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Multi-parameter homogenization by localization and blow-up

Felipe Alvarez
Centro de Modelamiento Matemático (CNRS UMR 2071),
Departamento de Ingeniería Matemática, Universidad de Chile,
Av. Blanco Encalada 2120, Santiago, Chile (falvarez@dim.uchile.cl)

Jean-Philippe Mandallena
Equipe de Mathématiques,
d'Informatique et Applications de Nîmes (EMIAN),
Centre Universitaire de Formation et de Recherche de Nîmes,
Site des Carmes, Place Gabriel Péri, Cedex 01, 30021 Nîmes, France
Institut de Mathématiques et Modélisation de Montpellier (I3M)
(CNRS UMR 5149), Université Montpellier II, Place Eugène Bataillon,
34090 Montpellier, France (jean-philippe.mandallena@unimes.fr)

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We give an alternative self-contained proof of the homogenization theorem for periodic multi-parameter integrals that was established by the authors. The proof in that paper relies on the so-called compactness method for Γ-convergence, while the one presented here is by direct verification: the candidate to be the limit homogenized functional is first exhibited and the definition of Γ-convergence is then verified. This is done by an extension of bounded gradient sequences using the Acerbi *et al.* extension theorem from connected sets, and by the adaptation of some localization and blow-up techniques developed by Fonseca and Müller, together with De Giorgi's slicing method.

1. Introduction

In a recent paper, we developed a framework to deal with some *multi-parameter* homogenization problems by establishing a general Γ-convergence result for sequences of periodic integral functionals [4, theorem 2.2]. We also gave applications to different 'degenerate' homogenization processes (soft inclusions, iterated homogenization, thin inclusions), showing the versatility of this unified approach. The proof of the abstract result that we gave there is based on the so-called *compactness method* of the general theory for variational functionals due to Dal Maso and Modica [10]. Generally speaking, this method relies both on a compactness theorem in De Giorgi's Γ-convergence sense and on an integral representation theorem for variational functionals. In order to apply it to the multi-parameter case, it is necessary to adapt certain techniques from [6]. Therefore, this proof uses various particular results that are not easily accessible for a non-specialist reader.

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In this article, we give a different proof of [4, theorem 2.2] (cf. theorem 2.2) by *direct verification* of Γ-convergence. More precisely, we first exhibit the candidate to be the limit homogenized functional, we then verify the definition of Γ-convergence. The sketch of this alternative proof is the following. We first prove that the effective domain of the $\Gamma\text{-}\liminf$ of the sequence is equal to the effective domain of the candidate functional (cf. proposition 3.1). To accomplish this, we assume a connectness condition that permits us to extend bounded energy sequences thanks to the Acerbi *et al*. extension theorem [3]. The second step consists of showing that the candidate functional is a lower bound of the $\Gamma\text{-}\liminf$ on this domain (cf. proposition 3.3). We adapt to this situation the *localization and blow-up* method developed by Fonseca and Müller [14,15] to deal with similar problems, which has been already applied to nonlinear homogenization problems by Michaille *et al*. [1, 17] and uses the well-known De Giorgi *cut-off and slicing* method [11]. The proof is then completed by a density argument (cf. proposition 3.5). Following [18], we first prove that the candidate functional is the upper bound of the $\Gamma\text{-}\limsup$ on a subspace of piecewise affine continuous functions and we then extend this property to the whole Sobolev space by approximation. In contrast to the original proof, the new one is self contained and no abstract result from Γ-convergence theory is required.

2. Multi-parameter homogenization theorem

We begin this section by recalling the definition of Γ-convergence. Let $\{F_n\}$ be a sequence of functionals defined on $L^p(\Omega;\mathbb{R}^m)$, where $\Omega \subset \mathbb{R}^N$ is a bounded domain, and, for each $u \in L^p(\Omega;\mathbb{R}^m)$, define

$$\Big(\Gamma\text{-}\liminf_{n\to\infty} F_n\Big)(u) := \inf\Big\{\liminf_{n\to\infty} F_n(u_n) : u_n \to u \text{ in } L^p(\Omega;\mathbb{R}^m)\Big\},$$
$$\Big(\Gamma\text{-}\limsup_{n\to\infty} F_n\Big)(u) := \inf\Big\{\limsup_{n\to\infty} F_n(u_n) : u_n \to u \text{ in } L^p(\Omega;\mathbb{R}^m)\Big\}.$$

Clearly,

$$\Gamma\text{-}\liminf_{n\to\infty} F_n \leqslant \Gamma\text{-}\limsup_{n\to\infty} F_n.$$

We say that $\{F_n\}$ Γ-converges to $\bar{F}$ as $n \to \infty$ with respect to the strong topology of $L^p(\Omega;\mathbb{R}^m)$ and we write

$$\bar{F} = \Gamma\text{-}\lim_{n\to\infty} F_n$$

whenever, for every $u \in L^p(\Omega;\mathbb{R}^m)$,

$$\bar{F}(u) = \Big(\Gamma\text{-}\liminf_{n\to\infty} F_n\Big)(u) = \Big(\Gamma\text{-}\limsup_{n\to\infty} F_n\Big)(u).$$

The following well-known result makes precise the variational nature of this notion of convergence; for deeper discussions of this theory, we refer the reader to [5, 7, 9].

THEOREM 2.1 (De Giorgi and Franzoni [12]). *Let $G : L^p(\Omega;\mathbb{R}^m) \to \mathbb{R}$ be continuous and assume that $\bar{F} = \Gamma\text{-}\lim_{n\to\infty} F_n$. For each $n \in \mathbb{N}$, let $\hat{u}_n \in L^p(\Omega;\mathbb{R}^m)$ be such that*

$$F_n(\hat{u}_n) + G(\hat{u}_n) \leqslant \inf\{F_n + G\} + \varepsilon_n,$$

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with $\varepsilon_n \to 0$ *as* $n \to \infty$*. Then*

$$\limsup_{n\to\infty}(\inf\{F_n + G\}) \leqslant \inf\{\bar{F} + G\}.$$

Moreover, if $\{\hat{u}_n\}$ *is relatively compact in* $L^p(\Omega;\mathbb{R}^m)$*, then*

$$\lim_{n\to\infty}(\inf\{F_n + G\}) = \inf\{\bar{F} + G\}$$

and every cluster point $\hat{u}$ *of* $\{\hat{u}_n\}$ *satisfies* $\bar{F}(\hat{u}) + G(\hat{u}) = \inf\{\bar{F} + G\}$*.*

Let m, N and k be positive integers and write Y for the unit cell $[0,1[^N$. Let Λ be a non-empty subset of $\mathbb{R}^k$ such that $0 \in \mathrm{cl}(\Lambda)$. Suppose that to every $\lambda \in \Lambda$ there corresponds a Carathéodory function $W_\lambda : \mathbb{R}^N \times \mathbb{R}^{mN} \to [0,+\infty[$ satisfying, for each $\xi \in \mathbb{R}^{mN}$,

(C_1) $W_\lambda(\cdot,\xi)$ is Y-periodic: $\forall (x,z) \in \mathbb{R}^N \times \mathbb{Z}^N$, $W_\lambda(x+z,\xi) = W_\lambda(x,\xi)$.

