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ROLLER-COASTER FAILURE RATES AND MEAN RESIDUAL LIFE FUNCTIONS WITH APPLICATION TO THE EXTENDED GENERALIZED INVERSE GAUSSIAN MODEL

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The investigation in this article was motivated by an extended generalized inverse Gaussian (EGIG) distribution, which has more than one turning point of the failure rate for certain values of the parameters. In order to study the turning points of a failure rate, we appeal to Glaser's eta function, which is much simpler to handle. We present some general results for studying the reationship among the change points of Glaser's eta function, the failure rate, and the mean residual life function (MRLF). Additionally we establish an ordering among the number of change points of Glaser's eta function, the failure rate, and the MRLF. These results are used to investigate, in detail, the monotonicity of the three functions in the case of the EGIG. The EGIG model has one additional parameter, δ, than the generalized inverse Gaussian (GIG) model's three parameters; see Jorgensen [7]. It has been observed that the EGIG model fits certain datasets better than the GIG of Jorgensen [7]. Thus, the purpose of this article is to present some general results dealing with the relationship among the change points of the three functions described earlier. The EGIG model is used as an illustration.

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1. INTRODUCTION

In reliability studies, the sense of variation of the failure rate is of major concern since it includes system wearout or burnin or, in some cases, a nonmonotonic failure rate. Generally, the survival and failure time data are frequently modeled by increasing or decreasing failure rates. This might be inappropriate when the course of the disease is such that the mortality reaches a peak after some finite period and then declines. In such a case, the failure rate is upside-down bathtub- shaped and the data are analyzed with appropriate models like log-normal, inverse Gaussian, log-logistic, and Burr-type XII distributions having nonmonotonic failure rates. In addition to bathtub- and upside-down bathtub-shaped failure rates, Wong [9–11] presented situations in which the failure rate curve is roller-coaster-shaped and remarked that the bathtub does not hold water anymore. His articles suggest some plausible physical reasons for the formulation of his roller-coaster type. He also remarked that believing in an erroneous failure rate can lead us into making wrong decisions and, worst of all, spending a great deal of effort in developing the wrong reliability methods.

Since most of the failure rates have complex expressions because of the integral in the denominator, the determination of the monotonicity is not straightforward. To alleviate this difficulty, Glaser [4] presented a method to determine the monotonicity of the failure rate with one turning point. Glaser's method uses the density function instead of the failure rate, which, in many cases, is much simpler. Glaser's method can be used in many complicated examples.

There are situations, as in the cases of the roller-coaster failure rates described earlier, in which there is more than one turning point of the failure rate. For such situations, Gupta and Warren [6] extended Glaser's techniques and discussed some examples. More specifically, they established the connection between the turning points of Glaser's eta function and the failure rate function and showed that the number of turning points of the failure rate function does not exceed the number of turning points of Glaser's eta function. They also investigated the location of the turning points of the failure rate in relation to the location of the turning points of the eta function.

Since the mean residual life function (MRLF) or life expectancy is intimately related to the failure rate function, various researchers have investigated the monotonicity of the MRLF in relation to the monotonicity of the failure rate. It is well known that in the case of increasing (decreasing) failure rate, the MRLF is decreasing (increasing); see Bryson and Siddiqui [3]. In the case of bathtub- and upside-down bathtub-shaped failure rates, Gupta and Akman [5] presented conditions for determining the shape of the MRLF. They also investigated the location of the turning points of the MRLF. Tang, Lu, and Chew [8] extended Gupta and Akman's [5] result for the case of multiple turning points. Bekker and Mi [2] presented an improved version of the proof of the Tang et al. [8] result.

The investigation in this article was motivated by an extended generalized inverse Gaussian (herein abbreviated EGIG) distribution that has more than one turning point of the failure rate for certain values of the parameters. We present some general results for studying the relationship among change points of Glaser's eta function, the failure rate, and the MRLF. Additionally, we establish an ordering among the number of change points of Glaser's eta function, the failure rate, and the MRLF. These results are used to investigate, in detail, the monotonicity of the three functions in the case of the EGIG.

The EGIG model has one additional parameter, δ, than the generalized inverse Gaussian (GIG) model's three parameters; see Jorgensen [7]. It has been observed that the EGIG model fits certain datasets better than the GIG model.

The presentation of this article is organized as follows: in Section 2, basic properties of the EGIG model are discussed. In Section 3 we present some definitions and the monotonicity of Glaser's eta function for the EGIG model. Section 4 contains some general results establishing the relationships among the turning points of Glaser's eta function, the failure rate, and the MRLF. These results are used in Section 5 to determine the shape of the MRLF in the case of one or two change points of Glaser's eta function. For the EGIG model, complete investigation of the monotonicity of the three functions is also provided. Finally, in Section 6 we present some conclusions and comments.

2. BASIC PROPERTIES OF THE EGIG MODEL

The probability density function of the EGIG model is given by

$$f(x) = \frac{1}{(2/\delta)(b/a)^{\lambda/2\delta} K_{\lambda/\delta}(2\sqrt{ab})} x^{\lambda-1} e^{-ax^{\delta}-bx^{-\delta}}, \qquad x > 0, \tag{2.1}$$

where $K_\nu(z)$ is a modified Bessel function of the third kind with index ν. Special cases of (2.1) are the generalized inverse Gaussian distribution, the inverse Gaussian distribution, the gamma distribution, the Weibull distribution, and the exponential distribution. In a recent article, Al-Zamel, Ali, and Kalla [1] proposed a unified form of the gamma-type and inverse Gaussian distribution, with (2.1) as a special case. We discard some of the restrictions set by Al-Zamel et al. [1] placed upon the parameters in (2.1) for the sake of generality. Instead, we adopt a similar domain of variation for the parameters to that given by Jorgensen [7]; that is

$$\lambda \in \mathbb{R}, \quad (a, b, \delta) \in \Omega_\lambda, \tag{2.2}$$

where

$$\Omega_\lambda = \begin{cases} (a, b, \delta) : a > 0, b \geq 0, \delta > 0 & \text{iff } \lambda > 0 \\ (a, b, \delta) : a > 0, b > 0, \delta > 0 & \text{iff } \lambda = 0 \\ (a, b, \delta) : a \geq 0, b > 0, \delta > 0 & \text{iff } \lambda < 0. \end{cases} \tag{2.3}$$

In the case that $a = 0$ or $b = 0$, the normalizing constants in (2.1) are given by

$$\int_0^\infty x^{\lambda-1} e^{-bx^{-\delta}} dx = \frac{1}{\delta}\Gamma\left(-\frac{\lambda}{\delta}\right) b^{\lambda/\delta}, \quad b, \delta, \lambda > 0 \tag{2.4}$$

and

$$\int_0^\infty x^{\lambda-1} e^{-ax^{\delta}} dx = \frac{1}{\delta}\Gamma\left(\frac{\lambda}{\delta}\right) a^{-\lambda/\delta}, \quad a, \delta, \lambda > 0. \tag{2.5}$$

