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Partial inverse problems for Sturm–Liouville operators on trees

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In this paper, inverse spectral problems for Sturm–Liouville operators on a tree (a graph without cycles) are studied. We show that if the potential on an edge is known *a priori*, then $b-1$ spectral sets uniquely determine the potential functions on a tree with b external edges. Constructive solutions, based on the method of spectral mappings, are provided for the considered inverse problems.

Keywords: quantum graphs; Sturm–Liouville operators; inverse spectral problems; method of spectral mappings

2010 *Mathematics subject classification:* Primary 34A55; 47E05; 34B45; 34L40

1. Introduction

This paper concerns the theory of inverse spectral problems for Sturm–Liouville operators on geometrical graphs. Inverse problems consist in recovering differential operators from their spectral characteristics. Differential operators on graphs (quantum graphs) have applications in various fields of science and engineering (mechanics, chemistry, electronics, nanoscale technology and others) and have attracted considerable attention from mathematicians in recent years. There is an extensive literature devoted to differential operators on graphs and their applications; we mention only some research papers and surveys [1, 2, 5–7, 10, 16].

There are a variety of different inverse problems studied for quantum graphs, one of which is to recover the coefficients of the operator when some information is known *a priori*. This paper is focused on the reconstruction of the potential of the Sturm–Liouville operator on a tree (a graph without cycles) with a prescribed structure and standard matching conditions in the vertices. Yurko [14, 15] studied such inverse problems on trees using the Weyl vector, the system of spectra or the spectral data as given spectral characteristics. These problems are generalizations of the well-studied inverse problems for Sturm–Liouville operators on a finite interval (see the monographs [3, 8, 9, 11] and references therein). By the method of spectral mappings [3, 13], Yurko proved uniqueness theorems and developed a constructive algorithm for the solution of inverse problems on trees.

In this paper we formulate and solve partial inverse problems for the Sturm–Liouville operator on the tree. We suppose that the Sturm–Liouville potential is known on part of the graph and show that we need less data to recover the potential on the remaining part. To our knowledge, the only work in this direction is that of Yang [12] where the potential is known on a half of one edge and completely on the other edges of the star-shaped graph, and the author solves the Hochstadt–Lieberman-type problem [4] by using a part of the spectrum.

In this paper we assume that the potential is known on one edge of a tree, then reconstruct the potential on the remaining part by using the system of spectra or the Weyl functions. By developing the ideas of Yurko [14, 15], we show that one needs one less spectral set or one less Weyl function for the solution of the partial inverse problem. We consider separately the cases of boundary and internal edges, present constructive solutions and corresponding uniqueness theorems for both of them.

The results of this paper can be generalized to the case in which the potential is known on several edges. However, in this case the number of given spectra sufficient to recover the potential on the whole graph depends not only on the number of these edges, but also on their location (see the example in § 5). We note that the method of spectral mappings works also for graphs with cycles (see [16]), so one can generalize our results in this direction.

The paper is organized as follows. In § 2 we introduce the notation and briefly describe the solution of inverse problems on trees by Yurko [14, 15]. In § 3 we formulate our main results and outline their constructive solutions. Proofs of the technical lemmas from § 3 are contained in § 4. In § 5 we illustrate our method by an example.

2. Inverse problems on a tree

In this section, we introduce the notation and provide the main results of Yurko on the inverse problems on trees (see [14, 15] for more details).

Consider a compact tree G with vertices $V = \{v_i\}_{i=1}^{m+1}$ and edges $E = \{e_j\}_{j=1}^{m}$. For each vertex $v \in V$, we denote the set of edges associated with v by E_v and call the size of E_v the *degree* of v. Assume that the tree G does not contain vertices of degree 2. The vertices of degree 1 are called *boundary vertices*. Denote the set of boundary vertices of the graph G by ∂G. For the sake of convenience, let each boundary vertex v_i be an end of the edge e_i; such edges are called *boundary edges*. All other vertices and edges are called *internal*. Let the vertex $v_r \in \partial G$ be *the root* of the tree.

Each edge $e_j \in E$ is viewed as a segment $[0, T_j]$ and is parametrized by the parameter $x_j \in [0, T_j]$. The value $x_j = 0$ correspond to one of the end vertices of the edge e_j, and $x_j = T_j$ corresponds to another one. For a boundary edge, the end $x_j = 0$ corresponds to the boundary vertex v_j.

A function on the tree G can be represented as a vector function $y = [y_j]_{j=1}^m$, where $y_j = y_j(x_j)$, $x_j \in [0, T_j]$, $j = \overline{1, m}$. Let $e_j = [v_i, v_k]$, i.e. the vertex v_i corresponds to the end $x_j = 0$ and the vertex v_k corresponds to $x_j = T_j$. Introduce the following notation:

$$y_j(v_i) = y_j(0), \qquad y_j(v_k) = y_j(T_j),$$
$$y_j'(v_i) = y_j'(0), \qquad y_j'(v_k) = -y_j'(T_j).$$

If $v_i \in \partial G$, we omit the index of the edge and write $y(v_i)$ and $y'(v_i)$.

Consider the Sturm–Liouville equation on G,

$$-y_j'' + q_j(x_j)y_j = \lambda y_j, \quad x_j \in [0, T_j], \ j = \overline{1, m}, \tag{2.1}$$

where λ is the spectral parameter, $q_j \in L[0, T_j]$. We call the function $q = [q_j]_{j=1}^m$ *the potential* on the graph G. The functions y_j, y_j' are absolutely continuous on the segments $[0, T_j]$ and satisfy the *standard matching conditions* in the internal vertices $v \in V \setminus \partial G$:

$$\left.\begin{aligned} y_j(v) = y_k(v), \quad e_j, e_k \in E_v \quad & \text{(continuity condition)}, \\ \sum_{e_j \in E_v} y_j'(v) = 0 \quad & \text{(Kirchhoff's condition)}. \end{aligned}\right\} \tag{2.2}$$

Let L_0 and L_k, $v_k \in \partial G$, be the boundary-value problem for system (2.1) with the matching conditions (2.2) and the following conditions in the boundary vertices:

$$L_0: \quad y(v_i) = 0, \quad v_i \in \partial G, \tag{2.3}$$
$$L_k: \quad y'(v_k) = 0, \quad y(v_i) = 0, \quad v_i \in \partial G \setminus \{v_k\}. \tag{2.4}$$

It is well known that the problems L_k have discrete spectra, which are the countable sets of eigenvalues $\Lambda_k = \{\lambda_{ks}\}_{s=1}^\infty$, $k = 0$ or $v_k \in \partial G$.

Fix a boundary vertex $v_k \in \partial G$. Let $\Psi_k = [\psi_{kj}]_{j=1}^m$, $\psi_{kj} = \psi_{kj}(x_j, \lambda)$, be the solution of the system (2.1) satisfying the matching conditions (2.2) and the boundary conditions

$$\psi_{kk}(0, \lambda) = 1, \qquad \psi_{kj}(0, \lambda) = 0, \qquad v_j \in \partial G \setminus \{v_k\}.$$

Define $M_k(\lambda) = \psi_{kk}'(0, \lambda)$. The functions Ψ_k and M_k are called *the Weyl solution* and *the Weyl function* of (2.1) with respect to the boundary vertex v_k, respectively. The notion of the Weyl function for the tree generalizes the notion of the Weyl function (m-function) for the classical Sturm–Liouville operator on a finite interval [3,9]. If the tree G consists of only one edge, then $M_k(\lambda)$ coincides with the classical Weyl function.

Consider the following inverse problems.

