$. Since $||S(x)||_{X_0} = ||x + y||_{X_0}$ and $S : Y_0\subseteq X_0\mapsto Y_0\subseteq X_0$, we have that $x + y$ in Y_0.

Recall that $S(x) = M^{-1}(N(x))$ where M^{-1} is linear and N is nonlinear. Since the function $-V'(x)$ is one-to-one on $\mathbb{R}$, the operator N is one-to-one also, implying, in conjunction with $M^{-1}(x)\neq 0$ for $x\neq 0$, that the compact operator $S = M^{-1}N$ is one-to-one on Y_0. Let $Z_0 = S(Y_0)\subset Y_0$. Then $T : Y_0\mapsto Z_0$, $T(x) = S(x)$ is bijective, that is,

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invertible. But as Y_0 is infinite dimensional and $T: Y_0 \subseteq X_0 \mapsto Z_0 \subseteq Y_0$ is compact, from Lemma III.3 follows that T is not invertible. Conclusively, the kernel of $S - I$ is not trivial, so that there is $x \neq 0$ solving $S(x) - x = 0$. Hence, the fixed-point equation (3.15) possesses at least one non-trivial solution. □

II. Second version of the proof:

For the application of the second version of Schauder's fixed-point theorem, we verify that the range of S is contained in a compact subset of Y_0. We consider the space

$$X_{0,0} = \left\{ u \in L^2_{per}([0,T]; l^2) \mid \int_0^T u_n(t)\,\mathrm{d}t = 0, \quad n \in \mathbb{Z}^d \right\}.$$

Representing $N(u) \in Y_0$ in terms of its spatial and temporal Fourier-transforms as

$$(N(u))_n(t) = \sum_{m\in\mathbb{Z}\setminus\{0\}} \frac{1}{(2\pi)^d} \int_0^{2\pi} \cdots \int_0^{2\pi} \hat{\hat{N}}_{k,m} \exp(ikn)\,\mathrm{d}k_1 \cdots \mathrm{d}k_d \exp(i\Omega m t), \quad (3.19)$$

the Fourier coefficients of $M^{-1}(N(u))$ fulfil for all $u \in Y_0$, for all $k \in [0,\, 2\pi]^d$, and for all $m \in \mathbb{Z}\setminus\{0\}$, the estimate

$$\left| \frac{\hat{\hat{N}}_{k,m}}{-\Omega^2 m^2 + \omega_0^2 + 4\kappa \sum_{j=1}^{d} \sin^2\left(\frac{k_j}{2}\right)} \right|^2 \leq \frac{\sum_{m\in\mathbb{Z}\setminus\{0\}} \frac{1}{(2\pi)^d} \int_0^{2\pi} \cdots \int_0^{2\pi} \left|\hat{\hat{N}}_{k,m}\right|^2 \mathrm{d}k_1 \cdots \mathrm{d}k_d}{m^4(\Omega^2 - (\omega_0^2 + 4\mathrm{d}\kappa))^2}$$
$$= \frac{\| N(u) \|^2_{X_{0,0}}}{m^4(\Omega^2 - (\omega_0^2 + 4\mathrm{d}\kappa))^2}. \quad (3.20)$$

Since $l^2_w \subset l^2$ and the inequality $||u||_{l^2} \leq ||u||_{l^2_w}$ holds, we get

$$\left| \frac{\hat{\hat{N}}_{k,m}}{-\Omega^2 m^2 + \omega_0^2 + 4\kappa \sum_{j=1}^{d} \sin^2\left(\frac{k_j}{2}\right)} \right|^2 \leq \frac{\overline{K}^2 R^{2(\alpha+1)}}{m^4(\Omega^2 - (\omega_0^2 + 4\mathrm{d}\kappa))^2} \leq \left(\frac{R}{m^2(1+\Omega^2)^2} \right)^2. \quad (3.21)$$