In a manner consistent with Jorgensen [7], we denote the function described by (2.1) and (2.3) by $N_\lambda^{-\delta}(a, b)$ and the class of such functions as

$$N_\lambda^{-\delta} = \{N_\lambda^{-\delta}(a, b) : (a, b, \delta) \in \Omega_\lambda, \ x > 0\}. \tag{2.6}$$

Note that

$$x \sim N_\lambda^{-\delta} \Longrightarrow X^\delta \sim N_{\lambda/\delta}^{-1}, \tag{2.7}$$

where $N_{\lambda/\delta}^{-1}$ is the class of generalized inverse Gaussian density functions; see Jorgensen [7]. It is readily seen that

$$E(X^k) = \left(\sqrt{\frac{b}{a}}\right)^{k/\delta} \frac{K_{(\lambda+k)/\delta}(2\sqrt{ab})}{K_{\lambda/\delta}(2\sqrt{ab})}, \quad k \in \mathbb{R}, \tag{2.8}$$

and

$$\operatorname{Var}(X) = \left(\frac{b}{a}\right)^{1/\delta} \left[\frac{K_{(\lambda+2)/\delta}(2\sqrt{ab})}{K_{\lambda/\delta}(2\sqrt{ab})} - \left(\frac{K_{(\lambda+1)/\delta}(2\sqrt{ab})}{K_{\lambda/\delta}(2\sqrt{ab})}\right)^2\right]. \tag{2.9}$$

Furthermore, in (2.8), we can define $k = \delta j$ for some j so that the results of Jorgensen [7] can be applied.

3. GLASER'S $\eta(x)$, FAILURE RATE, AND MRLF FUNCTIONS

In the introductory section of this article, many well-known distributions are listed as special cases of the extended generalized inverse Gaussian distribution. These aforementioned distributions are often used to model lifetime data. In this section we develop methods for determining the shape of the failure rate function and MRLF for a random variable following the extended generalized inverse Gaussian distribution.

3.1. Background

We introduce the following terminology used throughout the remainder of the text.

DEFINITION 3.1: *Suppose $g : \mathbb{R}^+ \mapsto \mathbb{R}^+$ is a differentiable function with $g'(t)$ continuous. Then the following hold:*

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1. *$g(t) \in \boldsymbol{I}$ (increasing) $\Leftrightarrow g'(t) > 0, \forall t > 0$.*
2. *$g(t) \in \boldsymbol{D}$ (decreasing) $\Leftrightarrow g'(t) < 0, \forall t > 0$.*
3. *$g(t) \in \boldsymbol{I}^*$ (nondecreasing) $\Leftrightarrow g'(t) \geq 0, \forall t > 0$, with equality on a set of measure 0.*
4. *$g(t) \in \boldsymbol{D}^*$ (nonincreasing) $\Leftrightarrow g'(t) \leq 0, \forall t > 0$, with equality on a set of measure 0.*
5. *$g(t) \in \boldsymbol{C}$ (constant) $\Leftrightarrow g'(t) = 0, \forall t > 0$.*
6. *$g(t) \in \boldsymbol{B}$ (bathtub) $\Leftrightarrow \exists t_1$ such that $g'(t) < 0$ for $t \in (0, t_1)$, and $g'(t_1) = 0$ and $g'(t) > 0$ for $t \in (t_1, \infty)$.*
7. *$g(t) \in \boldsymbol{U}$ (upside-down bathtub) $\Leftrightarrow \exists t_1$ such that $g'(t) > 0$ for $t \in (0, t_1)$, and $g'(t_1) = 0$ and $g'(t) < 0$ for $t \in (t_1, \infty)$.*
8. *$g(t) \in \boldsymbol{BU}$ (bathtub upside-down bathtub) $\Leftrightarrow \exists t_1, t_2$ such that $g'(t) < 0$ for $t \in (0, t_1), g'(t_1) = 0, g'(t) > 0$ for $t \in (t_1, t_2), g'(t_2) = 0$, and $g'(t) < 0$ for $t \in (t_2, \infty)$.*
9. *$g(t) \in \boldsymbol{UB}$ (upside-down bathtub bathtub) $\Leftrightarrow \exists t_1, t_2$ such that $g'(t) > 0$ for $t \in (0, t_1), g'(t_1) = 0, g'(t) < 0$ for $t \in (t_1, t_2), g'(t_2) = 0$, and $g'(t) > 0$ for $t \in (t_2, \infty)$.*
10. *$g(t) \in (\boldsymbol{B})\boldsymbol{UBU}\cdots$ (roller coaster) $\Leftrightarrow g(t)$ is a sequence of $\boldsymbol{BU}$, respectively, $\boldsymbol{UB}$, shape.*

DEFINITION 3.2: *Glaser's eta function, $\eta(x)$, for a random variable X with probability density function $f(x)$ is defined as $\eta(x) = -f'(x)/f(x)$.*

DEFINITION 3.3: *The failure rate function, $h(x)$, for a random variable X with probability density function $f(x)$ is defined as $h(x) = f(x)/S(x)$, where $S(x)$ is the survival function given by $S(x) = 1 - F(x)$.*

DEFINITION 3.4: *The mean residual life function, $\mu(x)$, for a random variable X with probability density function $f(x)$ and positive real support is defined as*

$$\mu(x) = E[X - x | X > x]. \tag{3.1}$$

An easily computed form is given by

$$\mu(x) = \frac{\int_x^\infty S(y)\,dy}{S(x)}. \tag{3.2}$$

The two aforementioned reliability functions $h(x)$ and $\mu(x)$ are connected through the well-known relation