Inverse problem 2.1. Given the spectra Λ_0, Λ_k, $v_k \in \partial G \setminus \{v_r\}$, construct the potential q on the tree G.

Inverse problem 2.2. Given the Weyl functions $M_k(\lambda)$, $v_k \in \partial G \setminus \{v_r\}$, construct the potential q on the tree G.

Note that if the number of boundary vertices is b, then one needs b spectra or $b-1$ Weyl functions to recover the potential. We do not require the data associated with the root v_r.

There is a close relation between inverse problems 2.1 and 2.2. The Weyl functions can be represented in the form

$$M_k(\lambda) = -\frac{\Delta_k(\lambda)}{\Delta_0(\lambda)}, \quad v_k \in \partial G, \tag{2.5}$$

where $\Delta_k(\lambda)$ are characteristic functions of the boundary-value problems L_k. If the eigenvalues Λ_k are known, one can construct characteristic functions as infinite products by Hadamard's theorem. Thus, with the system of spectra, one can obtain the Weyl functions and reduce inverse problem 2.1 to inverse problem 2.2.

Yurko has proved that inverse problems 2.1 and 2.2 are uniquely solvable, and provided a constructive algorithm for the solution by the method of spectral mappings [3]. In the remainder of this section, we shall briefly describe his algorithm. Let the Weyl functions $M_k(\lambda)$, $v_k \in \partial G \setminus \{v_r\}$ be given. Consider the following auxiliary problem.

Problem IP(k). Given $M_k(\lambda)$, construct the potential $q_k(x_k)$ on the edge e_k.

Note that this problem is not equivalent to the inverse problem on the finite interval, since the Weyl function $M_k(\lambda)$ contains information from the whole graph. However, it can be solved uniquely by the method of spectral mappings, and the potential on the boundary edges can be recovered. Then Yurko used the so-called μ-procedure to recover the potential on the internal edges. We reformulate these ideas in a form that is more convenient for us in the future.

Theorem 2.3. *Let v be an internal vertex connected to the set of boundary vertices $V' \subset \partial G \setminus \{v_r\}$ and only one other vertex. Suppose the potentials q_k on the edges e_k are known for all $v_k \in V'$, as well as a Weyl function $M_k(\lambda)$ for at least one vertex from the set V'. Denote by G' the graph obtained by removing the vertices $v_k \in V'$ together with the corresponding edges e_k from the graph G. Then the Weyl function for the graph G' with respect to the vertex v can be determined from the given information.*

Applying theorem 2.3, one can cut the boundary edges off until the potential is recovered on the whole graph.

3. Partial inverse problems

In this section the main results of the paper are formulated. We assume that the potential is known on one edge of the tree and formulate partial inverse problems. We consider separately the cases of boundary and internal edges. The first appears to be trivial. For the second we describe the procedure of the constructive solution. For the convenience of the reader, the proofs of the technical lemmas are provided in § 4.

Inverse problem 3.1. Let e_f be a boundary edge ($f \neq r$). Given the potential q_f on the edge e_f and the spectra Λ_0, Λ_k, $v_k \in \partial G \setminus \{v_f, v_r\}$, construct the potential q on the tree G.

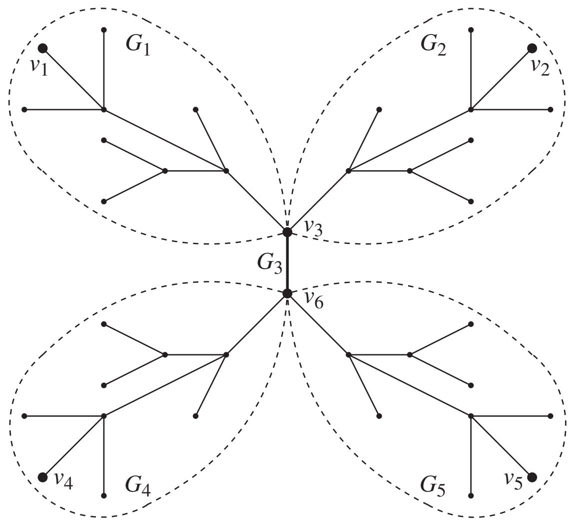

Figure 1. Splitting of the tree G.

The solution of inverse problem 3.1 is a slight modification of the method described in § 2. From Λ_0, Λ_k, $v_k \in \partial G \setminus \{v_f, v_r\}$, we easily construct the potentials q_k for $v_k \in \partial G \setminus \{v_f, v_r\}$. The potential q_f is known, so we can apply theorem 2.3 iteratively and recover the potential on G.

Now let e_f be an internal edge. If this edge is removed, the graph splits into two parts, call them P_1 and P_2. Let ∂P_1 and ∂P_2 be the sets of boundary vertices of P_1 and P_2, respectively. Fix two arbitrary vertices $v_{r1} \in \partial P_1$ and $v_{r2} \in \partial P_2$.

Inverse problem 3.2. Given the potential q_f on the internal edge e_f and the spectra Λ_0, Λ_k, $v_k \in \partial G \setminus \{v_{r1}, v_{r2}\}$, construct the potential q on the tree G.

Solution of inverse problem 3.2. For simplicity, we assume that the ends of the edge e_f have degree 3. The general case requires minor modifications. If one splits each of the ends of e_f into three vertices, the tree splits into five subtrees G_i, $i = \overline{1, 5}$, such that $v_{r1} \in G_2$, $v_{r2} \in G_5$, and G_3 contains the only edge e_f (see figure 1). Let v_1 and v_4 be arbitrary boundary vertices of the trees G_1 and G_4 (different from the ends of e_f), and $v_{r1} = v_2$, $v_{r2} = v_5$, $e_f = [v_3, v_6]$.

Step 1. Construct the characteristic functions $\Delta_k(\lambda)$ by using the given spectra Λ_k, $k = 0$ and $v_k \in \partial G \setminus \{v_2, v_5\}$. Find $M_k(\lambda)$ by using formula (2.5).

Step 2. Consider trees G_1 and G_4. Recover the potential q on the edges of G_1 and G_4 using the solutions of problem IP(k) for $v_k \in \partial G_1 \setminus \{v_3\}$ and $v_k \in \partial G_4 \setminus \{v_6\}$, and then applying theorem 2.3 iteratively.

Step 3. Introduce the characteristic functions of the boundary-value problems for the Sturm–Liouville equations (2.1) on the graphs G_1–G_5 with the standard matching conditions (2.2) in internal vertices and the following conditions in the boundary

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vertices:

$$\text{graph } G_1 \begin{cases} \Delta_1^{\mathrm{DD}}(\lambda): & y(v_k)=0, \quad v_k \in \partial G_1, \\ \Delta_1^{\mathrm{ND}}(\lambda): & y'(v_1)=0, \quad y(v_k)=0, \quad v_k \in \partial G_1 \setminus \{v_1\}, \\ \Delta_1^{\mathrm{DN}}(\lambda): & y'(v_3)=0, \quad y(v_k)=0, \quad v_k \in \partial G_1 \setminus \{v_3\}, \\ \Delta_1^{\mathrm{NN}}(\lambda): & y'(v_1)=0, \quad y'(v_3)=0, \quad y(v_k)=0, \quad v_k \in \partial G_1 \setminus \{v_1, v_3\}; \end{cases}$$

$$\text{graph } G_2 \begin{cases} \Delta_2^{\mathrm{D}}(\lambda): & y(v_k)=0, \quad v_k \in \partial G_2, \\ \Delta_2^{\mathrm{N}}(\lambda): & y'(v_3)=0, \quad y(v_k)=0, \quad v_k \in \partial G_2 \setminus \{v_3\}; \end{cases}$$

