Mathematical Proceedings of the Cambridge Philosophical Society

VOL. 172 MARCH 2022 PART 2

Math. Proc. Camb. Phil. Soc. (2022), **172**, 249–266

doi: 10.1017/S0305004121000219

First published online 5 April 2021

Braid groups, mapping class groups and their homology with twisted coefficients

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(*Received* 28 *January* 2019; *revised* 15 *December* 2020; *accepted* 14 *December* 2020)

Abstract

We consider the Birman–Hilden inclusion $\varphi\colon \mathfrak{Br}_{2g+1} \to \Gamma_{g,1}$ of the braid group into the mapping class group of an orientable surface with boundary, and prove that φ is stably trivial in homology with twisted coefficients in the symplectic representation $H_1(\Sigma_{g,1})$ of the mapping class group; this generalises a result of Song and Tillmann regarding homology with constant coefficients. Furthermore we show that the stable homology of the braid group with coefficients in $\varphi^*(H_1(\Sigma_{g,1}))$ has only 4-torsion.

2020 Mathematics Subject Classification: 55R20, 55R35, 55R37, 55R40, 55R80 (Primary); 20F36, 55N25 (Secondary).

1. *Introduction*

A theorem of Birman and Hilden [3, theorem 1] shows that the braid group $\mathfrak{Br}_{2g+1}$ on $2g+1$ strands can be identified with the hyperelliptic mapping class group $\Delta_{g,1} \subset \Gamma_{g,1}$: this is a certain subgroup of the mapping class group $\Gamma_{g,1}$ of an orientable surface $\Sigma_{g,1}$ of genus g with one parametrised boundary component (see Subsection 2·4).

It is natural to study the behaviour of the Birman–Hilden inclusion $\varphi\colon \mathfrak{Br}_{2g+1} \to \Gamma_{g,1}$ in homology. Song and Tillmann [14], show that the map φ_* is stably trivial in homology with constant coefficients.

THEOREM 1·1 (Song–Tillmann). *For any abelian group A the map*

$$\varphi_*\colon H_k(\mathfrak{Br}_{2g+1}; A) \to H_k(\Gamma_{g,1}; A)$$

is trivial in the range $k \leq 2g/3 - 2/3$.

The range $k \leq 2g/3 - 2/3$ is the best known stable range for the homology with constant coefficients of the mapping class group (see Theorem 2·1).

Segal and Tillmann [13] give an alternative and more geometric proof of Theorem 1·1; Bödigheimer and Tillmann [4] generalise Theorem 1·1 to other non-hyperelliptic embeddings of braid groups into mapping class groups.

In this article we prove an analogue of Theorem 1·1 for homology with symplectic twisted coefficients.

THEOREM 1·2. *Consider the symplectic representation* $\mathcal{H} := H_1(\Sigma_{g,1})$ *of the mapping class group* $\Gamma_{g,1}$*, and its pullback* $\varphi^*\mathcal{H}$*, which is a representation of* $\mathfrak{Br}_{2g+1}$*. The induced map in homology with twisted coefficients*

$$\varphi_*\colon H_k(\mathfrak{Br}_{2g+1}; \varphi^*\mathcal{H}) \longrightarrow H_k(\Gamma_{g,1}; \mathcal{H})$$

is trivial for $k \leq 2g/3 - 2/3 - 1$.

Our proof of Theorem 1·2 relies on a weak version of Harer's stability theorem: in particular we will not need to stabilise the mapping class groups with respect to the genus, but only with respect to the number of boundary components.

We expect Theorems 1·1 and 1·2 to be particular cases of a more general phenomenon: we conjecture that there are constants $A, B \geq 0$ such that for all $r \geq 0$ the map

$$\varphi_*\colon H_k(\mathfrak{Br}_{2g+1}; \varphi^*\mathcal{H}^{\otimes r}) \longrightarrow H_k(\Gamma_{g,1}; \mathcal{H}^{\otimes r})$$

is trivial in the range $k \leq 2g/3 - 2/3 - Ar - B$.

We also obtain a result concerning the homology $H_*(\mathfrak{Br}_{2g+1}; \varphi^*\mathcal{H})$ on its own:

THEOREM 1·3. *The homology* $H_*(\mathfrak{Br}_{2g+1}; \varphi^*\mathcal{H})$ *is* 4*-torsion, i.e. every homology class vanishes when multiplied by* 4.

The homology $H_*(\mathfrak{Br}_{2g+1}; \varphi^*\mathcal{H})$ arises naturally as a direct summand of $H_*(\varphi^*\mathcal{E}_{g,1})$. Here $\mathcal{E}_{g,1}$ denotes the total space of the tautological $\Sigma_{g,1}$-bundle $\mathcal{E}_{g,1} \to B\Gamma_{g,1}$ over the classifying space of the mapping class group, and $\varphi^*\mathcal{E}_{g,1}$ is its pullback over the classifying space of the braid group. Note indeed that $\varphi^*\mathcal{E}_{g,1}$, as every $\Sigma_{g,1}$-bundle, admits a section *at the boundary* (see Subection 2·2).

This paper contains the main results of my Master's thesis [2]. Recently Callegaro and Salvetti [6] have computed explicitly the homology $H_*(\mathfrak{Br}_{2g+1}; \varphi^*\mathcal{H})$, showing that it has only 2-torsion; the same authors [7] have also studied the analogue problem for totally ramified d-fold branched coverings of the disc. Their results are partially based on results of my Master's thesis, which are discussed in this paper.

2. *Preliminaries*

We recall some classical facts about braid groups and mapping class groups.

2·1. *Braid groups*

Let $\mathbb{D} = \{z : |z| < 1\} \subset \mathbb{C}$ denote the open unit disc, and let

$$F_n(\mathbb{D}) = \left\{(z_1, \ldots, z_n) \in \mathbb{D}^n : z_i \neq z_j \ \forall i \neq j\right\}$$

denote the *ordered configuration space* of n points in $\mathbb{D}$. The symmetric group $\mathfrak{S}_n$ acts on $F_n(\mathbb{D})$ by permuting the labels; the quotient space is the *unordered configuration space* of n points in $\mathbb{D}$, denoted

$$C_n(\mathbb{D}) = F_n(\mathbb{D})/\mathfrak{S}_n.$$

Artin's braid group $\mathfrak{Br}_n$ is defined as the fundamental group $\pi_1(C_n(\mathbb{D}))$; we recall that $C_n(\mathbb{D})$ is aspherical [9], and hence a classifying space for $\mathfrak{Br}_n$.

The braid group $\mathfrak{Br}_n$ has a presentation [1] with standard generators $\sigma_1, \ldots, \sigma_{n-1}$ and relations:

(i) $\sigma_i\sigma_j = \sigma_j\sigma_i$ for $|i - j| \geq 2$;
(ii) $\sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j$ for $|i - j| = 1$.

The space $C_n(\mathbb{D})$ has a natural structure of complex manifold, with local coordinates given by the positions $z_1, \ldots, z_n$ of the points in the configuration.

2·2. *Mapping class groups and surface bundles*

Let $\Sigma_{g,m}$ be a smooth, oriented, compact surface of genus g with $m \geq 1$ *parametrised* boundary components.

A parametrisation of the boundary is a diffeomorphism $\partial\Sigma_{g,m} \cong \{1, \ldots, m\} \times \mathbb{S}^1$, where $\mathbb{S}^1 \subset \mathbb{C}$ is the unit circle; the parametrisation must be compatible with the orientation induced by $\Sigma_{g,m}$ on the boundary. We choose as basepoint for $\Sigma_{g,m}$ the point $* \in \partial\Sigma_{g,m}$ corresponding to $(1, 1) \in \{1, \ldots, m\} \times \mathbb{S}^1$.

We consider the group $\mathrm{Diff}_{g,m}$ of diffeomorphisms $f : \Sigma_{g,m} \to \Sigma_{g,m}$ fixing some small collar neighbourhood of $\partial\Sigma_{g,m}$ in $\Sigma_{g,m}$. This is a topological group with the Whitney C^∞-topology. A result by Earle and Schatz [8] ensures that $\mathrm{Diff}_{g,m}$ has contractible connected components for all $g \geq 0$ and $m \geq 1$. In particular the tautological map $\mathrm{Diff}_{g,m} \to \pi_0(\mathrm{Diff}_{g,m})$ is a homotopy equivalence.

The discrete group $\pi_0(\mathrm{Diff}_{g,m})$ is called the *mapping class group* of $\Sigma_{g,m}$ and is denoted by $\Gamma_{g,m}$. Applying the bar construction we obtain a homotopy equivalence

$$B\,\mathrm{Diff}_{g,m} \xrightarrow{\simeq} B\Gamma_{g,m}.$$

The canonical action of $\mathrm{Diff}_{g,m}$ on $\Sigma_{g,m}$ yields, through the Borel construction, the map

$$E\,\mathrm{Diff}_{g,m} \times_{\mathrm{Diff}_{g,m}} \Sigma_{g,m} \longrightarrow B\,\mathrm{Diff}_{g,m} = E\,\mathrm{Diff}_{g,m} \,/\, \mathrm{Diff}_{g,m}\,.$$

This is a fibre bundle map with fibre $\Sigma_{g,m}$. The pullback bundle along the inverse homotopy equivalence $B\Gamma_{g,m} \to B\,\mathrm{Diff}_{g,m}$ is denoted by

$$\mathbf{p} : \mathcal{E}_{g,m} \longrightarrow B\Gamma_{g,m}.$$

The bundle $\mathbf{p}$ is a *universal* $\Sigma_{g,m}$*-bundle*: if $p : E \to X$ is a $\Sigma_{g,m}$-bundle over a paracompact space X (in particular its fibres are oriented surfaces and have parametrised boundary), then p arises as pullback of $\mathbf{p}$ along a map $\psi : X \to B\Gamma_{g,m}$ which is unique up to homotopy.

