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METRICAL RESULTS ON THE DISTRIBUTION OF FRACTIONAL PARTS OF POWERS OF REAL NUMBERS

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Abstract We establish several new metrical results on the distribution properties of the sequence $(\{x^n\})_{n\geq 1}$, where $\{\cdot\}$ denotes the fractional part. Many of them are presented in a more general framework, in which the sequence of functions $(x \mapsto x^n)_{n\geq 1}$ is replaced by a sequence $(f_n)_{n\geq 1}$, under some growth and regularity conditions on the functions f_n.

Keywords: fractional parts of powers; Diophantine approximation; Hausdorff dimension

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1. Introduction

Let $\{\cdot\}$ denote the fractional part and $\|\cdot\|$ the distance to the nearest integer. For a given real number $x > 1$, little is known about the distribution of the sequence $(\{x^n\})_{n\geq 1}$. For example, we still do not know whether 0 is a limit point of $(\{\mathrm{e}^n\})_{n\geq 1}$, nor of $(\{(\frac{3}{2})^n\})_{n\geq 1}$; see [**5**] for a survey of related results.

However, several metric statements have been established. The first one was obtained in 1935 by Koksma [**13**], who proved that for almost every $x > 1$ the sequence $(\{x^n\})_{n\geq 1}$ is uniformly distributed on the unit interval $[0, 1]$. Here and below, 'almost every' always refers to the Lebesgue measure. In 1967, Mahler and Szekeres [**15**] studied the quantity

$$P(x) := \liminf \|x^n\|^{1/n} \quad (x > 1).$$

They proved that if $P(x) = 0$ then x is transcendental, and $P(x) = 1$ for almost all $x > 1$. The function $x \mapsto P(x)$ was subsequently studied in 2008 by Bugeaud and Dubickas [**6**].

Among other results, it was shown in [6] that, for all $v > u > 1$ and $b > 1$, we have

$$\dim_H\{x \in (u, v) : P(x) \leq 1/b\} = \frac{\log v}{\log(bv)},$$

where $\dim_H$ denotes the Hausdorff dimension.

In a different direction, Pollington [16] showed in 1980 that there are many real numbers $x > 1$ such that $(\{x^n\})_{n\geq 1}$ is very far from being well distributed; namely, he established that, for any $\varepsilon > 0$, we have

$$\dim_H \left\{x > 1 : \{x^n\} < \varepsilon \text{ for all } n\right\} = 1.$$

This result was subsequently extended by Bugeaud and Moshchevitin [8] and, independently, by Kahane [11], who proved that for any $\varepsilon > 0$, for any sequence of real numbers $(y_n)_{n\geq 1}$, we have

$$\dim_H \left\{x > 1 : \|x^n - y_n\| < \varepsilon \text{ for all } n\right\} = 1.$$

In the present paper, we further investigate, from a metric point of view, the Diophantine approximation properties of the sequence $(\{x^n\})_{n\geq 1}$, where $x > 1$, and extend several known results to more general families of sequences $(\{f_n(x)\})_{n\geq 1}$, under some conditions on the sequence of functions $(f_n)_{n\geq 1}$.

As a consequence of our main theorem, we obtain an inhomogeneous version of the result of Bugeaud and Dubickas [6] mentioned above.

Theorem 1. *Let $b > 1$ be a real number and $y = (y_n)_{n\geq 1}$ an arbitrary sequence of real numbers in $[0, 1]$. Set*

$$E(b, y) := \{x > 1 : \|x^n - y_n\| < b^{-n} \text{ for infinitely many } n\}.$$

For every $v > 1$, we have

$$\lim_{\varepsilon\to 0} \dim_H([v - \varepsilon, v + \varepsilon] \cap E(b, y)) = \frac{\log v}{\log(bv)}.$$

In the homogeneous case (that is, the case where $y_n = 0$ for $n \geq 1$), Theorem 1 was proved in [6] by using a classical result of Koksma [14] and the mass transference principle developed by Beresnevich and Velani [3]. The method of [6] still works when y is a constant sequence, but one then needs to apply the inhomogeneous version of Koksma's theorem in [14]. Here, for an arbitrary sequence $(y_n)_{n\geq 1}$, we use a direct construction.

Letting v tend to infinity in Theorem 1, we obtain the following immediate corollary.

Corollary 2. *For an arbitrary sequence y of real numbers in $[0, 1]$ and any real number $b > 1$, the set $E(b, y)$ has full Hausdorff dimension.*

Theorem 1 gives, for every $v > 1$, the value of the localized Hausdorff dimension of $E(b, y)$ at the point v. We stress that in the present context, the localized Hausdorff dimension varies with v, while this is not at all the case for many classical results, including the Jarník–Besicovitch theorem and its extensions. Taking this point of view also allows us to place Theorem 1 in a more general context, where the family of functions $x \mapsto x^n$

is replaced by an arbitrary family of functions f_n satisfying some regularity and growth conditions.

We consider a family of strictly positive increasing C^1 functions $f = (f_n)_{n\geq 1}$ defined on an open interval $I \subset \mathbb{R}$ and such that $f_n(x), f_n'(x) > 1$ for all $x \in I$. For $\tau > 1$, define

$$E(f, y, \tau) := \{x \in I : \|f_n(x) - y_n\| < f_n(x)^{-\tau} \text{ for infinitely many } n\}.$$

For $v \in I$, put

$$u(v) := \limsup_{n\to\infty} \frac{\log f_n(v)}{\log f_n'(v)}, \qquad \ell(v) := \liminf_{n\to\infty} \frac{\log f_n(v)}{\log f_n'(v)}.$$

We will assume the regularity condition

$$\lim_{r\to 0} \limsup_{n\to\infty} \sup_{|x-y|<r} \frac{\log f_n'(x)}{\log f_n'(y)} = 1, \tag{1.1}$$

which guarantees the continuity of the functions u and ℓ.

For nonlinear functions f_n, i.e. when f_n is not of the form $f_n(x) = a_n \cdot x + b_n$, we also need the following condition:

$$M := \sup_{n\geq 1} \frac{\log f_{n+1}'(v)}{\log f_n'(v)} < \infty \quad \text{for all } v \in I. \tag{1.2}$$

Theorem 1 is a particular case of the following general statement.