$$\text{graph } G_3 \begin{cases} \Delta_3^{\mathrm{DD}}(\lambda): & y(v_3)=0, \quad y(v_6)=0, \\ \Delta_3^{\mathrm{ND}}(\lambda): & y'(v_3)=0, \quad y(v_6)=0, \\ \Delta_3^{\mathrm{DN}}(\lambda): & y(v_3)=0, \quad y'(v_6)=0, \\ \Delta_3^{\mathrm{NN}}(\lambda): & y'(v_3)=0, \quad y'(v_6)=0; \end{cases}$$

$$\text{graph } G_4 \begin{cases} \Delta_4^{\mathrm{DD}}(\lambda): & y(v_k)=0, \quad v_k \in \partial G_4, \\ \Delta_4^{\mathrm{ND}}(\lambda): & y'(v_4)=0, \quad y(v_k)=0, \quad v_k \in \partial G_4 \setminus \{v_4\}, \\ \Delta_4^{\mathrm{DN}}(\lambda): & y'(v_6)=0, \quad y(v_k)=0, \quad v_k \in \partial G_4 \setminus \{v_6\}, \\ \Delta_4^{\mathrm{NN}}(\lambda): & y'(v_4)=0, \quad y'(v_3)=0, \quad y(v_k)=0, \quad v_k \in \partial G_4 \setminus \{v_4, v_6\}; \end{cases}$$

$$\text{graph } G_5 \begin{cases} \Delta_5^{\mathrm{D}}(\lambda): & y(v_k)=0, \quad v_k \in \partial G_5, \\ \Delta_5^{\mathrm{N}}(\lambda): & y'(v_6)=0, \quad y(v_k)=0, \quad v_k \in \partial G_5 \setminus \{v_6\}. \end{cases}$$

LEMMA 3.3. *The following relation holds:*

$$\Delta_0(\lambda) = \begin{vmatrix} \Delta_1^{\mathrm{DD}}(\lambda) & -\Delta_2^{\mathrm{D}}(\lambda) & 0 & 0 & 0 & 0 \\ 0 & \Delta_2^{\mathrm{D}}(\lambda) & -1 & 0 & 0 & 0 \\ \Delta_1^{\mathrm{DN}}(\lambda) & \Delta_2^{\mathrm{N}}(\lambda) & 0 & -1 & 0 & 0 \\ 0 & 0 & \Delta_3^{\mathrm{ND}}(\lambda) & \Delta_3^{\mathrm{DD}}(\lambda) & -\Delta_4^{\mathrm{DD}}(\lambda) & 0 \\ 0 & 0 & 0 & 0 & \Delta_4^{\mathrm{DD}}(\lambda) & -\Delta_5^{\mathrm{D}}(\lambda) \\ 0 & 0 & \Delta_3^{\mathrm{NN}}(\lambda) & \Delta_3^{\mathrm{DN}}(\lambda) & \Delta_4^{\mathrm{DN}}(\lambda) & \Delta_5^{\mathrm{N}}(\lambda) \end{vmatrix}. \tag{3.1}$$

If we change $\Delta_1^{\mathrm{DD}}(\lambda)$ to $\Delta_1^{\mathrm{ND}}(\lambda)$ and $\Delta_1^{\mathrm{DN}}(\lambda)$ to $\Delta_1^{\mathrm{NN}}(\lambda)$, then we obtain the determinant equal to $\Delta_1(\lambda)$. Similarly, if we change $\Delta_4^{\mathrm{DD}}(\lambda)$ to $\Delta_4^{\mathrm{ND}}(\lambda)$ and $\Delta_4^{\mathrm{DN}}(\lambda)$ to $\Delta_4^{\mathrm{NN}}(\lambda)$, then we obtain $\Delta_4(\lambda)$.

STEP 4. Note that the functions $\Delta_0(\lambda)$, $\Delta_1(\lambda)$, $\Delta_4(\lambda)$ are known from step 1. Since we know the potential on the graphs G_1, G_4 (from step 2) and G_3 (given *a priori*), we can easily construct the characteristic functions for these graphs. Consider the relation (3.1) and similar relations for $\Delta_1(\lambda)$ and $\Delta_4(\lambda)$ as a system of equations with respect to $\Delta_2^{\mathrm{D}}(\lambda)$, $\Delta_2^{\mathrm{N}}(\lambda)$, $\Delta_5^{\mathrm{D}}(\lambda)$ and $\Delta_5^{\mathrm{N}}(\lambda)$ in the form

$$\left.\begin{aligned} a_{11}\Delta_2^{\mathrm{D}}\Delta_5^{\mathrm{D}} + a_{12}\Delta_2^{\mathrm{N}}\Delta_5^{\mathrm{D}} + a_{13}\Delta_2^{\mathrm{D}}\Delta_5^{\mathrm{N}} + a_{14}\Delta_2^{\mathrm{N}}\Delta_5^{\mathrm{N}} = \Delta_0, \\ a_{21}\Delta_2^{\mathrm{D}}\Delta_5^{\mathrm{D}} + a_{22}\Delta_2^{\mathrm{N}}\Delta_5^{\mathrm{D}} + a_{23}\Delta_2^{\mathrm{D}}\Delta_5^{\mathrm{N}} + a_{24}\Delta_2^{\mathrm{N}}\Delta_5^{\mathrm{N}} = \Delta_1, \\ a_{31}\Delta_2^{\mathrm{D}}\Delta_5^{\mathrm{D}} + a_{32}\Delta_2^{\mathrm{N}}\Delta_5^{\mathrm{D}} + a_{33}\Delta_2^{\mathrm{D}}\Delta_5^{\mathrm{N}} + a_{34}\Delta_2^{\mathrm{N}}\Delta_5^{\mathrm{N}} = \Delta_4, \end{aligned}\right\} \tag{3.2}$$

where $a_{ij} = a_{ij}(\lambda)$, $i = \overline{1,3}$, $j = \overline{1,4}$, are known coefficients.

STEP 5. Multiply the first equation of (3.2) by Δ_1 and subtract the second equation, multiplied by Δ_0. Apply similar transform to the first and the third equations. Then we obtain the system

$$b_{11}\Delta_2^{\mathrm{D}}\Delta_5^{\mathrm{D}} + b_{12}\Delta_2^{\mathrm{N}}\Delta_5^{\mathrm{D}} + b_{13}\Delta_2^{\mathrm{D}}\Delta_5^{\mathrm{N}} + b_{14}\Delta_2^{\mathrm{N}}\Delta_5^{\mathrm{N}} = 0,$$
$$b_{21}\Delta_2^{\mathrm{D}}\Delta_5^{\mathrm{D}} + b_{22}\Delta_2^{\mathrm{N}}\Delta_5^{\mathrm{D}} + b_{23}\Delta_2^{\mathrm{D}}\Delta_5^{\mathrm{N}} + b_{24}\Delta_2^{\mathrm{N}}\Delta_5^{\mathrm{N}} = 0,$$

where

$$b_{1i} = a_{1i}\Delta_1 - a_{2i}\Delta_0, \quad b_{2i} = a_{1i}\Delta_4 - a_{3i}\Delta_0, \quad i = \overline{1,4}. \tag{3.3}$$

Divide both equations by $\Delta_2^{\mathrm{D}}\Delta_5^{\mathrm{D}}$ to obtain

$$b_{i1} + b_{i2}\tilde{M}_2 + b_{i3}\tilde{M}_5 + b_{i4}\tilde{M}_2\tilde{M}_5 = 0, \quad i = 1, 2, \tag{3.4}$$

where

$$\tilde{M}_2(\lambda) = \frac{\Delta_2^{\mathrm{N}}(\lambda)}{\Delta_2^{\mathrm{D}}(\lambda)}, \quad \tilde{M}_5(\lambda) = \frac{\Delta_5^{\mathrm{N}}(\lambda)}{\Delta_5^{\mathrm{D}}(\lambda)}$$

are (up to the sign) the Weyl functions for the subtrees G_2 and G_5 associated with the vertices v_3 and v_6, respectively.