Consider a family of closed subsets $\{T_\lambda\}_{\lambda\in\Lambda} \subset Y$ and a function $r : \Lambda \to [0,\bar{r}]$ with $\bar{r} > 0$. Define $E_\lambda := Y \setminus T_\lambda + \mathbb{Z}^N$, and $r_\lambda(x) := \bar{r}$ if $x \in E_\lambda$ and $r_\lambda(x) := r(\lambda)$ if $x \in \mathbb{R}^N \setminus E_\lambda = T_\lambda + \mathbb{Z}^N$. Assume that there exist $p \in]1,+\infty[$ and $c_0 > 0$ such that[1]

(C_2) $\forall \lambda \in \Lambda$, $\forall x \in \mathbb{R}^N$, $\forall \xi', \xi \in \mathbb{R}^{mN}$, $r_\lambda(x)|\xi|^p \leqslant W_\lambda(x,\xi) \leqslant c_0 r_\lambda(x)(1+|\xi|^p)$.

We also require the following 'localization' condition: $\exists T \subset Y$ such that

(C_3) $\forall \lambda \in \Lambda$, $T_\lambda \subset T$ and $E := Y \setminus T + \mathbb{Z}^N$ is connected, open and ∂E is Lipschitz.

Let $\{\lambda_n\} \subset \Lambda$ be such that $\lambda_n \to 0$ as $n \to \infty$. For every $n \in \mathbb{N}$ and $\xi \in \mathbb{R}^{mN}$, we define

$$G_n^\xi(w;A) := \begin{cases} \displaystyle\int_A W_{\lambda_n}(x,\xi + \nabla w)\,\mathrm{d}x & \text{if } w\lfloor_A \in W_0^{1,p}(A;\mathbb{R}^m), \\ +\infty & \text{otherwise,} \end{cases}$$

where $w \in L^p_{\mathrm{loc}}(\mathbb{R}^N;\mathbb{R}^m)$ and A belongs to $\mathcal{U}_b(\mathbb{R}^N)$, the class of all bounded open subsets of $\mathbb{R}^N$. For every $\xi \in \mathbb{R}^{mN}$ and $n \in \mathbb{N}$, define $\bar{\mathcal{S}}^\xi, S_n^\xi : \mathcal{U}_b(\mathbb{R}^N) \to [0,\infty[$ by

$$\bar{\mathcal{S}}^\xi(A) := \inf\{\bar{G}^\xi(w;A) : w \in L^p(A;\mathbb{R}^m)\}$$

and

$$\mathcal{S}_n^\xi(A) := \inf\{G_n^\xi(w;A) : w \in L^p(A;\mathbb{R}^m)\},$$

respectively.

THEOREM 2.2. *Let* $\Omega \subset \mathbb{R}^N$ *be a bounded domain and assume that (*C_1*), (*C_2*) and (*C_3*) hold. Let the following conditions hold.*

(H_1) $\forall \xi \in \mathbb{R}^{mN}$, $\exists \bar{G}^\xi : L^p_{\mathrm{loc}}(\mathbb{R}^N;\mathbb{R}^m) \times \mathcal{U}_b(\mathbb{R}^N) \to [0,+\infty]$ *such that,* $\forall k \in \mathbb{N}^*$, $\forall v \in L^p_{\mathrm{loc}}(\mathbb{R}^N;\mathbb{R}^m)$,

$$\bar{G}^\xi(v;]0,k[^N) = \Gamma\text{-}\lim_{n\to\infty} G_n^\xi(v;]0,k[^N).$$

[1]This permits different types of singular behaviours, $r(\lambda) \to 0$ or $\mathrm{dist}(T_\lambda,\Sigma) \to 0$ as $\lambda \to 0$, where Σ is a submanifold of $\mathbb{R}^N$.

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(H$_2$) $\forall k_n \to \infty$, $\forall \xi \in \mathbb{R}^{mN}$,[2]

$$\lim_{n\to\infty} \frac{1}{k_n^N} \mathcal{S}_n^\xi(]0, k_n[^N) = \inf_{k\in\mathbb{N}^*} \left\{ \frac{1}{k^N} \bar{\mathcal{S}}^\xi(]0, k[^N) \right\}.$$

Then, for every $\varepsilon_n \to 0$*, the functionals* $F_n : L^p(\Omega;\mathbb{R}^m) \to [0,+\infty]$*, defined by*

$$F_n(u) := \begin{cases} \displaystyle\int_\Omega W_{\lambda_n}\left(\frac{x}{\varepsilon_n}, \nabla u\right) \mathrm{d}x & \text{if } u \in W^{1,p}(\Omega;\mathbb{R}^m), \\ +\infty & \text{otherwise,} \end{cases}$$

satisfy $\Gamma\text{-}\lim_{n\to\infty} F_n = F^{\mathrm{hom}}$*, where* $F^{\mathrm{hom}} : L^p(\Omega;\mathbb{R}^m) \to [0,+\infty]$ *is given by*

$$F^{\mathrm{hom}}(u) := \begin{cases} \displaystyle\int_\Omega W^{\mathrm{hom}}(\nabla u)\,\mathrm{d}x & \text{if } u \in W^{1,p}(\Omega;\mathbb{R}^m), \\ +\infty & \text{otherwise,} \end{cases}$$

with

$$W^{\mathrm{hom}}(\xi) := \inf_{k\in\mathbb{N}^*} \inf_v \left\{ \frac{1}{k^N} \bar{G}^\xi(v;]0, k[^N) : v \in L^p(]0, k[^N; \mathbb{R}^m) \right\}.$$

3. Proof of the theorem

3.1. Effective domain of $\Gamma\text{-}\liminf F_n$

The first step is to identify the effective domain of $\Gamma\text{-}\liminf_{n\to\infty} F_n$, which is defined by

$$\mathrm{dom}\left(\Gamma\text{-}\liminf_{n\to\infty} F_n\right) := \left\{ u \in L^p(\Omega;\mathbb{R}^m) : \left(\Gamma\text{-}\liminf_{n\to\infty} F_n\right)(u) < \infty \right\}.$$

The arguments used in the proof of the following proposition are standard. For more details we refer the reader to [7].

Proposition 3.1. *Under (C$_1$), (C$_2$) and (C$_3$),*

$$\mathrm{dom}\left(\Gamma\text{-}\liminf_{n\to\infty} F_n\right) = W^{1,p}(\Omega;\mathbb{R}^m).$$

Proof. Let $u \in \mathrm{dom}(\Gamma\text{-}\liminf_{n\to\infty} F_n)$. By definition, there is a sequence $u_n \to u$ in $L^p(\Omega;\mathbb{R}^m)$ such that, up to a subsequence, $\{F_n(u_n)\}$ is bounded. From the first inequality in (C$_2$), it follows that

$$\sup_{n\in\mathbb{N}} \int_{\Omega\cap\varepsilon_n E} |\nabla u_n|^p\,\mathrm{d}x < \infty.$$

By (C$_3$), $E = Y\setminus T + \mathbb{Z}^N$ is a periodic connected open set with Lipschitz boundary. If $E = \emptyset$, then, up to a subsequence, $u_n \rightharpoonup u$ weakly in $W^{1,p}$, hence $u \in W^{1,p}(\Omega;\mathbb{R}^m)$. When $E \neq \emptyset$, we extend u_n from $\Omega \cap \varepsilon_n E$ to the whole of Ω, keeping the above uniform boundedness property. This extension is not difficult to construct when the complement of E is disconnected (see [16]), and it is no longer possible in the general case, where $\Omega \cap \varepsilon E$ may be disconnected so that we cannot expect to control the $W^{1,p}$ norm of the extended function. This extension problem is considered in [3].