Hence, we conclude that S maps Y_0 into the subset

$$Y_{0,c} = \left\{ u = \{u\}_{n\in\mathbb{Z}^d} \in Y_0,\, u_n(t) = \sum_{m\in\mathbb{Z}\setminus\{0\}} \frac{1}{(2\pi)^d} \int_0^{2\pi} \cdots \int_0^{2\pi} \hat{\hat{u}}_{k,m} \exp(ikn)\,\mathrm{d}k_1 \cdots \mathrm{d}k_d \exp(i\Omega m t) : \left|\hat{\hat{u}}_{k,m}\right| \leq \frac{R}{m^2(1+\Omega^2)} \right\},$$

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which is compact in Y_0. That is, the operator S maps closed convex subsets $Y_0 \subset X_0 \subset L^2_{per}((0,T);l^2)$ into compact subsets $Y_{0,c}$ of Y_0. The second version of Schauder's fixed-point theorem implies then, that the fixed-point equation $u = S(u)$ has at least one non-trivial solution, and the proof is finished. □

b. A localization ring in X_0: upper and lower bounds for the norms of non-trivial solutions.

The proof of Theorem III.4 provides an upper bound for the norm of the breather solution. Strengthening the condition (3.7) to

$$\Omega^2 > \omega_0^2 + 4\mathrm{d}\kappa + \overline{K}, \tag{3.22}$$

enables for the identification of a ring in X_0 containing non-trivial solutions with frequencies satisfying (3.22). As in [33], we re-examine the definition of the map S in (3.15) and estimate (3.17) in order to derive a contraction regime for the map S. It follows that if we require

$$\frac{\overline{K}}{\Omega^2 - (\omega_0^2 + 4\mathrm{d}\kappa)} R^{\alpha+1} < 1,$$

then, we get that when

$$R \le \left[\frac{\Omega^2 - (\omega_0^2 + 4\mathrm{d}\kappa)}{\overline{K}}\right]^{\frac{1}{1+\alpha}} := R_{\mathrm{crit}}, \tag{3.23}$$

there exists as the unique solution, only the trivial one, $u = 0$. To be consistent with the bound R_{max} defined in (3.8), and in order to exclude the trivial solution, we need to assume the enhanced condition on the frequency (3.22). Then, under the extra condition (3.22), it holds that $R_{\mathrm{crit}} < R_{\mathrm{max}}$ and non-trivial solutions with frequencies satisfying (3.22) are located in the ring $\mathbf{R}_E$ of X_0, determined by

$$\mathbf{R}_{X_0} = \{u \in X_0 : R_{\mathrm{crit}} \le ||u||_{X_0} \le R_{\mathrm{max}}\}. \tag{3.24}$$

Remark III.5. 1. From (3.8) we infer that the amplitude of the breathers in systems with hard on-site potential goes to zero as their frequency approaches the upper edge of the phonon band, i.e. $\Omega \to (\sqrt{(\omega_0^2 + 4d\kappa})^+$. On the other hand, we observe that for fixed d and κ, in the limit $\overline{K} \to 0$,

$$\lim_{\overline{K}\to 0} R_{\mathrm{crit}} = \infty, \quad \lim_{\overline{K}\to 0} R_{\mathrm{max}} = \infty. \tag{3.25}$$

The limits (3.25) are physically relevant (see also [33] for the one-dimensional case): since for $\overline{K} \to 0$, the system approximates its linear limit, spatially extended 'almost harmonic' modes result instead of localized ones, implying the 'unboundness' of the weighted norms.

We also observe that for fixed $\overline{K}$, d and κ,

$$\lim_{\Omega\to\infty} R_{\mathrm{crit}} = \lim_{\Omega\to\infty} R_{\mathrm{max}} = \infty. \tag{3.26}$$

Yet the limits (3.26) are physically relevant in the following context: at least for hard interaction potentials, in the limit of arbitrary large frequency, a type of 'energy'

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of the solution, measured herein in the norm of X_0, should become also arbitrarily large. This growth of the weighted norm can be also associated with energy concentration phenomena due to enhanced localization, which may lead accordingly to phenomena of quasi-collapse. The quasi-collapse phenomenon is particularly relevant in higher-dimensional nonlinear lattices [15].
Both of the limiting examples (3.25) and (3.26) suggest a coherent dependence of Ω on R and the other parameters, as a result of the functional dependence of R on all the parameters in the inequality (3.17) which leads to the derivation of the upper and lower bounds in X_0.