$$\mu'(x) = h(x)\mu(x) - 1. \tag{3.3}$$

TABLE 1. Shapes of $\eta(x)$ for $f(x) \in N_\lambda^{-\delta}$

#	Type **L** Cases	$\eta(x) \in$	#	Type **Q** ($a > 0, \delta \neq 1$) Cases	$\eta(x) \in$
1	$a = 0$	***U***	1	$\Delta = -1$	***I***
2	$\delta = 1, \lambda < 1, b > 0$	***U***	2	$\Delta = 0, \lambda < 1$	***I******
3	$\delta = 1, \lambda < 1, b = 0$	***D***	3	$\Delta = 0, 0 < \delta < 1, \lambda = 1$	***D***
4	$\delta = 1, \lambda = 1, b > 0$	***I***	4	$\Delta = 0, \delta > 1, \lambda = 1$	***I***
5	$\delta = 1, \lambda = 1, b = 0$	***C***	5	$\Delta = 0, \lambda > 1$	***I***
6	$\delta = 1, \lambda > 1, b > 0$	***I***	6	$\Delta = 1, 0 < \delta < 1, \lambda < 1, b = 0$	***D***
7	$\delta = 1, \lambda > 1, b = 0$	***I***	7	$\Delta = 1, 0 < \delta < 1, \lambda < 1, b > 0$	***U***
			8	$\Delta = 1, 0 < \delta < 1, \lambda > 1, b = 0$	***U***
			9	$\Delta = 1, 0 < \delta < 1, \lambda > 1, b > 0$	***U***
			10	$\Delta = 1, \lambda = 1$	***U***
			11	$\Delta = 1, \delta > 1, \lambda < 1, b = 0$	***B***
			12	$\Delta = 1, \delta > 1, \lambda < 1, b > 0$	***UB***
			13	$\Delta = 1, \delta > 1, \lambda > 1, b = 0$	***I***
			14	$\Delta = 1, \delta > 1, \lambda > 1, b > 0$	***I***

3.2. Glaser's $\eta(x)$ for the EGIG Model

Suppose that $X \sim N_\lambda^{-\delta}$. Then

$$\eta(x) = -\frac{f'(x)}{f(x)} = \frac{1-\lambda}{x} + a\delta x^{\delta-1} - b\delta x^{-\delta-1} \tag{3.4}$$

and

$$\eta'(x) = \frac{1}{x^{\delta+2}}[a\delta(\delta-1)x^{2\delta} + (\lambda-1)x^\delta + b\delta(\delta+1)]. \tag{3.5}$$

By defining $g : \mathbb{R}^+ \mapsto \mathbb{R}^+$ as

$$g(x) = a\delta(\delta-1)x^{2\delta} + (\lambda-1)x^\delta + b\delta(\delta+1), \tag{3.6}$$

we have

$$\eta'(x) = \frac{g(x)}{x^{\delta+2}}, \tag{3.7}$$

where $g(x)$ is quadratic in x^δ. If $\delta \neq 1$ and $a \neq 0$ then $g(x)$ is a quadratic function in x^δ, a type **Q** case. Otherwise, if $\delta = 1$ or $a = 0$, then $g(x)$ is a linear function in x^δ, a type **L** case. Since $x > 0$, it follows that $\eta'(x)$ and $g(x)$ share the same sign. Table 1 displays the shape of $\eta(x)$ for all of Ω_λ. Note that Δ is defined as

$$\Delta := \begin{cases} -1 & \text{iff } (\lambda-1)^2 < 4\delta^2(\delta^2-1)ab \\ 0 & \text{iff } (\lambda-1)^2 = 4\delta^2(\delta^2-1)ab \\ 1 & \text{iff } (\lambda-1)^2 > 4\delta^2(\delta^2-1)ab. \end{cases}$$

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3.3. Monotonicity of The Failure Rate Function

We assume that $f(t)$ is twice-differentiable. We present the following extension of a result due to Glaser [4].

THEOREM 3.5: *Suppose* $\eta(x) \in \boldsymbol{I}^*(\boldsymbol{D}^*)$.

1. *If* $\eta(x) \in \boldsymbol{I}^*$*, then* $h(x) \in \boldsymbol{I}$.
2. *If* $\eta(x) \in \boldsymbol{D}^*$*, then* $h(x) \in \boldsymbol{D}$.

We now define $\epsilon = \lim_{x\to 0^+} f(x)$. The following result, due to Glaser [4], determines the monotonicity of $h(x)$ when $\eta(x) \in \boldsymbol{B}(\boldsymbol{U})$.

THEOREM 3.6: *Suppose* $\eta(x) \in \boldsymbol{B}(\boldsymbol{U})$.

1. *If* $\eta(x) \in \boldsymbol{B}$*, then we have the following:*
   (a) *if* $\epsilon = 0$*, then* $h(x) \in \boldsymbol{I}$.
   (b) *if* $\epsilon = \infty$*, then* $h(x) \in \boldsymbol{B}$.
2. *If* $\eta(x) \in \boldsymbol{U}$*, then we have the following:*
   (a) *if* $\epsilon = 0$*, then* $h(x) \in \boldsymbol{U}$.
   (b) *if* $\epsilon = \infty$*, then* $h(x) \in \boldsymbol{D}$.

When $\eta(x) \in \mathbf{UB}(\mathbf{BU})$, the following general results, due to Gupta and Warren [6], hold.

THEOREM 3.7: *Suppose* $\eta(x) \in \boldsymbol{UB}$ *with* x_1 *and* x_2 *such that* $\eta'(x_1) = \eta'(x_2) = 0$*. Then we have the following:*

1. *If* $\epsilon = 0$*, then*
   (a) *if* $h'(x_1) < 0$*, then* $h(x) \in \boldsymbol{UB}$.
   (b) *if* $h'(x_1) = 0$*, then* $h(x) \in \boldsymbol{I}^*$.
   (c) *if* $h'(x_1) > 0$*, then* $h(x) \in \boldsymbol{I}$.
2. *If* $h'(0) < 0$*, then* $h(x) \in \boldsymbol{B}$.

Suppose $\eta(x) \in \boldsymbol{BU}$ *with* x_1 *and* x_2 *such that* $\eta'(x_1) = \eta'(x_2) = 0$*. Then we have the following:*

1. *If* $h'(0) < 0$*, then*
   (a) *if* $h'(x_1) < 0$*, then* $h(x) \in \boldsymbol{D}$.
   (b) *if* $h'(x_1) = 0$*, then* $h(x) \in \boldsymbol{D}^*$.
   (c) *if* $h'(x_1) > 0$*, then* $h(x) \in \boldsymbol{BU}$.
2. *If* $\epsilon = 0$*, then* $h(x) \in \boldsymbol{U}$.

4. CHANGE POINTS OF $\eta(x), h(x)$, AND $\mu(x)$ (GENERAL RESULTS)

In this section, we seek to extend the results of Gupta and Warren [6] and Bekker and Mi [2] so that the shape of $\mu(x)$ can be inferred from the shape of $\eta(x)$. From Gupta and Warren [6], we begin with the following Theorem.

THEOREM 4.1: *Suppose that Glaser's $\eta(x)$ function has change points $x_1, x_2, \ldots, x_n$, is differentiable, and is strictly monotone on (x_n, ∞). Then the following hold:*

1. *$h(x)$ does not have a critical point on $[x_n, \infty)$.*
2. *$h(x)$ has at most one change point in each of (x_{j-1}, x_j), for $j = 1, 2, \ldots, n$, where $x_0 = 0$.*

From Bekker and Mi [2] we have the following Theorem.