STEP 6. From system (3.4) we easily derive

$$\tilde{M}_5 = -\frac{b_{i1} + b_{i2}\tilde{M}_2}{b_{i3} + b_{i4}\tilde{M}_2}, \quad i = 1, 2.$$

Hence,

$$(b_{11} + b_{12}\tilde{M}_2)(b_{23} + b_{24}\tilde{M}_2) = (b_{21} + b_{22}\tilde{M}_2)(b_{13} + b_{14}\tilde{M}_2).$$

Finally, we obtain the quadratic equation with respect to $\tilde{M}_2(\lambda)$,

$$A(\lambda)\tilde{M}_2^2(\lambda) + B(\lambda)\tilde{M}_2(\lambda) + C(\lambda) = 0, \tag{3.5}$$

with analytic coefficients $A(\lambda)$, $B(\lambda)$, $C(\lambda)$:

$$\left.\begin{aligned} A &= b_{12}b_{24} - b_{22}b_{14}, \\ B &= b_{11}b_{24} + b_{12}b_{23} - b_{21}b_{14} - b_{22}b_{13}, \\ C &= b_{11}b_{23} - b_{21}b_{13}. \end{aligned}\right\} \tag{3.6}$$

STEP 7. Consider the Sturm–Liouville equation (2.1) on the tree G with the potential $q = 0$. Implement steps 1–6 for this case and obtain the quadratic equation

$$A_0(\lambda)\tilde{M}_{20}^2(\lambda) + B_0(\lambda)\tilde{M}_{20}(\lambda) + C_0(\lambda) = 0, \tag{3.7}$$

which is analogous to (3.5). Define $\rho = \sqrt{\lambda}$, $\operatorname{Re}\rho \geqslant 0$, $S_\delta := \{\rho\colon \operatorname{Re}\rho \geqslant 0,\ |\operatorname{Im}\rho| \leqslant \delta\}$, $\delta > 0$, $[1] = 1 + O(\rho^{-1})$. Let $f(\rho^2)$ be an analytic function and let $\varepsilon > 0$. Define $Z_\varepsilon(f) := \{\rho\colon |f(\rho^2)| \geqslant \varepsilon\}$.

LEMMA 3.4. *The following asymptotic relations hold:*

$$A(\lambda) = A_0(\lambda)[1], \qquad B(\lambda) = B_0(\lambda)[1], \qquad C(\lambda) = C_0(\lambda)[1],$$
$$\rho \in S_\delta \cap Z_\varepsilon(A_0B_0C_0), \quad |\rho| \to \infty.$$

Consequently, $D(\lambda) = D_0(\lambda)[1]$ for $\rho \in S_\delta \cap Z_\varepsilon(D_0)$, $|\rho| \to \infty$, where $D(\lambda)$ and $D_0(\lambda)$ are discriminants of (3.5) and (3.7), respectively.

Lemma 3.5. *$A_0(\lambda) \not\equiv 0$, $D_0(\lambda) \not\equiv 0$.*

It follows from lemmas 3.4 and 3.5 that the quadratic equation (3.5) does not degenerate for $\rho \in S_\delta \cap Z_\varepsilon(A_0 D_0)$, and two roots of (3.5) are asymptotically different as $|\rho| \to \infty$. One can easily find an asymptotic representation of $\tilde{M}_2(\lambda)$ for any particular graph and choose the correct root of (3.5) on some region of S_δ for sufficiently large $|\rho|$. Then the function $\tilde{M}_2(\lambda)$ can be constructed for all $\lambda \in \mathbb{C}$ except its singularities by analytic continuation. Similarly one can find $\tilde{M}_5(\lambda)$.

Step 8. Consider the tree G_2 with the root v_2. Solve problem IP(k) by using $M_k(\lambda)$, $v_k \in \partial G_2 \setminus \{v_2, v_6\}$, and by using $\tilde{M}_2(\lambda)$ for v_3, obtain the potential on the boundary edges except e_2. Then apply the cutting of boundary edges by theorem 2.3 and recover the potential q on G_2. The subtree G_5 can be treated similarly.

Thus, we have recovered the potential q on the whole graph G. In parallel, we have proved the following uniqueness theorem.

Theorem 3.6. *Let the potential q_f on the edge e_f ($f \neq r$) be known.*

(i) *If e_f is a boundary edge, the spectra Λ_0, Λ_k, $v_k \in \partial G \setminus \{v_f, v_r\}$, uniquely determine the potential q on the whole graph G.*

(ii) *If e_f is an internal edge, the spectra Λ_0, Λ_k, $v_k \in \partial G \setminus \{v_{r1}, v_{r2}\}$, uniquely determine the potential q on the whole graph G.*

Using the described methods with some technical modifications, one can solve partial inverse problems using Weyl functions.

Inverse problem 3.7. Let e_f be a boundary edge ($f \neq r$). Given the potential q_f on the edge e_f and the Weyl functions $M_k(\lambda)$, $v_k \in \partial G \setminus \{v_f, v_r\}$, construct the potential q on the tree G.

Inverse problem 3.8. Given the potential q_f on the internal edge e_f and the Weyl functions $M_k(\lambda)$, $v_k \in \partial G \setminus \{v_{r1}, v_{r2}\}$, construct the potential q on the tree G.

Thus, if the number of boundary edges is b and the potential is known on one edge (boundary or internal), $b - 2$ Weyl functions are required to construct q on the whole graph.

4. Proofs

4.1. Proof of lemma 3.3

Consider the Sturm–Liouville equation (2.1) on the tree G. Let $C_j(x_j, \lambda)$ and $S_j(x_j, \lambda)$ be solutions of (2.1) on the edge e_j under initial conditions

$$C_j(0, \lambda) = S'_j(0, \lambda) = 1, \qquad C'_j(0, \lambda) = S_j(0, \lambda) = 0.$$

Any solution $y = [y_j]_{j=1}^m$ of the equation (2.1) on G admits the following representation:

$$y_j(x_j, \lambda) = M_j^0(\lambda) C_j(x_j, \lambda) + M_j^1(\lambda) S_j(x_j, \lambda), \quad j = \overline{1, m}, \ x_j \in [0, T_j]. \tag{4.1}$$

Let BC be some fixed boundary conditions in the vertices $v \in \partial G$ of the form $y(v) = 0$ or $y'(v) = 0$ (for instance, we consider conditions (2.3) for the problem L and (2.4) for the problem L_k). Denote by L the boundary-value problem for the Sturm–Liouville equation (2.1) with the standard matching conditions (2.2) and the boundary conditions BC. If y is a solution of a boundary-value problem L, substitute (4.1) into (2.2) and BC to obtain a linear algebraic system with respect to $M_j^0(\lambda)$, $M_j^1(\lambda)$. It is easy to check that the determinant of this system is a characteristic function $\Delta(\lambda)$ of the boundary-value problem L, i.e. zeros of $\Delta(\lambda)$ coincide with the eigenvalues of L.