Theorem 3. *Consider a family of strictly positive increasing C^1 functions $f = (f_n)_{n\geq 1}$ defined on an open interval $I \subset \mathbb{R}$ and such that $f_n(x), f_n'(x) > 1$ for all $x \in I$. Assume (1.1) and (1.2). If for all $x \in I$,*

$$\forall \varepsilon > 0, \qquad \sum_{n=1}^{\infty} f_n'(x)^{-\varepsilon} < \infty, \tag{1.3}$$

then, for any $v \in I$ and any $\tau > 1$, we have

$$\frac{1}{1 + \tau u(v)} \leq \lim_{\varepsilon\to 0} \dim_H([v - \varepsilon, v + \varepsilon] \cap E(f, y, \tau)) \leq \frac{1}{1 + \tau \ell(v)}.$$

If the functions f_n are linear then we do not need to assume (1.2), and the assertion is strengthened to

$$\lim_{\varepsilon\to 0} \dim_H([v - \varepsilon, v + \varepsilon] \cap E(f, y, \tau)) = \frac{1}{1 + \tau \ell(v)}.$$

We remark that the condition (1.3) is satisfied if

$$\forall x \in I, \qquad \lim_{n\to\infty} \frac{\log f_n'(x)}{\log n} = \infty. \tag{1.4}$$

We also observe that the condition (1.1) implies that $\ell(v) \geq 1$ for v in I. In many cases (in particular, for $f_n(x) = x^n$), we have $u(v) = \ell(v) = 1$ for v in I.

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It follows from the formulation of Theorem 3 that the real number τ can be replaced by a continuous function $\tau : I \to (0, \infty)$, in which case the set $E(f, y, \tau)$ is defined by

$$E(f, y, \tau) := \{x \in I : \|f_n(x) - y_n\| < f_n(x)^{-\tau(x)} \text{ for infinitely many } n\}.$$

We get at once the following localized version of Theorem 3. For the classical Jarník–Besicovitch theorem, such a localized theorem was obtained by Barral and Seuret [**2**], who were the first to consider localized Diophantine approximation.

Corollary 4. *With the above notation and under the hypotheses of Theorem 3, we have*

$$\frac{1}{1 + \tau(v)u(v)} \leq \lim_{\varepsilon \to 0} \dim_H([v - \varepsilon, v + \varepsilon] \cap E(f, y, \tau)) \leq \frac{1}{1 + \tau(v)\ell(v)}.$$

We illustrate Theorem 3 and Corollary 4 by some examples. If the family of functions $f = (f_n)_{n \geq 1}$ in Theorem 3 is such that, for every x in I, the sequence $(f_n(x))_{n \geq 1}$ increases sufficiently rapidly, then

$$\lim_{\varepsilon \to 0} \dim_H([v - \varepsilon, v + \varepsilon] \cap E(f, y, \tau)) = \frac{1}{1 + \tau},$$

independently of the family f. This applies, for example, to the families of functions $x^{n^2}, x^n, 2^n x$ and $x^{\sqrt{n}}$.

The case $f_n(x) = a_n x$, where $(a_n)_{n \geq 1}$ is an increasing sequence of positive integers, has been studied by Borosh and Fraenkel [**4**] (but only in the special case of a constant sequence y equal to 0). Let I be an open, non-empty, real interval. They proved that

$$\dim_H\{x \in I : \|a_n x\| < a_n^{-\tau} \text{ for infinitely many } n\} = \frac{1 + s}{1 + \tau},$$

where s (usually called the convergence exponent of the sequence $(a_n)_{n \geq 1}$) is the largest real number in $[0, 1]$ such that

$$\sum_{n \geq 1} a_n^{-s-\varepsilon} \quad \text{converges for any } \varepsilon > 0.$$

The case $s = 0$ of their result, which corresponds to rapidly growing sequences $(a_n)_{n \geq 1}$, follows from Theorem 3. The case $a_n = n$ for $n \geq 1$ corresponds to the Jarník–Besicovitch theorem. We stress that the assumption (1.3) is satisfied only if $(a_n)_{n \geq 1}$ increases sufficiently rapidly.

Questions of uniform Diophantine approximation were recently studied by Bugeaud and Liao [**7**] for the b-ary and β-expansions, and by Kim and Liao [**12**] for the irrational rotations. In this paper, we consider the uniform Diophantine approximation of the sequence $(\{x^n\})_{n \geq 1}$ with $x > 1$.

For a real number $B > 1$ and a sequence of real numbers $y = (y_n)_{n \geq 1}$ in $[0, 1]$, set

$$F(B, y) := \{x > 1 : \text{ for all large integers } N, \ \|x^n - y_n\| < B^{-N} \text{ has a solution } 1 \leq n \leq N\}.$$

Our next theorem gives a lower bound for the Hausdorff dimension of $F(B, y)$ intersected with a small interval.

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Theorem 5. *Let $B > 1$ be a real number and y an arbitrary sequence of real numbers in $[0, 1]$. For any $v > 1$, we have*

$$\lim_{\varepsilon\to 0} \dim_H([v-\varepsilon, v+\varepsilon] \cap F(B, y)) \geq \left(\frac{\log v - \log B}{\log v + \log B}\right)^2.$$

Unfortunately, we are unable to determine whether the inequality in Theorem 5 is an equality. Observe that the lower bound we obtain is the same as the one established in [7] for a question of uniform Diophantine approximation related to b-ary and β-expansions.

Letting v tend to infinity, we have the following corollary.

Corollary 6. *For an arbitrary sequence y of real numbers in $[0, 1]$ and any real number $B > 1$, the set $F(B, y)$ has full Hausdorff dimension.*

We end this paper with results on sequences $(\{x^n\})_{n\geq 1}$, with $x > 1$, which are badly distributed, in the sense that all of their points lie in a small interval. As above, we take a more general point of view. Consider a family of C^1 strictly positive increasing functions $f = (f_n)_{n\geq 1}$ defined on an open interval $I \subset \mathbb{R}$ such that $f_n(x), f_n'(x) > 1$ for all $x \in I$ and for all $n \geq 1$. Let $\delta = (\delta_n)_{n\geq 1}$ be a sequence of positive real numbers such that $\delta_n < 1/4$ for $n \geq 1$. Set

$$G(f, y, \delta) := \{x \in I : \|f_n(x) - y_n\| \leq \delta_n,\ \forall n \geq 1\}.$$

We need the following hypotheses:

$$\forall \varepsilon > 0, \forall n \geq 1, \qquad \frac{\inf_{x\in(v-\varepsilon,v+\varepsilon)} f_{n+1}'(x)}{\sup_{x\in(v-\varepsilon,v+\varepsilon)} f_n'(x)} \cdot \delta_n \geq 2, \tag{1.5}$$

$$\forall x \in I, \qquad \lim_{n\to\infty} \frac{\log f_{n+1}'(x)}{\log f_n'(x)} = \infty. \tag{1.6}$$

Our last main theorem is as follows.