2. The localized solutions on the infinite lattice $\mathbb{Z}^d$ are represented by (infinite) square-summable sequences, i.e. exponential decay of the solutions for $|n| \to \infty$ takes place in the sense of the exponentially weighted l^2 norm. Notably, for weight function $w_n \sim 1$, i.e. $\lambda \to 0$, our proof establishes the existence of general higher-dimensional localized patterns (e.g. multi-site breathers) [20, 23, 42, 43].

3. Since the obtained time-periodic H^2 fixed-point-solutions are by Sobolev embeddings C^1 in time and since the operator $x \mapsto V'(x)$ maps C^1 into itself, one concludes from Equation (3.9) that $\ddot{x} \in C^1$ are classical solutions.

4. Breathers in systems of coupled KG lattices

The fixed-point method can also be extended to prove the existence of (exponentially localized) breather solutions in systems of N diffusively coupled nonlinear KG lattices, of the form

$$\frac{d^2 u_n^l}{\mathrm{d}t^2} = \kappa\,(\Delta_d u^l)_n - U'(u_n^l) + \eta(u_n^{l+1} + u_n^{l-1} - 2u_n^l), \quad n \in \mathbb{Z}^d,\ 1 \le l \le N, \tag{4.1}$$

where η is the strength of the linear (diffusive) lattice-lattice interaction. For $\eta = 0$ system (4.1) decomposes into N, $d-$dimensional KG lattices each of them determined by (2.1).
We have the following statement:

Theorem IV.1. *Let condition* ***A*** *hold and suppose*

$$|\Omega| > \sqrt{\omega_0^2 + 4d\kappa + 4\eta}. \tag{4.2}$$

Then, there exist N non-zero sequences $x^l \equiv \{x_n^l\}_{n\in\mathbb{Z}^d} \in X_2$, $1 \le l \le N$, and $||x^l||_{X_0} \le R$, where

$$R \le \left(\frac{\Omega^2 - 4\mathrm{d}\kappa - 4\eta}{\overline{K}}\right)^{1/\alpha}.$$

The sequences x^l are solutions of system (4.1). They are exponentially localized along the N KG lattices, and time-periodic with period $T = \frac{2\pi}{\Omega}$.

We use the notation from § 3 and denote by X_0^N and X_2^N the extended function spaces $X_0^N = X_0 \times \ldots \times X_0$, $X_2^N = X_2 \times \ldots \times X_2$ on which we express system (4.1) as

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an operator equation $M_c u = N(u)$. The linear and nonlinear operator is determined by

$$M_c u_n^l = \ddot{u}_n^l + \omega_0^2 - \kappa\,(\Delta_d u^l)_n - \eta(u_n^{l+1} + u_n^{l-1} - 2u_n^l), \quad n \in \mathbb{Z}^d,\ 1 \le l \le N, \tag{4.3}$$

and

$$(N(u))_n^l = -U'(u_n^l), \quad n \in \mathbb{Z}^d,\ 1 \le l \le N, \tag{4.4}$$

respectively. We use the Fourier representation

$$u_n^l(t) = \sum_{m \neq 0} \frac{1}{(2\pi)^{1+d}} \int_0^{2\pi} \int_0^{2\pi} \cdots \int_0^{2\pi} \breve{u}_{k,m}^j \exp(i\,kn) \exp(i\,jl)\,\mathrm{d}k_1 \cdots \mathrm{d}k_d dj\ \exp(i\,m\Omega t). \tag{4.5}$$

Applying the linear operator M_c to the Fourier elements $\exp(i\,kn)\exp(i\,jl)\exp(i\,m\Omega t)$, gives

$$M_c \exp(i\,kn)\exp(i\,jl)\exp(i\,m\Omega t) = \left(-(m\Omega)^2 - 4\kappa \sum_{p=1}^d \sin^2\left(\frac{k_p}{2}\right) - 4\eta \sin^2\left(\frac{j}{2}\right)\right) \exp(i\,kn)\exp(i\,jl)\exp(i\,m\Omega t). \tag{4.6}$$