THEOREM 4.2: *Suppose that the failure rate function $h(x)$ has change points $x'_1, x'_2, \ldots, x'_m$, is differentiable, and is strictly monotone on (x'_m, ∞). Then the following hold:*

1. *$\mu(x)$ does not have a critical point on $[x'_m, \infty)$.*
2. *$\mu(x)$ has at most one change point in each of (x'_{k-1}, x'_k), for $k = 1, 2, \ldots, m$, where $x'_0 = 0$.*

As a consequence of the above two results, we may state the following Theorems.

THEOREM 4.3: *Suppose that Glaser's $\eta(x)$ function has change points $x_1, x_2, \ldots, x_n$, is differentiable, and is strictly monotone on (x_n, ∞). Then $\mu(x)$ has at most n change points and opposite monotonicity on $[x_n, \infty)$.*

THEOREM 4.4: *Suppose that Glaser's $\eta(x)$ function has change points $x_1, x_2, \ldots, x_n$, is differentiable, and is strictly monotone on (x_n, ∞). Then $\mu(x)$ has at most two change points in each of (x_{j-1}, x_j), for $j = 1, 2, \ldots, n$, where $x_0 = 0$.*

In what follows, we will denote the n change points of $\eta(x)$ by $x_1, x_2, \ldots, x_n$, the m change points of $h(x)$ by $x'_1, x'_2, \ldots, x'_m$, and the l change points of $\mu(x)$ by $x^*_1, x^*_2, \ldots, x^*_l$. Additionally, $x_0 = x'_0 = x^*_0 = 0$. Note that

$$l \leq m \leq n. \tag{4.1}$$

Therefore, although $\mu(x)$ has at most two change points in each of (x_{j-1}, x_j), certainly there cannot be two change points of $\mu(x)$ in each of these intervals or else inequality (4.1) is violated.

One can note that if $(x_{j-1}, x_j) \subset (x'_{k-1}, x'_k)$ (i.e., there is no change point of $h(x)$ in (x_{j-1}, x_j)) for some $j = 1, 2, \ldots, n$ and $k = 1, 2, \ldots, m$, then by Theorem 4.2, $\mu(x)$ has at most one change point in (x_{j-1}, x_j). Thus, we may state the following theorem.

THEOREM 4.5: *Suppose that* $h'(x_{j-1})h'(x_j) > 0$ *for some* $j = 1, 2, \ldots, n$. *Then the following hold:*

1. *If* $\mu'(x_{j-1})\mu'(x_j) > 0$, *then there exists no change points of* $\mu(x)$ *in* $[x_{j-1}, x_j]$ *and*
   (a) *if* $\mu'(x_{j-1}) > 0$, *then* $\mu(x) \in \boldsymbol{I}$ *in* $[x_{j-1}, x_j]$;
   (b) *if* $\mu'(x_{j-1}) < 0$, *then* $\mu(x) \in \boldsymbol{D}$ *in* $[x_{j-1}, x_j]$.
2. *If* $\mu'(x_{j-1})\mu'(x_j) < 0$, *then there exists a unique change point of* $\mu(x)$ *in* $[x_{j-1}, x_j]$ *and*
   (a) *if* $\mu'(x_{j-1}) > 0$, *then* $\mu(x) \in \boldsymbol{U}$ *in* $[x_{j-1}, x_j]$;
   (b) *if* $\mu'(x_{j-1}) < 0$, *then* $\mu(x) \in \boldsymbol{B}$ *in* $[x_{j-1}, x_j]$.

We now present three simplifying results for later usage. The first of which is due to Gupta and Warren [6].

LEMMA 4.6: *Suppose that* $\eta(x)$ *has n change points* $x_1, x_2, \ldots, x_n$. *Then the following hold:*

1. *If* $\eta'(x) > 0$ *for all* $x \in (x_{j-1}, x_j)$, *for some* $j = 1, 2, \ldots, n$, *and there exists* $x_j' \in (x_{j-1}, x_j)$ *such that* $h'(x_j') = 0$, *then* $h(x) \in \boldsymbol{U}$ *in* $[x_{j-1}, x_j]$.
2. *If* $\eta'(x) < 0$ *for all* $x \in (x_{j-1}, x_j)$, *for some* $j = 1, 2, \ldots, n$, *and there exists* $x_j' \in (x_{j-1}, x_j)$ *such that* $h'(x_j') = 0$, *then* $h(x) \in \boldsymbol{B}$ *in* $[x_{j-1}, x_j]$.

The following result is due to Bekker and Mi [2].

LEMMA 4.7: *Suppose that* $h(x)$ *has m change points* $x_1', x_2', \ldots, x_m'$. *Then the following hold:*

1. *If* $h'(x) > 0$ *for all* $x \in (x_{j-1}', x_j')$, *for some* $j = 1, 2, \ldots, m$, *and there exists* $x_j^* \in (x_{j-1}, x_j)$ *such that* $\mu'(x_j^*) = 0$, *then* $\mu(x) \in \boldsymbol{B}$ *in* $[x_{j-1}', x_j']$.
2. *If* $h'(x) < 0$ *for all* $x \in (x_{j-1}', x_j')$, *for some* $j = 1, 2, \ldots, m$, *and there exists* $x_j^* \in (x_{j-1}, x_j)$ *such that* $\mu'(x_j^*) = 0$, *then* $\mu(x) \in \boldsymbol{U}$ *in* $[x_{j-1}', x_j']$.

Finally, the last of our three simplifying results is given as follows.

LEMMA 4.8: *Suppose that* $\psi(x) = \mu'(x)S(x)$. *Then* $\psi(x)$ *and* $h(x)$ *have the same monotonicity.*

PROOF: It is readily seen that $\psi(x) = \mu'(x)S(x) = h(x)\int_x^\infty S(y)dy - S(x)$. By differentiating both sides, we have $\psi'(x) = h'(x)\int_x^\infty S(y)dy$. Therefore, $\psi'(x)$ and $h'(x)$ have the same sign and zeros and $\psi(x)$ and $h(x)$ have the same monotonicity. ■

For the case that there exists a change point x_k' of $h(x)$ in (x_{j-1}, x_j) for some $j = 1, 2, \ldots, n$ and $k = 1, 2, \ldots, m$, we state the following Theorem.

THEOREM 4.9: *Suppose that* $h'(x_{j-1})h'(x_j) < 0$ *for some* $j = 1, 2, \ldots, n$.