Example 4.1. Consider the problem L_0 for the star-type graph for $m = 3$. Then boundary conditions (2.3) yield $M_1^0(\lambda) = M_2^0(\lambda) = M_3^0(\lambda) = 0$. Consequently, from (2.2) we obtain the system with respect to $M_j^0(\lambda)$, $j = 1, 2, 3$, with the determinant

$$\Delta_0(\lambda) = \begin{vmatrix} S_1(T_1, \lambda) & -S_2(T_2, \lambda) & 0 \\ 0 & S_2(T_2, \lambda) & -S_3(T_3, \lambda) \\ S_1'(T_1, \lambda) & S_2'(T_2, \lambda) & S_3'(T_3, \lambda) \end{vmatrix}.$$

In the general case, the following assertion is valid.

Lemma 4.2. *Let $w \in V$ and let the degree of w be equal to n. Splitting the vertex w, we split G into n subtrees G_i, $i = \overline{1, n}$. For each $i = \overline{1, n}$ let $\Delta_i^{\mathrm{D}}(\lambda)$ and $\Delta_i^{\mathrm{N}}(\lambda)$ be characteristic functions for boundary-value problems for equation (2.1) on tree G_i with matching conditions (2.2), boundary conditions BC for $v \in \partial G \cap \partial G_i$, and the Dirichlet condition $y(u) = 0$ for $\Delta_i^{\mathrm{D}}(\lambda)$ and the Neumann condition $y'(u) = 0$ for $\Delta_i^{\mathrm{N}}(\lambda)$. Then the characteristic function $\Delta(\lambda)$ for G with the conditions (2.2) and BC admits the following representation:*

$$\Delta(\lambda) = \begin{vmatrix} \Delta_1^{\mathrm{D}}(\lambda) & -\Delta_2^{\mathrm{D}}(\lambda) & 0 & \cdots & 0 \\ 0 & \Delta_2^{\mathrm{D}}(\lambda) & -\Delta_3^{\mathrm{D}}(\lambda) & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & -\Delta_n^{\mathrm{D}}(\lambda) \\ \Delta_1^{\mathrm{N}}(\lambda) & \Delta_2^{\mathrm{N}}(\lambda) & \Delta_3^{\mathrm{N}}(\lambda) & \cdots & \Delta_n^{\mathrm{N}}(\lambda) \end{vmatrix}. \tag{4.2}$$

Indeed, if we write the determinant for $\Delta(\lambda)$ and analyse the participation of the edges of G_i in this determinant, we can easily see that $\Delta(\lambda) = \Delta_i^{\mathrm{D}}(\lambda) D_i(\lambda) + \Delta_i^{\mathrm{N}}(\lambda) E_i(\lambda)$, where the functions $D_i(\lambda)$ and $E_i(\lambda)$ do not depend on the subtree G_i. Thus, we can consider the simplest case of the star-type graph, when each G_i contains only one edge, and then change the multipliers, corresponding to subgraphs G_i, to $\Delta_i^{\mathrm{D}}(\lambda)$ and $\Delta_i^{\mathrm{N}}(\lambda)$. Thus, we directly obtain (4.2) from the formula for the star-type graph.

Lemma 3.3 follows from lemma 4.2 for the graph in figure 1. Alternatively, one can derive (3.1) from (5.2), replacing characteristic functions for one-edge subtrees with general characteristic functions.

4.2. Proof of lemma 3.4

Together with L, consider the boundary-value problem L^0 for equation (2.1) with $q \equiv 0$, the matching conditions (2.2) and the boundary conditions BC. If some symbol γ denotes the object related to L, we denote by the symbol γ^0 the similar object related to L^0. In particular, $\Delta^0(\lambda)$ is the characteristic function of L^0. Let the symbol $P(\rho)$ stand for different polynomials of $\sin \rho T_j$ and $\cos \rho T_j$, $j = \overline{1, m}$.

Lemma 4.3. *The characteristic function $\Delta(\lambda)$ has the asymptotic behaviour*

$$\Delta(\lambda) = \Delta^0(\lambda) + O(\rho^{-d}) = \frac{P(\rho)}{\rho^{d-1}} + O(\rho^{-d}), \quad \rho \in S_\delta, \ |\rho| \to \infty,$$

where $P(\rho) \not\equiv 0$ and $d = m - i - n$, where m is the number of edges, i is the number of internal vertices and n is the number of boundary vertices with the Neumann boundary condition $y'(v) = 0$.

Proof. The claim of the lemma immediately follows from the standard asymptotic formulae

$$C_j(x_j, \lambda) = \cos \rho x_j + O(\rho^{-1}), \qquad C_j'(x_j, \lambda) = -\rho \sin \rho x_j + O(1),$$
$$S_j(x_j, \lambda) = \frac{\sin \rho x_j}{\rho} + O(\rho^{-2}), \qquad S_j'(x, \lambda) = \cos \rho x_j,$$
$$\rho \in S_\delta, \qquad |\rho| \to \infty,$$

and the construction of $\Delta(\lambda)$. The relation $P(\rho) \not\equiv 0$ follows from the regularity of the standard matching conditions. □

Applying lemma 4.3 to the characteristic functions, defined on step 3 of the algorithm, we derive asymptotic representations for the coefficients $c = a_{ij}, b_{ij}, A, B, C$ in the form

$$c(\lambda) = c^0(\lambda) + O(\rho^{-d}) = \frac{P(\rho)}{\rho^{d-1}} + O(\rho^{-d}), \quad \rho \in S_\delta, \ |\rho| \to \infty,$$

where d stands for different integers. This relation yields lemma 3.4.

4.3. Proof of lemma 3.5

In this section we consider only the problem L^0 with $q \equiv 0$, so we omit the index 0 for brevity. For simplicity, let $T_f = 1$. Taking into account that

$$\Delta_3^{\mathrm{DD}} = \frac{\sin \rho}{\rho}, \qquad \Delta_3^{\mathrm{ND}} = \Delta_3^{\mathrm{DN}} = \cos \rho, \qquad \Delta_3^{\mathrm{NN}} = -\rho \sin \rho$$

and doing some algebra with the expressions (3.1), (3.3), (3.6), we derive

$$A(\lambda) = -F_1(\lambda) F_4(\lambda) \Delta_0(\lambda) \frac{\sin^2 \rho}{\rho^2} \Delta_4^{\mathrm{DD}}(\lambda) \Delta_5^{\mathrm{D}}(\lambda) \chi(\lambda), \tag{4.3}$$

$$B(\lambda) = -F_1(\lambda) F_4(\lambda) \frac{\sin \rho}{\rho} \Delta_0(\lambda) \times \left\{ \Delta_5^{\mathrm{D}}(\lambda) \Pi(\lambda) + \Delta_5^{\mathrm{D}}(\lambda) \frac{\sin \rho}{\rho} \xi(\lambda) - \Delta_4^{\mathrm{DD}}(\lambda) \Delta_5^{\mathrm{N}}(\lambda) \frac{\sin \rho}{\rho} \chi(\lambda) \right\}, \tag{4.4}$$

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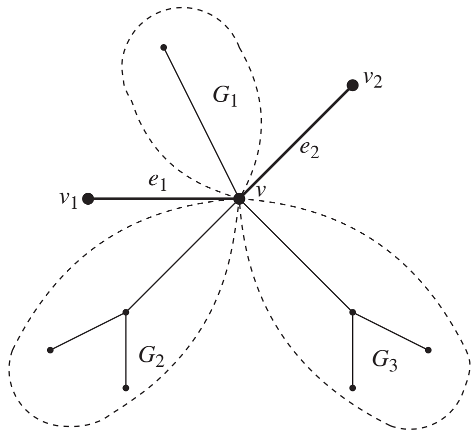

Figure 2. Illustration of the proof of lemma 4.4, step 2.