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The bundle $\mathbf{p}$ admits a global section *at the boundary* $\mathbf{s}_0\colon B\Gamma_{g,m} \to \mathcal{E}_{g,m}$, obtained by choosing the basepoint of each fibre (i.e. the point corresponding to $(1, 1) \in \{1, \ldots, m\} \times \mathbb{S}^1$ under the parametrisation). By abuse of notation, we will also regard

$$B\Gamma_{g,m} = \mathbf{s}_0(B\Gamma_{g,m}) \subset \mathcal{E}_{g,m}$$

as a subspace of $\mathcal{E}_{g,m}$.

Fibres of $\mathbf{p}$ are smooth surfaces, and we can assemble together their tangent bundles to obtain a vector bundle $\bar{\mathbf{p}}^v\colon \mathcal{V}_{g,m} \to \mathcal{E}_{g,m}$ with fibre $\mathbb{R}^2$, called the *vertical tangent bundle*. Choosing a Riemannian metric on $\bar{\mathbf{p}}^v$ and considering on each vector space its unit circle, we define an $\mathbb{S}^1$-bundle over $\mathcal{E}_{g,m}$, called the *unit vertical tangent bundle* and denoted

$$\mathbf{p}^v : \mathcal{UV}_{g,m} \longrightarrow \mathcal{E}_{g,m}.$$

We denote by $\partial\mathcal{E}_{g,m} \cong B\Gamma_{g,m} \times \left(\{1, \ldots, m\} \times \mathbb{S}^1\right)$ the subspace of $\mathcal{E}_{g,m}$ formed by all boundaries of fibres of $\mathbf{p}$. We can define a section of $\mathbf{p}^v$ over $\partial\mathcal{E}_{g,m}$ as follows: we assign to each point on the boundary of a fibre of $\mathbf{p}$ the unit vector which is tangent to that fibre, is orthogonal to the boundary of that fibre and points outwards. We restrict this section of $\mathbf{p}^v$ to $B\Gamma_{g,m} = \mathbf{s}_0(B\Gamma_{g,m}) \subset \partial\mathcal{E}_{g,m}$, obtaining a section $\mathbf{s}_0^v\colon B\Gamma_{g,m} = \mathbf{s}_0(B\Gamma_{g,m}) \to \mathcal{UV}_{g,m}$; again we regard

$$B\Gamma_{g,m} = \mathbf{s}_0^v(\mathbf{s}_0(B\Gamma_{g,m})) \subset \mathcal{UV}_{g,m}$$

as a subspace of $\mathcal{UV}_{g,m}$. See the following diagram

$$\begin{array}{ccc}
\mathbf{s}_0^v(B\Gamma_{g,m}) & \hookrightarrow & \mathcal{UV}_{g,m} \\
\Vert & & \downarrow \mathbf{p}^v \\
\mathbf{s}_0(B\Gamma_{g,m}) & \hookrightarrow & \mathcal{E}_{g,m} \\
& \searrow\!\!= & \downarrow \mathbf{p} \\
& & B\Gamma_{g,m}.
\end{array}$$

The previous constructions are natural with respect to pullbacks. Let $p\colon E \to X$ be a $\Sigma_{g,m}$-bundle over a paracompact space X, obtained as a pullback of $\mathbf{p}$ along a map $\psi\colon X \to B\Gamma_{g,m}$. We have a section $s_0 = \psi^*\mathbf{s}_0\colon X \to E$, a unit vertical tangent bundle $p^v = \psi^*\mathbf{p}^v\colon \psi^*\mathcal{UV} \to E$ and a section $s_0^v = \psi^*\mathbf{s}_0^v\colon X = s_0(X) \to \psi^*\mathcal{UV}$.

2·3. *Stabilisation maps*

We recall now the construction of the stabilisation maps

$$\alpha : \Gamma_{g,2} \longrightarrow \Gamma_{g+1,1}, \quad \beta : \Gamma_{g,1} \longrightarrow \Gamma_{g,2}, \quad \gamma : \Gamma_{g,2} \longrightarrow \Gamma_{g,1}$$

between different mapping class groups.

The map $\beta\colon \Gamma_{g,1} \to \Gamma_{g,2}$ is constructed as follows. First we decompose $\Sigma_{g,2}$ as the union of $\Sigma_{g,1}$ and a pair of pants $\Sigma_{0,3}$ along a boundary component. Each diffeomorphism of $\Sigma_{g,1}$ fixing a collar neighbourhood of $\partial\Sigma_{g,1}$ extends to a diffeomorphism of $\Sigma_{g,2}$, by prescribing the identity map on $\Sigma_{0,3}$: we obtain a homomorphism $\bar{\beta}\colon \mathrm{Diff}_{g,1} \to \mathrm{Diff}_{g,2}$, inducing a homomorphism β on mapping class groups. See Figure 1.

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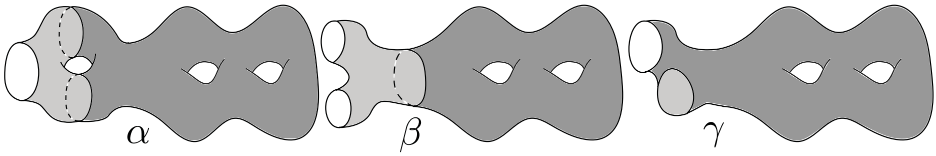

Fig. 1. Glueing surfaces in different ways yields homomorphisms α,β and γ between mapping class groups.

The map $\gamma\colon \Gamma_{g,2} \to \Gamma_{g,1}$ is constructed as follows. First, we decompose $\Sigma_{g,1}$ as the union of $\Sigma_{g,2}$ and a disc $\Sigma_{0,1}$ along a boundary component. Each diffeomorphism of $\Sigma_{g,2}$ fixing a neighbourhood of $\partial\Sigma_{g,2}$ extends to $\Sigma_{g,1}$ by prescribing the identity on $\Sigma_{0,1}$: we obtain a homomorphism $\bar{\gamma}\colon \mathrm{Diff}_{g,2} \to \mathrm{Diff}_{g,1}$, inducing γ on mapping class groups. The composition $\gamma \circ \beta\colon \Gamma_{g,1} \to \Gamma_{g,1}$ is the identity.

Finally, the map $\alpha\colon \Gamma_{g,2} \to \Gamma_{g+1,1}$ is constructed as follows. We decompose $\Sigma_{g+1,1}$ as the union of $\Sigma_{g,1}$ and a pair of pants $\Sigma_{0,3}$ along two boundary components. We extend diffeomorphisms of $\Sigma_{g,1}$ with the identity on $\Sigma_{0,3}$, obtaining a homomorphism $\mathrm{Diff}_{g,2} \to \mathrm{Diff}_{g+1,1}$, inducing a homomorphism α between mapping class groups.

We will state Harer's stability theorem in a form that suffices for our purposes, i.e. without mentioning mapping class groups of closed surfaces (see [**11**] for the original theorem and [**5**, **12**] for the improved stability ranges).

THEOREM 2·1 (Harer). *Let A be an abelian group. The maps α, β, γ described above induce isomorphisms in homology in a certain range:*

$$\alpha_*\colon H_k(\Gamma_{g,2};\, A) \cong H_k(\Gamma_{g+1,1};\, A) \quad \text{for } k \leq \frac{2}{3}g - \frac{2}{3};$$

$$\beta_*\colon H_k(\Gamma_{g,1};\, A) \cong H_k(\Gamma_{g,2};\, A) \quad \text{for } k \leq \frac{2}{3}g;$$

$$\gamma_*\colon H_k(\Gamma_{g,2};\, A) \cong H_k(\Gamma_{g,1};\, A) \quad \text{for } k \leq \frac{2}{3}g.$$

The proof of Theorem 1·1 uses the full statement of Theorem 2·1, in particular homology stability with respect to α. Conversely, we will only need homological stability for the maps β and γ to prove Theorem 1·2: these are the stabilisation maps that change the number of boundary components but not the genus.

We will also need the following classical result [**10**, propositions 3·19 and 4·6].

THEOREM 2·2. *The space $\mathcal{UV}_{g,1}$ is a classifying space for $\Gamma_{g,2}$, i.e. it is homotopy equivalent to $B\Gamma_{g,2}$.*

The map $\mathbf{s}_0^v \circ \mathbf{s}_0\colon B\Gamma_{g,1} \to \mathcal{UV}_{g,1}$ induces the map β on fundamental groups.

The map $\mathbf{p} \circ \mathbf{p}^v\colon \mathcal{UV}_{g,1} \to B\Gamma_{g,1}$ induces the map γ on fundamental groups.

2·4. *Hyperelliptic mapping class groups*

Fix a diffeomorphism J of $\Sigma_{g,1}$ with the following properites:

(i) J^2 is the identity of $\Sigma_{g,1}$;

(ii) J acts on $\partial\Sigma_{g,1} \cong \mathbb{S}^1$ as the rotation by an angle π;

(iii) J has exactly $2g+1$ fixed points in the interior of Σ.

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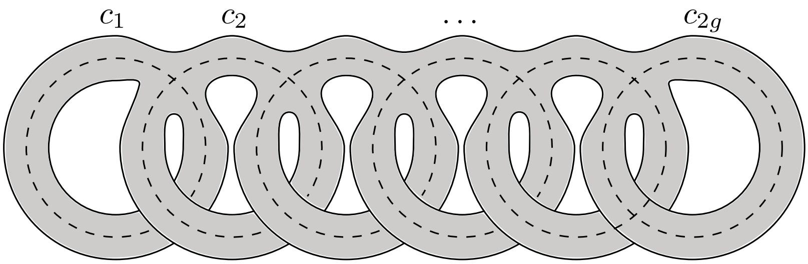

Fig. 2. A chain of $2g$ simple closed curves on $\Sigma_{g,1}$.

The quotient space $\Sigma_{g,1}/J$ is a disc and the map $\Sigma_{g,1} \to \Sigma_{g,1}/J$ is a 2-fold branched covering with $2g+1$ branching points. We say that J is a *hyperelliptic involution* of $\Sigma_{g,1}$.