(1) If $h'(x_{j-1}) > 0$, *then*
   (a) if $\mu'(x_{j-1}) > 0$ *and* $\mu'(x_j) > 0$, *then there exists no change points of* $\mu(x)$ *in* $[x_{j-1}, x_j]$ *and* $\mu(x) \in \boldsymbol{I}$ *in* $[x_{j-1}, x_j]$;
   (b) if $\mu'(x_{j-1}) > 0$ *and* $\mu'(x_j) < 0$, *then there exists a unique change point* x_i^*, *for some* $i = 1, 2, \ldots, l$, *in* (x_{j-1}, x_j). *In fact,* $x_i^* \in (x_k', x_j)$ *and* $\mu(x) \in \boldsymbol{U}$ *in* $[x_{j-1}, x_j]$;
   (c) if $\mu'(x_{j-1}) < 0$ *and* $\mu'(x_j) > 0$, *then there exists a unique change point* x_i^*, *for some* $i = 1, 2, \ldots, l$, *in* (x_{j-1}, x_j). *In fact,* $x_i^* \in (x_{j-1}, x_k')$ *and* $\mu(x) \in \boldsymbol{B}$ *in* $[x_{j-1}, x_j]$.

(2) If $h'(x_{j-1}) < 0$, *then*
   (a) if $\mu'(x_{j-1}) < 0$ *and* $\mu'(x_j) < 0$, *then there exists no change points of* $\mu(x)$ *in* $[x_{j-1}, x_j]$ *and* $\mu(x) \in \boldsymbol{D}$ *in* $[x_{j-1}, x_j]$;
   (b) if $\mu'(x_{j-1}) < 0$ *and* $\mu'(x_j) > 0$, *then there exists a unique change point* x_i^*, *for some* $i = 1, 2, \ldots, l$, *in* (x_{j-1}, x_j). *In fact,* $x_i^* \in (x_k', x_j)$ *and* $\mu(x) \in \boldsymbol{B}$ *in* $[x_{j-1}, x_j]$;
   (c) if $\mu'(x_{j-1}) > 0$ *and* $\mu'(x_j) < 0$, *then there exists a unique change point* x_i^*, *for some* $i = 1, 2, \ldots, l$, *in* (x_{j-1}, x_j). *In fact,* $x_i^* \in (x_{j-1}, x_k')$ *and* $\mu(x) \in \boldsymbol{U}$ *in* $[x_{j-1}, x_j]$.

PROOF:

1. Suppose that $h'(x_{j-1}) > 0$. This implies that $h'(x_j) < 0$. Thus, there exists a x_k' such that $x_k' \in (x_{j-1}, x_j)$ and $h'(x_k') = 0$, for some $k = 1, 2, \ldots, m$. Then $h'(x) > 0$ for $x \in [x_{j-1}, x_k'), h'(x_k') = 0$, and $h'(x) < 0$ for $x \in (x_k', x_j]$. By Lemma 4.8, we can say the same for $\psi'(x)$.
   (a) Suppose that $\mu'(x_{j-1}) > 0$ and $\mu'(x_j) > 0$. Then $\psi(x_{j-1}) > 0$ and $\psi(x_j) > 0$. Since $\psi'(x) > 0$ for $x \in [x_{j-1}, x_k')$, then $\psi(x) > 0$ for $x \in [x_{j-1}, x_k']$. Furthermore, since $\psi'(x) < 0$ for $x \in (x_k', x_j]$ and $\psi(x_j) > 0$, then $\psi(x) > 0$ for $x \in [x_k', x_j]$. Thus, $\psi(x) > 0$, and, therefore, $\mu'(x) > 0$, for $x \in [x_{j-1}, x_j]$. As a result, $\mu(x) \in \boldsymbol{I}$ in $[x_{j-1}, x_j]$.
   (b) Suppose that $\mu'(x_{j-1}) > 0$ and $\mu'(x_j) < 0$. Then $\psi(x_{j-1}) > 0$ and $\psi(x_j) < 0$. Since $\psi'(x) > 0$ for $x \in [x_{j-1}, x_k')$, then $\psi(x) > 0$ for $x \in [x_{j-1}, x_k']$. Furthermore, since $\psi'(x) < 0$ for $x \in (x_k', x_j]$ and $\psi(x_j) < 0$, then there exists a unique x_i^* such that $x_i^* \in (x_k', x_j)$ and $\mu'(x_i^*) = 0$, for some $i = 1, 2, \ldots, l$. Thus, $\psi(x) > 0$, and $\mu'(x) > 0$ for $x \in [x_{j-1}, x_i^*), \psi(x_i^*) = \mu'(x_i^*) = 0$, and $\psi(x) < 0$ and $\mu'(x) < 0$ for $x \in (x_i^*, x_j]$. As a result, $\mu(x) \in \boldsymbol{U}$.
   (c) Suppose that $\mu'(x_{j-1}) < 0$ and $\mu'(x_j) > 0$. A set of arguments similar to that in part (b) may be used.
2. Suppose that $h'(x_{j-1}) < 0$. A similar set of arguments to that of the above can be used.

■

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Our next theorem is the only result that requires the computation of the solutions of $h'(x) = 0$.

THEOREM 4.10: *Suppose that* $h'(x_{j-1})h'(x_j) < 0$ *for some* $j = 1, 2, \ldots, n$ *and that* x'_k *is the unique change point of* $h(x)$ *in* $[x_{j-1}, x_j]$.

1. *If* $h'(x_{j-1}) > 0, \mu'(x_{j-1}) < 0$, *and* $\mu'(x_j) < 0$, *then the following hold:*
   (a) *If* $\mu'(x'_k) < 0$, *then there exists no change points of* $\mu(x)$ *in* $[x_{j-1}, x_j]$ *and* $\mu(x) \in \boldsymbol{D}$ *in* $[x_{j-1}, x_j]$.
   (b) *If* $\mu'(x'_k) = 0$, *then there exists no change points of* $\mu(x)$ *in* $[x_{j-1}, x_j]$ *and* $\mu(x) \in \boldsymbol{D}^*$ *in* $[x_{j-1}, x_j]$.
   (c) *If* $\mu'(x'_k) > 0$, *then there exists two change points of* $\mu(x)$ *in* $[x_{j-1}, x_j]$. *In fact, one change point lies in* $[x_{j-1}, x'_k)$ *and the other in* $(x'_k, x_j]$ *and* $\mu(x) \in \boldsymbol{BU}$ *in* $[x_{j-1}, x_j]$.
2. *If* $h'(x_{j-1}) < 0, \mu'(x_{j-1}) > 0$, *and* $\mu'(x_j) > 0$, *then the following hold:*
   (a) *If* $\mu'(x'_k) > 0$, *then there exists no change points of* $\mu(x)$ *in* $[x_{j-1}, x_j]$ *and* $\mu(x) \in \boldsymbol{I}$ *in* $[x_{j-1}, x_j]$.
   (b) *If* $\mu'(x'_k) = 0$, *then there exists no change points of* $\mu(x)$ *in* $[x_{j-1}, x_j]$ *and* $\mu(x) \in \boldsymbol{I}^*$ *in* $[x_{j-1}, x_j]$.
   (c) *If* $\mu'(x'_k) < 0$, *then there exists two change points of* $\mu(x)$ *in* $[x_{j-1}, x_j]$. *In fact, one change point lies in* $[x_{j-1}, x'_k)$ *and the other in* $(x'_k, x_j]$ *and* $\mu(x) \in \boldsymbol{UB}$ *in* $[x_{j-1}, x_j]$.