$$C(\lambda) = F_1(\lambda)F_4(\lambda)\Delta_0(\lambda)\frac{\sin\rho}{\rho}\Delta_5^{\mathrm{N}}(\lambda)\biggl\{\Pi(\lambda) + \frac{\sin\rho}{\rho}\xi(\lambda)\biggr\}, \tag{4.5}$$

where

$$F_i(\lambda) = \Delta_i^{\mathrm{DD}}(\lambda)\Delta_i^{\mathrm{NN}}(\lambda) - \Delta_i^{\mathrm{DN}}(\lambda)\Delta_i^{\mathrm{ND}}(\lambda), \quad i = 1, 4,$$
$$\Pi(\lambda) = 2\Delta_1^{\mathrm{DD}}(\lambda)\Delta_2^{\mathrm{DD}}(\lambda)\Delta_4^{\mathrm{DD}}(\lambda),$$

and $\chi(\lambda)$ and $\xi(\lambda)$ are characteristic functions of the graphs $G_1 \cup G_2 \cup G_3$ and $G_1 \cup G_2 \cup G_3 \cup G_4$, respectively. Here we mean that the copies of the vertex v_3 (and v_6 in the second graph) are joined into one vertex with the standard matching conditions (2.2).

LEMMA 4.4. *Let v_1 and v_2 be two fixed vertices from ∂G. Denote by $\Delta^{\mathrm{DD}}(\lambda)$, $\Delta^{\mathrm{DN}}(\lambda)$, $\Delta^{\mathrm{ND}}(\lambda)$ and $\Delta^{\mathrm{NN}}(\lambda)$ the characteristic functions for equation (2.1) on the tree G with the matching conditions (2.2), the boundary conditions*

$$\Delta^{\mathrm{DD}}(\lambda): \quad y(v_1) = y(v_2) = 0,$$
$$\Delta^{\mathrm{DN}}(\lambda): \quad y(v_1) = y'(v_2) = 0,$$
$$\Delta^{\mathrm{ND}}(\lambda): \quad y'(v_1) = y(v_2) = 0,$$
$$\Delta^{\mathrm{NN}}(\lambda): \quad y'(v_1) = y'(v_2) = 0,$$

and with the conditions BC in the vertices $v \in \partial G \setminus \{v_1, v_2\}$. Then

$$\Delta^{\mathrm{DD}}(\lambda)\Delta^{\mathrm{NN}}(\lambda) - \Delta^{\mathrm{DN}}(\lambda)\Delta^{\mathrm{ND}}(\lambda) \not\equiv 0. \tag{4.6}$$

Proof. We shall divide the proof into the following steps.

(1) Let the tree G consist of the only edge $[v_1, v_2]$. Then one can check the relation (4.6) by direct calculation.

(2) Let the vertices v_1 and v_2 be connected by edges with the same vertex v, and let there also be subtrees G_i, $i = \overline{1, n}$, from the vertex v (see figure 2). Denote by $\Delta_i^{\mathrm{D}}(\lambda)$ and $\Delta_i^{\mathrm{N}}(\lambda)$ the characteristic functions for G_i with the matching conditions

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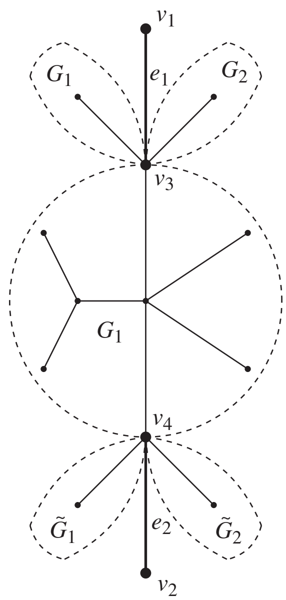

Figure 3. Illustration of the proof of lemma 4.4, step 3.

(2.2), the boundary conditions BC and $y(v) = 0$ for $\Delta_i^{\mathrm{D}}(\lambda)$ and $y'(v) = 0$ for $\Delta_i^{\mathrm{N}}(\lambda)$. According to lemma 4.2, the relation

$$\Delta^{\mathrm{DD}}(\lambda) = \frac{\sin\rho T_1 \sin\rho T_2}{\rho^2}\Delta^K(\lambda) + \frac{1}{\rho}(\sin\rho T_1 \cos\rho T_2 + \cos\rho T_1 \sin\rho T_2)\Delta^{\Pi}(\lambda)$$

holds, where

$$\Delta^{\Pi}(\lambda) = \prod_{i=1}^{n} \Delta_i^{\mathrm{D}}(\lambda),$$

$$\Delta^{K}(\lambda) = \Delta^{\Pi}(\lambda)\sum_{i=1}^{n}\frac{\Delta_i^{\mathrm{N}}(\lambda)}{\Delta_i^{\mathrm{D}}(\lambda)}.$$

Using similar representations for $\Delta^{\mathrm{NN}}(\lambda)$, $\Delta^{\mathrm{DN}}(\lambda)$ and $\Delta^{\mathrm{ND}}(\lambda)$, we derive

$$\Delta^{\mathrm{DD}}(\lambda)\Delta^{\mathrm{NN}}(\lambda) - \Delta^{\mathrm{DN}}(\lambda)\Delta^{\mathrm{ND}}(\lambda) = -(\Delta^{\Pi}(\lambda))^2 \not\equiv 0.$$

(3) Now let the vertices v_1 and v_2 be connected by the edges with v_3 and v_4, respectively. Let the tree G split by the vertices v_3 and v_4 into the subtrees G_i, $i = \overline{1, n_1}$, connected with v_3, the subtrees $\tilde{G}_j$, $j = \overline{1, n_2}$, connected with v_4, the subtree G_0, including both the vertices v_3 and v_4, and the edges e_1, e_2 (see figure 3). Denote by $\Delta_i^{\mathrm{D}}(\lambda)$, $\Delta_i^{\mathrm{N}}(\lambda)$, $i = \overline{1, n_1}$, and by $\tilde{\Delta}_j^{\mathrm{D}}(\lambda)$, $\tilde{\Delta}_j^{\mathrm{N}}(\lambda)$, $j = \overline{1, n_2}$, the characteristic functions for the subtrees G_i with the Dirichlet or Neumann boundary condition in v_3 and for the subtrees $\tilde{G}_i$ with the Dirichlet or Neumann boundary condition in v_4, respectively. Let $\Delta_0^{\mathrm{DD}}(\lambda)$, $\Delta_0^{\mathrm{DN}}(\lambda)$, $\Delta_0^{\mathrm{ND}}(\lambda)$ and $\Delta_0^{\mathrm{NN}}(\lambda)$ be characteristic

