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RESEARCH ARTICLE

# Improvement of some discrete Hardy inequalities with variants

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**Abstract**

In this paper, we establish a new version of one-dimensional discrete improved Hardy's inequality with shifts by introducing a shifting discrete Dirichlet's Laplacian. We prove that the general discrete Hardy's inequality as well as its variants in some special cases admit improvements. Further, it is proved that two-variable discrete $p$-Hardy inequality can also be improved via improved discrete $p$-Hardy inequality in one dimension. The result is also extended to the multivariable cases.

## 1. Introduction

Let $p > 1$ be a real number and $a = \{a_n\}$ be a sequence of complex numbers such that $a \in \ell_p$, a Banach space of all $p$-summable sequences. Then the classical discrete $p$-Hardy's inequality ([7], Theorem 326) in one dimension asserts that

$$\sum_{n=1}^{\infty} \left| \frac{1}{n} \sum_{k=1}^{n} a_k \right|^p < C_p \sum_{n=1}^{\infty} |a_n|^p, \tag{1.1}$$

holds unless $a_n$ is null for all $n \in \mathbb{N}$ and the associated constant term $C_p = \left(\frac{p}{p-1}\right)^p$ is best possible. Inequality (1.1) was developed in the twentieth century during the period $1906 - 1928$. On 21 June, 1921, Landau [18] wrote a letter to G. H. Hardy containing a proof of (1.1), but this letter officially published five years later than the letter of Landau to Schur [19]; however, the reason of this long delay is not clearly known. Apart from the development of the inequality by G. H. Hardy himself ([8, 9]), many other mathematicians such as E. Landau, G. Pólya, M. Riesz and I. Schur have played remarkable role for its development. Since Landau ([18, 19]) has great contribution for the development of inequality (1.1), so this inequality is sometimes called as 'Hardy–Landau' inequality. We refer a survey article [17] for a detailed history of the invention of inequality (1.1). By denoting $A = \{A_n\} \in C_c(\mathbb{N}_0)$ with $A_0 = 0$, where $C_c(\mathbb{N}_0)$ is a space of all finitely supported functions defined on $\mathbb{N}_0$, we observe that inequality (1.1) can be written equivalently as below:

$$\sum_{n=1}^{\infty} |A_n - A_{n-1}|^p \geq \sum_{n=1}^{\infty} \frac{|A_n|^p}{C_p n^p}. \tag{1.2}$$

The particular case of $p = 2$ in each of the inequalities (1.1) and (1.2) is of great interest. Note that since the constant in (1.1) is sharp, so the reduced constant term '$\frac{1}{4}$' in (1.2) is also sharp, or in other words, optimal. Due to the sharpness of the constant term, inequalities (1.1) and (1.2) and its continuous analogues have wide applications in different sections of mathematics such as differential equations, graph theory and spectral theory. In 2018, a surprising discovery by Keller, Pinchover and Pogorzelski [13] (see also [14]) suggests that although the constant term '$\frac{1}{4}$' is sharp, the whole weight '$w_n^H = \frac{1}{4n^2}$' in inequality (1.2) is not optimal. They proved that there exists a weight sequence $w_n^{KPP} = 2 - \left(1 - \frac{1}{n}\right)^{1/2} - \left(1 + \frac{1}{n}\right)^{1/2} > \frac{1}{4n^2} = w_n^H$, $n \in \mathbb{N}$ for which inequality (1.2) admits an improvement as below:

$$\sum_{n=1}^{\infty} |A_n - A_{n-1}|^2 \geq \sum_{n=1}^{\infty} w_n^{KPP} |A_n|^2 > \frac{1}{4} \sum_{n=1}^{\infty} \frac{|A_n|^2}{n^2}. \tag{1.3}$$

This improvement result has stimulated the interest in research in this direction by many mathematicians and researchers. Two of the authors ([2, 3]), Gerhat et al. [5], Krejčiřík and Štampach [15] and Krejčiřík et al. [16] have studied and obtained a general improvement of discrete Hardy's inequality by using an elementary technique and factorisation method. The authors in [6] and [12] have also studied the improvement of Hardy's inequality with power weights and Hardy–Rellich inequality, respectively.

Recently, Fischer, Keller and Pogorzelski [4] have extended the above inequality (1.3) for any real $p > 1$ and proved the following improved inequality:

$$\sum_{n=1}^{\infty} |A_n - A_{n-1}|^p \geq \sum_{n=1}^{\infty} w_n^{FKP}(p) |A_n|^p > \sum_{n=1}^{\infty} \frac{|A_n|^p}{C_p n^p}, \tag{1.4}$$

where for each $n \in \mathbb{N}$, the improved weight sequence $w_n^{FKP}(p)$ is defined by

$$w_n^{FKP}(p) = \left(1 - (1 - \frac{1}{n})^{\frac{p-1}{p}}\right)^{p-1} - \left((1 + \frac{1}{n})^{\frac{p-1}{p}} - 1\right)^{p-1} > \frac{1}{C_p n^p}$$

Due to the enormous application, inequality (1.1) has been extended in many ways. One of such extension to inequality (1.1) for the case $p = 2$ was established by G. H. Hardy [9] himself in 1925. To state his inequality, we suppose that $\{q_n\}$ is any sequence of real numbers such that $q_n > 0$ and denote $A_n = q_1 a_1 + q_2 a_2 + \ldots + q_n a_n$ and $Q_n = q_1 + q_2 + \ldots + q_n$ for $n \in \mathbb{N}$. If $\{\sqrt{q_n} a_n\} \in \ell_2$, then

$$\sum_{n=1}^{\infty} q_n Q_n^{-2} |A_n|^2 \leq 4 \sum_{n=1}^{\infty} q_n |a_n|^2. \tag{1.5}$$

Also the constant term '4' is sharp. One can notice that inequality (1.5) has the following equivalent form:

$$\sum_{n=1}^{\infty} \frac{|A_n - A_{n-1}|^2}{q_n} \geq \frac{1}{4} \sum_{n=1}^{\infty} \frac{q_n}{Q_n^2} |A_n|^2. \tag{1.6}$$

It is interesting to know whether inequality (1.6) admits any improvement or not. In our previous article ([3], Corollary 2.1), we were able to achieve an improvement of inequality (1.6) in a particular case when $q_n = n$, $n \in \mathbb{N}$ only by obtaining a suitable improved weight sequence. Then we have a natural question as below:

*Q(a): Are there any other choices of $q_n$ for which an improvement of (1.6) is possible?*

Keeping this question in mind, here we choose two important cases (i) $q_n = n^2$ and (ii) $q_n = n^3$. In fact, when we choose (i) $q_n = n^2$, then inequality (1.6) reduces to

$$\sum_{n=1}^{\infty} \frac{|A_n - A_{n-1}|^2}{n^2} \geq 9 \sum_{n=1}^{\infty} \frac{|A_n|^2}{(n+1)^2(2n+1)^2}. \tag{1.7}$$

Again when we consider (ii) $q_n = n^3$, then inequality (1.6) becomes

$$\sum_{n=1}^{\infty} \frac{|A_n - A_{n-1}|^2}{n^3} \geq 4 \sum_{n=1}^{\infty} \frac{|A_n|^2}{n(n+1)^4}. \tag{1.8}$$

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Therefore, in particular, we first investigate the following question:

*Q(b): Do the Hardy inequalities* (1.7) *and* (1.8) *admit any improvement?*

In 1919, Hardy [10] first established the dual inequality of (1.1) in the case when $p = 2$. It states that

$$\sum_{n=1}^{\infty}\left|\sum_{k=n}^{\infty}\frac{a_k}{k}\right|^2 \leq 4\sum_{n=1}^{\infty}|a_n|^2, \tag{1.9}$$

holds and the associated constant term '4' is sharp and equality holds good when all $a_n$ are null. Later in 1927, Copson [1] by adapting Elliott's proof and dual Hardy's inequality (1.9) introduced and studied a general form of the variant of inequality (1.5) as below:

$$\sum_{n=1}^{\infty}q_n\left|\sum_{k=n}^{\infty}\frac{q_k a_k}{Q_k}\right|^2 \leq 4\sum_{n=1}^{\infty}q_n|a_n|^2, \tag{1.10}$$

where the attached constant term is best possible. It is pertinent to mention here that Hardy [11] first observe the 'dual concept' between inequalities (1.5) and (1.10). In the case of $q_n = 1$ for each $n \in \mathbb{N}$, one has (1.9). Since Copson [1] and Hardy [10] both have significant roles to obtain inequality (1.10), so this inequality sometimes called as Copson–Hardy inequality [17]. Let us substitute $A_n = \sum_{k=n}^{\infty}\frac{q_k a_k}{Q_k}$ in (1.10), and then it is equivalent to the following inequality:

$$\sum_{n=2}^{\infty}\frac{Q_{n-1}^2}{q_{n-1}}|A_n - A_{n-1}|^2 \geq \frac{1}{4}\sum_{n=2}^{\infty}q_n|A_n|^2, \tag{1.11}$$

with $A_0 = A_1 = 0$. If we put $q_n = 1$ for each $n \in \mathbb{N}$, then one obtains an equivalent version of (1.9) as given below:

$$\sum_{n=2}^{\infty}(n-1)^2|A_n - A_{n-1}|^2 \geq \sum_{n=2}^{\infty}\frac{1}{4}|A_n|^2. \tag{1.12}$$

Similarly for $q_n = n, n^2, n^3$ for each $n \in \mathbb{N}$, then we get the reduced forms of (1.11) as below:

$$\sum_{n=2}^{\infty}(n-1)n^2|A_n - A_{n-1}|^2 \geq \sum_{n=2}^{\infty}n|A_n|^2, \tag{1.13}$$

$$\sum_{n=2}^{\infty}n^2(2n-1)^2|A_n - A_{n-1}|^2 \geq 9\sum_{n=2}^{\infty}n^2|A_n|^2, \tag{1.14}$$

$$\sum_{n=2}^{\infty}(n-1)n^4|A_n - A_{n-1}|^2 \geq 4\sum_{n=2}^{\infty}n^3|A_n|^2, \tag{1.15}$$

respectively. It is observed that no such study has been carried out on the improvement of variant Hardy inequalities as compared to the Hardy inequalities. Here for the first time, we will examine the improvement of variant Hardy inequalities. In fact, we answer the following question in the sequel:

*Q(c): Can the variant Hardy inequalities* (1.12)–(1.15) *be improved?*

On the other hand, Pachpatte [20] first considered the multivariable discrete $p$-Hardy's inequality for $p > 1$, and later Salem et al. [21] obtained a sharp discrete $p$-Hardy's inequality for a double sequence $\{a_{mn}\}$ of real or complex numbers indexed by $m, n \in \mathbb{N}$ as well as for multivariable sequences. The discrete $p$-Hardy's inequality for two variables asserts that

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left|\frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n}a_{ij}\right|^p \leq (C_p)^2\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}|a_{mn}|^p, \tag{1.16}$$

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where the constant term $(C_p)^2 = \left(\frac{p}{p-1}\right)^{2p}$ is sharp. We denote $A_{mn} = \sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}$. Then inequality (1.16) takes the following form:

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{(C_p)^2 m^p n^p}|A_{mn}|^p \leq \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}|a_{mn}|^p. \tag{1.17}$$

Again for multivariables $m_1, m_2, \ldots, m_r$ (say) the discrete Hardy's inequality ([20, 21]) states that

$$\sum_{m_1=1}^{\infty}\sum_{m_1=1}^{\infty}\cdots\sum_{m_r=1}^{\infty}\left|\frac{1}{m_1 m_2 \ldots m_r}\sum_{i_1=1}^{m_1}\sum_{i_2=1}^{m_2}\cdots\sum_{i_r=1}^{m_r} a_{i_1 i_2 \ldots i_r}\right|^p \leq (C_p)^r \sum_{m_1=1}^{\infty}\sum_{m_1=1}^{\infty}\cdots\sum_{m_r=1}^{\infty}|a_{m_1 m_2 \ldots m_r}|^p \tag{1.18}$$

holds, where $p > 1$, $\{a_{m_1 m_2 \ldots m_r}\}$ is an $r$-fold sequence of complex numbers and the constant term $(C_p)^r = \left(\frac{p}{p-1}\right)^{rp}$ is best possible. Inequality (1.18) can be written equivalently as below:

$$\sum_{m_1=1}^{\infty}\sum_{m_1=1}^{\infty}\cdots\sum_{m_r=1}^{\infty}\frac{1}{(C_p)^r\big(m_1 m_2 \ldots m_r\big)^p}|A_{m_1 m_2 \ldots m_r}|^p \leq \sum_{m_1=1}^{\infty}\sum_{m_1=1}^{\infty}\cdots\sum_{m_r=1}^{\infty}|a_{m_1 m_2 \ldots m_r}|^p, \tag{1.19}$$

where $A_{m_1 m_2 \ldots m_r} = \sum_{i_1=1}^{m_1}\sum_{i_2=1}^{m_2}\cdots\sum_{i_r=1}^{m_r} a_{i_1 i_2 \ldots i_r}$.

We then have the following question to answer:

*Q(d): Is it possible to improve both nequalities* (1.17) *and* (1.19)?

Therefore, the main objective of this present paper is to deliver the possible answer to the queries $Q(a) - Q(d)$ raised above. To reach our objectives, we first establish a general improved one-dimensional discrete Hardy's inequality with shifts by introducing a $m$th shift discrete Dirichlet's Laplacian operator $(-\Delta_m^{(r,s)})$. So, we derive multiple new and previously enhanced Hardy inequalities incorporating shifts. This will also help us for achieving answers to the questions $Q(a)$– $Q(c)$. In response to $Q(b)$, we prove that both the discrete Hardy inequalities (1.7) and (1.8) admit improvements, and as a consequence, we give an affirmative answer to $Q(a)$. In a reply to $Q(c)$, we show that there exist weight sequences for which improvement of all the variant Hardy inequalities (1.12)–(1.15) is possible. Finally, we prove that one-dimensional discrete improved Hardy's inequalities lead to the improvement of the multivariate Hardy inequalities (1.17) and (1.19), which provides an answer to $Q(d)$. It is pertinent to mention here that the optimality of the improved weights for each of the addressed inequalities is still open.

The paper is structured as follows. Section 2 introduces a general discrete improved Hardy's inequality with shifts. Section 3 focuses on enhancing specific discrete Hardy inequalities by selecting particular values of $q_n$, $n \in \mathbb{N}$. The study of improving all considered variant Hardy inequalities is discussed in Section 4. Lastly, Section 5 explores enhancements of the multivariable discrete Hardy's inequality.

## 2. An improved Hardy's inequality with shifts

We begin this section by introducing shifting *backward* and *forward difference operators* acting on a sequence $A = \{A_n\}$ of real or complex numbers. Let us choose $m, n \in \mathbb{N}$ and $r$, $s$ be two real numbers such that $rs > 0$. Then the $m$th shifting *backward* $T_m^{(r,s)}$ and *forward difference operators* $T_m^{*(r,s)}$ acting on a sequence $\{A_n\}$ are defined as follows:

$$(T_m^{(r,s)}A)_n = \begin{cases} -sA_n & \text{if } n = 1, 2, \ldots, m, \\ rA_{n-m} - sA_n & \text{if } \; n > m, \end{cases}$$

and for $n \geq m$

$$(T_m^{*(r,s)}A)_n = rA_{n+m} - sA_n.$$

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Using the above difference operators, we introduce the $m$th shift discrete Dirichlet's Laplacian operator $(-\Delta_m^{(r,s)})$ acting on $A=\{A_n\}$ as below:

$$(-\Delta_m^{(r,s)}A)_n=(T_m^{*(r,s)}T_m^{(r,s)}A)_n=(r^2+s^2)A_n-rsA_{n-m}-rsA_{n+m}.$$