Theorem 7. *Keep the above notation. Under the hypotheses (1.1), (1.5) and (1.6), for all $v \in I$, we have*

$$\lim_{\varepsilon\to 0} \dim_H([v-\varepsilon, v+\varepsilon] \cap G(f, y, \delta)) = \liminf_{n\to\infty} \frac{\log f_n'(v) + \sum_{j=1}^{n-1} \log \delta_j}{\log f_n'(v) - \log \delta_n}. \tag{1.7}$$

We remark that our result extends a recent result of Baker [1]. In fact, in [1], the author studied the special case $f_n(x) = x^{q_n}$, with $(q_n)_{n\geq 1}$ being a strictly increasing sequence of real numbers such that

$$\lim_{n\to\infty} (q_{n+1} - q_n) = +\infty.$$

Our result also gives the following corollary.

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Corollary 8. *Let $(a_n)_{n\geq 1}$ be a sequence of positive real numbers such that*

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = +\infty.$$

Then, for any sequence $(y_n)_{n\geq 1}$ of real numbers, we have

$$\dim_H\{x \in \mathbb{R} : \lim_{n\to+\infty} \|a_n x - y_n\| = 0\} = 1.$$

2. Basic tools

We present two lemmas which serve as important tools for estimating the Hausdorff dimension of the sets studied in this paper.

Let $[0,1] = E_0 \supset E_1 \supset E_2 \supset \cdots$ be a decreasing sequence of sets, with each E_k a finite union of disjoint closed intervals. The components of E_k are called kth-level basic intervals. Set $F = \cap_{k=0}^{\infty} E_k$. We do not assume that each basic interval in E_{k-1} contains the same number of next-level basic intervals, nor that they are of the same length, nor that the gaps between two consecutive basic intervals are equal. Instead, for $x \in E_{k-1}$, we denote by $m_k(x)$ the number of kth-level basic intervals contained in the $(k-1)$th-level basic interval containing x, and by $\tilde{\varepsilon}_k(x)$ the minimal distance between two of them. Set

$$\varepsilon_k(x) = \min_{i\leq k} \tilde{\varepsilon}_i(x).$$

In the following, we generalize a lemma from Falconer's book [9, Example 4.6].

Lemma 9. *For any open interval $I \subset [0,1]$ intersecting F, we have*

$$\dim_H(I \cap F) \geq \inf_{x\in I\cap F} \liminf_{k\to\infty} \frac{\log(m_1(x)\cdots m_{k-1}(x))}{-\log(m_k(x)\varepsilon_k(x))}.$$

Proof. The proof is similar to that in the book of Falconer. We define a probability measure μ on F by assigning the mass evenly. Precisely, for $k \geq 1$, let $I_k(x)$ be the kth-level interval containing x. For $x \in F$ and $k \geq 1$, we assign a mass $(m_1(x)\cdots m_k(x))^{-1}$ to the interval $I_k(x)$. Note that any two kth-level basic intervals contained in the same $(k-1)$th-level interval have the same measure. One can check that the measure μ is well defined. Now let us calculate the local dimension at the point x. Let $B(x,r)$ be the ball of radius r centred at x. Suppose that $\varepsilon_k(x) \leq 2r < \varepsilon_{k-1}(x)$. The number of kth-level intervals intersecting $B(x,r)$ is at most

$$\min\left\{m_k(x),\ \frac{2r}{\varepsilon_k(x)}+1\right\} \leq \min\left\{m_k(x),\ \frac{4r}{\varepsilon_k(x)}\right\} \leq m_k(x)^{1-s}\left(\frac{4r}{\varepsilon_k(x)}\right)^s,$$

for any $s \in [0,1]$. Thus

$$\mu(B(x,r)) \leq m_k(x)^{1-s}\left(\frac{4r}{\varepsilon_k(x)}\right)^s \cdot (m_1(x)\cdots m_k(x))^{-1}.$$

Hence

$$\frac{\log\mu(B(x,r))}{\log r} \geq \frac{s\log m_k(x)\varepsilon_k(x) - s\log(4r) + \log(m_1(x)\cdots m_{k-1}(x))}{-\log r}.$$

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Let s be in $(0,1)$ such that

$$s < \inf_{z\in I\cap F} \liminf_{k\to\infty} \frac{\log(m_1(z)\cdots m_{k-1}(z))}{-\log m_k(z)\varepsilon_k(z)} \le \liminf_{k\to\infty} \frac{\log(m_1(x)\cdots m_{k-1}(x))}{-\log m_k(x)\varepsilon_k(x)}.$$

Then

$$s\log m_k(x)\varepsilon_k(x) - s\log 4 + \log(m_1(x)\cdots m_{k-1}(x)) \ge 0,$$

for k large enough. Therefore

$$\liminf_{r\to 0} \frac{\log \mu(B(x,r))}{\log r} \ge s.$$

The proof is completed by applying the mass distribution principle (see [**10**, Proposition 2.3]). □

We also have an upper bound for the dimension of the set $I\cap F$. Denote by $|I_k(x)|$ the length of the kth-level basic interval $I_k(x)$ containing x.

Lemma 10. *For any open interval $I \subset [0,1]$ intersecting F, we have*

$$\dim_H(I\cap F) \le \sup_{x\in I\cap F} \liminf_{k\to\infty} \frac{\log(m_1(x)\cdots m_k(x))}{-\log|I_k(x)|}.$$

Proof. We define the same probability measure μ as in Lemma 9, i.e. the interval $I_k(x)$ has measure $(m_1(x)\cdots m_k(x))^{-1}$. Then

$$\liminf_{r\to 0} \frac{\log \mu(B(x,r))}{\log r} \le \liminf_{k\to\infty} \frac{\log \mu(I_k(x))}{-\log|I_k(x)|} = \liminf_{k\to\infty} \frac{\log(m_1(x)\cdots m_k(x))}{-\log|I_k(x)|}.$$

We finish the proof by again applying the mass distribution principle (see [**10**, Proposition 2.3]). □

3. Asymptotic approximation

In this section, we prove Theorem 3. To see that Theorem 1 is a special case of it, take the family of functions f defined by

$$f_n(x) = x^n, \qquad \forall n \ge 1.$$

We have $u(v) = \ell(v) = 1$ and

$$\begin{aligned}[v-\varepsilon, v+\varepsilon] \cap E\left(f, y, \frac{\log b}{\log(v+\varepsilon)}\right) &\subset [v-\varepsilon, v+\varepsilon]\cap E(b,y)\\ &\subset [v-\varepsilon, v+\varepsilon] \cap E\left(f, y, \frac{\log b}{\log(v-\varepsilon)}\right).\end{aligned}$$

Then, Theorem 1 follows directly from Theorem 3.