[2]This hypothesis is the most difficult to verify in practice (see [4] for some examples).

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THEOREM 3.2 (Acerbi *et al.* [3]). *Let E be a periodic connected open subset of $\mathbb{R}^N$ with Lipschitz boundary. There exist constants $k_0, k_1, k_2 > 0$ such that, for every bounded open set $\Omega \subset \mathbb{R}^N$ and $\varepsilon > 0$, there exists a linear and continuous extension operator $P_\varepsilon : W^{1,p}(\Omega \cap \varepsilon E; \mathbb{R}^m) \to W^{1,p}_{\mathrm{loc}}(\Omega; \mathbb{R}^m)$ with*

(a) $P_\varepsilon u = u$ *a.e. in* $\Omega \cap \varepsilon E$;

(b) $\displaystyle\int_{\Omega(\varepsilon k_0)} |P_\varepsilon u|^p \,\mathrm{d}x \leqslant k_1 \int_{\Omega\cap\varepsilon E} |u|^p \,\mathrm{d}x$;

(c) $\displaystyle\int_{\Omega(\varepsilon k_0)} |\nabla(P_\varepsilon u)|^p \,\mathrm{d}x \leqslant k_2 \int_{\Omega\cap\varepsilon E} |\nabla u|^p \,\mathrm{d}x$

for every $u \in W^{1,p}(\Omega \cap \varepsilon E; \mathbb{R}^m)$, where $\Omega(\alpha) := \{x \in \Omega : \mathrm{dist}(x, \partial\Omega) > \alpha\}$.

For each $n \in \mathbb{N}$, we define

$$v_n := P_{\varepsilon_n}(u_n|_{\Omega\cap\varepsilon_n E}).$$

We deduce that, for every n', $\{v_n : n \geqslant n'\}$ is bounded in $W^{1,p}(\Omega'; \mathbb{R}^m)$ for every open set $\Omega' \subset \Omega$ with $\mathrm{dist}(\Omega', \partial\Omega) > \varepsilon_{n'} k_0$. Let us consider an increasing sequence $\{\Omega_i\}$ of open subsets of Ω with Lipschitz boundaries and such that, in the limit, we obtain Ω. Let Ω_i belong to this sequence. We assume, moreover, that $\mathrm{dist}(\Omega_i, \partial\Omega) > 0$. Thus, by the reflexivity of $W^{1,p}$ and the Rellich theorem, there exist $v \in W^{1,p}(\Omega_i; \mathbb{R}^m)$ and a subsequence of $\{v_n\}$ that converges to v strongly in $L^p(\Omega_i; \mathbb{R}^m)$ and weakly in $W^{1,p}(\Omega_i; \mathbb{R}^m)$. We can extract a diagonal subsequence, still denoted by $\{v_n\}$, which converges to a function $v \in W^{1,p}_{\mathrm{loc}}(\Omega; \mathbb{R}^m)$ strongly in $L^p_{\mathrm{loc}}(\Omega; \mathbb{R}^m)$ and weakly in $W^{1,p}_{\mathrm{loc}}(\Omega; \mathbb{R}^m)$. Now let $\Omega' \subset\subset \Omega$ be arbitrary. For every n, we have, in particular, that $u_n = v_n$ a.e. in $\Omega' \cap \varepsilon_n E$. Since $1_{\Omega'\cap\varepsilon_n E} \rightharpoonup \mathcal{L}_N(Y \setminus T)$ weakly in $L^p(\Omega')$, we deduce that $\mathcal{L}_N(Y \setminus T)u = \mathcal{L}_N(Y \setminus T)v$. As $\mathcal{L}_N(Y \setminus T) > 0$, we have that $u = v$ a.e. in Ω', for every $\Omega' \subset\subset \Omega$. Hence $u = v$ a.e. in Ω. Thus

$$\|\nabla u\|_{p,\Omega'} \leqslant \liminf_{n\to\infty} \|\nabla v_n\|_{p,\Omega'} \leqslant c,$$

for every $\Omega' \subset\subset \Omega$, with the constant c being independent of Ω'. Consequently, $u \in W^{1,p}(\Omega; \mathbb{R}^m)$. Hence $\mathrm{dom}(\Gamma\text{-}\liminf_{n\to\infty} F_n) \subset W^{1,p}(\Omega; \mathbb{R}^m)$. Finally, by the second inequality in (C_2), it follows easily that equality holds in the previous inclusion. □

3.2. Lower bound on the Γ-$\liminf F_n$

We have to prove that

$$\Big(\Gamma\text{-}\liminf_{n\to\infty} F_n\Big)(u) \geqslant F^{\mathrm{hom}}(u)$$

for every $u \in L^p(\Omega; \mathbb{R}^m)$. By proposition 3.1 above, this is trivially satisfied when $u \notin W^{1,p}(\Omega; \mathbb{R}^m)$. Let $u \in W^{1,p}(\Omega; \mathbb{R}^m)$ and consider a sequence $u_n \to u$ in $L^p(\Omega; \mathbb{R}^m)$. Without loss of generality, we can suppose that $\{F_n(u_n)\}$ is bounded. We are thus reduced to proving

$$\int_\Omega W^{\mathrm{hom}}(\nabla u(x)) \,\mathrm{d}x \leqslant \liminf_{n\to\infty} \int_\Omega W_{\lambda_n}\Big(\frac{x}{\varepsilon_n}, \nabla u_n(x)\Big) \,\mathrm{d}x. \tag{3.1}$$

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Proposition 3.3. *If (C_1), (C_2), (H_1) and (H_2) are satisfied, then (3.1) holds.*

Proof. We denote by $M(\Omega)$ the set of all Radon measures in Ω and define

$$M^+(\Omega) := \{\nu \in M(\Omega) : \nu \geqslant 0\}.$$

Consider the sequence $\{\mu_n\} \subset M^+(\Omega)$ defined by

$$\mu_n := W_{\lambda_n}\left(\frac{\cdot}{\varepsilon_n}, \nabla u_n\right) \mathrm{d}x.$$

By assumption, $\{\mu_n\}$ is uniformly bounded in $M^+(\Omega)$, and hence there exists $\mu \in M^+(\Omega)$ such that, up to a subsequence, $\mu_n \rightharpoonup \mu$ weakly in $M(\Omega)$. Let $\mu^{\mathrm{hom}} \in M^+(\Omega)$ be defined by $\mu^{\mathrm{hom}} := W^{\mathrm{hom}}(\nabla u)\mathrm{d}x$. The idea is to compare the limit measure μ with μ^{hom}. Since $\mu(\Omega) \leqslant \liminf_{n\to\infty}\mu_n(\Omega)$, it suffices to prove that $\mu^{\mathrm{hom}}(\Omega) \leqslant \mu(\Omega)$.