Note that the 1st shift discrete Dirichlet's Laplacian operator $(-\Delta_1^{(r,s)})$ with $r=s=1$ coincides with the well-known discrete Dirichlet's Laplacian operator $(-\Delta)$ and has been considered earlier in [5, 16] (see also [3]) in connection with the study of Hardy and Rellich inequalities. Now for a strictly positive sequence $\mu=\{\mu_n\}$ of real numbers, we define a weight sequence $w_n(r,s,\mu)$ as

$$w_n(r,s,\mu)=\frac{(-\Delta_m^{(r,s)}\mu)_n}{\mu_n}=r^2+s^2-rs\Big(\frac{\mu_{n+m}}{\mu_n}+\frac{\mu_{n-m}}{\mu_n}\Big),\quad m\in\mathbb{N}.$$

Suppose further that $\lambda=\{\lambda_n\}$ is a strictly positive sequence of real numbers. Then we introduce a more accurate and general form of the weight sequence $w_n(r,s,\mu)$ as follows:

$$w_n(\lambda,r,s,\mu)=\frac{1}{\lambda_n}\Big(r^2-rs\frac{\mu_{n-m}}{\mu_n}\Big)+\frac{1}{\lambda_{n+m}}\Big(s^2-rs\frac{\mu_{n+m}}{\mu_n}\Big),\quad m\in\mathbb{N}. \tag{2.1}$$

With this definition, we have the following result, which gives a general improved discrete Hardy's inequality with shifts. Let us begin with the following statement.

**Theorem 2.1.** *Let $A=\{A_n\}$ be any sequence of complex numbers such that $A\in C_c(\mathbb{N}_0)$ with $A_n=0$ for $0\le n\le m-1$ and $r,s\in\mathbb{R}$ be such that $rs>0$. Then we have*

$$\sum_{n=m}^{\infty}\frac{|rA_n-sA_{n-m}|^2}{\lambda_n}\ge\sum_{n=m}^{\infty}w_n(\lambda,r,s,\mu)|A_n|^2, \tag{2.2}$$

*where the sequence $w_n(\lambda,r,s,\mu)$ is defined in (2.1).*

*Proof.* We first calculate the following sum and proceed with an approach initiated by [15]. Let us assume that $A_n=0$, $0\le n\le m-1$ and $\frac{|A_0|^2}{\lambda_m\mu_0}$ is zero. Then

$$\begin{aligned}&\sum_{n=m}^{\infty}w_n(\lambda,r,s,\mu)|A_n|^2\\&=\sum_{n=m}^{\infty}\Big(\frac{r^2}{\lambda_n}+\frac{s^2}{\lambda_{n+m}}\Big)|A_n|^2-\sum_{n=m}^{\infty}\Big(\frac{rs}{\lambda_{n+m}}\frac{\mu_{n+m}}{\mu_n}+\frac{rs}{\lambda_n}\frac{\mu_{n-m}}{\mu_n}\Big)|A_n|^2\\&=\sum_{n=m}^{\infty}\frac{r^2}{\lambda_n}|A_n|^2+\sum_{n=2m}^{\infty}\frac{s^2}{\lambda_n}|A_{n-m}|^2-rs\sum_{n=2m}^{\infty}\frac{\mu_n}{\lambda_n\mu_{n-m}}|A_{n-m}|^2-rs\sum_{n=m}^{\infty}\frac{\mu_{n-m}}{\lambda_n\mu_n}|A_n|^2\\&=\sum_{n=m}^{\infty}\Big(\frac{r^2}{\lambda_n}|A_n|^2+\frac{s^2}{\lambda_n}|A_{n-m}|^2\Big)-rs\sum_{n=m}^{\infty}\Big(\frac{\mu_n}{\lambda_n\mu_{n-m}}|A_{n-m}|^2+\frac{\mu_{n-m}}{\lambda_n\mu_n}|A_n|^2\Big).\end{aligned}$$

Further assume that $\mu_n=0$ for $0\le n\le m-1$ and term $\sqrt{\frac{\mu_m}{\lambda_m\mu_0}}A_0$ is also zero. Since $rs>0$, so the following computation of differences gives

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$$\sum_{n=m}^{\infty}\frac{|rA_n - sA_{n-m}|^2}{\lambda_n} - \sum_{n=m}^{\infty} w_n(\lambda, r, s, \mu)|A_n|^2$$

$$= \sum_{n=m}^{\infty}\Big(\frac{r^2}{\lambda_n}|A_n|^2 + \frac{s^2}{\lambda_n}|A_{n-m}|^2 - \frac{rs}{\lambda_n}2\mathbb{R}(A_n\bar{A}_{n-m})\Big)$$

$$- \sum_{n=m}^{\infty}\Big(\frac{r^2}{\lambda_n}|A_n|^2 + \frac{s^2}{\lambda_n}|A_{n-m}|^2 - rs\frac{\mu_n}{\lambda_n\mu_{n-m}}|A_{n-m}|^2 - rs\frac{\mu_{n-m}}{\lambda_n\mu_n}|A_n|^2\Big)$$

$$= rs\sum_{n=m}^{\infty}\Big(\frac{\mu_n}{\lambda_n\mu_{n-m}}|A_{n-m}|^2 + \frac{\mu_{n-m}}{\lambda_n\mu_n}|A_n|^2\Big) - \frac{rs}{\lambda_n}\sum_{n=m}^{\infty}2\mathbb{R}(A_n\bar{A}_{n-m})$$

$$= rs\sum_{n=2m}^{\infty}\frac{1}{\lambda_n}\,|\,\Big(\frac{\mu_{n-m}}{\mu_n}\Big)^{1/2}A_n - \Big(\frac{\mu_n}{\mu_{n-m}}\Big)^{1/2}A_{n-m}\,|^2 \geq 0.$$

This proves the desired inequality:

$$\sum_{n=m}^{\infty}\frac{|rA_n - sA_{n-m}|^2}{\lambda_n} \geq \sum_{n=m}^{\infty} w_n(\lambda, r, s, \mu)|A_n|^2.$$

Hence the theorem. □

Observe that Theorem 2.1 establishes several important results and some of them are given in the following corollaries.

**Corollary 2.1.** *Suppose that $\lambda_n = 1$ and $\mu_n = n^\alpha$, $\alpha \in (0, 1)$ for each $n \in \mathbb{N}$. Then $w_n(\lambda, r, s, \mu)$ reduces to $w_n(r, s, \alpha)$ (say), where*

$$w_n(r, s, \alpha) = r^2 + s^2 - rs\Big(1 - \frac{m}{n}\Big)^\alpha - rs\Big(1 + \frac{m}{n}\Big)^\alpha.$$

*Expanding $w_n(r, s, \alpha)$, one gets*

$$w_n(r, s, \alpha) = \begin{cases} r^2 + s^2 - 2^\alpha rs & \text{if } n = m,\\ (r-s)^2 + 2rs\alpha\sum_{k=1}^{\infty}\Big(\frac{1}{(2k!)}\frac{m^{2k}}{n^{2k}}\Big(\prod_{i=1}^{2k-1}(i-\alpha)\Big)\Big) & \text{if } n \geq m+1,\ n, m \in \mathbb{N}.\end{cases}$$

*Hence we have $w_n(r, s, \alpha) > rs\alpha(1-\alpha)\frac{m^2}{n^2}$. Therefore we obtain a different look of improved discrete Hardy's inequality from (2.2) as below:*

$$\sum_{n=m}^{\infty}|rA_n - sA_{n-m}|^2 \geq \sum_{n=m}^{\infty} w_n(r, s, \alpha)_n|A_n|^2 > \sum_{n=m}^{\infty} rs\alpha(1-\alpha)\frac{m^2}{n^2}|A_n|^2. \tag{2.3}$$

Note that in the case of $m = 1$, $r = s$ and $\alpha = \frac{1}{2}$, inequality (2.3) is nothing but an improved discrete Hardy's inequality (1.3) established by Keller, Pinchover and Pogorzelski [13].