By hypothesis (4.2), it is assured that M_c possesses a left inverse M_c^{-1}, determined by

$$M_c^{-1} \exp(i\,kn)\exp(i\,jl)\exp(i\,m\Omega t) = \frac{1}{\nu_m^l(k)} \exp(i\,kn)\exp(i\,jl)\exp(i\,m\Omega t), \tag{4.7}$$

with

$$\nu_m^l(k) = (m\Omega)^2 - 4\kappa \sum_{p=1}^d \sin^2\left(\frac{k_p}{2}\right) - 4\eta \sin^2\left(\frac{j}{2}\right). \tag{4.8}$$

Then, the remainder of the proof of Theorem IV.1 follows the lines in the proof of Theorem III.4.

Remark IV.2. In the case of system (4.1), a localization ring similar to (3.24) can be identified in X_0 for exponentially breathers solutions with frequencies satisfying $\Omega^2 > 4\mathrm{d}\kappa + 4\eta + \overline{K}$.

5. Nonlinear interactions and superexponential localization

In this section, we treat higher-dimensional KG lattices with purely nonlinear interaction terms, of the form

$$\frac{d^2 u_n}{\mathrm{d}t^2} = \sum_j W'(u_{n+j} - u_n) - \sum_j W'(u_n - u_{n-j}), \quad n \in \mathbb{Z}^d, \tag{5.1}$$

The function $W(x)$ describes the anharmonic interaction potential between nearest neighbours. In the case $d = 1$, examples of discrete KG systems for which superexponentially localized breathers (i.e. solutions decaying faster than any exponential) exist, involve anharmonic interaction forces of the form $(x_{n+1} - x_n)^3 - (x_n - x_{n-1})^3$ [30, 41].

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Superexponentially localized travelling (solitary) waves for DGK systems with anharmonic interaction forces $|x_{n+1}-x_n|^m\,\mathrm{sign}(x_{n+1}-x_n)-|x_n-x_{n-1}|^m\,\mathrm{sign}(x_n-x_{n-1})$ with $m\geq 1$, were found in [3].

Concerning the interaction potential, we make the following assumption:

- **B:** The anharmonic interaction potential $W(x)$ has a minimum at $x=0$ and is at least twice continuously differentiable on $\mathbb{R}$, with $W(0)=W'(0)=W''(0)=0$. We further assume that W satisfies for some constant $\overline{K}_s>0$ and $\gamma>0$, the relation

$$|W'(x)|\leq \overline{K}_s|x|^{1+\gamma},\quad \forall x\in\mathbb{R}. \tag{5.2}$$

As a consequence of the absence of linear interaction terms, plane wave (phonon) solutions to the linearized system do not exist. That is, the linearized system has no continuous spectrum. This fact excludes resonances with the internal breather frequency, allowing for the occurrence of superexponential localization. Therefore, what used to be the non-resonance condition in the presence of a continuous spectrum in the previous sections, namely (3.7) and (4.2) for the existence of breather solutions with frequency Ω, changes here to $|\Omega|\neq 0$.

We use the same functional analysis set-up as in § 3, except for

$$l_s^2=\left\{u_n\in\mathbb{R}:||u||_{l_s^2}^2:=\sum_n s_n|u_n-u_{n-1}|^2\right\}, \tag{5.3}$$

with the superexponential weight

$$s_n=\exp(\lambda|n|\ln(1+\mu|n|)),\quad \lambda\geq 0,\ \mu\geq 0. \tag{5.4}$$

We use the spaces

$$X_0^s=\left\{u\in L^2_{per}([0,T];l_s^2)\ :\ \int_0^T u_n(t)\,\mathrm{d}t=0,\quad n\in\mathbb{Z}\right\},$$

and

$$X_2^s=\left\{u\in H^2_{per}([0,T];l_s^2):\int_0^T u_n(t)\,\mathrm{d}t=0,\quad n\in\mathbb{Z}\right\}.$$