Now we prove Theorem 3.

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Proof of Theorem 3. Lower bound: We can assume that $u(v)$ is finite, since otherwise there is nothing to prove. Let us start with a simple observation about the condition (1.1). Given an integer $n \geq 1$, set

$$\eta(n) = \sup\left\{\frac{\log f_n'(w)}{\log f_n'(z)} - 1; w, z \in [v-\varepsilon, v+\varepsilon], |f_n(w) - f_n(z)| \leq 1\right\}. \tag{3.1}$$

Lemma 11. *If (1.1) and (1.3) hold, then*

$$\lim_{n\to\infty} \eta(n) = 0.$$

Proof. Assume this is not true. Then there exists a sequence of integers (n_i) and a sequence of pairs of points (w_i, z_i) such that $|f_{n_i}(w_i) - f_{n_i}(z_i)| \leq 1$ and

$$\frac{\log f_{n_i}'(w_i)}{\log f_{n_i}'(z_i)} > Z > 1.$$

By compactness of $[v-\varepsilon, v+\varepsilon]$, taking a subsequence if necessary, we can assume that $(w_i)_{i\geq 1}$ converges to some point w_0.

By (1.3), $f_n'(v) \to \infty$. Hence, (1.1) gives us

$$\lim_{n\to\infty} \inf_{x\in[v-\varepsilon,v+\varepsilon]} f_n'(x) = \infty.$$

This implies that

$$|w_i - z_i| \leq \frac{1}{\inf_{x\in[v-\varepsilon,v+\varepsilon]} f_{n_i}'(x)} \to 0$$

as $i \to \infty$, and hence any neighbourhood of w_0 contains all except finitely many points w_i, z_i. Thus, in any neighbourhood U of w_0, we have

$$\limsup_{n\to\infty} \sup_{w,z\in U} \frac{\log f_n'(w)}{\log f_n'(z)} > Z,$$

which contradicts (1.1). □

Now we construct a nested Cantor set, which is the intersection of unions of subintervals at level n_i, where $(n_i)_{i\geq 1}$ is an increasing sequence of positive integers that will be defined precisely later. Suppose we have already well chosen this subsequence. Let us describe the nested family of subintervals. For each level i, we need to consider the set of points x such that

$$\|f_{n_i}(x) - y_{n_i}\| \leq f_{n_i}(x)^{-\tau}.$$

By the property $\|f_{n_1}(x) - y_{n_1}\| \leq f_{n_1}(x)^{-\tau}$, we take the intervals at level 1 as

$$I_1(k, v, f, y, \tau) := [f_{n_1}^{-1}(k + y_{n_1} - f_{n_1}(v+\varepsilon)^{-\tau}), \quad f_{n_1}^{-1}(k + y_{n_1} + f_{n_1}(v+\varepsilon)^{-\tau})],$$

with k being an integer in $[f_{n_1}(v-\varepsilon) + 1, f_{n_1}(v+\varepsilon) - 1]$.

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Suppose we have constructed the intervals at level $i-1$. Let $[c_{i-1}, d_{i-1}]$ be an interval at this level. A subinterval of $[c_{i-1}, d_{i-1}]$ at level i is such that

$$[f_{n_i}^{-1}(k + y_{n_i} - f_{n_i}(d_{i-1})^{-\tau}), \quad f_{n_i}^{-1}(k + y_{n_i} + f_{n_i}(d_{i-1})^{-\tau})],$$

with k being an integer in $[f_{n_i}(c_{i-1}) + 1, \; f_{n_i}(d_{i-1}) - 1]$. By continuing this construction, we obtain intervals $I_i(\cdot)$ for all levels.

Finally, the intersection F of these nested intervals is obviously a subset of $[v - \varepsilon, v + \varepsilon] \cap E(f, y, \tau)$.

Let $z \in F$ and $[c_i(z), d_i(z)]$ be the ith-level interval containing z. Then we have

$$m_{i+1}(z) \geq f'_{n_{i+1}}(w_i) \cdot (d_i - c_i) - 2 \geq f'_{n_{i+1}}(w_i) \cdot \frac{2 f_{n_i}(d_i)^{-\tau}}{f'_{n_i}(z_i)} - 2, \tag{3.2}$$

where $w_i, z_i \in [c_i(z), d_i(z)]$. Furthermore,

$$\varepsilon_{i+1}(z) \geq \frac{1 - 2 f_{n_{i+1}}(c_i(z))^{-\tau}}{f'_{n_{i+1}}(u_i)} \geq \frac{1}{2 f'_{n_{i+1}}(u_i)}, \tag{3.3}$$

where $u_i \in [c_i(z), d_i(z)]$.

Now we are going to define the subsequence $(n_i)_{i \geq 1}$.

Lemma 12. *Assume (1.1) and (1.2). For any $\gamma > 0$, we can find a subsequence $(n_i)_{i \geq 1}$ such that*

$$\frac{f'_{n_{i+1}}(w)}{f'_{n_{i+1}}(u)} \leq f'_{n_i}(z)^{\gamma} \quad \forall w, u \in [c_i(z), d_i(z)], \tag{3.4}$$

and for any small $\varepsilon > 0$, we have

$$\forall x \in (v - \varepsilon, v + \varepsilon), \qquad \lim_{i \to \infty} \frac{\log f'_{n_i}(x)}{\log f'_{n_{i-1}}(x)} = \lim_{i \to \infty} \frac{\log f'_{n_i}(x)}{\log f_{n_{i-1}}(x)} = \infty, \tag{3.5}$$

and

$$\frac{\inf_{x \in (v-\varepsilon, v+\varepsilon)} f'_{n_{i+1}}(x)}{\sup_{x \in (v-\varepsilon, v+\varepsilon)} f'_{n_i}(x) \cdot f_{n_i}(x)^{\tau}} \geq 2. \tag{3.6}$$

If f_n are linear then we do not need to assume (1.2); moreover, we can choose (n_i) in such a way that we have (in addition to the other parts of the assertion)

$$\lim_{i \to \infty} \frac{\log f_{n_i}(v)}{\log f'_{n_i}(v)} = \ell(v). \tag{3.7}$$

Proof. In the linear case (3.4) is automatically true, and to have (3.5) and (3.6) we just need that $(n_i)_{i \geq 1}$ increases sufficiently fast (as will be clear from the proof for the general case). Hence, we will be free to choose (n_i), satisfying in addition (3.7).