**Corollary 2.2.** *Let us choose $\lambda_n = 1$ for each $n \in \mathbb{N}$, $m = 1$ and $r = s = 1$ in inequality (2.2). Then we have the following inequality:*

$$\sum_{n=1}^{\infty}|A_n - A_{n-1}|^2 \geq \sum_{n=1}^{\infty} w_n(\mu)|A_n|^2, \tag{2.4}$$

*where $w_n(\mu) = 2 - \frac{\mu_{n+1}}{\mu_n} + \frac{\mu_{n-1}}{\mu_n}$.*

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The above inequality (2.4) was established by Krejčiřík et al. [16], and an elementary proof of (2.4) for $\mu_n = \sqrt{n}$ was presented by Krejčiřík and Štampach [15].

**Corollary 2.3.** *If one chooses $\lambda_n = \frac{1}{n^\alpha}$, $\alpha \in \mathbb{R}$ and $\mu_n = n^\beta$, $\beta \in \mathbb{R}$, then the corresponding weight sequence $w_n(\lambda, r, s, \mu)$ reduces to $w_n(\alpha, r, s, \beta)$, where*

$$w_n(\alpha, r, s, \beta) = n^\alpha \Big(r^2 + s^2\Big(1 + \frac{m}{n}\Big)^\alpha - rs\Big(1 - \frac{m}{n}\Big)^\beta - rs\Big(1 + \frac{m}{n}\Big)^{\alpha+\beta}\Big),$$

*and inequality (2.2) becomes a power type improved Hardy's inequality with shifts as below:*

$$\sum_{n=m}^{\infty} n^\alpha |ru_n - su_{n-m}|^2 \geq \sum_{n=m}^{\infty} w(\alpha, r, s, \beta)_n |u_n|^2. \tag{2.5}$$

Note that the above inequality (2.5) strengthen the inequality of Gupta [6], who obtained this inequality (2.5) for the case when $m = 1$ and $r = s$.

## 3. Improvement of some general Hardy inequalities

This section is devoted to the study of improvement of general Hardy's inequality (1.6) in some particular cases. In fact, we study the improvement of inequalities (1.7) and (1.8) and prove that these sharp inequalities are not optimal; that is, they admit improvement. Indeed, we prove the following two successive theorems.

**Theorem 3.1.** *Let $A = \{A_n\}$ be any sequence of complex numbers such that $A \in C_c(\mathbb{N}_0)$ with $A_0 = 0$. Then we have*

$$\sum_{n=1}^{\infty} \frac{|A_n - A_{n-1}|^2}{n^2} \geq \sum_{n=1}^{\infty} \eta_n^{(1)} |A_n|^2 > 9 \sum_{n=1}^{\infty} \frac{|A_n|^2}{(2n+1)^2(n+1)^2}, \tag{3.1}$$

*where the weight sequence $\eta_n^{(1)}$ is defined as below:*

$$\eta_n^{(1)} = \frac{1}{n^2} + \frac{1}{(n+1)^2} - \frac{1}{n^2}\Big(\frac{(n-1)(2n-1)}{(n+1)(2n+1)}\Big)^{\frac{1}{2}} - \frac{1}{(n+1)^2}\Big(\frac{(n+2)(2n+3)}{n(2n+1)}\Big)^{\frac{1}{2}}.$$

**Theorem 3.2.** *Suppose that $A = \{A_n\} \in C_c(\mathbb{N}_0)$ be such that $A_0 = 0$. Then*

$$\sum_{n=1}^{\infty} \frac{|A_n - A_{n-1}|^2}{n^3} \geq \sum_{n=1}^{\infty} \eta_n^{(2)} |A_n|^2 > 4 \sum_{n=1}^{\infty} \frac{|A_n|^2}{n(n+1)^4}, \tag{3.2}$$

*holds, where the improved weight $\eta_n^{(2)}$ is read as follows:*

$$\eta_n^{(2)} = \frac{1}{n^3} + \frac{1}{(n+1)^3} - \frac{1}{n^3}\Big(\frac{n-1}{n+1}\Big) - \frac{1}{(n+1)^3}\Big(\frac{n+2}{n}\Big).$$

We first begin with the proof of Theorem 3.1. Before proceed to prove it, we need to establish some results given in the form of lemma. Let us start with the following lemma.

**Lemma 3.3.** *Suppose that $n \in \mathbb{N}$. Then*

$$4n^4 + 6n^3 - \frac{n^2}{2} + 5n + 1 > (2n^2 + 3n + 1)\sqrt{4n^4 - 5n^2 + 1} \text{ holds.}$$

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*Proof.* Let us denote $T_1 = 4n^4 + 6n^3 - \frac{n^2}{2} + 5n + 1$ and $T_2 = (2n^2 + 3n + 1)\sqrt{4n^4 - 5n^2 + 1}$. Then the difference of the squares of $T_1$ and $T_2$ gives

$$T_1^2 - T_2^2 = \frac{n}{4}(280n^4 + 501n^3 + 100n^2 + 64n + 16) > 0,$$

which implies that $\big(T_1 + T_2\big)\big(T_1 - T_2\big) > 0$. Since $(T_1 + T_2) > 0$ always holds, so we conclude that $(T_1 - T_2) > 0$, which leads to the desired inequality. □

We have another lemma to prove.

**Lemma 3.4.** *Suppose that* $n \in \mathbb{N}$. *Then*

$$4n^4 + 10n^3 + \frac{11n^2}{2} + n > (2n^2 + n)\sqrt{4n^4 + 16n^3 + 19n^2 + 6n} \ \text{ holds.}$$

*Proof.* Here we denote $S_1 = 4n^4 + 10n^3 + \frac{11n^2}{2} + n$ and $S_2 = (2n^2 + n)\sqrt{4n^4 + 16n^3 + 19n^2 + 6n}$. Then the following computation gives

$$S_1^2 - S_2^2 = \frac{n^2}{4}(8n^3 + 29n^2 + 20n + 4) > 0,$$

which means $\big(S_1 + S_2\big)\big(S_1 - S_2\big) > 0$. Since $(S_1 + S_2) > 0$, so we get $(S_1 - S_2) > 0$. This proves the desired inequality. □

The next lemma is quite important and is mainly concerned with the improvement of inequality (1.7) via the weight sequence $\eta_n^{(1)}$. Indeed, we prove the following point-wise result.

**Lemma 3.5.** *Let* $n \in \mathbb{N}$. *Then* $\eta_n^{(1)} > \frac{9}{(2n+1)^2(n+1)^2}$ *holds for each* $n \in \mathbb{N}$.

*Proof.* We denote $\tilde{f}(n) = \eta_n^{(1)} - \frac{9}{(2n+1)^2(n+1)^2}$. Then it is enough to show that $\tilde{f}(n) > 0$ holds for each $n \in \mathbb{N}$. We have

$$\begin{aligned}
&\tilde{f}(n)\\
&= \eta_n^{(1)} - \frac{9}{(2n+1)^2(n+1)^2}\\
&= \frac{1}{n^2} + \frac{1}{(n+1)^2} - \frac{1}{n^2}\left(\frac{(n-1)(2n-1)}{(n+1)(2n+1)}\right)^{\frac{1}{2}} - \frac{1}{(n+1)^2}\left(\frac{(n+2)(2n+3)}{n(2n+1)}\right)^{\frac{1}{2}} - \frac{9}{(2n+1)^2(n+1)^2}\\
&= \frac{H(n)}{n^2(n+1)^2(2n+1)^2},
\end{aligned}$$

where we denote

$$\begin{aligned}
H(n) =& (n+1)^2(2n+1)^2 + n^2(2n+1)^2 - (n+1)(2n+1)\sqrt{(n-1)(2n-1)(n+1)(2n+1)}\\
&- n(2n+1)\sqrt{n(n+2)(2n+3)(2n+1)} - 9n^2.
\end{aligned}$$