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functions for the subtree G_0 with the boundary conditions

$$\Delta_0^{\mathrm{DD}}(\lambda): \quad y(v_3) = y(v_4) = 0, \qquad \Delta_0^{\mathrm{DN}}(\lambda): \quad y(v_3) = y'(v_4) = 0,$$
$$\Delta_0^{\mathrm{ND}}(\lambda): \quad y'(v_3) = y(v_4) = 0, \qquad \Delta_0^{\mathrm{NN}}(\lambda): \quad y'(v_3) = y'(v_4) = 0,$$

and BC in other boundary vertices. Define the functions

$$\Delta_1^{\Pi}(\lambda) = \prod_{i=1}^{n_1} \Delta_i^{\mathrm{D}}(\lambda), \quad \Delta_2^{\Pi}(\lambda) = \prod_{j=1}^{n_2} \tilde{\Delta}_j^{\mathrm{D}}(\lambda),$$
$$\Delta_1^{K}(\lambda) = \Delta_1^{\Pi}(\lambda) \sum_{i=1}^{n} \frac{\Delta_i^{\mathrm{N}}(\lambda)}{\Delta_i^{\mathrm{D}}(\lambda)}, \quad \Delta_2^{K}(\lambda) = \Delta_2^{\Pi}(\lambda) \sum_{j=1}^{n} \frac{\tilde{\Delta}_j^{\mathrm{N}}(\lambda)}{\tilde{\Delta}_j^{\mathrm{D}}(\lambda)}.$$

$$\left.\begin{aligned} \Delta^{KK}(\lambda) =& \Delta_0^{\mathrm{DD}}(\lambda)\Delta_1^{K}(\lambda)\Delta_2^{K}(\lambda) + \Delta_0^{\mathrm{ND}}(\lambda)\Delta_1^{\Pi}(\lambda)\Delta_2^{K}(\lambda) \\ &+ \Delta_0^{\mathrm{DN}}(\lambda)\Delta_1^{K}(\lambda)\Delta_2^{\Pi}(\lambda) + \Delta_0^{\mathrm{NN}}(\lambda)\Delta_1^{\Pi}(\lambda)\Delta_2^{\Pi}(\lambda), \\ \Delta^{\Pi K}(\lambda) =& \Delta_0^{\mathrm{DD}}(\lambda)\Delta_1^{\Pi}(\lambda)\Delta_2^{K}(\lambda) + \Delta_0^{\mathrm{DN}}(\lambda)\Delta_1^{\Pi}(\lambda)\Delta_2^{\Pi}(\lambda), \\ \Delta^{K\Pi}(\lambda) =& \Delta_0^{\mathrm{DD}}(\lambda)\Delta_1^{K}(\lambda)\Delta_2^{\Pi}(\lambda) + \Delta_0^{\mathrm{ND}}(\lambda)\Delta_1^{\Pi}(\lambda)\Delta_2^{\Pi}(\lambda), \\ \Delta^{\Pi\Pi}(\lambda) =& \Delta_0^{\mathrm{DD}}(\lambda)\Delta_1^{\Pi}(\lambda)\Delta_2^{\Pi}(\lambda). \end{aligned}\right\} \tag{4.7}$$

In view of lemma 4.2, the following relation holds:

$$\Delta^{\mathrm{DD}}(\lambda) = \frac{\sin\rho T_1 \sin\rho T_2}{\rho^2}\Delta^{KK}(\lambda) + \frac{\cos\rho T_1 \sin\rho T_2}{\rho}\Delta^{\Pi K}(\lambda) + \frac{\sin\rho T_1 \cos\rho T_2}{\rho}\Delta^{K\Pi}(\lambda) + \cos\rho T_1 \cos\rho T_2 \Delta^{\Pi\Pi}(\lambda).$$

Together with the similar relations for $\Delta^{\mathrm{DN}}(\lambda)$, $\Delta^{\mathrm{ND}}(\lambda)$ and $\Delta^{\mathrm{NN}}(\lambda)$, it yields

$$\Delta^{\mathrm{DD}}(\lambda)\Delta^{\mathrm{NN}}(\lambda) - \Delta^{\mathrm{DN}}(\lambda)\Delta^{\mathrm{ND}}(\lambda) = \Delta^{\Pi\Pi}(\lambda)\Delta^{KK}(\lambda) - \Delta^{\Pi K}(\lambda)\Delta^{K\Pi}(\lambda).$$

Taking (4.7) into account, we obtain

$$\Delta^{\Pi\Pi}(\lambda)\Delta^{KK}(\lambda) - \Delta^{\Pi K}(\lambda)\Delta^{K\Pi}(\lambda) = (\Delta_0^{\mathrm{DD}}(\lambda)\Delta_0^{\mathrm{NN}}(\lambda) - \Delta_0^{\mathrm{DN}}(\lambda)\Delta_0^{\mathrm{ND}}(\lambda))(\Delta_1^{\Pi}(\lambda)\Delta_2^{\Pi}(\lambda))^2.$$

By virtue of lemma 4.3, $\Delta_i^{\Pi}(\lambda) \not\equiv 0$, $i = 1, 2$. Therefore, relation (4.6) holds for the tree G if and only if it holds for the subtree G_0. By induction, the claim of the lemma is valid for any tree G. □

By virtue of lemmas 4.3 and 4.4, and (4.3), $A(\lambda) \not\equiv 0$. It follows from (4.3)–(4.5) that

$$\begin{aligned} D(\lambda) &= B^2(\lambda) - 4A(\lambda)C(\lambda) \\ &= F_1^2(\lambda)F_4^2(\lambda)\frac{\sin^2\rho}{\rho^2}\Delta_0(\lambda) \\ &\quad \times \left\{\Delta_5^{\mathrm{D}}(\lambda)\Pi(\lambda) + \Delta_5^{\mathrm{D}}(\lambda)\frac{\sin\rho}{\rho}\xi(\lambda) + \Delta_4^{\mathrm{DD}}(\lambda)\Delta_5^{\mathrm{N}}(\lambda)\frac{\sin\rho}{\rho}\chi(\lambda)\right\}^2. \end{aligned}$$

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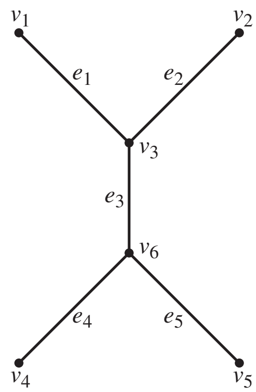

Figure 4. Illustration of the example.

Note that the expression in the bracket above is equal to

$$\Delta_5^{\mathrm{D}}(\lambda)\Pi(\lambda) + \frac{\sin\rho}{\rho}\Delta_0(\lambda).$$

Similarly to lemma 4.3, the asymptotic formulae

$$\Delta_5^{\mathrm{D}}(\lambda)\Pi(\lambda) = C_1 r^{-p}\exp(r(T-1))[1], \qquad \frac{\sin\rho}{\rho}\Delta_0(\lambda) = C_2 r^{-q}\exp(r(T+1))[1]$$

can be obtained, where $\rho = \mathrm{i}r$, $r \to +\infty$, $T = \sum_{j=1}^{m} T_j$, C_1, C_2, p and q are some constants. Clearly, the second term grows faster than the first one. Therefore, $\Delta_0(\lambda) \not\equiv 0$ implies $D(\lambda) \not\equiv 0$. The proof of lemma 3.5 is finished.

Using lemma 4.3, one can also check that $B(\lambda)$ and $\sqrt{D(\lambda)}$ have the same power of ρ in the denominator, so the roots of (3.5) have different asymptotic behaviour.

5. Example

In this section we provide the solution of inverse problem 3.2 for the example of the graph in the figure 4. For simplicity, let $T_j = 1$, $j = \overline{1,5}$. Let $x_3 = 0$ correspond to the vertex v_3 and let $x_3 = 1$ correspond to v_6. For the boundary edges, $x_j = 0$ correspond to the boundary vertices. The matching conditions (2.2) take the form

$$\left.\begin{aligned} v_3: \quad & y_1(1) = y_2(1) = y_3(0), \quad y_1'(1) + y_2'(1) - y_3'(0) = 0, \\ v_6: \quad & y_3(1) = y_4(1) = y_5(1), \quad y_3'(1) + y_4'(1) + y_5'(1) = 0. \end{aligned}\right\} \tag{5.1}$$

For this example, each subtree G_i consists of only one edge e_i, $i = \overline{1,5}$. Let the spectra Λ_0, Λ_1, Λ_4 and the potential q_3 be given. Using the given spectra, one can easily find the characteristic functions $\Delta_0(\lambda)$, $\Delta_1(\lambda)$, $\Delta_4(\lambda)$ and the Weyl functions $M_1(\lambda)$, $M_4(\lambda)$. Solving problems IP(1) and IP(4) recover q_1 and q_3.