Consider the group $\mathrm{Diff}^J_{g,1}$ of diffeomorphisms $f: \Sigma_{g,1} \to \Sigma_{g,1}$ that preserve the orientation and restrict on a neighbourhood of $\partial\Sigma_{g,1}$ either to the identity or to J. The corresponding group of connected components, denoted $\Gamma^J_{g,1} = \pi_0(\mathrm{Diff}^J_{g,1})$, is the *$J$-extended mapping class group*: it is a split extension of $\mathbb{Z}_2 = \langle [J] \rangle$ with kernel $\Gamma_{g,1}$.

The *hyperelliptic mapping class group* $\triangle_{g,1} \subset \Gamma_{g,1}$ is defined as the intersection in $\Gamma^J_{g,1}$ between $\Gamma_{g,1}$ and the centraliser of $[J]$. By [3, theorem 1], the group $\triangle_{g,1}$ is also isomorphic to the group of connected components of the centralizer of J in $\mathrm{Diff}^J_{g,1}$; see also [10, theorem 9·2].

3. *Definition of the map* φ

We consider on $\Sigma_{g,1}$ a chain of $2g$ simple closed curves $c_1, \ldots, c_{2g}$, such that $c_i \cap c_j = \emptyset$ for $|i-j| \geq 2$, whereas c_i and c_j intersect transversely in one point if $|i-j| = 1$. Note that a small neighbourhood in $\Sigma_{g,1}$ of the union of these curves is itself diffeomorphic to $\Sigma_{g,1}$. See Figure 2.

Denote by $D_i \in \Gamma_{g,1}$ the Dehn twist about the curve c_i; then the following relations hold in $\Gamma_{g,1}$ [10, fact 3·9 and proposition 3·11]:

(i) $D_i D_j = D_j D_i$ for $|i-j| \geq 2$;
(ii) $D_i D_j D_i = D_j D_i D_j$ for $|i-j| = 1$.

Therefore there is an induced morphism of groups

$$\varphi: \mathfrak{Br}_{2g+1} \longrightarrow \Gamma_{g,1}$$

which is defined by sending the generator $\sigma_i \in \mathfrak{Br}_{2g+1}$ to the Dehn twist $D_i \in \Gamma_{g,1}$. This map is called the *Birman–Hilden inclusion*: it is injective and its image is the hyperelliptic mapping class group (see [3, theorem 1] and [10, theorem 9·2]).

From now on let n denote the number $2g+1$, in particular n is odd. We give now a geometric description of the $\Sigma_{g,1}$-bundle $\varphi^*\mathcal{E}_{g,1}$ over $C_n(\mathbb{D}) \simeq B\mathfrak{Br}_n$ (see also [13]).

Consider, in the complex manifold with boundary $C_n(\mathbb{D}) \times \overline{\mathbb{D}} \times \mathbb{C}$, the subspace

$$\mathcal{V}_n = \left\{ (\{z_1, \ldots, z_n\}, x, y) \; : \; y^2 = \prod_{i=1}^{n} (x - z_i) \right\}.$$

Here $\overline{\mathbb{D}}$ is the closed unit disc in $\mathbb{C}$.

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LEMMA 3·1. *The space $\mathcal{V}_n$ is a smooth manifold with boundary, and is transverse to $C_n(\mathbb{D}) \times \partial\overline{\mathbb{D}} \times \mathbb{C}$ inside $C_n(\mathbb{D}) \times \overline{\mathbb{D}} \times \mathbb{C}$. The natural map*

$$\pi : \mathcal{V}_n \longrightarrow C_n(\mathbb{D}), \quad (\{z_1, \ldots, z_n\}, x, y) \longmapsto \{z_1, \ldots, z_n\}$$

is a submersion; in particular its fibres are smooth.

Proof. Consider on $C_n(\mathbb{D}) \times \overline{\mathbb{D}} \times \mathbb{C}$ the holomorphic function

$$f(\{z_i\}, x, y) = y^2 - \prod_i (x - z_i),$$

and note that $\mathcal{V}_n$ is the zero locus of f. The partial derivatives of f with respect to x and y are given by the formulas:

$$\frac{df}{dx}(\{z_1, \ldots, z_n\}, x, y) = -\sum_{i=1}^{n} \prod_{j \neq i} (x - z_j);$$

$$\frac{df}{dy}(\{z_1, \ldots, z_n\}, x, y) = 2y.$$

We claim that for each point $v = (\{\hat{z}_1, \ldots, \hat{z}_n\}, \hat{x}, \hat{y}) \in \mathcal{V}_n \subset C_n(\mathbb{D}) \times \overline{\mathbb{D}} \times \mathbb{C}$ at least one of the partial derivatives $df(v)/dx$ and $df(v)/dx$ does not vanish. For this, suppose that $df(v)/dy$ vanishes: then $\hat{y} = 0$. Since $f(v) = 0$, we have that $\hat{x} = \hat{z}_i$ for exactly one value of i. Then all the summands but exactly one in the sum for $df(v)/dx$ vanish, and therefore $df(v)/dx \neq 0$. We have thus shown that $\mathcal{V}_n$ is smooth, as df never vanishes on $\mathcal{V}_n$. Moreover, since at least one of the partial derivatives $df(v)/dx$ and $df(v)/dx$ does not vanish, df is not in the image of the map $pi^*\colon T^*_{\pi(v)}(C_n(\mathbb{D})) \to T^*_v(C_n(\mathbb{D}) \times \overline{\mathbb{D}} \times \mathbb{C})$: this image is spanned by $dz_1, \ldots, dz_n$. Therefore $\pi_*\colon T_v(C_n(\mathbb{D}) \times \overline{\mathbb{D}} \times \mathbb{C}) \to T_{\pi(v)}(C_n(\mathbb{D}))$ is surjective, i.e. π is a submersion.

To see that $\mathcal{V}_n$ is transverse to $C_n(\mathbb{D}) \times \partial\overline{\mathbb{D}} \times \mathbb{C}$, note that if $|\hat{x}| = 1$ then $\hat{x} \neq \hat{z}_i$ for all i and we can rewrite

$$\frac{df}{dx}(v) = -\left(\prod_{i=1}^{n} (\hat{x} - \hat{z}_i)\right)\left(\sum_{i=1}^{n} \frac{1}{\hat{x} - \hat{z}_i}\right) \neq 0,$$

where the sum in the second factor is non-zero because each summand has a non-trivial component in the direction of $1/\hat{x}$.

Since $\pi : \mathcal{V}_n \to C_n(\mathbb{D})$ is surjective, it is a fibre bundle with smooth Riemann surfaces as fibres.

LEMMA 3·2. *The fibres of π are diffeomorphic to $\Sigma_{g,1}$. The boundaries of the fibres can be parametrised continuously with respect to $C_n(\mathbb{D})$.*

Proof. Let $q = \{\hat{z}_1, \ldots, \hat{z}_n\} \in C_n(\mathbb{D})$. The projection $(q, x, y) \mapsto x$ exhibits $\pi^{-1}(q)$ as a double covering of $\overline{\mathbb{D}}$, branched over n points. Thus the Euler characteristic of $\pi^{-1}(q)$ is $2 \cdot \chi(\overline{\mathbb{D}}) - n = 1 - 2g$. The boundary of $\pi^{-1}(q)$ is naturally identified with $\left\{(x, y) \in \mathbb{C}^2 : \ |x| = 1,\ y^2 = \prod_{i=1}^{n}(x - \hat{z}_i)\right\}$, and the projection π restricts to a double covering $\partial\pi^{-1}(q) \to \mathbb{S}^1$. Since n is odd the total space $\partial\pi^{-1}(q)$ of the covering is connected. Therefore $\pi^{-1}(q)$ is diffeomorphic to $\Sigma_{g,1}$.

We want now to parametrise the boundary of each fibre of π. For any $q = \{\hat{z}_1, \ldots, \hat{z}_n\}$ we can consider the equation $y^2 = \prod_{i=1}^{g}(1 - t\hat{z}_i)$, for t ranging in $[0, 1]$. If $t = 1$ the two solutions for y give rise to two points $p_1, p_2 \in \partial\pi^{-1}(q)$, by setting $x = 1$; if $t = 0$ the two solutions for y are ± 1. Since $\prod_{i=1}^{g}(1 - t\hat{z}_i) \neq 0$ for all t, the two values of y are always different and change continuously while t ranges from 0 to 1. This gives a bijection between the sets $\{p_1, p_2\}$ and $\{\pm 1\}$. Assume that p_1 corresponds to $+1$; then we parametrise $\partial\pi^{-1}(q)$ with the unique continuous function $\lambda\colon \partial\pi^{-1}(q) \to \mathbb{S}^1 \subset \mathbb{C}$ taking the value $+1$ on p_1, and satisfying the equality

$$\lambda(q, x, y)^2 = x$$

for all points (q, x, y) of $\partial\pi^{-1}(q)$. The existence and uniqueness of the function λ is granted by the fact that $\partial\pi^{-1}(q)$ is a connected double covering of $\mathbb{S}^1 = \partial\mathbb{D}$, and x is the coordinate of $\mathbb{D}$. The construction is continuous in $q \in C_n(\mathbb{D})$.

We have therefore constructed a $\Sigma_{g,1}$-bundle over $C_n(\mathbb{D})$, and this yields a classifying map $C_n(\mathbb{D}) \to B\Gamma_{g,1}$ which in turn gives a map $\mathfrak{Br}_n \to \Gamma_{g,1}$ between fundamental groups: the induced map is precisely φ [13, proposition 2·1].

The construction above can be generalised by fixing a natural number $d \geq 3$ and replacing the equation $y^2 = \prod_{i=1}^{n}(x - z_i)$ by the equation $y^d = \prod_{i=1}^{n}(x - z_i)$ when defining $\mathcal{V}_n$: one obtains the universal family of *superelliptic curves* of degree d.

A superelliptic curve of degree d is a d-fold covering of $\overline{\mathbb{D}}$ branched over n points $z_1, \ldots, z_n \in \mathbb{D}$, which satisfies the following properites:

(i) the group of deck transformations is cyclic of order d, in particular it acts transitively on all fibres;

(ii) the fibre over each branching point consists of only one point, and all other fibres consist of d points;

(iii) for every regular point $z \in \overline{D}$, every $1 \leq i < j \leq n$ and every couple of small, simple loops $\omega_i, \omega_j \subset \overline{D} \setminus \{z_1, \ldots, z_n\}$ based at z and spinning clockwise around z_i and z_j respectively, the monodromies along ω_i and ω_j are *the same permutation* of the fibre over z.