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Simplifying $H(n)$, one gets

$$\begin{aligned}H(n) =& 8n^4+16n^3+5n^2+6n+1-(2n^2+3n+1)\sqrt{4n^4-5n^2+1}\\ &-(2n^2+n)\sqrt{4n^4+16n^3+19n^2+6n}\\ =&\left(4n^4+6n^3-\frac{n^2}{2}+5n+1-(2n^2+3n+1)\sqrt{4n^4-5n^2+1}\right)\\ &+\left(4n^4+10n^3+\frac{11n^2}{2}+n-(2n^2+n)\sqrt{4n^4+16n^3+19n^2+6n}\right)\\ &=I_1+I_2,\end{aligned}$$

where

$$\begin{aligned}I_1&=4n^4+6n^3-\frac{n^2}{2}+5n+1-(2n^2+3n+1)\sqrt{4n^4-5n^2+1},\\ I_2&=4n^4+10n^3+\frac{11n^2}{2}+n-(2n^2+n)\sqrt{4n^4+16n^3+19n^2+6n}.\end{aligned}$$

Using Lemma 3.3 and Lemma 3.4, we get $I_1>0$ and $I_2>0$. Therefore, $H(n)>0$ for all $n\in\mathbb{N}$. This shows that $\tilde{f}(n)>0$ for all $n\in\mathbb{N}$. This completes proof of the lemma. □

*Proof.* (of **Theorem** 3.1):

The L.H.S. part of the inequality is easily derived by choosing $r=s$, $m=1$, $\lambda_n=n^2$ and $\mu_n=\left(\frac{n(n+1)(2n+1)}{6}\right)^{\frac{1}{2}}$ in inequality (2.2). The R.H.S. part of the inequality is an immediate consequence of Lemma 3.5. This finishes proof of the theorem. □

We now proceed to prove Theorem 3.2. The following result is required to prove the theorem.

**Lemma 3.6.** *Let $n\in\mathbb{N}$. Then $\eta_n^{(2)}>\frac{4}{n(n+1)^4}$ holds for each $n\in\mathbb{N}$.*

*Proof.* Denote $\tilde{\tilde{f}}(n)=\eta_n^{(2)}-\frac{4}{n(n+1)^4}$. Then it is sufficient to prove that $\tilde{\tilde{f}}(n)>0$ for each $n\in\mathbb{N}$. We write

$$\begin{aligned}\tilde{\tilde{f}}(n)&=\eta_n^{(2)}-\frac{4}{n(n+1)^4}\\ &=\frac{1}{n^3}+\frac{1}{(n+1)^3}-\frac{1}{n^3}\left(\frac{n-1}{n+1}\right)-\frac{1}{(n+1)^3}\left(\frac{n+2}{n}\right)-\frac{4}{n(n+1)^4}\\ &=\frac{2}{n^3(n+1)}-\frac{2}{n(n+1)^3}-\frac{4}{n(n+1)^4}.\end{aligned}$$

Simplifying the terms, one gets

$$\tilde{\tilde{f}}(n)=\frac{6n+2}{n^3(n+1)^4}>0.$$

This proves the lemma. □

*Proof.* (of **Theorem** 3.2):

Let us choose $r=s$, $m=1$, $\lambda_n=n^3$ and $\mu_n=\frac{n(n+1)}{2}$ in inequality (2.2). Then we immediately establish the L.H.S. part of the inequality. On the other hand, the R.H.S. follows easily from Lemma 3.6. This completes proof of the theorem. □

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## 4. Improvement of the variant Hardy inequalities

In this section, an improvement study is being considered for the variant inequalities (1.12)–(1.15). We begin with the study of improvement for inequality (1.12), and subsequently, we consider the other inequalities. We prove that each of these inequalities (1.12)–(1.15) admits an improvement for a corresponding weight sequence. Let us state our first result in this section.

**Theorem 4.1.** *Suppose that $A = \{A_n\}$ be any sequence of complex numbers such that $A \in C_c(\mathbb{N}_0)$ with $A_0 = A_1 = 0$. Then*

$$\sum_{n=2}^{\infty} (n-1)^2 |A_n - A_{n-1}|^2 \geq \sum_{n=2}^{\infty} \beta_n |A_n|^2 > \sum_{n=2}^{\infty} \frac{1}{4} |A_n|^2 \text{ holds} \tag{4.1}$$

*where the improved weight $\beta_n$ for $n \geq 2$ is defined as below:*

$$\beta_n = n^2 \Big[ 1 + \Big(1 - \frac{1}{n}\Big)^2 - \Big(1 + \frac{1}{n}\Big)^{-\frac{1}{2}} - \Big(1 - \frac{1}{n}\Big)^{\frac{3}{2}} \Big].$$

Before proving the above theorem, we need to establish the following lemma, statement of which is presented below.

**Lemma 4.2.** *Let $n \in \mathbb{N}$. Then $\beta_n > \frac{1}{4}$ holds for each $n \geq 2$.*

*Proof.* Note that $\beta_2 = \frac{5\sqrt{6} - 2\sqrt{3} - 8}{\sqrt{6}} > \frac{1}{4}$ and $\beta_3 = \frac{25\sqrt{3} - 12\sqrt{2} - 27}{2\sqrt{3}} > \frac{1}{4}$. Hence $\beta_n > \frac{1}{4}$ for $n = 2, 3$. Let us now choose $n \geq 4$. We choose a function $Q(x)$ as below:

$$Q(x) = 1 + (1-x)^2 - (1-x)^{\frac{3}{2}} - (1+x)^{-\frac{1}{2}} - \frac{x^2}{4} \tag{4.2}$$

where $x \in [0, \frac{1}{4}]$. Now successive differentiation of $Q(x)$ gives the following:

$$Q'(x) = \frac{3x}{2} - 2 + \frac{3}{2}(1-x)^{\frac{1}{2}} + \frac{1}{2}(1+x)^{\frac{-3}{2}},$$

$$Q''(x) = \frac{3}{2} - \frac{3}{4}(1-x)^{-\frac{1}{2}} - \frac{3}{4}(1+x)^{-\frac{5}{2}},$$

$$Q'''(x) = -\frac{3}{8}(1-x)^{-\frac{3}{2}} + \frac{15}{8}(1+x)^{-\frac{7}{2}},$$

$$Q''''(x) = -\frac{9}{16}(1-x)^{-\frac{5}{2}} - \frac{105}{16}(1+x)^{-\frac{9}{2}}.$$

It is easy to observe that for any $x \in [0, \frac{1}{4}]$, we have $Q''''(x) < 0$. This shows that $Q'''(x)$ is decreasing in $x \in [0, \frac{1}{4}]$. Since $Q'''(1/4) > 0$, so we conclude that $Q'''(x) > 0$ for $x \in [0, \frac{1}{4}]$, and hence $Q''(x)$ is increasing in $[0, \frac{1}{4}]$. Again since $Q''(0) = 0$, so $Q''(x) > 0$ for $x \in (0, \frac{1}{4}]$.This shows that $Q'(x)$ is increasing in $[0, \frac{1}{4}]$. Further, we see that $Q'(0) = 0$ and $Q(0) = 0$. Therefore, we get $Q(x) \geq 0$ for $x \in [0, \frac{1}{4}]$. Hence $Q(x) > 0$ in $x \in (0, \frac{1}{4}]$.

Let us now put $x = \frac{1}{n}$ for $0 < x \leq \frac{1}{4}$, and inserting it in $Q(x)$, we get

$$Q\Big(\frac{1}{n}\Big) = 1 + (1 - \frac{1}{n})^2 - (1 + \frac{1}{n})^{-\frac{1}{2}} - (1 - \frac{1}{n})^{\frac{3}{2}} - \frac{1}{4n^2} > 0 \text{ for } n \geq 4.$$

Since $\beta_n - \frac{1}{4} = n^2 Q\Big(\frac{1}{n}\Big)$, so we get $\beta_n > \frac{1}{4}$ for $n \geq 4$. Hence for each $n \geq 2$, we have $\beta_n > \frac{1}{4}$. This proves the lemma. □

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*Proof.* (of **Theorem** 4.1):

The L.H.S. of the above inequality is easily followed from inequality (2.2) by choosing $r = s$, $m = 1$, $\lambda_n = \frac{1}{(n-1)^2}$ and $\mu_n = \frac{1}{\sqrt{n}}$. To establish the R.H.S part, it is sufficient to prove that $\beta_n > \frac{1}{4}$ holds point-wise for $n \geq 2$, $n \in \mathbb{N}$ and which is enclosed in Lemma 4.2. This completes proof of the theorem. □

In the next theorem, we show that inequality (1.13) gets an improvement for a suitable weight sequence $\beta_n^{(1)}$ for $n \geq 2$. In fact, we have the following improved inequality.