Localization

We write $\mathcal{L}_N$ for the Lebesgue measure in $\mathbb{R}^N$ as well as for its restriction to Ω. Consider the Lebesgue decomposition of the limit measure $\mu = \mu^{\mathrm{a}} + \mu^{\mathrm{s}}$, where μ^{a} and μ^{s} are, respectively, the absolutely continuous and the singular part of μ with respect to $\mathcal{L}_N$. Thus there exists $f \in L^1(\Omega; \mathbb{R}_+)$ such that $\mu^{\mathrm{a}} = f\,\mathrm{d}x$ and the Besicovitch differentiation theorem ensures that

$$f(x_0) = \lim_{\rho\to 0^+} \frac{\mu^{\mathrm{a}}(\mathcal{Q}_\rho(x_0))}{\mathcal{L}_N(\mathcal{Q}_\rho(x_0))} = \lim_{\rho\to 0^+} \frac{\mu(\mathcal{Q}_\rho(x_0))}{\mathcal{L}_N(\mathcal{Q}_\rho(x_0))}$$

for $\mathcal{L}_N$-almost every $x_0 \in \Omega$. Here, $\mathcal{Q}_\rho(x_0)$ is the open cube centred at x_0 and of side ρ in all directions. Fix x_0 such that the previous equality holds. Since $\mu_n \rightharpoonup \mu$ in $M(\Omega)$, the Alexandroff theorem yields, in particular, that

$$\mu(\mathcal{Q}_\rho(x_0)) = \lim_{n\to\infty} \mu_n(\mathcal{Q}_\rho(x_0))$$

whenever $\mu(\partial\mathcal{Q}_\rho(x_0)) = 0$. As $\mu(\Omega) < \infty$, the latter holds for every $\rho \in \,]0, \rho_0] \setminus D$, where D is a countable set. In the sequel, we will take ρ such that $\mu(\partial\mathcal{Q}_\rho(x_0)) = 0$. Consequently, it suffices to prove that

$$W^{\mathrm{hom}}(\nabla u(x_0)) \leqslant \lim_{\rho\to 0^+} \lim_{n\to\infty} \frac{1}{\rho^N} \int_{\mathcal{Q}_\rho(x_0)} W_{\lambda_n}\left(\frac{x}{\varepsilon_n}, \nabla u_n(x)\right) \mathrm{d}x. \tag{3.2}$$

Assume first that $u_n \in \bar{u} + W_0^{1,p}(\mathcal{Q}_\rho(x_0); \mathbb{R}^m)$, where $\bar{u} : \mathbb{R}^N \to \mathbb{R}^m$ is the affine function defined by $\bar{u}(x) := u(x_0) + \nabla u(x_0)\cdot(x - x_0)$. Then

$$\int_{\mathcal{Q}_\rho(x_0)} W_{\lambda_n}\left(\frac{x}{\varepsilon_n}, \nabla u_n\right) \mathrm{d}x \geqslant \varepsilon_n^N \mathcal{S}_n^{\nabla u(x_0)}\left(\frac{1}{\varepsilon_n}\mathcal{Q}_\rho(x_0)\right).$$

Lemma 3.4. *Let* $\mathrm{Cub}(\mathbb{R}^N)$ *be the class of all open cubes in* $\mathbb{R}^N$. *If* (C_1), (C_2), (H_1) *and* (H_2) *hold, then* $\forall\xi \in \mathbb{R}^{mN}$, $\forall\mathcal{Q} \in \mathrm{Cub}(\mathbb{R}^N)$,

$$\lim_{n\to\infty} \frac{\mathcal{S}_n^\xi((1/\varepsilon_n)\mathcal{Q})}{\mathcal{L}_N((1/\varepsilon_n)\mathcal{Q})} = W^{\mathrm{hom}}(\xi).$$

Proof. Fix $\xi \in \mathbb{R}^{mN}$ and $Q \in \mathrm{Cub}(\mathbb{R}^N)$. Given $k \in \mathbb{N}^*$ and $n \in \mathbb{N}$ large enough, let $k_n \in \mathbb{N}^*$ be the largest integer such that $(k_n - 2)]0, k[^N + k(z_n + \hat{e}) \subset (1/\varepsilon_n)Q$ for an appropriate $z_n \in \mathbb{Z}^N$, where $\hat{e} := (1, 1, \dots, 1)$. From (C_1) and (C_2), it follows that $\mathcal{S}_n^\xi$ is a subadditive and $\mathbb{Z}^N$-invariant set function satisfying

$$0 \leqslant \mathcal{S}_n^\xi(A) \leqslant c_0 \bar{r}(1 + |\xi|^p)\mathcal{L}_N(A)$$

for all $A \in \mathcal{U}_b(\mathbb{R}^N)$. Therefore,

$$\mathcal{S}_n^\xi\left(\frac{1}{\varepsilon_n}Q\right) \leqslant (k_n - 2)^N \mathcal{S}_n^\xi(]0, k[^k) + \mathcal{S}_n^\xi\left(\frac{1}{\varepsilon_n}Q \setminus [(k_n - 2)[0, k]^N + k(z_n + \hat{e})]\right).$$

Since, up to a set of zero Lebesgue measure, the set

$$(1/\varepsilon_n)Q \setminus [(k_n - 2)[0, k]^N + k(z_n + \hat{e})]$$

may be written as the disjoint union of $k_n^N - (k_n - 2)^N$ integer translations of open sets contained in $]0, k[^N$, we deduce that

$$\mathcal{S}_n^\xi\left(\frac{1}{\varepsilon_n}Q\right) \leqslant (k_n - 2)^N \mathcal{S}_n^\xi(]0, k[^N) + (k_n^N - (k_n - 2)^N)ck^N,$$

where $c = c_0 \bar{r}(1 + |\xi|^p)$. We thus obtain the estimate

$$\frac{\mathcal{S}_n^\xi((1/\varepsilon_n)Q)}{\mathcal{L}_N((1/\varepsilon_n)Q)} \leqslant \frac{\mathcal{S}_n^\xi(]0, k[^N)}{k^N} + \frac{k_n^N - (k_n - 2)^N}{(k_n - 2)^N}c.$$

From (H_1), we have that

$$\limsup_{n \to \infty} \mathcal{S}_n^\xi(]0, k[^N) \leqslant \bar{\mathcal{S}}^\xi(]0, k[^N)$$

for every $k \in \mathbb{N}^*$. Since $k_n \to \infty$ as $n \to \infty$,

$$\limsup_{n \to \infty} \frac{\mathcal{S}_n^\xi((1/\varepsilon_n)Q)}{\mathcal{L}_N((1/\varepsilon_n)Q)} \leqslant \inf_{k \in \mathbb{N}^*}\left\{\frac{\bar{\mathcal{S}}^\xi(]0, k[^N)}{k^N}\right\} = W^{\mathrm{hom}}(\xi).$$