Let us proceed with the general case. For any $\gamma > 0$, by Lemma 11, there exists $n_0 \in \mathbb{N}$ such that

$$\forall n \geq n_0, \qquad \eta(n) < \frac{\gamma}{2M},$$

where M is the constant in assumption (1.2).

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Starting with this n_0, by assumption (1.2), we can construct a subsequence $(n_i)_{i\geq 1}$ satisfying

$$\frac{\gamma}{2\eta(n_i)\cdot M} \leq \frac{\log f'_{n_{i+1}}(v)}{\log f'_{n_i}(v)} \leq \frac{\gamma}{2\eta(n_i)}. \tag{3.8}$$

Observe that as $\eta(n_i) \to 0$ by Lemma 11, the left-hand side of (3.8) implies the first part of (3.5). As $u < \infty$, the second part of (3.5) follows. The condition (3.6) will also follow, provided that n_0 was selected large enough.

We need now to prove (3.4). By (3.1), for any w, u in the interval $[c_i(z), d_i(z)]$,

$$\frac{f'_{n_{i+1}}(w)}{f'_{n_{i+1}}(u)} \leq \frac{f'_{n_{i+1}}(z)^{1+\eta(n_i)}}{f'_{n_{i+1}}(z)^{1-\eta(n_i)}} = f'_{n_{i+1}}(z)^{2\eta(n_i)}. \tag{3.9}$$

Combining (3.8) and (3.9), we get (3.4). □

We continue the proof of the lower bound of Theorem 3. By (3.2) and (3.6),

$$m_{i+1}(z) \geq f'_{n_{i+1}}(w_i)\cdot \frac{2f_{n_i}(d_i)^{-\tau}}{f'_{n_i}(z_i)} - 2 \geq 2,$$

which implies that F is non-empty. Further, by (3.4), for any $\gamma > 0$,

$$m_{i+1}(z) \geq f'_{n_{i+1}}(z)\cdot f'_{n_i}(z)^{-\gamma}\cdot \frac{f_{n_i}(d_i)^{-\tau}}{f'_{n_i}(z_i)}. \tag{3.10}$$

By (3.2)–(3.4),

$$m_{i+1}(z)\varepsilon_{i+1}(z) \geq f'_{n_i}(z)^{-\gamma}\cdot \frac{f_{n_i}(d_i)^{-\tau}}{2f'_{n_i}(z_i)}. \tag{3.11}$$

Thus, (1.3) and (3.5) imply that $-\log m_{i+1}(z)\varepsilon_{i+1}(z)$ is unbounded. So, by (3.10), (3.11) and (3.5),

$$\begin{aligned}
&\liminf_{i\to\infty} \frac{\log(m_2(z)\cdots m_i(z))}{-\log m_{i+1}(z)\varepsilon_{i+1}(z)} \\
&\quad\geq \liminf_{i\to\infty} \frac{\sum_{j=2}^{i}(\log f'_{n_j}(z) - \gamma\log f'_{n_{j-1}}(z) - \tau\log f_{n_{j-1}}(d_j) - \log f'_{n_{j-1}}(z_j))}{\log 2 + \log f'_{n_i}(z_i) + \gamma\log f'_{n_i}(z) + \tau\log f_{n_i}(d_i)} \\
&\quad= \liminf_{i\to\infty} \frac{\log f'_{n_i}(z)}{\log f'_{n_i}(z_i) + \gamma\log f'_{n_i}(z) + \tau\log f_{n_i}(d_i)}.
\end{aligned}$$

Hence, by the definition of $\eta(n_i)$, we have

$$\begin{aligned}
&\liminf_{i\to\infty} \frac{\log(m_2(z)\cdots m_i(z))}{-\log m_{i+1}(z)\varepsilon_{i+1}(z)} \\
&\quad\geq \frac{1}{\limsup_{i\to\infty}(1+\eta(n_i)+\gamma+\tau(1+\eta(n_i))\cdot(\log f_{n_i}(d_i))/(\log f'_{n_i}(d_i)))}.
\end{aligned}$$

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In the linear case, $\log f_{n_i}(d_i)/\log f'_{n_i}(d_i)$ converges to $\ell(\lim_{i\to\infty} d_i)$. In the general situation, we have

$$\limsup_{i\to\infty} \frac{\log f_{n_i}(d_i)}{\log f'_{n_i}(d_i)} \le u(\lim_{i\to\infty} d_i).$$

As γ can be chosen arbitrarily small, $\eta(n_i) \to 0$ by Lemma 11 and

$$\lim_{i\to\infty} d_i \in [v-\varepsilon, v+\varepsilon],$$

the lower bound is obtained by applying Lemma 9.

Upper bound: Since for all $x \in [v-\varepsilon, v+\varepsilon] \cap E(f, y, \tau)$, we have

$$\|f_n(x) - y_n\| < f_n(x)^{-\tau}$$

for infinitely many $n \ge 1$. Then the set $[v-\varepsilon, v+\varepsilon] \cap E(f, y, \tau)$ is covered by the union of the family of intervals

$$I_n(k) := \left[f_n^{-1}(k + y_n - f_n(v-\varepsilon)^{-\tau}), f_n^{-1}(k + y_n + f_n(v-\varepsilon)^{-\tau})\right],$$

where $k \in [f_n(v-\varepsilon),\ f_n(v+\varepsilon)]$ is an integer. Note that the length of the interval $I_n(k)$ satisfies

$$|I_n(k)| \le \frac{2f_n(v-\varepsilon)^{-\tau}}{f'_n(z)} \quad \text{for some } z \in (v-\varepsilon, v+\varepsilon).$$

The number of intervals at level n is less than

$$f_n(v+\varepsilon) - f_n(v-\varepsilon) \le 2\varepsilon f'_n(w) \quad \text{for some } w \in (v-\varepsilon, v+\varepsilon).$$