**Theorem 4.3.** *Let $A = \{A_n\}$ be any sequence of complex numbers such that $A \in C_c(\mathbb{N}_0)$ with $A_0 = A_1 = 0$. Then*

$$\sum_{n=2}^{\infty} (n-1)n^2|A_n - A_{n-1}|^2 \geq \sum_{n=2}^{\infty} \beta_n^{(1)}|A_n|^2 > \sum_{n=2}^{\infty} n|A_n|^2 \tag{4.3}$$

*holds, where the improved weight is defined as below:*

$$\beta_n^{(1)} = n^2(n-1) + n(n+1)^2 - n^2(n-1)\Big(\frac{n-1}{n+1}\Big)^{-\frac{1}{2}} - n(n+1)^2\Big(\frac{n+2}{n}\Big)^{-\frac{1}{2}}.$$

We establish this theorem through some important inequalities stated in the following lemmas.

**Lemma 4.4.** *Suppose that $n \in \mathbb{N}$. Then we have*

$$(a)\ (n^3 + n^2 + \frac{n}{2})\sqrt{n+2} > n\sqrt{n}(n+1)^2,$$

$$(b)\ n^3 - \frac{n}{2} > n^2\sqrt{n^2 - 1}.$$

*Proof.* $(a)$ Let us denote $L_1 = (n^3 + n^2 + \frac{n}{2})\sqrt{n+2}$ and $L_2 = n\sqrt{n}(n+1)^2$. Then difference of squares of $L_1$ and $L_2$ gives

$$L_1^2 - L_2^2 = \frac{n^2}{4}\Big(4n^2 + 5n + 2\Big) > 0.$$

This implies that $\big(L_1 + L_2\big)\big(L_1 - L_2\big) > 0$. Since $(L_1 + L_2) > 0$ always holds for any $n \in \mathbb{N}$, so we conclude that $(L_1 - L_2) > 0$. This proves the desired inequality in $(a)$.

(b) Here we first denote $J_1 = n^3 - \frac{n}{2}$ and $J_2 = n^2\sqrt{n^2 - 1}$. Then direct computation gives

$$J_1^2 - J_2^2 = \frac{n^2}{4} > 0,$$

which means $\big(J_1 + J_2\big)\big(J_1 - J_2\big) > 0$. Since for each $n \in \mathbb{N}$, $(J_1 + J_2) > 0$, so we have $(J_1 - J_2) > 0$. Hence we have the desired inequality in $(b)$. This also completes proof of the lemma. □

**Lemma 4.5.** *Let $n \in \mathbb{N}$ be such that $n \geq 2$. Then $\beta_n^{(1)} > n$ holds for each $n \geq 2$.*

*Proof.* We denote $G(n) = \beta_n^{(1)} - n$. It is then enough to prove that $G(n) > 0$ for each $n \geq 2$. Note that $G(n)$ can be simplified into the following form:

$$G(n) = n^2(n-1) + n(n+1)^2 - n^2(n-1)\Big(\frac{n-1}{n+1}\Big)^{-\frac{1}{2}} - n(n+1)^2\Big(\frac{n+2}{n}\Big)^{-\frac{1}{2}} - n = \frac{D(n)}{\sqrt{n+2}},$$

where $D(n) = (2n^3 + n^2)\sqrt{n+2} - n^2\sqrt{n^2-1}\sqrt{n+2} - n^{\frac{3}{2}}(n+1)^2$. Again $D(n)$ can be further simplified into the following $D(n) = \Big((n^3 + n^2 + \frac{n}{2})\sqrt{n+2} - n\sqrt{n}(n+1)^2\Big) + \sqrt{n+2}\Big(n^3 - \frac{n}{2} - n^2\sqrt{n^2-1}\Big) = k_1 + \sqrt{n+2}k_2$ (say), where $k_1 = (n^3 + n^2 + \frac{n}{2})\sqrt{n+2} - n\sqrt{n}(n+1)^2$ and

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$k_2 = n^3 - \frac{n}{2} - n^2\sqrt{n^2-1}$. Using Lemma 4.4, we get $k_1 > 0$ and $k_2 > 0$. Therefore, $D(n) > 0$ for all $n \in \mathbb{N}$. This shows that $G(n) > 0$ for all $n \geq 2$. This completes proof of the lemma. □

*Proof.* (of **Theorem** 4.3):

Let us put $r = s$, $m = 1$, $\lambda_n = \frac{1}{(n-1)n^2}$ and $\mu_n = \left(\frac{n(n+1)}{2}\right)^{-\frac{1}{2}}$ in inequality (2.2). Then we instantly get the L.H.S. part of the inequality. The R.H.S. part is an immediate consequence of Lemma 4.5. □

Our next result demonstrates that inequality (1.14) achieves an improvement. In fact, we have the following result.

**Theorem 4.6.** *Suppose $A = \{A_n\} \in C_c(\mathbb{N}_0)$ such that $A_0 = A_1 = 0$. Then*

$$\sum_{n=2}^{\infty} (2n-1)^2 n^2 |A_n - A_{n-1}|^2 \geq \sum_{n=2}^{\infty} \beta_n^{(2)} |A_n|^2 > 9 \sum_{n=2}^{\infty} n^2 |A_n|^2 \text{ holds} \tag{4.4}$$

*where the improved variant Hardy weight $\beta_n^{(2)}$, $n \geq 2$ is defined as below:*

$$\beta_n^{(2)} = n^2(2n-1)^2 + (2n+1)^2(n+1)^2 - n^2(2n-1)^2\left(\frac{(n-1)(2n-1)}{(n+1)(2n+1)}\right)^{-\frac{1}{2}} - (2n+1)^2(n+1)^2\left(\frac{(n+2)(2n+3)}{n(2n+1)}\right)^{-\frac{1}{2}}.$$

It is required to establish a lemma before proving Theorem 4.6.

**Lemma 4.7.** *Suppose that $n \in \mathbb{N}$ such that $n \geq 2$. Then we have*

$$(a)\ \left(4n^4 + 2n^3 - \frac{n^2}{2} + n + 1\right)\sqrt{2n^2 - 3n + 1} > n^2(2n-1)^2\sqrt{2n^2+3n+1},$$

$$(b)\ \left(4n^4 + 6n^3 + \frac{11n^2}{2} + 5n\right)\sqrt{2n^2+7n+6} > (n+1)^2(2n+1)^2\sqrt{2n^2+n}.$$

*Proof.* (*a*) We denote $R_1 = (4n^4 + 2n^3 - \frac{n^2}{2} + n + 1)\sqrt{2n^2-3n+1}$ and $R_2 = n^2(2n-1)^2\sqrt{2n^2+3n+1}$. Squaring $R_1$ and $R_2$ and then subtracting them, we get

$$R_1^2 - R_2^2 = \frac{(2n-1)}{4}\Big(8n^4(n^2-4) + 12(n^5 - n^3) + (n^5 - n^4) + (8n^2 - 4n - 4)\Big) > 0 \ \textit{for} \ n \geq 2.$$

which is positive, and this implies that $\big(R_1 + R_2\big)\big(R_1 - R_2\big) > 0$. Hence we have $(R_1 - R_2) > 0$ since $(R_1 + R_2) > 0$ for each $n \in \mathbb{N}$. This proves the inequality.