The boundary of a superelliptic curve is in general disconnected, but it admits a canonical parametrisation, similarly as in the case $d = 2$ described above. The family of superelliptic curves was studied by Callegaro and Salvetti in [7].

4. *Unit vertical vector fields*

Our next aim is to construct on the $\Sigma_{g,1}$-bundle $\mathcal{V}_n \to C_n(\mathbb{D})$ a unit vertical vector field, i.e. a section of the $\mathbb{S}^1$-bundle $\varphi^*\mathcal{UV}_{g,1} \to \mathcal{V}_n = \varphi^*\mathcal{E}_{g,1}$. To do so consider on the entire manifold $C_n(\mathbb{D}) \times \overline{\mathbb{D}} \times \mathbb{C}$ the holomorphic vector field

$$\vec{\mathbf{v}}(\{z_1, \ldots, z_n\}, x, y) = \frac{df}{dy} \cdot \frac{\partial}{\partial x} - \frac{df}{dx} \cdot \frac{\partial}{\partial y} = 2y \cdot \frac{\partial}{\partial x} - \left(\sum_{i=1}^{n} \prod_{j \neq i} (x - z_j)\right) \cdot \frac{\partial}{\partial y}.$$

LEMMA 4·1. *For $v \in \mathcal{V}_n$ the vector $\vec{\mathbf{v}}(v)$ is tangent to $\mathcal{V}_n$, does not vanish and is vertical with respect to π.*

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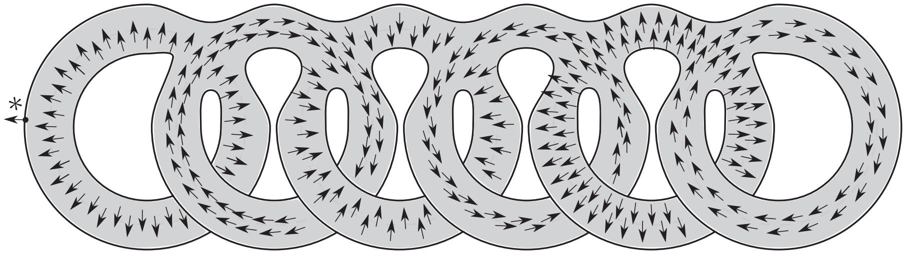

Fig. 3. The vector field $\vec{\mathfrak{v}}$.

Proof. We have already seen in the proof of Lemma 3·1 that for each $v \in \mathcal{V}_n$ at least one of the x- and y-partial derivatives of $y^2 - \prod_{i=1}^n (x - z_i)$ does not vanish: this implies that $\vec{\mathbf{v}}(v) \neq 0$. Moreover $\vec{\mathbf{v}}$ is tangent to $\mathcal{V}_n$, since it annihilates df; and $\vec{\mathbf{v}}$ is *vertical*, as it is a linear combination of $\partial/\partial x$ and $\partial/\partial y$, i.e. it vanishes under the differential of π.

Up to the canonical identification between the holomorphic tangent bundle and the real tangent bundle of a smooth complex manifold, and up to normalisation, we have found a unit vertical vector field on $\mathcal{V}_n$, i.e. a section of the $\mathbb{S}^1$-bundle $\varphi^*\mathcal{UV}_{g,1} \to \varphi^*\mathcal{E}_{g,1}$. By abuse of notation we will denote by $\vec{\mathbf{v}}$ also this unit vertical vector field on $\mathcal{V}_n$.

On the subspace $C_n(\mathbb{D}) = \varphi^*\mathbf{s}_0(C_n(\mathbb{D})) \subset \varphi^*\mathcal{E}_{g,1}$ we already had a unit vertical vector field, namely $\varphi^*\mathbf{s}_0^v$; the ratio $(\varphi^*\mathbf{s}_0^v)/\vec{\mathbf{v}}$ (in the sense of ratio between sections of a principal $\mathbb{S}^1$-bundle) is given by a map $\theta\colon C_n(\mathbb{D}) \to \mathbb{S}^1$. We can then consider the unit vertical vector field $\vec{\mathbf{w}} := (\theta \circ \pi) \cdot \vec{\mathbf{v}}$ on $\mathcal{V}_n$, which restricts to $\varphi^*\mathbf{s}_0^v$ over the subspace $\varphi^*\mathbf{s}_0(C_n(\mathbb{D})) \subset \mathcal{V}_n$. We obtain the following theorem.

THEOREM 4·1. *There is a unit vertical vector field $\vec{\mathbf{w}}$ on $\mathcal{V}_n$ which restricts to $\varphi^*\mathbf{s}_0^v$ over $\varphi^*\mathbf{s}_0(C_n(\mathbb{D})) \subset \mathcal{V}_n$.*

In the rest of the section we present an alternative argument to prove Theorem 4·1. Let $\vec{\mathfrak{v}}$ be a vector field on $\Sigma_{g,1}$ as in Figure 3: it is orthogonal to the curve c_1, parallel to c_2, again orthogonal to c_3 and so on; moreover if $* \in \Sigma_{g,1}$ denotes the basepoint, then $\vec{\mathfrak{v}}(*)$ is exactly the unit tangent vector at $*$ that is orthogonal to $\partial\Sigma_{g,1}$ and points outwards.

Let $\mathbb{V}$ be the space of all vector fields $\vec{w}$ on $\Sigma_{g,1}$ that satisfy $\vec{w}(*) = \vec{\mathfrak{v}}(*)$ and that have no zeroes on $\Sigma_{g,1}$ (we say briefly that they are *non-vanishing*).

LEMMA 4·2. *The space $\mathbb{V}$ is homotopy equivalent to $Map_*(\Sigma_{g,1}; \mathbb{S}^1)$, and is thus a disjoint union contractible components; $\pi_0(\mathbb{V})$ is a $H^1(\Sigma_{g,1})$-torsor.*

Proof. Fix a Riemannian metric on $\Sigma_{g,1}$; then normalisation of vector fields gives a deformation retraction of $\mathbb{V}$ onto its subspace $\mathcal{U}\mathbb{V}$ of sections of the unit tangent bundle of $\Sigma_{g,1}$ with value $\vec{\mathfrak{v}}(*)$ on $*$. Since $\Sigma_{g,1}$ is a surface with boundary, its unit tangent bundle is a trivial $\mathbb{S}^1$-bundle, hence $\mathcal{U}\mathbb{V} \cong Map_*(\Sigma_{g,1}, \mathbb{S}^1)$, where the homeomorphism is neither canonical nor unique, but respect the action of $Map_*(\Sigma_{g,1}, \mathbb{S}^1)$ by pointwise multiplication (in other words, it is a homeomorphism of $Map_*(\Sigma_{g,1}, \mathbb{S}^1)$-torsors). It follows that $\pi_0(\mathbb{V})$ is a $H^1(\Sigma_{g,1})$-torsor, since $\pi_0(Map_*(\Sigma_{g,1}, \mathbb{S}^1))$ is canonically isomorphic, as abelian group, to $H^1(\Sigma_{g,1})$.

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The group $\mathrm{Diff}_{g,1}$ acts on $\mathbb{V}$ through differentials of diffeomorphisms: this action is well-defined thanks to the hypothesis that differomorphisms in $\mathrm{Diff}_{g,1}$ restrict to the identity on a neighbourhood of $\partial\Sigma_{g,1}$, so that in particular their differential fixes the vector $\vec{\mathfrak{v}}(*)$.

There is an induced action of the mapping class group $\Gamma_{g,1}$ on $\pi_0(\mathbb{V})$, and the *framed mapping class group* associated with $\vec{\mathfrak{v}}$, denoted $\Gamma^{fr}_{g,1}(\vec{\mathfrak{v}})$, is by definition the stabiliser of $[\vec{\mathfrak{v}}] \in \pi_0(\mathbb{V})$.

Consider the pullback $\mathfrak{i}^*\mathcal{E}_{g,1} \to B\Gamma^{fr}_{g,1}(\vec{\mathfrak{v}})$ of the universal $\Sigma_{g,1}$-bundle $\mathbf{p}\colon \mathcal{E}_{g,1} \to B\Gamma_{g,1}$ along the inclusion $\mathfrak{i}\colon B\Gamma^{fr}_{g,1}(\vec{\mathfrak{v}}) \to B\Gamma_{g,1}$. Using that connected components of $\mathbb{V}$ are contractible one can construct a unit vertical vector field $\mathfrak{v}$ on $\mathfrak{i}^*\mathcal{E}_{g,1}$ that restricts to $\mathfrak{i}^*s_0^v$ on the section at the boundary $\mathfrak{i}^*s_0(B\Gamma^{fr}_{g,1}(\vec{\mathfrak{v}}))$.

The key remark is that the image of the Birman–Hilden inclusion $\varphi\colon \mathfrak{Br}_{2g+1} \to \Gamma_{g,1}$ lies inside $\Gamma^{fr}_{g,1}(v_0)$: it suffices to note that $\vec{\mathfrak{v}}$ is preserved, up to isotopy through vector fields in $\mathbb{V}$, by the differential of all Dehn twists about the curves c_i.

Therefore the map $\varphi\colon C_n(\mathbb{D}) \to B\Gamma_{g,1}$ factors up to homotopy through $B\Gamma^{fr}_{g,1}(\vec{\mathfrak{v}})$, and we can now pull back the unit vertical vector field $\mathfrak{v}$ over $\mathfrak{i}^*\mathcal{E}_{g,1}$ to a unit vertical vector field $\vec{\mathbf{w}}$ over $\varphi^*\mathcal{E}_{g,1}$ with all the desired properties.