Regarding the existence of superexponentially localized breather solutions in system (5.1), we have the following theorem:

Theorem V.1. *Let conditions* ***A*** *and* ***B*** *hold, and suppose*

$$|\Omega|\neq 0. \tag{5.5}$$

System (5.1) possesses at least one superexponentially localized time-periodic solution, represented by a non-zero sequence $x\equiv\{x_n\}_{n\in\mathbb{Z}^d}\in X_2^s$ *and* $||x||_{X_0^s}\leq R$*, where*

$$R\leq\left(\frac{\Omega^2}{2^{2+\gamma}\,d\,\overline{K}_s}\right)^{1/\gamma} \tag{5.6}$$

and period $T=\frac{2\pi}{\Omega}$.

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Proof. The proof utilizes the first version of Schauder's fixed-point theorem and proceeds analogously to the proof of Theorem III.4, so that we only present the essential steps. We cast system (5.1) in the from:

$$\ddot{u}_n = \sum_j W'(u_{n+j} - u_n) - \sum_j W'(u_n - u_{n-j}), \tag{5.7}$$

so that the left-hand side of (5.7) is related to the linear mapping: $M_s : X_2^s \to X_0^s$:

$$M_s(u_n) = \ddot{u}_n.$$

Then, applying the operator M_s to the Fourier elements $\exp(i\,\Omega m t)$ of the representation (3.2), (3.3), we get that

$$M_s \exp(i\,\Omega m t) = \nu_m \exp(i\,\Omega m t), \quad \nu_m = -\Omega^2\, m^2,\ m \in \mathbb{Z}\backslash\{0\}.$$

The operator M_s possesses an inverse M_s^{-1}, obeying $M_s^{-1}\exp(i\,\Omega m t) = (1/\nu_m)\exp(i\,\Omega m t)$. For the norm of the linear operator $M_s^{-1} : X_0^s \to X_2^s$ one derives the upper bound:

$$||M_s^{-1}||^2_{X_0^s, X_2^s} \leq= \frac{1+\Omega^2+\Omega^4}{\Omega^4} \leq \frac{(1+\Omega^2)^2}{\Omega^4} < \infty, \tag{5.8}$$

verifying the boundedness of M_s^{-1}. When we consider M_s^{-1} as a map $M_s^{-1} : X_0^s \to X_0^s$, we have the estimate

$$||M_s^{-1}||_{X_0^s, X_0^s} \leq \frac{1}{\Omega^2}. \tag{5.9}$$

We associate with the right-hand side of (5.7) the nonlinear operator $N : X_0^s \to X_0^s$, as

$$(N(u))_n = \sum_j W'(u_{n+j} - u_n) - \sum_j W'(u_n - u_{n-j}).$$

Continuity of the operator N on X_0 is proven in an analogous vein as in the proof of Theorem III.4.

We proceed by considering the closed convex subset Y_0^s of X_0^s determined by its closed ball centred at 0 of radius R,

$$Y_0^s = \{u \in X_0^s : ||u||_{X_0} \leq R\}.$$

With the aid of the continuous embeddings

$$l_s^p \subset l_s^q,\ \ ||x||_{l_s^q} \leq ||x||_{l_s^p},\ \ 1 \leq p \leq q \leq \infty,$$

the Banach algebra property of l_s^2, i.e. $||xy||_{l_s^2} \leq ||y||_{l_s^2}\,||y||_{l_s^2}$ for all $x,\, y \in l_s^2$, and the monotonicity $\exp(\lambda|n|\ln(1+\mu|n|)) < \exp(\lambda|n+1|\ln(1+\mu|n+1|))$, we obtain for the