(*b*) Similarly, we denote $P_1 = (4n^4 + 6n^3 + \frac{11n^2}{2} + 5n)\sqrt{2n^2+7n+6}$ and $P_2 = (n+1)^2(2n+1)^2\sqrt{2n^2+n}$. The following computation gives

$$P_1^2 - P_2^2 = \frac{n}{4}\Big(368n^6 + 1602n^5 + 2787n^4 + 2690n^3 + 1676n^2 + 544n - 4\Big) > 0,$$

which finally gives $P_1 > P_2$ as $(P_1 + P_2) > 0$ for any $n \in \mathbb{N}$. This proves the desired inequality and hence the lemma. □

**Lemma 4.8.** *Let $n \geq 2$ be a natural number. Then $\beta_n^{(2)} > 9n^2$ holds for each $n \geq 2$.*

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*Proof.* Let us denote $M(n) = \beta_n^{(2)} - 9n^2$. Then it is enough to show that $M(n) > 0$ holds for each $n \geq 2$. A direct computation gives

$$M(n) = n^2(2n-1)^2 + (2n+1)^2(n+1)^2 - n^2(2n-1)^2\left(\frac{(n-1)(2n-1)}{(n+1)(2n+1)}\right)^{-\frac{1}{2}}$$
$$- (2n+1)^2(n+1)^2\left(\frac{(n+2)(2n+3)}{n(2n+1)}\right)^{-\frac{1}{2}} - 9n^2$$
$$= \frac{R(n)}{\sqrt{(n+2)(n-1)(2n-1)(2n+3)}},$$

where simplifying $R(n)$, we get

$$R(n) = \sqrt{(n+2)(n-1)(2n-1)(2n+3)}(8n^4 + 8n^3 + 5n^2 + 6n + 1)$$
$$- n^2(2n-1)^2\sqrt{(n+2)(n+1)(2n+1)(2n+3)}$$
$$- (n+1)^2(2n+1)^2\sqrt{n(n-1)(2n+1)(2n-1)}$$
$$= C_1\sqrt{(n+2)(2n+3)} + C_2\sqrt{(n-1)(2n-1)},$$

where $C_1$ and $C_2$ are defined as below:

$$C_1 = (4n^4 + 2n^3 - \frac{n^2}{2} + n + 1)\sqrt{2n^2 - 3n + 1} - n^2(2n-1)^2\sqrt{2n^2 + 3n + 1},$$
$$C_2 = (4n^4 + 6n^3 + \frac{11n^2}{2} + 5n)\sqrt{2n^2 + 7n + 6} - (n+1)^2(2n+1)^2\sqrt{2n^2 + n}.$$

Using Lemma 4.7, we observed that $C_1 > 0$ and $C_2 > 0$. Therefore, $R(n) > 0$ for all $n \geq 2$. This proves that $M(n) > 0$ for all $n \geq 2$. This proves the lemma. □

*Proof.* (of **Theorem** 4.6):

By choosing $r = s$, $m = 1$, $\lambda_n = \frac{1}{(2n-1)^2 n^2}$ and $\mu_n = \left(\frac{n(n+1)(2n+1)}{6}\right)^{-\frac{1}{2}}$ in inequality (2.2), we can easily establish the L.H.S. of the desired inequality. Since by Lemma 4.8, we have $\beta_n^{(2)} > 9n^2$ point-wise, so R.H.S. of the inequality easily follows. □

Now we prove a last result in this section. It is shown that the improvement of inequality (1.15) is possible. We have the following result.

**Theorem 4.9.** *Let* $A = \{A_n\} \in C_c(\mathbb{N}_0)$ *be such that* $A_0 = A_1 = 0$. *Then*

$$\sum_{n=2}^{\infty} (n-1)n^4|A_n - A_{n-1}|^2 \geq \sum_{n=2}^{\infty} \beta_n^{(3)}|A_n|^2 > 4\sum_{n=2}^{\infty} n^3|A_n|^2 \text{ holds} \tag{4.5}$$

*where for* $n \geq 2$

$$\beta_n^{(3)} = n^4(n-1) + n(n+1)^4 - n^4(n-1)\left(\frac{n-1}{n+1}\right)^{-1} - n(n+1)^4\left(\frac{n+2}{n}\right)^{-1}.$$

*Proof.* Suppose that $r = s$, $m = 1$, $\lambda_n = \frac{1}{(n-1)n^4}$ and $\mu_n = \left(\frac{n(n+1)}{2}\right)^{-1}$, and inserting them in inequality (2.2), one immediately obtains the L.H.S. part of the desired inequality. To establish the R.H.S. part of the same, it is sufficient to prove that $V(n) = \beta_n^{(3)} - 4n^3 > 0$ for each $n \geq 2$. The term $V(n)$ can be

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simplified into the following form:

$$V(n) = n^4(n-1) + n(n+1)^4 - n^4(n-1)\left(\frac{n-1}{n+1}\right)^{-1} - n(n+1)^4\left(\frac{n+2}{n}\right)^{-1} - 4n^3$$
$$= \frac{2n(2n^2+4n+1)}{n+2} > 0$$

for each $n \geq 2$. Hence the proof. □

## 5. An improved multivariate discrete $p$-Hardy's inequality

In the following subsection, we prove that inequality (1.17) admits an improvement. For the sake of clarity, let us first recall inequality (1.4) as below:

$$\sum_{n=1}^{\infty} |A_n - A_{n-1}|^p \geq \sum_{n=1}^{\infty} w_n^{FKP}(p)|A_n|^p > \sum_{n=1}^{\infty} \frac{|A_n|^p}{C_p n^p}.$$

Now onwards, $w_n^{FKP}(p)$ and $w_n(p)$ will share the same expression. For brevity, we will use the latter.

### *5.1. Improved Hardy's inequality for two variables*

Let us begin with the following theorem.

**Theorem 5.1.** *Let $\{a_{mn}\}$ be any sequence of complex numbers and inequality (1.4) of one dimension holds. Then*

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{1}{(C_p)^2 m^p n^p}|A_{mn}|^p < \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} w_{mn}(p)|A_{mn}|^p \leq \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} |a_{mn}|^p, \tag{5.1}$$

*where the sequence $w_{mn}(p)$ is defined as below:*

$$w_{mn}(p) = w_m(p)w_n(p) = \prod_{\theta=m,n}\left\{\left(1-\left(1-\frac{1}{\theta}\right)^{\frac{p-1}{p}}\right)^{p-1} - \left(\left(1+\frac{1}{\theta}\right)^{\frac{p-1}{p}}-1\right)^{p-1}\right\} > \frac{1}{(C_p)^2 m^p n^p}.$$

*Proof.* Since $w_{mn}(p) > \frac{1}{(C_p)^2 m^p n^p}$ holds point-wise for each $m, n \in \mathbb{N}$ so L.H.S. of inequality (5.1) is established. To prove the R.H.S. of inequality (5.1), we begin with

$$\begin{aligned}
&\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} w_{mn}(p)|A_{mn}|^p \\
&= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} w_m(p)w_n(p)\left|\sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}\right|^p \\
&= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} w_m(p)w_n(p)\left|\sum_{j=1}^{n} B_j\right|^p \quad \textit{where} \ \ B_j = \sum_{i=1}^{m} a_{ij} \\
&= \sum_{m=1}^{\infty} w_m(p)\sum_{n=1}^{\infty} w_n(p)\left|\sum_{j=1}^{n} B_j\right|^p \leq \sum_{m=1}^{\infty} w_m(p)\sum_{n=1}^{\infty} |B_n|^p,
\end{aligned}$$

where the last inequality easily follows from $\sum_{n=1}^{\infty} w_n(p)\left|\sum_{j=1}^{n} B_j\right|^p \leq \sum_{n=1}^{\infty} |B_n|^p$.

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Therefore, we have

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} w_{mn}(p)|A_{mn}|^p \le \sum_{m=1}^{\infty} w_m(p) \sum_{n=1}^{\infty}\Big|\sum_{i=1}^{m} a_{in}\Big|^p,$$

and hence

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} w_{mn}(p)|A_{mn}|^p \le \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} w_m(p)\Big|\sum_{i=1}^{m} a_{in}\Big|^p \le \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} |a_{mn}|^p.$$

This proves the desired inequality and completes the proof of the theorem. □

### *5.2. Improved Hardy's inequality for multivariables*

We now give an improved version of inequality (1.19) in the following theorem.