5. *Stable vanishing of φ_**

The proof of Theorem 1·2 consists of two steps. In the first step we formulate the problem in an alternative way, namely we replace the map

$$\varphi_*\colon H_k(\mathfrak{Br}_{2g+1}; \varphi^*\mathcal{H}) \longrightarrow H_k(\Gamma_{g,1}; \mathcal{H})$$

with the map

$$\varphi_*\colon H_{k+1}(\varphi^*\mathcal{E}_{g,1}, B\mathfrak{Br}_{2g+1}) \longrightarrow H_{k+1}(\mathcal{E}_{g,1}, B\Gamma_{g,1}).$$

Recall that $B\mathfrak{Br}_{2g+1}$ can be regarded as a subspace of $\varphi^*\mathcal{E}_{g,1}$ through the section at the boundary. The second map has the advantage of dealing only with homology with constant coefficients, although we have now more complicated spaces.

In the second step we factor the map φ_* through the homology group

$$H_{k+1}(\mathcal{UV}_{g,1}, B\Gamma_{g,1})$$

which is the trivial group for $k \le 2/3g - 2/3 - 1$. This will conclude the proof that φ_* is trivial in the stable range.

The strategy of the proof is summarized in the following diagram

$$\begin{array}{ccc}
H_k(\mathfrak{Br}_{2g+1}; \varphi^*\mathcal{H}) & \xrightarrow{\quad\varphi_*\quad} & H_k(\Gamma_{g,1}; \mathcal{H}) \\
\Big\downarrow{\scriptstyle\cong} & & \Big\downarrow{\scriptstyle\cong} \\
H_{k+1}(\varphi^*\mathcal{E}_{g,1}, B\mathfrak{Br}_{2g+1}) & \xrightarrow{\quad\varphi_*\quad} & H_{k+1}(\mathcal{E}_{g,1}, B\Gamma_{g,1}) \\
& \searrow \qquad \nearrow {\scriptstyle \mathbf{p}^v_*} & \\
& H_{k+1}(\mathcal{UV}_{g,1}, B\Gamma_{g,1}) = 0. &
\end{array}$$

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5·1. *The reformulation of the problem*

The bundle $\mathcal{E}_{g,1} \to B\Gamma_{g,1}$, together with the global section $\mathbf{s}_0$, can be regarded as a pair of bundles $(\mathcal{E}_{g,1}, B\Gamma_{g,1}) \to B\Gamma_{g,1}$ with fibre the pair $(\Sigma_{g,1}, *)$. There is an associated Serre spectral sequence whose second page contains the homology groups

$$E^2_{p,q} = H_p(B\Gamma_{g,1}; H_q(\Sigma_{g,1}, *))$$

and whose limit is the homology of the pair $(\mathcal{E}_{g,1}, B\Gamma_{g,1})$. Note that the homology group $H_q(\Sigma_{g,1}, *)$ is non-trivial only for $q = 1$, in which case it is exactly the symplectic representation $\mathcal{H}$ of $\Gamma_{g,1}$. So the second page of the spectral sequence has only one non-vanishing row and therefore coincides with its limit, i.e.

$$H_{p+1}(\mathcal{E}_{g,1}, B\Gamma_{g,1}) = H_p(B\Gamma_{g,1}; \mathcal{H}).$$

The entire construction is natural with respect to pullbacks. Again let $n = 2g + 1$: the natural map $\varphi\colon (\varphi^*\mathcal{E}_{g,1}, B\mathfrak{Br}_n) \to (\mathcal{E}_{g,1}, B\Gamma_{g,1})$ is a map of pairs of bundles, i.e. it covers the map $\varphi\colon B\mathfrak{Br}_n \to B\Gamma_{g,1}$. The fibre of the pair of bundles $(\varphi^*\mathcal{E}_{g,1}, B\mathfrak{Br}_n) \to B\mathfrak{Br}_n$ is still the pair $(\Sigma_{g,1}, *)$, so its homology is concentrated in degree one and the corresponding spectral sequence gives again an isomorphism

$$H_{p+1}(\varphi^*\mathcal{E}_{g,1}, B\mathfrak{Br}_n) = H_p(B\mathfrak{Br}_n; \varphi^*\mathcal{H}).$$

The induced map between the second pages of the spectral sequences is the map

$$\varphi_*\colon H_k(\mathfrak{Br}_n; \varphi^*\mathcal{H}) \longrightarrow H_k(\Gamma_{g,1}; \mathcal{H}),$$

appearing in Theorem 1·2; the induced map on the limit is the map

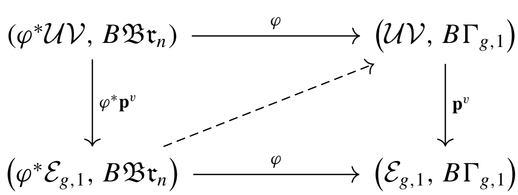
$$\varphi_*\colon H_{k+1}(\varphi^*\mathcal{E}_{g,1}, B\mathfrak{Br}_n) \longrightarrow H_{k+1}(\mathcal{E}_{g,1}, B\Gamma_{g,1}).$$

Hence we can study the latter map, thus reducing the problem to understanding the behaviour of the map of pairs $\varphi\colon (\varphi^*\mathcal{E}_{g,1}, B\mathfrak{Br}_n) \to (\mathcal{E}_{g,1}, B\Gamma_{g,1})$ in homology with constant coefficients.

5·2. *The factorisation through $H_{k+1}(\mathcal{UV}_{g,1}, B\Gamma_{g,1})$.*

By Theorem 4·1 there is a unit vertical vector field $\vec{\mathbf{w}}$ on $\mathcal{V}_n = \varphi^*\mathcal{E}_{g,1}$ extending the canonical vector field $\varphi^*\mathbf{s}_0^v S$ on the subspace $B\mathfrak{Br}_n \subset \varphi^*\mathcal{E}_{g,1}$.

This means that in the following diagram



there is a dashed diagonal arrow lifting the bottom horizontal map, so that the lower right triangle commutes. In particular the map

$$\varphi_*\colon H_{k+1}(\varphi^*\mathcal{E}_{g,1}, B\mathfrak{Br}_n) \longrightarrow H_{k+1}(\mathcal{E}_{g,1}, B\Gamma_{g,1})$$

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factors through the homology group $H_{k+1}\left(\mathcal{UV},\, B\Gamma_{g,1}\right)$. Since by Theorem 2·1 the inclusion $\mathbf{s}_0^v \circ \mathbf{s}_0 \colon B\Gamma_{g,1} \to \mathcal{UV}_{g,1}$ is a homology-isomorphism in degree $\le 2g/3$, we deduce that $H_{k+1}\left(\mathcal{UV},\, B\Gamma_{g,1}\right) = 0$ for $k+1 \le 2g/3$, and therefore for $k \le 2g/3 - 1$ the map

$$\varphi_* \colon H_{k+1}(\varphi^*\mathcal{E}_{g,1},\, B\mathfrak{Br}_n) \longrightarrow H_{k+1}(\mathcal{E}_{g,1},\, B\Gamma_{g,1})$$

is the zero map. This completes the proof of Theorem 1·2.

6. *Torsion property of* $H_*(\mathfrak{Br}_n;\, \varphi^*\mathcal{H})$.

The aim of this section is to prove Theorem 1·3. Using the isomorphism

$$H_k(\mathfrak{Br}_n;\, \varphi^*\mathcal{H}) \simeq H_{k+1}(\mathcal{V}_n,\, C_n(\mathbb{D}))$$

we reduce to proving that the second group is 4-torsion. We will define a submanifold $\mathcal{Z}_n \subset \mathcal{V}_n$ disjoint from $C_n(\mathbb{D}) \subset \mathcal{V}_n$, and use a Mayer-Vietoris argument on a splitting of the pair $(\mathcal{V}_n, C_n(\mathbb{D}))$ as union of $(\mathcal{V}_n \setminus \mathcal{Z}_n, C_n(\mathbb{D}))$ and a tubular neighbourhood $N(\mathcal{Z}_n)$ of $\mathcal{Z}_n$.

On the complex manifold $\mathcal{V}_n$ we consider the coordinate function y, which is a holomorphic function. We denote by $\mathcal{Z}_n \subset \mathcal{V}_n$ the zero locus of y.

LEMMA 6·1. *The subspace $\mathcal{Z}_n \subset \mathcal{V}_n$ is a smooth complex manifold. The normal bundle $\mathfrak{N}_{\mathcal{V}_n}(\mathcal{Z}_n)$ of $\mathcal{Z}_n$ in $\mathcal{V}_n$ is trivial.*

Proof. Recall the vector field $\vec{\mathbf{v}}$ already considered at the beginning of Section 4, given on $\mathcal{V}_n$ by the formula

$$\vec{\mathbf{v}}(\{z_1, \ldots, z_n\}, x, y) = 2y \cdot \frac{\partial}{\partial x} - \left(\sum_{i=1}^{n} \prod_{j \neq i} (x - z_j)\right) \cdot \frac{\partial}{\partial y},$$

and recall that for all $v \in \mathcal{V}_n$ we have $\vec{\mathbf{v}}(v) \neq 0$. If $v \in \mathcal{Z}_n$ the $\partial/\partial x$-component of $\vec{\mathbf{v}}(v)$ is zero, hence the $\partial/\partial y$-component of $\vec{\mathbf{v}}(v)$ must be non-zero; this witnesses the non-vanishing of $dy|_{\mathcal{V}_n}$ on $\mathcal{Z}_n$.

Note now that dy gives a non-vanishing section of the dual $\mathfrak{N}^*_{\mathcal{V}_n}(\mathcal{Z}_n)$ of the $\mathbb{C}$-bundle $\mathfrak{N}_{\mathcal{V}_n}(\mathcal{Z}_n)$: therefore $\mathfrak{N}^*_{\mathcal{V}_n}(\mathcal{Z}_n)$ is a trivial $\mathbb{C}$-bundle over $\mathcal{Z}_n$, and hence also $\mathfrak{N}^*_{\mathcal{V}_n}(\mathcal{Z}_n)$ is a trivial $\mathbb{C}$-bundle over $\mathcal{Z}_n$.