Consider the boundary-value problem L. Represent the solution y in the form (4.1) and substitute it into (2.2) and (2.3). From (2.3), one gets $M_1^0(\lambda) = M_2^0(\lambda) =$

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$M_4^0(\lambda) = M_5^0(\lambda) = 0$. Then matching conditions (5.1) yield the system

$$\begin{pmatrix} S_1 & -S_2 & 0 & 0 & 0 & 0 \\ 0 & S_2 & -1 & 0 & 0 & 0 \\ S_1' & S_2' & 0 & -1 & 0 & 0 \\ 0 & 0 & C_3 & S_3 & -S_4 & 0 \\ 0 & 0 & 0 & 0 & S_4 & -S_5 \\ 0 & 0 & C_3' & S_3' & S_4' & S_5' \end{pmatrix} \begin{pmatrix} M_1^1 \\ M_2^1 \\ M_3^0 \\ M_3^1 \\ M_4^1 \\ M_5^1 \end{pmatrix} = 0. \tag{5.2}$$

Here we omit arguments $(1, \lambda)$ and (λ) for brevity. The characteristic function $\Delta_0(\lambda)$ equals the determinant of (5.2). Since we know q_1, q_3 and q_4, we can solve (2.1) and obtain the functions $S_j(x_j, \lambda)$ and $C_j(x_j, \lambda)$ for $j = 1, 3, 4$. Therefore, the determinant admits the representation

$$\Delta_0 = a_{11} S_2 S_5 + a_{12} S_2' S_5 + a_{13} S_2 S_5' + a_{14} S_2' S_5',$$

where

$$a_{11} = S_1' \begin{vmatrix} S_3 & -S_4 \\ S_3' & S_4' \end{vmatrix} + S_1 \begin{vmatrix} C_3 & -S_4 \\ C_3' & S_4' \end{vmatrix}, \qquad a_{12} = S_1 \begin{vmatrix} S_3 & -S_4 \\ S_3' & S_4' \end{vmatrix},$$
$$a_{13} = (S_1' S_3 + S_1 C_3) S_4, \qquad a_{14} = S_1 S_3 S_4.$$

If we change S_1 to C_1 or S_4 to C_4, then we obtain analogous relations for $\Delta_1(\lambda)$ and $\Delta_4(\lambda)$, respectively. Thus, we arrive at system (3.2).

Let $q \equiv 0$ on G. Then

$$C_j^0(x_j, \lambda) = \cos \rho x_j, \qquad S_j^0(x_j, \lambda) = \frac{\sin \rho x_j}{\rho},$$
$$a_{11}^0 = \frac{\sin 3\rho}{\rho}, \qquad a_{12}^0 = a_{13}^0 = \frac{\sin 2\rho \sin \rho}{\rho^2}, \qquad a_{14}^0 = \frac{\sin^3 \rho}{\rho^3},$$
$$a_{21}^0 = a_{31}^0 = \cos 3\rho, \qquad a_{22}^0 = a_{33}^0 = \frac{\sin 2\rho \cos \rho}{\rho},$$
$$a_{23}^0 = a_{32}^0 = \frac{\cos 2\rho \sin \rho}{\rho}, \qquad a_{24}^0 = a_{34}^0 = \frac{\cos \rho \sin^2 \rho}{\rho^2}.$$
$$\Delta_0^0 = \frac{-9 \sin 5\rho + 13 \sin 3\rho + 6 \sin \rho}{16 \rho^3}, \qquad \Delta_1^0 = \Delta_4^0 = \frac{-9 \cos 5\rho + 7 \cos 3\rho + 2 \cos \rho}{16 \rho^2}.$$

Using (3.3), we obtain

$$b_{11}^0 = b_{21}^0 = \frac{-3 \sin 6\rho - 2 \sin 4\rho + 13 \sin 2\rho}{16 \rho^3},$$
$$b_{12}^0 = b_{23}^0 = \frac{-3 \cos 6\rho + 6 \cos 4\rho + 3 \cos 2\rho - 6}{16 \rho^4},$$
$$b_{13}^0 = b_{22}^0 = \frac{3 \cos 6\rho - 10 \cos 4\rho + 13 \cos 2\rho - 6}{32 \rho^4},$$
$$b_{14}^0 = b_{24}^0 = \frac{-3 \sin 6\rho + 12 \sin 4\rho - 15 \sin 2\rho}{32 \rho^5}.$$

Substitute these formulae into (3.6) to obtain

$$A_0 = \frac{-27\sin 12\rho + 174\sin 10\rho - 420\sin 8\rho + 378\sin 6\rho + 153\sin 4\rho - 468\sin 2\rho}{2048\rho^9},$$

$$B_0 = \frac{1}{2048\rho^8}(-27\cos 12\rho + 84\cos 10\rho + 106\cos 8\rho - 764\cos 6\rho + 1099\cos 4\rho - 344\cos 2\rho - 154),$$

$$C_0 = \frac{-27\sin 12\rho + 48\sin 10\rho + 140\sin 8\rho - 336\sin 6\rho - 71\sin 4\rho + 512\sin 2\rho}{1024\rho^7}.$$

Calculate the discriminant of equation (3.7) to obtain

$$\begin{aligned} D_0 &= B_0^2 - 4A_0C_0 \\ &= \frac{1}{8388608\rho^{16}}(6561\cos 24\rho - 52488\cos 22\rho + 128628\cos 20\rho + 83592\cos 18\rho \\ &\quad - 987134\cos 16\rho + 1543976\cos 14\rho + 702372\cos 12\rho \\ &\quad - 4646312\cos 10\rho + 3755087\cos 8\rho + 3053616\cos 6\rho \\ &\quad - 4805144\cos 4\rho - 4176688\cos 2\rho + 5393934). \end{aligned}$$

We used wxMaxima v. 12.04.0 for calculations.

Obviously, $A_0(\lambda) \neq 0$, $D_0(\lambda) \neq 0$, so according to lemma 3.4 the roots of equation (3.5) in the general case have different asymptotics:

$$\tilde{M}_2^1(\lambda) = \frac{\rho\cos\rho}{\sin\rho}[1], \qquad \tilde{M}_2^2(\lambda) = -\frac{1 + 6\cos^2\rho}{3\sin\rho\cos\rho}[1].$$

Since $\tilde{M}_2(\lambda) = S_2'(1,\lambda)/S_2(1,\lambda)$, only the root $M_2^1(\lambda)$ is the required one.

Finally, one can easily find $\tilde{M}_5(\lambda)$ and solve classical Sturm–Liouville inverse problems using Weyl functions on the edges e_2 and e_5.

Now let us consider the case in which the potential is known *a priori* on two edges. If they are e_1 and e_4, then only two spectra Λ_0 and Λ_2 are sufficient to recover the potential on the whole graph. Indeed, one can solve IP(2), then apply theorem 2.3 to the vertex v_3, find q_3 and then similarly find q_5. However, the knowledge of q_1 and q_2 do not allow us to recover the potential from two spectra by our method. If we have only Λ_0 and Λ_4, we cannot recover q_3. Similarly, if we know q_3 initially, the knowledge of the potential on one of the boundary edges does not allow us to reduce the number of given spectra. Thus, if the potential is known on multiple edges, the number of required spectra depends on the location of these edges.

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