PROOF:

1. Suppose that $h'(x_{j-1}) > 0$. This implies that $h'(x_j) < 0$. Thus, there exists a x'_k such that $x'_k \in (x_{j-1}, x_j)$ and $h'(x'_k) = 0$, for some $k = 1, 2, \ldots, m$. Then $h'(x) > 0$ for $x \in [x_{j-1}, x'_k), h'(x'_k) = 0$, and $h'(x) < 0$ for $x \in (x'_k, x_j]$. By Lemma 4.8, we can say the same for $\psi'(x)$. Furthermore, since $\mu'(x_{j-1}) < 0$ and $\mu'(x_j) < 0$, then $\psi(x_{j-1}) < 0$ and $\psi(x_j) < 0$. Thus, $\psi(x) > 0$ for some $x \in [x_{j-1}, x_j]$ if and only if $\psi(x'_k) > 0$. In this case, $\psi(x) < 0$ and $\mu'(x) < 0$ for $x \in [x_{j-1}, x^*_{i-1}) \cup (x^*_i, x_j], \psi(x^*_{i-1}) = \psi(x^*_i) = \mu'(x^*_i) = \mu'(x^*_{i-1}) = 0$, and $\psi(x) > 0$ and $\mu'(x) > 0$ for $x \in (x^*_{i-1}, x^*_i)$, for some $i = 1, 2, \ldots, l$. As a result, we have the following:
   (a) If $\psi(x'_k) < 0$, and therefore $\mu'(x'_k) < 0$, then $\mu(x) \in \boldsymbol{D}$.
   (b) If $\psi(x'_k) = 0$, and therefore $\mu'(x'_k) = 0$, then $\mu(x) \in \boldsymbol{D}^*$.
   (c) If $\psi(x'_k) > 0$, and therefore $\mu'(x'_k) > 0$, then $\mu(x) \in \boldsymbol{BU}$.
2. Suppose that $h'(x_{j-1}) < 0, \mu'(x_{j-1}) > 0$, and $\mu'(x_j) > 0$. A set of arguments similar to that of the above can be used. ■

We must consider the case that $\eta(x)$ and $\mu(x)$ share a common change point and either $h'(x) \neq 0$ for all $x \in [x_{j-1}, x_{j+1}]$ or $h'(x_j) = 0$. To this end, we state the following preliminary result due to Gupta and Warren [6].

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LEMMA 4.11: *Suppose $\eta'(x)$ has n zeroes $x_1, x_2, \ldots, x_n$ and that $\eta(x) \in \boldsymbol{B}(\boldsymbol{U})$ on the interval (x_{j-1}, x_{j+1}) for some $j = 1, 2, \ldots, n$. Then x_j is a common critical point of $\eta(x)$ and $h(x)$ if and only if $h(x) \in \boldsymbol{D}^*(\boldsymbol{I}^*)$ on (x_{j-1}, x_{j+l}).*

Thus, we may state the following Theorem.

THEOREM 4.12: *Suppose $\eta'(x_j) = 0$ and $\mu'(x_j) = 0$ for some $j = 1, 2, \ldots, n-1$.*

1. *If $h'(x) \neq 0$ for all $x \in [x_{j-1}, x_{j+1}]$, then x_j is the only change point of $\mu(x)$ in $[x_{j-1}, x_{j+1}]$ and*
   (a) *if $\mu'(x_{j-1}) < 0$, then $\mu(x) \in \boldsymbol{B}$ in $[x_{j-1}, x_{j+1}]$;*
   (b) *if $\mu'(x_{j+1}) > 0$, then $\mu(x) \in \boldsymbol{U}$ in $[x_{j-1}, x_{j+1}]$.*
2. *If $h'(x) = 0$, then x_j is the only change point of $\mu'(x)$ in $[x_{j-1}, x_{j+1}]$ and*
   (a) *if $h(x) \in \boldsymbol{D}^*$ in $[x_{j-1}, x_{j+1}]$ and $\mu'(x_{j-1}) < 0$, then $\mu(x) \in \boldsymbol{D}$ in $[x_{j-1}, x_{j+1}]$;*
   (b) *if $h(x) \in \boldsymbol{D}^*$ in $[x_{j-1}, x_{j+1}]$ and $\mu'(x_{j-1}) > 0$, then $\mu(x) \in \boldsymbol{U}$ in $[x_{j-1}, x_{j+1}]$;*
   (c) *if $h(x) \in \boldsymbol{I}^*$ in $[x_{j-1}, x_{j+1}]$ and $\mu'(x_{j-1}) < 0$, then $\mu(x) \in \boldsymbol{B}$ in $[x_{j-1}, x_{j+1}]$;*
   (d) *if $h(x) \in \boldsymbol{I}^*$ in $[x_{j-1}, x_{j+1}]$ and $\mu'(x_{j-1}) > 0$, then $\mu(x) \in \boldsymbol{I}$ in $[x_{j-1}, x_{j+1}]$;*

PROOF:

1. Suppose that $h'(x) \neq 0$ for all $x \in [x_{j-1}, x_{j+1}]$. The result follows from Theorem 4.5.
2. Suppose that $h'(x_j) = 0$. The result from Lemma 4.11 and Theorem 4.2. ■

Finally, we consider the case that $\eta(x)$ and $\mu(x)$ share a common change point and $h(x)$ has a change point in $[x_{j-1}, x_j)$ and/or $(x_j, x_{j+1}]$. To this end, we generalize Theorem 4.9 as follows:

THEOREM 4.13: *Suppose $\eta'(x_j) = 0$ and $\mu'(x_j) = 0$ for some $j = 1, 2, \ldots, n-1$ and that $h'(x_j)h'(x_{j+1}) < 0$. Let x'_k be the zero of $h'(x)$ in (x_j, x_{j+1}).*