Similarly, for every $n \in \mathbb{N}$, let $k_n \in \mathbb{N}^*$ be such that $(1/\varepsilon_n)Q \subset]0, k_n[^N + z_n$ for a suitable $z_n \in \mathbb{Z}^N$. We then have

$$\mathcal{S}_n^\xi(]0, k_n[^N) \leqslant \mathcal{S}_n^\xi((1/\varepsilon_n)Q) + \mathcal{S}_n^\xi((]0, k_n[^N + z_n) \setminus (1/\varepsilon_n)\bar{Q}),$$

and so

$$\frac{\mathcal{S}_n^\xi(]0, k_n[^N)}{k_n^N} \leqslant \frac{\mathcal{S}_n^\xi((1/\varepsilon_n)Q)}{\mathcal{L}_N((1/\varepsilon_n)Q)} + \frac{k_n^N - (k_n - 2)^N}{(k_n)^N}c.$$

From (H_2), we see that

$$W^{\mathrm{hom}}(\xi) = \lim_{n \to \infty} \frac{1}{k_n^N}\mathcal{S}_n^\xi(]0, k_n[^N) \leqslant \liminf \frac{\mathcal{S}_n^\xi((1/\varepsilon_n)Q)}{\mathcal{L}_N((1/\varepsilon_n)Q)},$$

which completes the proof. □

By lemma 3.4, we have

$$W^{\hom}(\nabla u(x_0)) \leqslant \lim_{n\to\infty} \frac{1}{\rho^N} \int_{\mathcal{Q}_\rho(x_0)} W_{\lambda_n}\left(\frac{x}{\varepsilon_n}, \nabla u_n\right) \mathrm{d}x,$$

and we thus get inequality (3.2). We next indicate how to remove the restriction $u_n \in \bar{u} + W_0^{1,p}(\mathcal{Q}_\rho(x_0);\mathbb{R}^m)$ by the application of a well-known technique introduced by De Giorgi in [11].

Cut-off and slicing method of De Giorgi

We say that a function φ is a cut-off function between A' and A, with $A' \subset\subset A \in \mathcal{U}_b(\mathbb{R}^N)$, if $\varphi \in \mathcal{D}(A)$, $0 \leqslant \varphi \leqslant 1$ and $\varphi \equiv 1$ on A'. Let $\alpha \in]0,1]$ and $l \in \mathbb{N}^*$. For each $i \in \{0,\dots,l\}$, define $\mathcal{Q}_i := \mathcal{Q}_{(1-\alpha+i\alpha/l)\rho}(x_0)$ and consider a cut-off function φ_i between $\mathcal{Q}_{i-1}$ and $\mathcal{Q}_i$ ($i \geqslant 1$) such that $\|\nabla\varphi_i\|_\infty \leqslant 2l/\alpha\rho$. Setting

$$u_n^i(x) := \bar{u}(x) + \varphi_i(x)(u_n(x) - \bar{u}(x)),$$

we obtain $u_n^i \in \bar{u} + W_0^{1,p}(\mathcal{Q}_\rho(x_0);\mathbb{R}^m)$, with

$$\nabla u_n^i = \begin{cases} \nabla u_n & \text{on } \mathcal{Q}_{i-1}, \\ \nabla u(x_0) + (u_n - \bar{u}) \otimes \nabla\varphi_i + \varphi_i(\nabla u_n - \nabla u(x_0)) & \text{on } \mathcal{Q}_i \setminus \mathcal{Q}_{i-1}, \\ \nabla u(x_0) & \text{on } Q_\rho(x_0) \setminus \mathcal{Q}_i. \end{cases}$$

We have the following estimates,

$$\frac{1}{\rho^N} \int_{\mathcal{Q}_\rho(x_0)} W_{\lambda_n}\left(\frac{x}{\varepsilon_n}, \nabla u_n^i\right) \mathrm{d}x \leqslant \frac{1}{\rho^N} \int_{\mathcal{Q}_\rho(x_0)} W_{\lambda_n}\left(\frac{x}{\varepsilon_n}, \nabla u_n\right) \mathrm{d}x + E_{l,\alpha}^i(\rho,n),$$

where

$$E_{l,\alpha}^i(\rho,n) := \frac{1}{\rho^N} \int_{\mathcal{Q}_i \setminus \mathcal{Q}_{i-1}} W_{\lambda_n}\left(\frac{x}{\varepsilon_n}, \nabla u_n^i\right) \mathrm{d}x + c_0\bar{r}(1 + |\nabla u(x_0)|^p)(1 - (1-\alpha)^N).$$

Noticing that

$$\int_{\mathcal{Q}_\rho(x_0)} W_{\lambda_n}(x/\varepsilon_n, \nabla u_n^i)\,\mathrm{d}x = \varepsilon_n^N \int_{1/\varepsilon_n \mathcal{Q}_\rho(x_0)} W_{\lambda_n}(x, \nabla u(x_0) + \nabla w_n^i)\,\mathrm{d}x,$$

with $w_n^i \in W_0^{1,p}(1/\varepsilon_n \mathcal{Q}_\rho(x_0);\mathbb{R}^m)$, we conclude that, for every $i \in \{1,\dots,l\}$,

$$\frac{\mathcal{S}_n^{\nabla u(x_0)}((1/\varepsilon_n)\mathcal{Q}_\rho(x_0))}{\mathcal{L}_N((1/\varepsilon_n)\mathcal{Q}_\rho(x_0))} \leqslant E_{l,\alpha}^i(\rho,n) + \frac{1}{\rho^N} \int_{\mathcal{Q}_\rho(x_0)} W_{\lambda_n}\left(\frac{x}{\varepsilon_n}, \nabla u_n\right) \mathrm{d}x.$$

Consequently, averaging these inequalities over the layers $\mathcal{Q}_i \setminus \mathcal{Q}_{i-1}$, we obtain

$$\frac{\mathcal{S}_n^{\nabla u(x_0)}((1/\varepsilon_n)\mathcal{Q}_\rho(x_0))}{\mathcal{L}_N((1/\varepsilon_n)\mathcal{Q}_\rho(x_0))} \leqslant \bar{E}_{l,\alpha}(\rho,n) + \frac{1}{\rho^N} \int_{\mathcal{Q}_\rho(x_0)} W_{\lambda_n}\left(\frac{x}{\varepsilon_n}, \nabla u_n\right) \mathrm{d}x, \tag{3.3}$$

where

$$\bar{E}_{l,\alpha}(\rho,n) := \frac{1}{l} \sum_{i=1}^{l} E_{l,\alpha}^i(\rho,n).$$

From (C_2) and the definition of u_n^i, it follows that there exists a constant $c > 0$ such that

$$W_{\lambda_n}(\cdot, \nabla u_n^i) \leqslant c[1 + |\nabla u(x_0)|^p + (2l/\alpha\rho)^p |u_n - \bar{u}|^p + r_{\lambda_n}(\cdot)|\nabla u_n|^p].$$