**Theorem 5.2.** *Let $\{a_{m_1 m_2 \cdots m_r}\}$ be a r-fold sequence of complex numbers. Then*

$$\sum_{m_1=1}^{\infty}\sum_{m_1=1}^{\infty}\cdots\sum_{m_r=1}^{\infty} \frac{1}{(C_p)^r\Big(m_1 m_2 \ldots m_r\Big)^p} \mid A_{m_1 m_2 \ldots m_r} \mid^p$$

$$< \sum_{m_1=1}^{\infty}\sum_{m_1=1}^{\infty}\cdots\sum_{m_r=1}^{\infty} w_{m_1 m_2 \ldots m_r}(p) \mid A_{m_1 m_2 \ldots m_r} \mid^p \le \sum_{m_1=1}^{\infty}\sum_{m_1=1}^{\infty}\cdots\sum_{m_r=1}^{\infty} \mid a_{m_1 m_2 \ldots m_r} \mid^p,$$

*where the weight sequence $\{w_{m_1 m_2 \ldots m_r}(p)\}$ is defined as below:*

$$w_{m_1 m_2 \ldots m_r}(p) = \prod_{\theta = m_1, m_2, \ldots, m_r} w_\theta(p) > \frac{1}{(C_p)^r (m_1 m_2 \ldots m_r)^p}.$$

*Proof.* The proof of this theorem will be established by an induction process. Since Theorem 5.2 is true for $r = 1$ and $r = 2$, which gives inequalities (1.4) and (5.1), respectively, so we assume that Theorem 5.2 is true for $r$-fold series. We now prove that the above theorem is true for $(r + 1)$-fold series. Before moving further, we first define

$$P_{i_2 \ldots i_r i_{r+1}} = \sum_{i_1=1}^{m_1} a_{i_1 i_2 \ldots i_r i_{r+1}} \tag{5.2}$$

and

$$Q_{m_2 m_3 \ldots m_r m_{r+1}} = \sum_{i_2=1}^{m_2}\cdots\sum_{i_r=1}^{m_r}\sum_{i_{r+1}=1}^{m_{r+1}} P_{i_2 \ldots i_r i_{r+1}}. \tag{5.3}$$

Using (5.2), we have

$$\sum_{m_1=1}^{\infty}\sum_{m_2=1}^{\infty}\cdots\sum_{m_r=1}^{\infty}\sum_{m_{r+1}=1}^{\infty} w_{m_1 m_2 \ldots m_{r+1}}(p)|A_{m_1 m_2 \ldots m_r m_{r+1}}|^p$$

$$= \sum_{m_1=1}^{\infty}\sum_{m_2=1}^{\infty}\cdots\sum_{m_r=1}^{\infty}\sum_{m_{r+1}=1}^{\infty}\Big(\prod_{\theta=m_1,m_2,\ldots,m_{r+1}} w_\theta(p)\Big)\Big|\sum_{i_1=1}^{m_1}\sum_{i_2=1}^{m_2}\cdots\sum_{i_r=1}^{m_r}\sum_{i_{r+1}=1}^{m_{r+1}} a_{i_1 i_2 \ldots i_r i_{r+1}}\Big|^p$$

$$= \sum_{m_1=1}^{\infty}\sum_{m_2=1}^{\infty}\cdots\sum_{m_r=1}^{\infty}\sum_{m_{r+1}=1}^{\infty}\Big(\prod_{\theta=m_1,m_2,\ldots,m_{r+1}} w_\theta(p)\Big)\Big|\sum_{i_2=1}^{m_2}\cdots\sum_{i_r=1}^{m_r}\sum_{i_{r+1}=1}^{m_{r+1}} P_{i_2 \ldots i_r i_{r+1}}\Big|^p.$$

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Again by using (5.3), one gets

$$\sum_{m_1=1}^{\infty}\sum_{m_2=1}^{\infty}\cdots\sum_{m_r=1}^{\infty}\sum_{m_{r+1}=1}^{\infty} w_{m_1m_2\ldots m_{r+1}}(p)|A_{m_1m_2\ldots m_rm_{r+1}}|^p$$

$$=\sum_{m_1=1}^{\infty} w_{m_1}(p)\Big(\sum_{m_2=1}^{\infty}\cdots\sum_{m_r=1}^{\infty}\sum_{m_{r+1}=1}^{\infty}\Big(\prod_{\theta=m_2,\ldots,m_{r+1}} w_\theta(p)\Big)|Q_{m_2m_3\cdots m_rm_{r+1}}|^p\Big)$$

$$\le\sum_{m_1=1}^{\infty} w_{m_1}(p)\sum_{m_2=1}^{\infty}\cdots\sum_{m_r=1}^{\infty}\sum_{m_{r+1}=1}^{\infty}|P_{m_2m_3\ldots m_rm_{r+1}}|^p \text{ (by (1.19))}$$

$$=\sum_{m_1=1}^{\infty} w_{m_1}(p)\sum_{m_2=1}^{\infty}\cdots\sum_{m_r=1}^{\infty}\sum_{m_{r+1}=1}^{\infty}\Big|\sum_{i_1=1}^{m_1} a_{i_1m_2\cdots m_rm_{r+1}}\Big|^p \text{ (using (5.2)).}$$

Thus Fubini's theorem implies that

$$\sum_{m_1=1}^{\infty}\sum_{m_2=1}^{\infty}\cdots\sum_{m_r=1}^{\infty}\sum_{m_{r+1}=1}^{\infty} w_{m_1m_2\ldots m_{r+1}}(p)|A_{m_1m_2\ldots m_rm_{r+1}}|^p$$

$$\le\sum_{m_2=1}^{\infty}\cdots\sum_{m_r=1}^{\infty}\sum_{m_{r+1}=1}^{\infty}\Big(\sum_{m_1=1}^{\infty} w_{m_1}(p)\Big|\sum_{i_1=1}^{m_1} a_{i_1m_2\ldots m_rm_{r+1}}\Big|^p\Big)$$

$$\le\sum_{m_2=1}^{\infty}\cdots\sum_{m_r=1}^{\infty}\sum_{m_{r+1}=1}^{\infty}\sum_{m_1=1}^{\infty}|a_{m_1m_2\ldots m_rm_{r+1}}|^p \text{ (by 1-fold improved Hardy's inequality)}$$

$$=\sum_{m_1=1}^{\infty}\sum_{m_2=1}^{\infty}\cdots\sum_{m_r=1}^{\infty}\sum_{m_{r+1}=1}^{\infty}|a_{m_1m_2\ldots m_rm_{r+1}}|^p.$$

This proves that Theorem 5.2 is true for $(r+1)$-fold series. Hence, by principal of induction, we conclude that the result is true for all $r\in\mathbb{N}$. This finishes proof of the theorem. □

## 6. Conclusion

In conclusion, this research article has provided affirmative responses to four pivotal questions posed by the authors regarding the improvements of Hardy inequalities. The paper begins by presenting a new version of the one-dimensional discrete improved Hardy's inequality with m-shifts, introducing a shifting discrete Dirichlet's Laplacian. This progression has enabled us to address $Q(a)$–$Q(c)$ comprehensively. Specifically, in our response to $Q(b)$, we establish an improvement of both discrete Hardy inequalities by selecting $q_n$ as some integral powers of $n$, which contributes to affirming the validity of $Q(a)$. In a reply to $Q(c)$, we show that there exist weight sequences for which improvement of the variant Hardy inequalities is possible. Finally, our findings extend beyond the one-dimensional realm, demonstrating improved two-variable discrete $p$-Hardy inequalities and further extending to multivariable cases, thereby confirming $Q(d)$. This research opens up new avenues for exploration and refinement in the field of discrete inequalities.

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