Thus, for $s > 0$

$$\sum_{n=1}^{\infty} \sum_{k\in[f_n(v-\varepsilon), f_n(v+\varepsilon)]} |I_n(k)|^s \le \sum_{n=1}^{\infty} 2\varepsilon f'_n(w) \cdot \left(\frac{2f_n(v-\varepsilon)^{-\tau}}{f'_n(z)}\right)^s. \tag{3.12}$$

By the definition of $\ell(v)$, for any $\eta > 0$, there exists $n_0 = n_0(\eta) \in \mathbb{N}$ such that for any $n \ge n_0$

$$f_n(v-\varepsilon) > f'_n(v-\varepsilon)^{\ell(v-\varepsilon)-\eta}.$$

Thus, by ignoring the first n_0 terms, we have that (3.12) is bounded by

$$2^{1+s}\varepsilon \sum_{n=n_0}^{\infty} f'_n(w) \cdot f'_n(z)^{-s} \cdot (f'_n(v-\varepsilon))^{-\tau s(\ell(v-\varepsilon)-\eta)}. \tag{3.13}$$

Hence, by the assumption (1.3), if

$$s > \limsup_{n\to\infty} \frac{\log f'_n(w)}{\log f'_n(z) + \tau(\ell(v-\varepsilon)-\eta)\log f'_n(v-\varepsilon)}$$

the sum in (3.12) converges. By (1.1),

$$\lim_{n\to\infty} \frac{\log f'_n(w)}{\log f'_n(z)} = 1, \qquad \lim_{n\to\infty} \frac{\log f'_n(w)}{\log f'_n(v-\varepsilon)} = 1.$$

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Therefore

$$\lim_{\varepsilon\to 0} \dim_H [v-\varepsilon, v+\varepsilon] \cap E(f,y,\tau) \le \frac{1}{1+\tau\ell(v)}. \qquad \square$$

4. Uniform Diophantine approximation

In this section, we study the uniform Diophantine approximation of the sequence $(\{x^n\})_{n\ge 1}$ with $x>1$.

Recall that for any sequence of real numbers $y=(y_n)_{n\ge 1}$ in $[0,1]$, we are interested in the set

$$F(B,y) := \{x>1 : \text{ for all large integers } N,\ \|x^n - y_n\| < B^{-N} \text{ has a solution } 1\le n\le N\}.$$

For any $v\in F(B,y)$, for any $\varepsilon>0$, we will give a lower bound for the Hausdorff dimension of $[v-\varepsilon, v+\varepsilon]\cap F(B,y)$. To this end, we investigate the uniform Diophantine approximation and asymptotic Diophantine approximation together. We consider the following subset of $[v-\varepsilon, v+\varepsilon]\cap F(B,y)$

$$F(v,\varepsilon,b,B,y) := \{z\in[v-\varepsilon,v+\varepsilon] : \|z^n - y_n\| < b^{-n} \text{ for infinitely many } n, \text{ and } \forall N \gg 1, \|z^n - y_n\| < B^{-N} \text{ has a solution } 1\le n\le N\}.$$

The proof of Theorem 5 will be completed by maximizing the lower bounds of $F(v,\varepsilon,b,B,y)$ with respect to $b>B$.

Proof of Theorem 5. We first construct a subset $F\subset F(v,\varepsilon,b,B,y)$. Suppose that $b=B^\theta$ with $\theta>1$. Let $n_k=\lfloor\theta^k\rfloor$. Consider the points z such that

$$\|z^{n_k} - y_{n_k}\| < b^{-n_k}.$$

Then one can check that

$$z\in F(v,\varepsilon,b,B,y) = F(v,\varepsilon,B^\theta,B,y) = F(v,\varepsilon,b,b^{1/\theta},y).$$

By the same construction as in Section 3, we obtain a Cantor set $F\subset F(v,\varepsilon,b,b^{1/\theta},y)$, which is the intersection of a nested family of intervals with

$$m_k(z) = \frac{2n_{k+1}c_k(z)^{n_{k+1}-1}}{n_k b^{n_k} d_k(z)^{n_k-1}}$$

and

$$\varepsilon_k(z) = \left(1-\frac{2}{b^{n_{k+1}}}\right)\frac{1}{n_{k+1}d_k(z)^{n_{k+1}-1}},$$

where $[c_k(z), d_k(z)]$ is the kth-level interval containing z.

By the choice of n_k, we have the following estimations:

$$m_k(z) \geq \frac{2(\theta^{k+1}-1)c_k(z)^{\theta^{k+1}-1}}{\theta^k b^{\theta^k} d_k(z)^{\theta^k-1}} \geq \theta \cdot b^{-\theta^k} \cdot \left(\frac{c_k(z)}{d_k(z)}\right)^{\theta^k} \cdot c_k(z)^{\theta^k(\theta-1)}$$

and

$$\varepsilon_k(z) \geq \frac{1}{2\theta^{k+1}} \cdot d_k(z)^{-\theta^{k+1}}.$$

Since

$$d_k(z) - c_k(z) \leq \frac{b^{-n_k}}{n_k c_k(z)^{n_k-1}} \leq b^{-\theta^k}$$

is much more smaller than $1/\theta^k$,

$$\left(\frac{c_k(z)}{d_k(z)}\right)^{\theta^k} = \left(1 - \frac{d_k(z)-c_k(z)}{d_k(z)}\right)^{\theta^k} \geq \frac{1}{2}.$$

Then

$$m_k(z) \geq \frac{\theta}{2} \cdot \left(\frac{c_k(z)^{\theta-1}}{b}\right)^{\theta^k} \geq \frac{\theta}{2^{\theta+1}}\left(\frac{z^{\theta-1}}{b}\right)^{\theta^k}$$

and

$$m_k(z)\varepsilon_k(z) \geq \frac{1}{4\theta^k} \cdot \left(\frac{c_k(z)^{\theta-1}}{b \cdot d_k(z)^{\theta}}\right)^{\theta^k} \geq \frac{1}{2^{\theta+3}\theta^k}\left(\frac{1}{bz}\right)^{\theta^k}.$$