1. *If $h'(x_j) > 0$, then the following hold:*
   (a) *If $\mu'(x_{j+1}) > 0$, then there exists no change points of $\mu(x)$ in $(x_j, x_{j+1}]$ and $\mu(x) \in \boldsymbol{I}$ in $(x_j, x_{j+1}]$.*
   (b) *If $\mu'(x_{j+1}) < 0$, then there exists a unique change point x^*_i for some $i = 1, 2, \ldots, l$, in (x_j, x_{j+1}). In fact, $x^*_i \in (x'_k, x_{j+1})$ and $\mu(x) \in \boldsymbol{U}$ in $(x_j, x_{j+1}]$.*
2. *If $h'(x_j) < 0$, then the following hold:*
   (a) *If $\mu'(x_{j+1}) < 0$, then there exists no change points of $\mu(x)$ in $(x_j, x_{j+1}]$ and $\mu(x) \in \boldsymbol{D}$ in $(x_j, x_{j+1}]$.*
   (b) *If $\mu'(x_{j+1}) > 0$, then there exists a unique change point x^*_i, for some $i = 1, 2, \ldots, l$, in (x_j, x_{j+1}). In fact, $x^*_i \in (x'_k, x_{j+1})$ and $\mu(x) \in \boldsymbol{B}$ in $(x_j, x_{j+1}]$.*

THEOREM 4.14: *Suppose $\eta'(x_j) = 0$ and $\mu'(x_j) = 0$ for some $j = 1, 2, \ldots, n-1$ and that $h'(x_{j-1})h'(x_j) < 0$. Let x'_k be the zero of $h'(x)$ in (x_{j-1}, x_j).*

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1. *If $h'(x_j) > 0$, then the following hold:*
   (a) *If $\mu'(x_{j-1}) < 0$, then there exists no change points of $\mu(x)$ in $[x_{j-1}, x_j)$ and $\mu(x) \in \boldsymbol{D}$ in $[x_{j-1}, x_j)$.*
   (b) *If $\mu'(x_{j-1}) > 0$, then there exists a unique change point x_i^*, for some $i = 1, 2, \ldots, l$, in (x_{j-1}, x_j). In fact, $x_i^* \in (x_{j-1}, x_k')$ and $\mu(x) \in \boldsymbol{U}$ in $[x_{j-1}, x_j)$.*
2. *If $h'(x_j) < 0$, then the following hold:*
   (a) *If $\mu'(x_{j-1}) > 0$, then there exists no change points of $\mu(x)$ in $[x_{j-1}, x_j)$ and $\mu(x) \in \boldsymbol{I}$ in $[x_{j-1}, x_j)$.*
   (b) *If $\mu'(x_{j-1}) < 0$, then there exists a unique change point x_i^*, for some $i = 1, 2, \ldots, l$, in (x_{j-1}, x_j). In fact, $x_i^* \in (x_{j-1}, x_k')$ and $\mu(x) \in \boldsymbol{B}$ in $[x_{j-1}, x_j)$.*

Remark 4.15: We can combine the results of Theorems 4.13 and 4.14 to determine the shape of $\mu(x)$ in the interval $[x_{j-1}, x_{j+1}]$.

The power of the methods outlined above is realized by the diminished number of solutions of $h'(x) = 0$ needing to be computed; that is, the methods developed by Bekker and Mi [2] require one to compute all of the solutions of $h'(x) = 0$ and the value of $\mu'(x)$ at these solutions to determine the shape of $\mu(x)$. Alternatively, in the methods above, one must usually only compute the value of $h'(x)$ and $\mu'(x)$ at the solutions of $\eta'(x) = 0$ to determine the shape of $\mu(x)$. Certainly, the solutions of $\eta'(x) = 0$ are much easier to obtain than the solutions of $h'(x) = 0$. Thus, we can assert that our methods are more computationally tractable.

5. CHANGE POINTS OF $\mu(x)$ WHEN $\eta(x)$ HAS ONE OR TWO CHANGE POINTS

We employ the results stated in the previous section to determine the shape of $\mu(x)$ when $\eta(x)$ has $n = 1$ or 2 change points. Furthermore, we extend the well-known case that $h(x) \in \boldsymbol{I}(\boldsymbol{D})$. We first state the shape of $\mu(x)$ when $\eta(x)$ has $n = 1$ change point. We omit the case that $h(x)$ has no change points since Theorem 4.5 deals with this situation. Recall that $\epsilon = \lim_{x \to 0^+} f(x)$ and $\mu = \mu(0)$ is the mean.

THEOREM 5.1: *Suppose that $\eta(x)$ has $n = 1$ change point (namely x_1) and that $h'(0)h'(x_1) < 0$. Then the following hold:*

1. *If $h'(0) > 0$ and*
   (a) *$\epsilon\mu \geq 1$, then $\mu'(x_1) > 0$ and $\mu(x) \in \boldsymbol{I}$;*
   (b) *$\epsilon\mu < 1$, then $\mu'(x_1) > 0$ and $\mu(x) \in \boldsymbol{B}$.*
2. *If $h'(0) < 0$ and*
   (a) *$\epsilon\mu \leq 1$, then $\mu'(x_1) < 0$ and $\mu(x) \in \boldsymbol{D}$;*
   (b) *$\epsilon\mu > 1$, then $\mu'(x_1) < 0$ and $\mu(x) \in \boldsymbol{U}$.*

PROOF: The proof follows from Theorem 4.9. ■

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Next, we state the shape of $\mu(x)$ when $\eta(x)$ has $n = 2$ change points. In the following result, we omit the case that $h(x) \in \boldsymbol{I}(\boldsymbol{D})$ and $h(x) \in \boldsymbol{B}(\boldsymbol{U})$ since each is easily seen from Theorems 4.2 and 5.1, respectively.

THEOREM 5.2: *Suppose that $\eta(x)$ has $n = 2$ change points (namely x_1 and x_2) and $h(x)$ has two change points (namely x'_1 and x'_2).*

1. *If $h'(0) > 0$, then*
   (a) *if $\epsilon\mu \geq 1$ and $\mu'(x_1) > 0$, then $\mu(x) \in \boldsymbol{U}$;*
   (b) *if $\epsilon\mu \geq 1$ and $\mu'(x_1) < 0$, then $\mu(x) \in \boldsymbol{U}$;*
   (c) *if $\epsilon\mu < 1$ and $\mu'(x_1) > 0$, then $\mu(x) \in \boldsymbol{BU}$;*
   (d) *if $\epsilon\mu < 1$ and $\mu'(x_1) < 0$, then*
      (i) *if $\mu'(x'_1) < 0$, then $\mu(x) \in \boldsymbol{D}$;*
      (ii) *if $\mu'(x'_1) = 0$, then $\mu(x) \in \boldsymbol{D}^*$;*
      (iii) *if $\mu'(x'_1) > 0$, then $\mu(x) \in \boldsymbol{BU}$.*
2. *If $h'(0) < 0$, then*
   (a) *if $\epsilon\mu \leq 1$ and $\mu'(x_1) < 0$, then $\mu(x) \in \boldsymbol{B}$;*
   (b) *if $\epsilon\mu \leq 1$ and $\mu'(x_1) > 0$, then $\mu(x) \in \boldsymbol{B}$;*
   (c) *if $\epsilon\mu > 1$ and $\mu'(x_1) < 0$, then $\mu(x) \in \boldsymbol{UB}$;*
   (d) *if $\epsilon\mu > 1$ and $\mu'(x_1) > 0$, then*
      (i) *if $\mu'(x'_1) < 0$, then $\mu(x) \in \boldsymbol{I}$;*
      (ii) *if $\mu'(x'_1) = 0$, then $\mu(x) \in \boldsymbol{I}^*$;*
      (iii) *if $\mu'(x'_1) > 0$, then $\mu(x) \in \boldsymbol{UB}$.*

PROOF: The proof follows from Theorems 4.9 and 4.10. ■

We now extend the well-known case that $n = 0$.