Then we deduce that

$$\bar{E}_{l,\alpha}(\rho, n) \leqslant c\biggl[R_{l,\alpha} + \biggl(\frac{2l}{\alpha\rho}\biggr)^p \frac{1}{\rho^N} \int_{\mathcal{Q}_\rho(x_0)} |u_n - \bar{u}|^p \,\mathrm{d}x + \frac{1}{l\rho^N} \int_{\mathcal{Q}_\rho(x_0)} r_{\lambda_n}\biggl(\frac{x}{\varepsilon_n}\biggr)|\nabla u_n|^p \,\mathrm{d}x\biggr],$$

where $R_{l,\alpha} := (1 - (1-\alpha)^N) + 1/l$. By the coercivity condition,

$$r_{\lambda_n}(x/\varepsilon_n)|\nabla u_n(x)|^p \leqslant W_{\lambda_n}(x/\varepsilon_n, \nabla u_n(x)),$$

and since

$$\frac{1}{\rho^N} \int_{\mathcal{Q}_\rho(x_0)} W_{\lambda_n}(x/\varepsilon_n, \nabla u_n) \,\mathrm{d}x \leqslant K,$$

with K being a constant independent of ρ and n, we deduce that, for a suitable constant $c' > 0$, we have

$$\bar{E}_{l,\alpha}(\rho, n) \leqslant c'\biggl[R_{l,\alpha} + \biggl(\frac{2l}{\alpha\rho}\biggr)^p \frac{1}{\rho^N} \int_{\mathcal{Q}_\rho(x_0)} |u_n - \bar{u}|^p \,\mathrm{d}x\biggr].$$

Hence

$$\limsup_{n\to\infty} \bar{E}_{l,\alpha}(\rho, n) \leqslant c'\biggl[R_{l,\alpha} + \biggl(\frac{2l}{\alpha\rho}\biggr)^p \frac{1}{\rho^N} \int_{\mathcal{Q}_\rho(x_0)} |u - \bar{u}|^p \,\mathrm{d}x\biggr].$$

Let us recall that every function $u \in W^{1,p}(\Omega; \mathbb{R}^m)$ satisfies the following weak differentiability property,

$$\lim_{\rho\to 0^+} \frac{1}{\rho^N} \int_{\mathcal{Q}_\rho(x_0)} \frac{1}{\rho^p} |u(x) - u(x_0) - \nabla u(x_0) \cdot (x - x_0)|^p \,\mathrm{d}x = 0, r$$

for $\mathcal{L}_N$-almost every $x_0 \in \Omega$ (see [19, theorem 3.4.2]). Thus, letting $\rho \to 0$, we have that

$$\limsup_{\rho\to 0} \limsup_{n\to\infty} \bar{E}_{l,\alpha}(\rho, n) \leqslant c' R_{l,\alpha}.$$

We conclude from (3.3) and lemma 3.4 that

$$W^{\mathrm{hom}}(\nabla u(x_0)) \leqslant c' R_{l,\alpha} + \lim_{\rho\to 0^+} \lim_{n\to\infty} \frac{1}{\rho^N} \int_{\mathcal{Q}_\rho(x_0)} W_{\lambda_n}\biggl(\frac{x}{\varepsilon_n}, \nabla u_n\biggr) \,\mathrm{d}x.$$

Finally, we let $l \to \infty$ and $\alpha \to 0$ to prove our claim. □

3.3. Upper bound on the Γ- lim sup F_n

We prove that, for every $u \in L^p(\Omega; \mathbb{R}^m)$,

$$F^{\mathrm{hom}}(u) \geqslant \Bigl(\Gamma\text{-}\limsup_{n\to\infty} F_n\Bigr)(u).$$

By definition of F^{hom}, this is trivially satisfied when $u \notin W^{1,p}(\Omega; \mathbb{R}^m)$.

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Proposition 3.5. *If* (C_1), (C_2) *and* (H_1) *hold, then* $\forall u \in W^{1,p}(\Omega;\mathbb{R}^m)$, $\exists u_n \to u$ *in* $L^p(\Omega;\mathbb{R}^m)$ *such that* $\lim_{n\to\infty} F_n(u_n) = F^{\hom}(u)$.

Proof. We divide the proof into two parts.

Part 1 (piecewise affine continuous functions). Let us denote by $\mathrm{Aff}(\Omega;\mathbb{R}^m)$ the subspace of piecewise affine continuous functions.

Lemma 3.6. *If* (C_1), (C_2) *and* (H_1) *hold, then* $\forall u \in \mathrm{Aff}(\Omega;\mathbb{R}^m)$, $\exists u_n \to u$ *in* $L^p(\Omega;\mathbb{R}^m)$ *with* $u_n \in u + W_0^{1,p}(\Omega;\mathbb{R}^m)$ *such that* $\lim_{n\to\infty} F_n(u_n) = F^{\hom}(u)$.

Proof. We begin by proving the lemma for an arbitrary linear function. The proof is adapted from [18, lemma 2.1 (a)]. Let $\xi \in \mathbb{R}^{mN}$. By the definition of $W^{\hom}$, for every $\delta > 0$, there exist $k \in \mathbb{N}^*$ and $\psi^\delta \in L^p(]0,k[^N;\mathbb{R}^m)$ such that

$$W^{\hom}(\xi) \leqslant \frac{1}{k^N}\bar{G}^\xi(\psi^\delta;]0,k[^N) < W^{\hom}(\xi) + \delta.$$

Fix $\delta > 0$. According to (H_1), there exists a sequence $\{\psi_n^\delta\} \subset W_0^{1,p}(]0,k[^N;\mathbb{R}^m)$ such that $\lim_{n\to\infty} \|\psi_n^\delta - \psi^\delta\|_{p,]0,[^N} = 0$ and

$$\lim_{n\to\infty} G_n^\xi(\psi_n^\delta;]0,k[^N) = \bar{G}^\xi(\psi^\delta;]0,k[^N). \tag{3.4}$$

We extend ψ_n^δ from $]0,k[^N$ to $\mathbb{R}^N$ by kY-periodicity, and, for each $n \in \mathbb{N}$, we define

$$u_n^\delta(x) := \begin{cases} \xi\cdot x + \varepsilon_n \psi_n^\delta(x/\varepsilon_n) & \text{if } x \in \Omega^{\varepsilon_n k},\\ \xi \cdot x & \text{if } x \in \Omega\setminus\Omega^{\varepsilon_n k}, \end{cases}$$

where $\Omega^{\varepsilon_n k}$ is the union of all the cubes of side $\varepsilon_n k$ that are contained in Ω. Of course, $u_n^\delta - \xi\cdot x \in W_0^{1,p}(\Omega;\mathbb{R}^m)$. Since

$$\|u_n^\delta - \xi\cdot x\|_{p,\Omega} \leqslant \varepsilon_n \frac{\mathcal{L}_N(\Omega)}{k^N}\|\psi_n^\delta\|_{p,]0,k[^N},$$

we have that $\lim_{n\to\infty}\|u_n^\delta - \xi\cdot x\|_{p,\Omega} = 0$. By definition of F_n and u_n^δ,

$$F_n(u_n^\delta) = \int_{\Omega^{\varepsilon_n k}} W_{\lambda_n}\left(\frac{x}{\varepsilon_n}, \xi + \nabla\psi_n^\delta\left(\frac{x}{\varepsilon_n}\right)\right)\mathrm{d}x + \int_{\Omega\setminus\Omega^{\varepsilon_n k}} W_{\lambda_n}\left(\frac{x}{\varepsilon_n},\xi\right)\mathrm{d}x.$$