Thus, by Lemma 9, for any $z \in F$, we have

$$\begin{aligned}
&\liminf_{k\to\infty} \frac{\log(m_1(z)\cdots m_{k-1}(z))}{-\log m_k(z)\varepsilon_k(z)} \\
&\quad \geq \liminf_{k\to\infty} \frac{((\theta-1)\log z - \log b)\sum_{j=1}^{k-1}\theta^j}{\theta^k \log bz} \\
&\quad = \liminf_{k\to\infty} \frac{(\theta-1)\log z - \log b}{(\theta-1)\log bz} \cdot \frac{\theta^{k-1}-1}{\theta^{k-1}} \\
&\quad = \frac{(\theta-1)\log z - \log b}{(\theta-1)\log bz}.
\end{aligned}$$

Hence, by the relation $b = B^\theta$, we deduce that the Hausdorff dimension of the set $F(v,\varepsilon,b,B,y) = F(v,\varepsilon,B^\theta,B,y)$ is at least equal to

$$\frac{(\theta-1)\log(v-\varepsilon) - \log b}{(\theta-1)\log(b(v-\varepsilon))} = \frac{\log(v-\varepsilon) - (\theta/(\theta-1))\log B}{\log(v-\varepsilon) + \theta\log B}.$$

Taking $\theta \to \infty$ on the left-hand side of the equality, we get the lower bound $\log(v-\varepsilon)/\log(b(v-\varepsilon))$ for the Hausdorff dimension of the set considered in Theorem 1:

$$\begin{aligned}
&[v-\varepsilon, v+\varepsilon] \cap E(b,y) \\
&\quad = \{v-\varepsilon \leq x \leq v+\varepsilon : \|x^n - y_n\| < b^{-n} \text{ for infinitely many } n\}.
\end{aligned}$$

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By maximizing the right-hand side of the equality with respect to $\theta > 1$, we obtain the lower bound

$$\left(\frac{\log(v-\varepsilon)-\log B}{\log(v-\varepsilon)+\log B}\right)^2$$

for the Hausdorff dimension of the set

$$\begin{aligned}&[v-\varepsilon, v+\varepsilon] \cap F(B,y)\\ &\quad = \{v-\varepsilon \le x \le v+\varepsilon : \forall N \gg 1, ||x^n - y|| < B^{-N} \text{ has a solution } 1 \le n \le N\}.\end{aligned}$$

Letting ε tend to 0, this completes the proof of Theorem 5. □

5. Bad approximation

In this section, we study the bad approximation properties of the sequence $(\{x^n\})_{n\ge1}$, where $x > 1$.

Let $q = (q_n)_{n\ge1}$ be a sequence of positive real numbers and $y = (y_n)_{n\ge1}$ an arbitrary sequence of real numbers in $[0, 1]$. Define

$$G(q,y) = \{x > 1 : \lim_{n\to\infty} \|x^{q_n} - y_n\| = 0\}$$

and, for $v > 1$, define

$$G(v,q,y) = \{1 < x < v : \lim_{n\to\infty} \|x^{q_n} - y_n\| = 0\}.$$

Recently, Baker [1] showed that if $q = (q_n)_{n\ge1}$ is strictly increasing and

$$\lim_{n\to\infty} (q_{n+1} - q_n) = \infty,$$

then the set $G(q, y)$ has Hausdorff dimension 1.

We want to generalize Baker's result. Consider a family of C^1 functions $f = (f_n)_{n\ge1}$ from an interval $I \subset \mathbb{R}$ to $\mathbb{R}$, such that $f_n'(x) \ge 1$ for all $x \in I$ and for all $n \ge 1$. Let $\delta = (\delta_n)_{n\ge1}$ be a sequence of positive real numbers tending to 0. For $\varepsilon > 0$, set

$$G(\varepsilon, v, f, y, \delta) := \{v-\varepsilon < x < v+\varepsilon : \|f_n(x) - y_n\| \le \delta_n,\ \forall n \ge 1\}.$$

To prove Theorem 7, we need to estimate $\dim_H G(\varepsilon, v, f, y, \delta)$.

Sketch proof of Theorem 7. Lower bound. We do the same construction as in the proof of the lower bound in Theorem 3. If the right-hand-side inequality in (3.8) is satisfied, that is, if

$$\frac{\log f_{n+1}'(v)}{\log f_n'(v)} \le \frac{\gamma}{2\eta(n)}, \tag{5.1}$$

for some $\gamma > 0$, for large enough n and for η defined in (3.1), then the distortion estimation (3.4) holds and we estimate the dimension in exactly the same way as in Theorem 3.

If, however, (5.1) is not satisfied, that is, in some place f_n' is too sparse, with $\log f_{n+1}'(v) \gg \log f_n'(v)$, we can apply the idea of Baker [1, p. 69]. We add some new functions $\tilde{f}_m$ between f_n and f_{n+1} in such a way that the resulting, expanded sequence of their logarithms of derivatives is not too sparse anymore. We also add some $\tilde{\delta}_m = 1$ for each added $\tilde{f}_m$. Observe that the right-hand side of (1.7) does not change. Naturally, the resulting set $G(\varepsilon, v, \tilde{f}, y, \tilde{\delta})$ is exactly the same as $G(\varepsilon, v, f, y, \delta)$. So, for the lower bound, we need only to estimate the lower bound of $\dim_H G(\varepsilon, v, \tilde{f}, y, \tilde{\delta})$. This means we can freely assume that (5.1) holds.

We next construct a subset of $G(\varepsilon, v, f, y, \delta)$ which is the intersection of a nested family of subintervals $I_n(\cdot)$.

For $n = 1$, by the property $\|f_1(x) - y_1\| \le \delta_1$, we take the intervals at level 1 as

$$I_1(k, v, f, y, \delta) := [f_1^{-1}(k + y_1 - \delta_1), f_1^{-1}(k + y_1 + \delta_1)],$$

with k being an integer in $[f_1(v - \varepsilon) + 1, f_1(v + \varepsilon) - 1]$.

Suppose we have constructed the intervals at level $n - 1$. Let $[c_{n-1}, d_{n-1}]$ be an interval at this level. A subinterval of $[c_{n-1}, d_{n-1}]$ at level n is

$$[f_n^{-1}(k + y_n - \delta_n), f_n^{-1}(k + y_n + \delta_n)],$$

with k being an integer in $[f_n(c_{n-1}) + 1, f_n(d_{n-1}) - 1]$. By continuing this construction, we obtain intervals $I_n(\cdot)$ for all levels. Finally, the intersection F of these nested intervals is obviously a subset of $G(\varepsilon, v, f, y, \delta)$.