We define the *configuration space* of $n-1$ *black* and one *white* points in the disc as

$$C_{n-1,1}(\mathbb{D}) = \{(\{z_1, \ldots, z_{n-1}\}, x) \in C_{n-1}(\mathbb{D}) \times \mathbb{D} \colon\ x \neq z_i\ \forall 1 \le i \le n-1\}.$$

LEMMA 6·2. *The spaces $\mathcal{Z}_n$ and $C_{n-1,1}(\mathbb{D})$ are homeomorphic.*

Proof. Let $v = (\left\{\hat{z}_1, \ldots, \hat{z}_n\right\}, \hat{x}, \hat{y}) \in \mathcal{Z}_n$. Since $\hat{y} = 0$, the equation $y^2 = \prod_{i=1}^n (x - z_i)$ defining $\mathcal{V}_n$ tells us that $\hat{x}$ must coincide with one, and exactly one, of the numbers $\hat{z}_i$; hence a point $v \in \mathcal{Z}_n$ can be recovered from an unordered configuration $\left\{\hat{z}_1, \ldots, \hat{z}_n\right\}$ of n points in $\mathbb{D}$, one of which is special (and we say, it is *white*) because it coincides with $\hat{x}$.

We denote by $\mathcal{T}_n$ the (open) complement of $\mathcal{Z}_n$ in $\mathcal{V}_n$. We fix a small, closed tubular neighbourhood $N(\mathcal{Z}_n)$ of $\mathcal{Z}_n$ in $\mathcal{V}_n$. Since the normal bundle of $\mathcal{Z}_n$ in $\mathcal{V}_n$ is trivial, we

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have $N(\mathcal{Z}_n) \cong \mathcal{Z}_n \times \overline{\mathbb{D}}$, and $N(\mathcal{Z}_n) \cap \mathcal{T}_n \simeq \partial N(\mathcal{Z}_n) \cong \mathcal{Z}_n \times \mathbb{S}^1$. By construction the copy of $C_n(\mathbb{D})$ contained in $\mathcal{V}_n$, i.e. the image of the section $\varphi^*\mathbf{s}_0$, is also contained in $\mathcal{V}_n \setminus N(\mathcal{Z}_n) = \mathcal{T}_n \setminus N(\mathcal{Z}_n)$. We have a Mayer–Vietoris sequence

$$\ldots \longrightarrow H_k\left(N(\mathcal{Z}_n) \cap \mathcal{T}_n\right) \longrightarrow H_k\left(\mathcal{Z}_n\right) \oplus H_k\left(\mathcal{T}_n, C_n(\mathbb{D})\right) \longrightarrow H_k\left(\mathcal{V}_n, C_n(\mathbb{D})\right) \longrightarrow \ldots$$

from which we derive the following lemma, after an application of the Künneth formula for the homology of $\mathcal{Z}_n \times \mathbb{S}^1$.

LEMMA 6·3. *There is a long exact sequence*

$$\begin{aligned} \cdots \longrightarrow H_k\left(C_{n-1,1}(\mathbb{D})\right) \oplus H_{k-1}\left(C_{n-1,1}(\mathbb{D})\right) \otimes H_1(\mathbb{S}^1) \stackrel{\iota}{\longrightarrow} \\ \stackrel{\iota}{\longrightarrow} H_k\left(C_{n-1,1}(\mathbb{D})\right) \oplus H_k\left(\mathcal{T}_n, C_n(\mathbb{D})\right) \longrightarrow H_k\left(\mathcal{V}_n, C_n(\mathbb{D})\right) \longrightarrow \cdots \end{aligned}$$

Our goal is to obtain information about the homology of $(\mathcal{V}_n, C_n(\mathbb{D}))$ by knowing the other homologies and the behaviour of the maps in the previous sequence. In particular we need some results about the space $\mathcal{T}_n$.

There is a double convering map $\mathrm{Sq}\colon \mathcal{T}_n \to C_{n,1}(\overline{\mathbb{D}})$, where

$$C_{n,1}(\overline{\mathbb{D}}) := \left\{(\{z_1, \ldots, z_n\}, x) \in C_n(\mathbb{D}) \times \overline{\mathbb{D}} :\ x \neq z_i\ \forall 1 \leq i \leq n\right\}$$

is the *configuration space* of $n-1$ *black* points in $\mathbb{D}$ and one *white* point in $\overline{\mathbb{D}}$. The map Sq is given by forgetting the value of y and interpreting x as the *white*, distinguished point. We have introduced $C_{n,1}(\overline{\mathbb{D}})$ because for a configuration $(\{\hat{z}_1, \ldots, \hat{z}_n\}, \hat{x}, \hat{y}) \in \mathcal{T}_n$ it may happen that $\hat{x} \in \mathbb{S}^1$, whereas the numbers $\hat{z}_i$ are always in the interior of the unit disc; nevertheless note that the inclusion $C_{n,1}(\mathbb{D}) \subset C_{n,1}(\overline{\mathbb{D}})$ is a homotopy equivalence.

The 2-fold covering $\mathrm{Sq}\colon \mathcal{T}_n \to C_{n,1}(\overline{\mathbb{D}})$ has a nontrivial deck transformation $\varepsilon\colon \mathcal{T}_n \to \mathcal{T}_n$, which corresponds to changing the sign of y.

LEMMA 6·4. *The map ε is homotopic to the identity of $\mathcal{T}_n$.*

Proof. First we define a homotopy $H_\varepsilon\colon C_{n,1}(\overline{\mathbb{D}}) \times [0, 1] \to C_{n,1}(\overline{\mathbb{D}})$. For $q \in C_{n,1}(\overline{\mathbb{D}})$ and $t \in [0, 1]$ we set $H_\varepsilon(q, t) = e^{2\pi i t} \cdot q$: that is, at time t we rotate the configuration q by an angle $2\pi t$ counterclockwise. Thus H_ε is a homotopy from the identity of $C_{n,1}(\overline{\mathbb{D}})$ to the identity of $C_{n,1}(\overline{\mathbb{D}})$.

We lift this homotopy to a homotopy $\tilde{H}_\varepsilon\colon \mathcal{T}_n \times [0, 1] \to \mathcal{T}_n$, starting from the identity of $\mathcal{T}_n$ at time $t = 0$. At time $t = 1$ any point $v \in \mathcal{T}_n$ is mapped to a point v' lying over the same point $q \in C_{n,1}(\overline{\mathbb{D}})$, i.e., $\mathrm{Sq}(v) = \mathrm{Sq}(v')$.

For fixed $v = (\{\hat{z}_1, \ldots, \hat{z}_n\}, \hat{x}, \hat{y}) \in \mathcal{T}_n$, the y-coordinate of $\tilde{H}_\varepsilon(v, t)$ changes continuously in $t \in [0, 1]$ and its square equals $e^{2\pi t}\hat{y}^2 \neq 0$: it follows that the y-coordinate of $\tilde{H}_\varepsilon(v, t)$ is equal to $e^{2\pi i n t/2}\hat{y}$. In particular, for $t = 1$, the y-coordinate of $\tilde{H}_\varepsilon(v, 1)$ is $-\hat{y} \neq \hat{y}$, therefore $v' = \varepsilon(v)$. In the argument we have used that n is odd.

We obtain the following corollary in homology:

COROLLARY 6·5. *The map* $\mathrm{Sq}_*\colon H_*(\mathcal{T}_n) \to H_*\left(C_{n,1}(\overline{\mathbb{D}})\right)$ *has the following properties*:

(i) *every element in the kernel of* Sq_* *has order 2 in* $H_*(\mathcal{T}_n)$;

(ii) *every element of the form* $2c$ *with* $c \in H_*\left(C_{n,1}(\overline{\mathbb{D}})\right)$ *is in the image of* Sq_*.

Similarly, the map $\mathrm{Sq}^!\colon H_n(C_{n,1}(\overline{\mathbb{D}})) \to H_n(\mathcal{T}_n)$ *has the following properties*:

(iii) *every element in the kernel of* $\mathrm{Sq}^!$ *has order 2 in* $H_*\left(C_{n,1}(\overline{\mathbb{D}})\right)$;

(iv) *every element of the form* $2c$ *with* $c \in H_*(\mathcal{T}_n)$ *is in the image of* $\mathrm{Sq}^!$.

Proof. Note that the composition $\mathrm{Sq}_* \circ \mathrm{Sq}^!$ is multiplication by 2 on $H_*\left(C_{n,1}(\overline{\mathbb{D}})\right)$, since 2 is the degree of the covering Sq. Note also that since Sq is a normal covering (it has degree 2), the composition $\mathrm{Sq}^! \circ \mathrm{Sq}_*$ is the sum of the maps induced by the deck transformations of Sq, i.e. it is the sum of the identity of $H_*(\mathcal{T}_n)$ and ε_*. By Lemma 6·4 we know that ε_* is the identity on $H_*(\mathcal{T}_n)$; hence the composition $\mathrm{Sq}^! \circ \mathrm{Sq}_*$ is also multiplication by 2 on $H_*(\mathcal{T}_n)$. The result follows immediately.

There is a copy of $C_n(\mathbb{D})$ embedded in $C_{n,1}(\overline{\mathbb{D}})$, given by selecting $1 \in \mathbb{S}^1$ as white point: this is exactly the image under Sq of the copy of $C_n(\mathbb{D})$ embedded in $\mathcal{T}_n$ along $\varphi^*\mathbf{s}_0$. We denote by $\mathbf{s}\colon C_n(\mathbb{D}) \to C_{n,1}(\overline{\mathbb{D}})$ this inclusion.