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range of N on Y_0^s, the estimate

$$||N(u)||_{X_0^s} = \left(\frac{1}{T}\int_0^T \sum_{n\in\mathbb{Z}} w_n \sum_j |W'(u_{n+j}(t) - u_n(t)\right) - \sum_j W'(u_n(t) - u_{n-j}(t))|^2 \,\mathrm{d}t)^{1/2}$$
$$\leq 2d\overline{K}_s(2R)^\gamma ||u||_{X_0^s} \leq 2d\overline{K}_s(2R)^{1+\gamma}, \quad \forall u \in Y_0^s. \tag{5.10}$$

The final step of the proof is to express problem (5.7) as a fixed-point equation, in terms of a mapping $Y_0^s \to Y_0^s$:

$$x = M_s^{-1} \circ N(x) \equiv S(x).$$

S is continuous on X_0^s as its ingredients M_s^{-1} and N are continuous. Now we confirm that $S(Y_0^s) \subseteq Y_0^s$. With the help of (5.9) and (5.10), one gets

$$||S(x)||_{X_0^s} = ||M_s^{-1}(N(x))||_{X_0^s} \leq ||M_s^{-1}||_{X_0^s, X_0^s} \cdot ||N(x)||_{X_0^s}$$
$$\leq \frac{2d\overline{K}_s(2R)^{1+\gamma}}{\Omega^2} \leq R, \quad \forall x \in Y_0^s, \tag{5.11}$$

affirming by assumption (5.6), that

$$S(Y_0^s) \subseteq Y_0^s.$$

Since M_s^{-1} maps X_0^s to $X_2^s \Subset X_0^s$, it follows that $S(Y_0^s) \subseteq Y_0^s \cap X_2^s$. Thus, when considered as a map $S : Y_0^s \subseteq X_0^s \mapsto Y_0^s \subseteq X_0^s$, $S = M_s^{-1} \circ N$ is compact. By the first version of Schauder's fixed-point theorem and the argumentation regarding the existence of non-trivial solutions in the proof of Theorem III.4, the fixed-point equation $x = S(x)$ has at least one non-zero solution. □

Remark V.2. In contrast to the superexponential decay of the breather solutions in this section, in the previous § 3 and § 4, where the linearized system possesses a continuous spectrum (phonon band), the breather solutions decay 'only' exponentially because their frequencies are situated in the discrete (point) spectrum outside the continuous spectrum. Solutions associated with frequencies in the discrete spectrum are exponentially localized. Hence, the superexponential weight (5.4) does not work in § 3 and § 4.

Remark V.3. We remark that a localization ring similar to (3.24) can be identified in X_0^s for superexponentially breathers solutions with frequencies satisfying $\Omega^2 > 2^{2+\gamma} d\overline{K}_s$.

6. Conclusions

In summary, we have proved the existence of non-trivial, exponentially and superexponentially breather solutions for general nonlinear KG systems on the infinite lattice $\mathbb{Z}^d$, $d \geq 1$. The existence problem has been reformulated as a fixed-point equation (involving the relevant linear and nonlinear operators associated with the system), on weighted sequence spaces. This fixed-point problem has been solved utilizing two versions of Schauder's Fixed-Point Theorem, combined with a contradiction argument for the invertibility of the 'fixed-point map', in order to prove the existence of at least one non-zero solution. The

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method provides under meaningful non-resonant conditions, physically relevant energy bounds for the solutions. The class of systems on which the method was implemented in the present paper suggests its potential wide applicability. For example, it can also be applied to establish the existence of breather solutions in systems with linear and/or nonlinear long-range interactions

$$\frac{d^2u_n}{dt^2} = \sum_{m>0} \kappa_m (W'(u_{n+m} - u_n) + W'(u_{n-m} - u_n)) - (U'(u))_n, \quad n \in \mathbb{Z}^d.$$

For further use of our methods, one may consider Fermi–Pasta–Ulam systems and Discrete Nonlinear Schrödinger models with various types of nonlinearities, other degrees of localization such as algebraic (which can be relevant for the existence of discrete rational solutions [4, 5]) lattices in more complicated geometries [8, 9, 17, 45, 46], as well as, the existence of compact-like discrete breathers [3, 30, 41]. Such investigations are in progress and will be reported elsewhere.

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