Let $z \in F$ and $[c_n(z), d_n(z)]$ be the nth-level interval containing z. Then, by (1.5),

$$m_{n+1}(z) \ge f'_{n+1}(w_n) \cdot (d_n - c_n) - 2 \ge f'_{n+1}(w_n) \cdot \frac{2\delta_n}{f'_n(z_n)} - 2 \ge 2$$

and

$$\varepsilon_{n+1}(z) \ge \frac{1 - 2\delta_{n+1}}{f'_{n+1}(u_n)} \ge \frac{1}{2f'_{n+1}(u_n)},$$

with $w_n, z_n, u_n \in [c_n(z), d_n(z)]$. As we are assuming (5.1), we have (3.4), and then for any $\gamma > 0$

$$m_{n+1}(z) \ge f'_{n+1}(z) \cdot f'_n(z)^{-\gamma} \cdot \frac{\delta_n}{f'_n(z_n)}$$

and

$$m_{n+1}(z)\varepsilon_{n+1}(z) \ge f'_n(z)^{-\gamma} \cdot \frac{\delta_n}{2f'_n(z_n)}.$$

Thus,

$$\begin{aligned} &\frac{\log(m_2(z) \cdots m_n(z))}{-\log m_{n+1}(z)\varepsilon_{n+1}(z)} \\ &\quad \ge \frac{\log f'_n(z) - \log f'_1(z) + \sum_{j=1}^{n-1} \log \delta_j + \sum_{j=1}^{n-1} \log((f'_j(z)^{-\gamma})/(f'_j(z_j)))}{\log 2 + \log f'_n(z_n) + \gamma \log f'_n(z) - \log \delta_n}. \end{aligned}$$

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By (1.6), we have

$$\liminf_{n\to\infty} \frac{\log(m_2(z)\cdots m_n(z))}{-\log m_{n+1}(z)\varepsilon_{n+1}(z)}$$
$$\geq \liminf_{n\to\infty} \frac{\log f_n'(z) + \sum_{j=1}^{n-1} \log \delta_j}{\log f_n'(z_n) + \gamma \log f_n'(z) - \log \delta_n}.$$

Since γ can be chosen to be arbitrary small and z_n tends to z, by (1.1) we have

$$\liminf_{n\to\infty} \frac{\log(m_2(z)\cdots m_n(z))}{-\log m_{n+1}(z)\varepsilon_{n+1}(z)} \geq \liminf_{n\to\infty} \frac{\log f_n'(z) + \sum_{j=1}^{n-1} \log \delta_j}{\log f_n'(z) - \log \delta_n}.$$

Hence, the lower bound of Theorem 7 is obtained by Lemma 9.

Upper bound. We will apply Lemma 10. For each basic interval $I_n(z)$, by (1.1) we have, for any γ and for n large enough

$$\frac{\delta_n}{f_n'(z) f_{n-1}'(z)^\gamma} \leq |I_n(z)| \leq \frac{\delta_n f_{n-1}'(z)^\gamma}{f_n'(z)}.$$

Thus,

$$m_n(z) \leq |I_{n-1}(z)| \cdot f_n'(z) f_{n-1}'(z)^\gamma \leq \frac{\delta_{n-1} f_{n-2}'(z)^\gamma}{f_{n-1}'(z)} f_n'(z) f_{n-1}'(z)^\gamma.$$

Hence,

$$\prod_{j=2}^{n} m_j(z) \leq f_n'(z) \cdot \prod_{j=1}^{n-1} \delta_j \cdot \frac{\prod_{j=1}^{n-1} f_j'(z)^{2\gamma}}{f_1'(z)}.$$

Therefore, by (1.6),

$$\liminf_{n\to\infty} \frac{\log(m_1(z)\cdots m_n(z))}{-\log |I_n(z)|} \leq \liminf_{n\to\infty} \frac{\log f_n'(z) + \sum_{j=1}^{n-1} \log \delta_j}{\log f_n'(z) - \log \delta_n}.$$

By Lemma 10, we conclude the proof. □

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References

1. S. Baker, On the distribution of powers of real numbers modulo 1, *Unif. Distrib. Theory* **10**(2) (2015), 67–75.
2. J. Barral and S. Seuret, A localized Jarnik–Besicovich theorem, *Adv. Math.* **226**(4) (2011), 3191–3215.
3. V. Beresnevich and S. Velani, A mass transference principle and the Duffin–Schaeffer conjecture for Hausdorff measures, *Ann. Math. (2)* **164**(3) (2006), 971–992.
4. I. Borosh and A. S. Fraenkel, A generalization of Jarník's theorem on Diophantine approximations, *Indag. Math.* **34** (1972), 193–201.

5. Y. Bugeaud, *Distribution modulo one and Diophantine approximation*. Cambridge Tracts in Mathematics, Volume 193 (Cambridge University Press, 2012).
6. Y. Bugeaud and A. Dubickas, On a problem of Mahler and Szekeres on approximation by roots of integers, *Michigan Math. J.* **56** (2008), 703–715.
7. Y. Bugeaud and L. Liao, Uniform Diophantine approximation related to b-ary and β-expansions, *Ergodic Theory Dynam. Systems* **36**(1) (2016), 1–22.
8. Y. Bugeaud and N. Moshchevitin, On fractional parts of powers of real numbers close to 1, *Math. Z.* **271**(3–4) (2012), 627–637.
9. K. J. Falconer, *Fractal geometry, mathematical foundations and application* (Wiley, 1990).
10. K. J. Falconer, *Techniques in fractal geometry* (Wiley, 1997).
11. J. P. Kahane, Sur la répartition des puissances modulo 1, *C. R. Math. Acad. Sci. Paris* **352**(5) (2014), 383–385.
12. D. H. Kim and L. Liao, Dirichlet uniformly well-approximated numbers, *Int. Math. Res. Not. IMRN* (2018), in press.
13. J. F. Koksma, Ein mengentheoretischer Satz über die Gleichverteilung modulo Eins, *Compos. Math.* **2** (1935), 250–258.
14. J. F. Koksma, Sur la théorie métrique des approximations diophantiques, *Indag. Math.* **7** (1945), 54–70.
15. K. Mahler and G. Szekeres, On the approximation of real numbers by roots of integers, *Acta Arith.* **12** (1967), 315–320.
16. A. D. Pollington, The Hausdorff dimension of certain sets related to sequences which are not dense mod 1, *Quart. J. Math. Oxford Ser.* **31**(2) (1980), 351–361.