LEMMA 6·6. *For all* $k \geq 0$ *there is a diagram of split short exact sequences*

$$\begin{array}{ccccc}
H_k(C_n(\mathbb{D})) & \xrightarrow{(\varphi^*\mathbf{s}_0)_*} & H_k(\mathcal{T}_n) & \longrightarrow & H_k(\mathcal{T}_n, C_n(\mathbb{D})) \\
\Big\| & & \Big\downarrow {\scriptstyle \mathrm{Sq}_*} & & \Big\downarrow {\scriptstyle \mathrm{Sq}_*} \\
H_k(C_n(\mathbb{D})) & \xrightarrow{\mathbf{s}_*} & H_k(C_{n,1}(\overline{\mathbb{D}})) & \longrightarrow & H_k(C_{n,1}(\overline{\mathbb{D}}), C_n(\mathbb{D})).
\end{array}$$

Moreover there is a map $\mathrm{Sq}^!\colon H_k(C_{n,1}(\overline{\mathbb{D}}), C_n(\mathbb{D})) \to H_k(\mathcal{T}_n, C_n(\mathbb{D}))$ *such that the following diagram commutes*:

$$\begin{array}{ccccc}
H_k(C_n(\mathbb{D})) & \xrightarrow{\mathbf{s}_*} & H_k(C_{n,1}(\overline{\mathbb{D}})) & \longrightarrow & H_k(C_{n,1}(\overline{\mathbb{D}}), C_n(\mathbb{D})) \\
\Big\downarrow {\scriptstyle \cdot 2} & & \Big\downarrow {\scriptstyle \mathrm{Sq}^!} & & \Big\downarrow {\scriptstyle \mathrm{Sq}^!} \\
H_k(C_n(\mathbb{D})) & \xrightarrow{(\varphi^*\mathbf{s}_0)_*} & H_k(\mathcal{T}_n) & \longrightarrow & H_k(\mathcal{T}_n, C_n(\mathbb{D})).
\end{array}$$

Proof. Both $\mathcal{T}_n$ and $C_{n,1}(\overline{D})$ retract onto $C_n(\mathbb{D})$: the retraction is given in both cases by forgetting all data but the position of the points z_i. Note in particular that the map $\mathrm{Sq}\colon \mathcal{T}_n \to C_{n,1}(\overline{D})$ is compatible with these retractions. This implies that the long exact homology sequences associated with the couples $(\mathcal{T}_n, C_n(\mathbb{D}))$ and $(C_{n,1}(\overline{D}), C_n(\mathbb{D}))$ give rise to split exact sequences as in the statement of the lemma. The splittings are compatible with the vertical maps in the left square of the first diagram.

For the second diagram, we first prove commutativity of the left square. Note that the covering Sq is trivial over the subspace $C_n(\mathbb{D}) = \mathbf{s}(C_n(\mathbb{D}) \subset C_{n,1}(\overline{\mathbb{D}})$: the two sections of this covering are $\varphi^*\mathbf{s}_0$ and $\epsilon \circ \varphi^*\mathbf{s}_0$. It follows that $\mathrm{Sq}^! \circ \mathbf{s}_* = (\varphi^*\mathbf{s}_0)_* + (\epsilon \circ \varphi^*\mathbf{s}_0)_*$, as maps $H_k(C_n(\mathbb{D})) \to H_k(\mathcal{T}_n)$, and the second map can be rewritten by Lemma 6·4 as $2 \cdot (\varphi^*\mathbf{s}_0)_*$. We can now define $\mathrm{Sq}^!\colon H_k(C_{n,1}(\overline{\mathbb{D}}), C_n(\mathbb{D})) \to H_k(\mathcal{T}_n, C_n(\mathbb{D}))$ as the map $H_k(C_{n,1}(\overline{\mathbb{D}}))/H_k(C_n(\mathbb{D})) \to H_k(\mathcal{T}_n)/H_k(C_n(\mathbb{D}))$ induced from the left square on the quotients.

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It follows that the properties listed in Corollary 6·5 hold also for $\mathrm{Sq}_*\colon H_*(\mathcal{T}_n, C_n) \to H_*(C_{n,1}(\overline{\mathbb{D}}), C_n(\mathbb{D}))$ and $\mathrm{Sq}^!\colon H_*(C_{n,1}(\overline{\mathbb{D}}), C_n(\mathbb{D})) \to H_*(\mathcal{T}_n, C_n)$.

Let $\mu\colon C_{n-1,1}(\mathbb{D}) \times \mathbb{S}^1 \to C_{n,1}(\mathbb{D})$ be the following map:

$$\mu\left(\left(\{z_1, \ldots, z_{n-1}\}, x\right), \theta\right) = \left(\{z_1, \ldots, z_{n-1}, x + \delta\theta\}, x\right),$$

where

$$\delta = \delta(\{z_1, \ldots, z_{n-1}\}, x) = \frac{1}{2} \min\left(\{1 - |x|\} \cup \{|z_i - x| \,:\, 1 \le i \le n-1\}\right) > 0.$$

Roughly speaking, μ transforms a configuration of one white point x and $n-1$ black points $z_1, \ldots, z_{n-1}$ into a configuration with one more black point, by adding a new black point near x, in the direction of θ. Note that if we regard $\mathbb{S}^1$ as a homotopy equivalent replacement of $C_{1,1}$, then μ is up to homotopy a special case of the multiplication $\mu\colon C_{1,h} \times C_{1,k} \to C_{1,h+k}$ making $\coprod_{k\ge 0} C_{1,k}$ into a H-space; this general construction was described in [**16**].

We recall also the following result, that can be found in [**15**]

LEMMA 6·7. *Let ν be the composition*

$$H_{k-1}\left(C_{n-1,1}(\mathbb{D})\right) \otimes H_1(\mathbb{S}^1) \subset H_k(C_{n-1,1}(\mathbb{D}) \times \mathbb{S}^1) \xrightarrow{\mu_*} H_k(C_{n,1}(\mathbb{D})) \simeq H_k(C_{n,1}(\overline{\mathbb{D}}))$$

Then ν is an isomorphism of $H_{k-1}\left(C_{n-1,1}(\mathbb{D})\right) \otimes H_1(\mathbb{S}^1)$ with the kernel of the retraction $H_k(C_{n,1}(\overline{\mathbb{D}})) \to H_k(C_n(\mathbb{D}))$; this kernel is also isomorphic to $H_k(C_{n,1}(\overline{\mathbb{D}}), C_n(\mathbb{D}))$.

The following lemma analyses the behaviour of the map ι appearing in the Mayer–Vietoris sequence of Lemma 6·3.

LEMMA 6·8. *Let ι be the map in the Mayer Vietoris sequence of Lemma 6·3. We consider the restriction of ι to the two summands of its domain, and its projection to the two summands of its codomain*:

(i) *ι induces an isomorphism $H_k(C_{n-1,1}(\mathbb{D})) \to H_k(C_{n-1,1}(\mathbb{D}))$;*
(ii) *ι induces the zero map $H_{k-1}(C_{n-1,1}(\mathbb{D})) \otimes H_1(\mathbb{S}^1) \to H_k(C_{n-1,1}(\mathbb{D}))$;*
(iii) *ι induces the following map $H_{k-1}(C_{n-1,1}(\mathbb{D})) \otimes H_1(\mathbb{S}^1) \to H_k(\mathcal{T}_n, C_n(\mathbb{D}))$*

$$H_{k-1}(C_{n-1,1}(\mathbb{D})) \otimes H_1(\mathbb{S}^1) \xrightarrow{\nu} H_k(C_{n,1}(\overline{\mathbb{D}})) \xrightarrow{\mathrm{Sq}^!} H_k(\mathcal{T}_n) \longrightarrow H_k(\mathcal{T}_n, C_n(\mathbb{D})).$$

Proof. The first two points of the statement come from the behaviour of the map $\iota\colon H_k(C_{n-1,1}(\mathbb{D}) \times \mathbb{S}^1) \to H_k(C_{n-1,1}(\mathbb{D}) \times \overline{\mathbb{D}})$ on Künneth summands.

For the third point, recall that $C_{n-1,1}(\mathbb{D}) \times \mathbb{S}^1$ represents $\partial N(\mathcal{Z}_n)$, where $N(\mathcal{Z}_n) \cong \mathcal{Z}_n \times \overline{\mathbb{D}}$ is a tubular neighbourhood of $\mathcal{Z}_n \cong C_{n-1,1}(\mathbb{D})$ in $\mathcal{V}_n$. Note that the map $\mathrm{Sq}\colon \mathcal{T}_n \to C_{n,1}(\overline{\mathbb{D}})$ extends to a map (which is no longer a covering) $\mathrm{Sq}\colon \mathcal{V}_n \to C_n(\mathbb{D}) \times \overline{\mathbb{D}}$: this map still consists in forgetting y. Let $\mathcal{Z}'_n \subset C_n \times \overline{\mathbb{D}}$ be the subspace of configurations for which the white point coincides with one of the n black points; then again $\mathcal{Z}'_n \simeq C_{n-1,1}(\mathbb{D})$ and $\mathcal{Z}'_n$ has a small, closed tubular neighbourhood $N(\mathcal{Z}'_n) \simeq \mathcal{Z}'_n \times \overline{\mathbb{D}} \subset C_n(\mathbb{D}) \times \overline{\mathbb{D}}$.

We can assume to have chosen $N(\mathcal{Z}_n)$ to be $\mathrm{Sq}^{-1}(N(\mathcal{Z}'_n)) \subset \mathcal{V}_n$; the map $\mathrm{Sq}\colon N(\mathcal{Z}_n) \to N(\mathcal{Z}'_n)$ is a 2-fold branched covering, and it is branched exactly over $\mathcal{Z}'_n$, which is homeomorphically covered by $\mathcal{Z}_n$. The restriction $\mathrm{Sq}\colon \partial N(\mathcal{Z}_n) \to \partial N(\mathcal{Z}'_n)$ is a genuine 2-fold covering. Let $\pi_{N'}\colon N(\mathcal{Z}'_n) \to \mathcal{Z}'_n$ be a projection of the tubular neighbourhood $N(\mathcal{Z}'_n)$

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onto $\mathcal{Z}'_n$, i.e. $\pi_{N'}$ exhibits $N(\mathcal{Z}'_n)$ as a $\overline{\mathbb{D}}$-bundle over $\mathcal{Z}'_n$; then the composition $\pi_N = \pi_{N'} \circ \mathrm{Sq}\colon N(\mathcal{Z}'_n) \to \mathcal{Z}'_n \cong \mathcal{Z}_n$ is a projection of the tubular neighbourhood $N(\mathcal{Z}_n)$ onto $\mathcal{Z}_n$. We have a commutative diagram

$$\begin{array}{ccc} \partial N(\mathcal{Z}_n) \cong \mathcal{Z}_n \times \mathbb{S}^1 & \xrightarrow{\pi_N} & \mathcal{Z}_n \cong C_{n-1,1}(\mathbb{D}) \\ \downarrow{\scriptstyle \mathrm{Sq}} & & \| \\ \partial N(\mathcal{Z}'_n) \cong \mathcal{Z}'_n \times \mathbb{S}^1 & \xrightarrow{\pi_{N'}} & \mathcal{Z}'_n \cong C_{n-1,1}(\mathbb{D}). \end{array}$$

In particular in homology we can express the Gysin map $\pi_N^!$ as $\mathrm{Sq}^! \circ \pi_{N'}^!$.