THEOREM 5.3: *Suppose that $h(x) \in \boldsymbol{I}^*(\boldsymbol{D}^*)$.*

1. *If $h(x) \in \boldsymbol{I}^*$, then $\mu(x) \in \boldsymbol{D}$.*
2. *If $h(x) \in \boldsymbol{D}^*$, then $\mu(x) \in \boldsymbol{I}$.*

PROOF: The proof is similar to that of Theorem 3.5. ■

The Shape of $\eta(x), h(x)$, and $\mu(x)$ for the EGIG Model

Table 2 displays the shapes of $\eta(x), h(x)$, and $\mu(x)$ when $f(x) \in N_\lambda^{-\delta}$. The case numbers correspond to those in Table 1.

When $\eta(x)$ has at most one change point, the results follow from Theorems 5.1 and 5.3. When $\eta(x)$ has two change points, the monotonicity of $h(x)$ and $\mu(x)$ can be seen as follows.

Table 2. Shapes of $\eta(x), h(x)$, and $\mu(x)$ for $f(x) \in N_{\lambda}^{-\delta}$

L#	$\eta(x) \in$	$h(x) \in$	$\mu(x) \in$	Q#	$\eta(x) \in$	$h(x) \in$	$\mu(x)$
1	***U***	***U***	***B***	1	***I***	***I***	***D***
2	***U***	***U***	***B***	2	$\boldsymbol{I}^*$	***I***	***D***
3	***D***	***D***	***I***	3	***D***	***D***	***I***
4	***I***	***I***	***D***	4	***I***	***I***	***D***
5	***C***	***C***	***C***	5	***I***	***I***	***D***
6	***I***	***I***	***D***	6	***D***	***D***	***I***
7	***I***	***I***	***D***	7	***U***	***U***	***B***
				8	***U***	***U***	***B***
				9	***U***	***U***	***B***
				10	***U***	***U***	***B***
				11	***B***	***B***	***U***
				12_{a1}	***UB***	***UB***	***BU***
				12_{a2}	***UB***	***UB***	$\boldsymbol{D}^*$
				12_{a3i}	***UB***	***UB***	***D***
				12_{a3ii}	***UB***	***UB***	$\boldsymbol{D}^*$
				12_{a3iii}	***UB***	***UB***	***BU***
				12_{b}	***UB***	$\boldsymbol{I}^*$	***D***
				12_{c}	***UB***	***I***	***D***
				13	***I***	***I***	***D***
				14	***I***	***I***	***D***

Suppose that $\Delta = 1, \delta > 1, \lambda < 1$, and $b > 0$. Then $g(x)$ has roots $x_1 = \alpha^{l/\delta}$ and $x_2 = \gamma^{1/\delta}$, where

$$\alpha = \frac{1 - \lambda + \sqrt{(\lambda - 1)^2 - 4\delta^2(\delta^2 - 1)ab}}{2a\delta(\delta - 1)},$$

$$\gamma = \frac{1 - \lambda + \sqrt{(\lambda - 1)^2 - 4\delta^2(\delta^2 - 1)ab}}{2a\delta(\delta - 1)}.$$

It can be verified that $\alpha > 0$ and $\gamma > 0$. Since the leading coefficient of $g(x)$ is positive, it follows that $\eta(x) \in \boldsymbol{UB}$. In reference to Theorem 3.7, it follows that (a) if $h'(x_1) < 0$, then $h(x) \in \boldsymbol{UB}$, (b) if $h'(x_1) = 0$, then $h(x) \in \boldsymbol{I}^*$, and (c) if $h'(x_1) > 0$, then $h(x) \in \boldsymbol{I}$. We first consider the case that $h(x) \in \boldsymbol{UB}$. Certainly, $\mu'(x_2) < 0$ and since $\epsilon = 0$, then $\mu'(0) < 0$. Thus, (a_1) if $\mu'(x_1) > 0$, then by Theorem 4.9 $\mu(x) \in \boldsymbol{BU}$. Alternatively, ($a_2$) if $\mu'(x_1) = 0$, then by Theorems 4.13 and 4.14, $\mu(x) \in \boldsymbol{D}^*$. Finally, ($a_3$) if $\mu'(x_1) < 0$, then by Theorem 4.1, $\mu(x) \in \boldsymbol{D}$ in $[x_1, \infty)$. However, in the interval $[0, x_1)$, we must apply Theorem 4.10. Thus, ($a_{3\mathrm{i}}$) if $\mu'(x_1') < 0$, then $\mu(x) \in \boldsymbol{D}$, ($a_{3\mathrm{ii}}$) if $\mu'(x_1') = 0$, then $\mu(x) \in \boldsymbol{D}^*$, and ($a_{3\mathrm{iii}}$) if $\mu'(x_1) > 0$, then $\mu(x) \in \boldsymbol{BU}$. We next consider the case that $h(x) \in \boldsymbol{I} \cup \boldsymbol{I}^*$. By Theorem 5.3, it follows that $\mu(x) \in \boldsymbol{D}$.

6. CONCLUSION

Apart from the probability density functions, the failure rate and the MRLF are two basic functions for analyzing survival or reliability data. The monotonicity of these functions describe the aging characteristic of the underlying distribution. It has been observed that, in some cases, there is more than one turning point of the failure rate. In this article, we have dealt with the monotonicity of three functions: (1) Glaser's eta function, (2) the failure rate, and (3) the MRLF for roller-coaster failure rates. We have presented some general results dealing with the change points of these functions. Additionally, we have established an ordering between the number of change points of these functions. These results are used to investigate, in detail, the monotonicity of the three functions in the case of the EGIG model. For certain values of the parameters, the EGIG model has two turning points of the failure rate. However, because of the complexity of the model, it was not feasible to obtain analytic expressions for these turning points.

The statistical inference of the EGIG model and the importance of the additional parameter δ will be presented elsewhere.

We hope that our investigation will be useful for reliability theoreticians interested in determining the monotonicity of the failure rate and the MRLF when these functions have more than one turning point. Additionally, the EGIG model will prove to be more useful for data analysts using the GIG or even the inverse Gaussian model.

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