By kY-periodicity, we obtain

$$\int_{\Omega^{\varepsilon_n k}} W_{\lambda_n}\left(\frac{x}{\varepsilon_n}, \xi + \nabla\psi_n^\delta\left(\frac{x}{\varepsilon_n}\right)\right)\mathrm{d}x = \frac{\mathcal{L}_N(\Omega^{\varepsilon_n k})}{k^N}\int_{]0,k[^N} W_{\lambda_n}(y, \xi + \nabla\psi_n^\delta(y))\,\mathrm{d}y.$$

By (3.4), we deduce that there exists $n_0 \in \mathbb{N}$ such that

$$W^{\hom}(\xi) - \delta < \frac{1}{k^N}\int_{]0,k[^N} W_{\lambda_n}(y,\xi+\nabla\psi_n^\delta)\,\mathrm{d}y < W^{\hom}(\xi) + \delta$$

for every $n \geqslant n_0$. We thus have the following estimates,

$$\mathcal{L}_N(\Omega^{\varepsilon_n k})[W^{\hom}(\xi) - \delta] \leqslant F_n(u_n^\delta) \leqslant \mathcal{L}_N(\Omega^{\varepsilon_n k})[W^{\hom}(\xi)+\delta] + c'\mathcal{L}_N(\Omega\setminus\Omega^{\varepsilon_n k})$$

for every $n \geqslant n_0$, where $c' = c_0\bar{r}(1+|\xi|^p)$. Consequently, for every $\delta > 0$,

$$F^{\text{hom}}(\xi \cdot x) - \delta\mathcal{L}_N(\Omega) \leqslant \liminf_{n\to\infty} F_n(u_n^\delta) \leqslant \limsup_{n\to\infty} F_n(u_n^\delta) \leqslant F^{\text{hom}}(\xi \cdot x) + \delta\mathcal{L}_N(\Omega).$$

By a standard diagonalization argument [5, corollary 1.16], we obtain a mapping $n \mapsto \delta_n$ such that $\delta_n \to 0$ as $n \to \infty$, $\lim_{n\to\infty} \|u_n^{\delta_n} - \xi \cdot x\|_{p,\Omega} = 0$ and $\lim_{n\to\infty} F_n(u_n^{\delta_n}) = F^{\text{hom}}(\xi \cdot x)$. Finally, setting $u_n := u_n^{\delta_n}$, we obtain the required sequence. The case of an arbitrary $u \in \text{Aff}(\Omega;\mathbb{R}^m)$ follows by a straightforward generalization of the above construction. □

Part 2 (density argument). Before dealing with a general $u \in W^{1,p}(\Omega;\mathbb{R}^m)$, we establish the following properties of the homogenized integrand.

Lemma 3.7. *Under (C_1), (C_2) and (C_3), we have the following.*

(i) *If (H_1) holds, then $\exists c_1 > 0$ such that $\forall \xi \in \mathbb{R}^{mN}$*

$$c_1|\xi|^p \leqslant W^{\text{hom}}(\xi) \leqslant c_0\bar{r}(1+|\xi|^p).$$

(ii) *If (H_1) and (H_2) hold, then $\exists c_2 > 0$ such that $\forall \xi', \xi \in \mathbb{R}^{mN}$*

$$|W^{\text{hom}}(\xi') - W^{\text{hom}}(\xi)| \leqslant c_2(1 + |\xi'|^{p-1} + |\xi|^{p-1})|\xi' - \xi|.$$

Proof. (i) From (H_1), it follows easily that $\bar{G}^\xi(0;]0,k[^N) \leqslant k^N c_0\bar{r}(1+|\xi|^p)$ for every $\xi \in \mathbb{R}^{mN}$. Hence the upper estimate for W^{hom} follows. For the coercivity condition, we may argue as in [3, proposition 3.3]. By lemma 3.6, there exists a sequence $u_n \to \xi \cdot x$ in L^p with $u_n \in \xi \cdot x + W_0^{1,p}(\Omega;\mathbb{R}^m)$ and such that

$$\lim_{n\to\infty} F_n(u_n) = F^{\text{hom}}(\xi \cdot x) = W^{\text{hom}}(\xi)\mathcal{L}_N(\Omega).$$

Let $\Omega' \subset \mathbb{R}^N$ be an open set with $\Omega \subset\subset \Omega'$. Letting $u_n = \xi \cdot x$ outside of Ω, we extend it to Ω'. Consider the extension operator

$$P_{\varepsilon_n} : W^{1,p}(\Omega' \cap \varepsilon_n E;\mathbb{R}^m) \to W^{1,p}_{\text{loc}}(\Omega';\mathbb{R}^m)$$

given by the theorem of Acerbi *et al*. [3]. For every $n \in \mathbb{N}$ with ε_n small enough such that $\Omega \subset \Omega'(\varepsilon_n k_0)$, we have

$$\|P_{\varepsilon_n} u_n\|^p_{p,\Omega} \leqslant k_1\|u_n\|^p_{p,\Omega\cap\varepsilon_n E} + k_1\|\xi \cdot x\|^p_{\Omega'\setminus\Omega}$$

and

$$\|\nabla(P_{\varepsilon_n} u_n)\|^p_{p,\Omega} \leqslant k_2\|\nabla u_n\|^p_{p,\Omega\cap\varepsilon E} + k_2|\xi|^p\mathcal{L}_N(\Omega' \setminus \Omega).$$

Using the inequality $\bar{r}\|\nabla u_n\|^p_{p,\Omega\cap\varepsilon_n E} \leqslant F_n(u_n)$, together with arguments similar to the proof of proposition 3.1, we deduce that, up to a subsequence, $P_{\varepsilon_n} u_n \rightharpoonup \xi \cdot x$ in $W^{1,p}(\Omega;\mathbb{R}^m)$. Hence, by weak lower-semicontinuity, we obtain

$$\liminf_{n\to\infty} \|\nabla(P_{\varepsilon_n} u_n)\|^p_{p,\Omega} \geqslant |\xi|^p\mathcal{L}_N(\Omega),$$

and from

$$\liminf_{n\to\infty} \|\nabla(P_{\varepsilon_n} u_n)\|^p_{p,\Omega} \leqslant \frac{k_2}{\bar{r}} W^{\text{hom}}(\xi)\mathcal{L}_N(\Omega) + k_2|\xi|^p\mathcal{L}_N(\Omega' \setminus \Omega),$$

it follows that

$$|\xi|^p \mathcal{L}_N(\Omega) \leqslant \frac{k_2}{\bar{r}} W^{\mathrm{hom}}(\xi)\mathcal{L}_N(\Omega) + k_2|\xi|^p \mathcal{L}_N(\Omega' \setminus \Omega).$$

Since $\Omega' \supset\supset \Omega$ is arbitrary, the lower estimate for $W^{\mathrm{hom}}(\xi)$ follows.