We now observe that $\pi_N^!\colon H_{k-1}(\mathcal{Z}_n) \to H_k(\partial N(\mathcal{Z}_n))$ is injective, and its image is the summand $H_{k-1}(C_{n-1,1}) \otimes H_1(\mathbb{S}^1) \subset H_k(\partial N(\mathcal{Z}_n))$, where we use the Künneth formula for $\mathcal{Z}_n \times \mathbb{S}^1$ and identify $C_{n-1,1} \cong \mathcal{Z}_n$ by Lemma 6·2. The restriction of ι to $H_{k-1}(C_{n-1,1}) \otimes H_1(\mathbb{S}^1)$ can thus be identified with the composition

$$H_{k-1}(\mathcal{Z}_n) \xrightarrow{\pi_N^!} H_k(\partial N(\mathcal{Z}_n)) \longrightarrow H_k(\mathcal{T}_n),$$

where the second map is induced by the inclusion $\partial N(\mathcal{Z}_n) \hookrightarrow \mathcal{T}_n$.

On the other hand, the map $\pi_{N'}^!\colon H_{k-1}(\mathcal{Z}'_n) \to H_k(\partial N(\mathcal{Z}'_n))$ is injective, and its image is the summand $H_{k-1}(C_{n-1,1}(\mathbb{D})) \otimes H_1(\mathbb{S}^1) \subset H_k(\partial N(\mathcal{Z}'_n))$, were again we use the Künneth formula for $\mathcal{Z}'_n \times \mathbb{S}^1$ and identify $C_{n-1,1} \cong \mathcal{Z}_n \cong \mathcal{Z}'_n$.

Therefore the composition

$$H_{k-1}(C_{n-1,1}(\mathbb{D})) \otimes H_1(\mathbb{S}^1) \xrightarrow{\nu} H_k(C_{n,1}(\overline{\mathbb{D}})) \xrightarrow{\mathrm{Sq}^!} H_k(\mathcal{T}_n).$$

can be identified with the composition

$$H_{k-1}(\mathcal{Z}'_n) \xrightarrow{\pi_{N'}^!} H_k(\partial N(\mathcal{Z}'_n)) \longrightarrow H_k(C_{n,1}(\overline{\mathbb{D}})) \xrightarrow{\mathrm{Sq}^!} H_k(\mathcal{T}_n),$$

where the middle map is induced by the inclusion $\partial N(\mathcal{Z}'_n) \hookrightarrow C_{n,1}(\overline{\mathbb{D}})$.

The statement is now a consequence of the following commutative diagram

$$\begin{array}{ccccc} H_{k-1}(\mathcal{Z}'_n) & \xrightarrow{\pi_{N'}^!} & H_k(\partial N(\mathcal{Z}'_n)) & \longrightarrow & H_k(C_{n,1}(\overline{\mathbb{D}})) \\ \| & & \downarrow{\scriptstyle \mathrm{Sq}^!} & & \downarrow{\scriptstyle \mathrm{Sq}^!} \\ H_{k-1}(\mathcal{Z}_n) & \xrightarrow{\pi_N^!} & H_k(\partial N(\mathcal{Z}_n)) & \longrightarrow & H_k(\mathcal{T}_n). \end{array}$$

Proof of Theorem 1·3. We fix a class $a \in H_k(\mathcal{V}_n, C_n(\mathbb{D}))$, and map it to $H_{k-1}(\partial N(\mathcal{Z}_n))$ along the long exact sequence of Lemma 6·3; we obtain a class of the form $b + c$, where $b \in H_{k-1}(C_{n-1,1}(\mathbb{D}))$ and $c \in H_{k-2}(C_{n-1,1}(\mathbb{D})) \otimes H_1(\mathbb{S}^1)$. Then $\iota(b + c)$ must be zero, hence its first component, lying in $H_{k-1}(C_{n-1,1}(\mathbb{D}))$, must be zero; therefore $b = 0$ by the first two points of Lemma 6·8.

We deduce that $\iota(c) = 0$, and in particular the second component of $\iota(c)$ is $0 \in H_{k-1}(\mathcal{T}_n, C_n(\mathbb{D}))$. This implies by the third point of Lemma 6·8 that c is mapped to 0 also under the following composition

$$\begin{aligned} H_{k-2}(C_{n-1,1}(\mathbb{D})) \otimes H_1(\mathbb{S}^1) &\xrightarrow{\nu} H_{k-1}(C_{n,1}(\overline{\mathbb{D}})) \xrightarrow{\mathrm{Sq}^!} H_{k-1}(\mathcal{T}_n) \\ &\longrightarrow H_{k-1}(\mathcal{T}_n, C_n(\mathbb{D})) \xrightarrow{\mathrm{Sq}_*} H_{k-1}(C_{n,1}(\overline{\mathbb{D}}), C_n(\mathbb{D})). \end{aligned}$$

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Note that we have added the map $\mathrm{Sq}_*\colon H_{k-1}(\mathcal{T}_n, C_n(\mathbb{D})) \to H_{k-1}(C_{n,1}(\overline{\mathbb{D}}), C_n(\mathbb{D}))$ at the end of the composition from Lemma 6·8. We can rewrite the last composition of maps as follows, using Lemma 6·6

$$H_{k-2}(C_{n-1,1}(\mathbb{D})) \otimes H_1(\mathbb{S}^1) \xrightarrow{\nu} H_{k-1}(C_{n,1}(\overline{\mathbb{D}})) \longrightarrow$$
$$\longrightarrow H_{k-1}(C_{n,1}(\overline{\mathbb{D}}), C_n(\mathbb{D})) \xrightarrow{\mathrm{Sq}^!} H_{k-1}(\mathcal{T}_n, C_n(\mathbb{D})) \xrightarrow{\mathrm{Sq}_*} H_{k-1}(C_{n,1}(\overline{\mathbb{D}}), C_n(\mathbb{D}))$$

As observed in the proof of Corollary 6·5, the composition $\mathrm{Sq}_* \circ \mathrm{Sq}^!$ is multiplication by 2; on the other hand, the first two maps compose to an isomorphism by Lemma 6·7. Since we know that c is mapped to 0 along the entire composition, we obtain $2c = 0$.

It follows that $2a$ is in the kernel of the map $H_k(\mathcal{V}_n, C_n(\mathbb{D})) \to H_{k-1}(\partial N(\mathcal{Z}_n))$, so it is in the image of the map $H_k(C_{n-1,1}(\mathbb{D})) \oplus H_k(\mathcal{T}_n, C_n(\mathbb{D})) \to H_k(\mathcal{V}_n, C_n(\mathbb{D}))$.

Let $d + e \mapsto 2a$, where $d \in H_k(C_{n-1,1}(\mathbb{D}))$ and $e \in H_k(\mathcal{T}_n, C_n(\mathbb{D}))$: we want now to show that $2d + 2e$ is in the image of ι. By Lemma 6·8 we have that $\iota(d + 0) = d + h$ for some $h \in H_k(\mathcal{T}_n, C_n(\mathbb{D}))$; as a consequence we have $\iota(2d + 0) = 2d + 2h$, so it suffices to find $i \in H_{k-1}(C_{n-1,1}(\mathbb{D})) \otimes H_1(\mathbb{S}^1)$ such that $\iota(i)$ has second component equal to $2e - 2h$ (the first component of $\iota(i)$ is automatically 0 by the second point of Lemma 6·8).

Since $2e - 2h = 2(e - h)$ is twice a homology class, by Lemmas 6·6 and Corollary 6·5 there is $j \in H_k(C_{n,1}(\overline{\mathbb{D}}), C_n(\mathbb{D}))$ with $\mathrm{Sq}^!(j) = 2e - 2h \in H_k(\mathcal{T}_n, C_n(\mathbb{D}))$. By Lemma 6·7 there is $i \in H_{k-1}(C_{n-1,1}(\mathbb{D})) \otimes H_1(\mathbb{S}^1)$ such that $\nu(i)$ is mapped to j along the map $H_k(C_{n-1,1}(\overline{\mathbb{D}})) \to H_k(C_{n-1,1}(\overline{\mathbb{D}}), C_n(\mathbb{D}))$. It follows from the second point of Lemma 6·8 that $\iota(i) = 2e - 2h$.

In particular the class $2d + 2e$ is in the image of ι and must therefore also be in the kernel of the map $H_k(C_{n-1,1}(\mathbb{D})) \oplus H_k(\mathcal{T}_n, C_n(\mathbb{D})) \to H_k(\mathcal{V}_n, C_n(\mathbb{D}))$: this exactly means that $4a = 0 \in H_k(\mathcal{V}_n, C_n(\mathbb{D}))$, and Theorem 1·3 follows from the isomorphism $H_k(\mathcal{V}_n, C_n(\mathbb{D})) \simeq H_{k-1}(\mathfrak{Br}_n; \varphi^*\mathcal{H})$.

Acknowledgements. I would like to thank Ulrike Tillmann, Mario Salvetti and Filippo Callegaro for their supervision, their help and their encouragement during the preparation of my Master's thesis, and Carl–Friedrich Bödigheimer for helpful discussions and detailed comments on a first draft of this paper. The research leading to this article was conducted while I was visiting the St John's College in Oxford, after an agreement between the College and the Scuola Normale Superiore in Pisa. I was partially supported by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) under Germany's Excellence Strategy - EXC-2047/1 - 390685813.

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