(ii) First, observe that, for every $A \in \mathcal{U}_b(\mathbb{R}^N)$,

$$\mathcal{S}_n^\xi(A) = \inf\biggl\{\int_A \mathcal{Q}W_{\lambda_n}(x, \nabla w)\,\mathrm{d}x : w \in \xi\cdot x + W_0^{1,p}(A;\mathbb{R}^m)\biggr\},$$

where $\mathcal{Q}W_{\lambda_n}$ is the quasi-convexification of W_{λ_n} (see [2, 8]). Fix $\xi', \xi \in \mathbb{R}^{mN}$. For every $n \in \mathbb{N}$, consider a function $w_n \in \xi \cdot x + W_0^{1,p}((1/\varepsilon_n)]0,1[^N;\mathbb{R}^m)$ such that

$$\int_{A_n} \mathcal{Q}W_{\lambda_n}(x, \nabla w_n)\,\mathrm{d}x \leqslant \mathcal{S}_n^\xi(A_n) + \delta_n,$$

with $A_n := (1/\varepsilon_n)]0,1[^N$ and $\delta_n := \varepsilon_n r(\lambda_n) \to 0$ as $n \to \infty$. We have

$$\mathcal{S}_n^{\xi'}(A_n) - \mathcal{S}_n^\xi(A_n) \leqslant \int_{A_n} |\mathcal{Q}W_{\lambda_n}(x, \xi' - \xi + \nabla w_n) - \mathcal{Q}W_{\lambda_n}(x, \nabla w_n)|\,\mathrm{d}x + \delta_n.$$

By [8, ch. 4, lemma 2.2], it follows from (C$_2$) that, for a suitable constant $c > 0$,

$$\begin{aligned}|\mathcal{Q}W_{\lambda_n}(\cdot, \xi' - \xi + \nabla w_n) - \mathcal{Q}W_{\lambda_n}(\cdot, \nabla w_n)|&\\ \leqslant c r_{\lambda_n}(1 + |\xi'|^{p-1} + |\xi|^{p-1} + |\nabla w_n|^{p-1})|\xi' - \xi|.&\end{aligned}$$

Then we have to estimate the integral

$$\int_{A_n} r_{\lambda_n}(x)|\nabla w_n|^{p-1}\,\mathrm{d}x = \bar{r}\int_{A_n \cap E_{\lambda_n}} |\nabla w_n|^{p-1}\,\mathrm{d}x + r(\lambda_n)\int_{A_n \setminus E_{\lambda_n}} |\nabla w_n|^{p-1}\,\mathrm{d}x.$$

On the one hand, Hölder's inequality yields

$$\begin{aligned}\int_{A_n \cap E_{\lambda_n}} |\nabla w_n|^{p-1}\,\mathrm{d}x &\leqslant \mathcal{L}_N(A_n \cap E_{\lambda_n})^{1/p}\biggl(\int_{A_n \cap E_{\lambda_n}} |\nabla w_n|^p\,\mathrm{d}x\biggr)^{(p-1)/p}\\ &\leqslant \frac{1}{\varepsilon_n^{N/p}}\|\nabla w_n\|^{p-1}_{p, A_n \cap E_{\lambda_n}}.\end{aligned}$$

On the other hand, using the coercivity condition in (C$_2$), we can deduce that

$$\int_{A_n} r_{\lambda_n}(x)|\nabla w_n|^p\,\mathrm{d}x \leqslant \mathcal{S}_n^\xi(A_n) + \varepsilon_n r(\lambda_n) \leqslant c_0\bar{r}(1 + |\xi|^p)\frac{1}{\varepsilon_n^N} + \delta_n,$$

which gives, in particular,

$$\|\nabla w_n\|^{p-1}_{p, A_n \cap E_{\lambda_n}} \leqslant \biggl[c_0(1 + |\xi|^p)\frac{1}{\varepsilon_n^N} + \delta_n/\bar{r}\biggr]^{(p-1)/p}.$$

Consequently, there exists a constant c such that

$$\int_{A_n \cap E_{\lambda_n}} |\nabla w_n|^{p-1}\,\mathrm{d}x \leqslant \frac{c}{\varepsilon_n^N}[(1 + |\xi|^{p-1}) + \varepsilon_n^{(N+1)(p-1)/p}].$$

By similar arguments, we obtain

$$\int_{A_n\setminus E_{\lambda_n}} |\nabla w_n|^{p-1}\,\mathrm{d}x \leqslant \frac{c}{\varepsilon_n^N}\left[(1+|\xi|^{p-1})\frac{1}{r(\lambda_n)^{(p-1)/p}} + \varepsilon_n^{(N+1)(p-1)/p}\right].$$

We thus deduce that

$$\int_{A_n} |\nabla w_n|^{p-1}\,\mathrm{d}x \leqslant \frac{c}{\varepsilon_n^N}[(\bar r + \bar r^{1/p})(1+|\xi|^{p-1}) + 2\bar r\varepsilon_n^{(N+1)(p-1)/p}].$$

Therefore, there exists a constant c such that

$$\varepsilon_n^N \mathcal{S}_n^{\xi'}(A_n) - \varepsilon_n^N \mathcal{S}_n^{\xi}(A_n) \leqslant c(1+|\xi'|^{p-1}+|\xi|^{p-1}+\varepsilon_n^{(N+1)(p-1)/p})|\xi'-\xi| + \varepsilon_n^N\delta_n.$$

Letting $n\to\infty$, we get

$$W^{\mathrm{hom}}(\xi') - W^{\mathrm{hom}}(\xi) \leqslant c(1+|\xi'|^{p-1}+|\xi|^{p-1})|\xi'-\xi|.$$

□

Now we can complete the proof by a standard density argument. First, note that F^{hom} is a continuous function on $W^{1,p}(\Omega;\mathbb{R}^m)$. In fact, from lemma 3.7 (ii), it follows that

$$|F^{\mathrm{hom}}(u) - F^{\mathrm{hom}}(v)| \leqslant c(1+\|\nabla u\|^p_{p,\Omega} + \|\nabla v\|^p_{p,\Omega})^{(p-1)/p}\|\nabla u - \nabla v\|_{p,\Omega},$$

for every $u, v \in W^{1,p}(\Omega;\mathbb{R}^m)$. Since Ω has Lipschitz boundary, the space $\mathrm{Aff}(\Omega;\mathbb{R}^m)$ is dense in $W^{1,p}(\Omega;\mathbb{R}^m)$ for the strong topology (see [13]). Let $u\in W^{1,p}(\Omega;\mathbb{R}^m)$ and consider $\{u^k\}\subset \mathrm{Aff}(\Omega;\mathbb{R}^m)$ such that $u^k\to u$ as $k\to\infty$ strongly in $W^{1,p}(\Omega;\mathbb{R}^m)$. Then $\lim_{k\to\infty} F^{\mathrm{hom}}(u^k) = F^{\mathrm{hom}}(u)$. By lemma 3.6, $\forall k\in\mathbb{N}$, $\exists\{u_n^k\}\subset W^{1,p}(\Omega;\mathbb{R}^m)$ such that $u_n^k\to u^k$ in $L^p(\Omega;\mathbb{R}^m)$ as $n\to\infty$ and $\lim_{n\to\infty}F_n(u_n^k) = F^{\mathrm{hom}}(u^k)$. Setting

$$f(k,n) := |F_n(u_n^k) - F^{\mathrm{hom}}(u)| + \|u_n^k - u\|_{p,\Omega},$$

we have

$$\lim_{k\to\infty}\lim_{n\to\infty} f(k,n) = 0.$$

By diagonalization (see [5, corollary 1.16]), there exists a mapping $n\to k_n$, increasing to ∞ as $n\to\infty$, such that $\lim_{n\to\infty} f(k_n,n)=0$. Defining $u_n := u_n^{k_n}$, we have thus proved the result. □

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