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Solvency requirements for pension annuities

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Abstract

This paper deals with solvency requirements for life annuities portfolios and funded pension plans. Particular emphasis is devoted to longevity risk, i.e. the risk arising from uncertainty in future mortality trends. This risk must be faced by insurance companies and pension plans that have guaranteed lifelong payoffs.

Solvency is investigated referring to immediate annuities, and hence the so-called decumulation phase is addressed. To assess solvency, assets are compared with the random present value of liabilities. Several requirements are considered, each leading to a required asset level that must be financed both with premiums (or contributions) and capital allocation.

1 Introduction

A rapidly moving scenario is the current framework of life insurance business and pension schemes. In particular, new types of risks affect the management of pension annuities. A very important example is provided by the so-called longevity risk, arising from the uncertainty in long-term mortality trends at adult and old ages. Actually, this risk must be faced by insurance companies and pension plans that have guaranteed lifelong payoffs. So, efficient tools must be used for monitoring the capability to meet future obligations.

Meeting future obligations requires an appropriate funding, i.e. appropriate assets. When an insurance portfolio is concerned, it is necessary: (i) to set up the portfolio reserve, (ii) to allocate the solvency margin (or risk-based capital). Analogous requirements hold for a (funded) pension plan.

The traditional approach to reserving focuses on the expected value of future obligations, which is assessed adopting a prudent valuation basis, representing

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interest rates and mortality from the time of valuation onwards. Usual assumptions consist in deterministic interest rates and in mortality levels obtained from past data via mortality projections (when annuities or other lifelong living benefits are concerned).

Solvency investigations are usually based on a comparison between the random profile of the assets and the random profile of the reserve. However, this approach might be lacking when lifetime living benefits are involved. In this case, owing to the uncertainty in mortality trends at adult and old ages and in the future performance of financial markets, it is difficult to judge on the appropriateness of a reserve profile simply based on a deterministic view of the future scenario. Hence, comparing the asset level and the reserve could be meaningless, in particular as far as the capability of the assets to meet future obligations on realistic grounds is concerned.

In this paper, we investigate solvency comparing the assets with the random present value of future obligations, hence without explicit reference to a reserve, whatever the relevant valuation basis might be. Several requirements are considered, each requirement leading to a 'required asset level' that must be financed with both premiums (or contributions) and capital allocation. In a traditional perspective, the total actual amount of assets (which should be greater or equal to the required amount) can be split into an amount backing the reserve (calculated with a given valuation basis) and an amount representing the solvency margin (as a residual). Implications of this approach on the management of pension annuities are discussed.

The paper is organized as follows. In section 2 risks inherent in life insurance products and pension schemes and the relevant solvency requirements are discussed. A model for risk investigations and solvency assessment is presented in section 3. A procedure, based on stochastic simulation, for assessing the mortality risk, including random fluctuations risk as well as longevity risk, is described in section 4 where pension annuities are specifically focussed. Results coming from numerical investigations embedding the financial risk are presented and discussed in section 5. Finally, section 6 presents some concluding remarks and suggestions for future research.

Some bibliographic remarks can help in understanding the genesis of the present paper. First, some references concerning mortality follow. Mortality trends at old-adult ages reveal decreasing annual probabilities of death; the reader can refer to Benjamin and Soliman (1993) also for a list of references. Population mortality trends are investigated in many countries; see, for example, MacDonald (1997), MacDonald *et al.* (1998), Rüttermann (1999). Specific references about longevity risk and related solvency requirements are still rather scanty. However, it is worth stressing that uncertainty in future mortality improvements should be carefully considered when pricing and reserving for life annuities; to this purpose see, for example, Marocco and Pitacco (1998) and Olivieri (2001). More generally, mortality trends and the relevant uncertainty affect any insurance cover providing some kind of 'living benefits', such as long-term care benefits or lifetime sickness benefits; see, for example, Olivieri and Pitacco (2002), Pitacco (2002).

The approach to solvency evaluation, leading to the definition of a required asset level and disregarding in principle the concept of reserve, adopted in the present article, has been suggested by the paper of the Faculty of Actuaries' Solvency Working Party (1986). Readers can refer to this paper also for an extensive list of references, including in particular the first studies on solvency in the European Union.

In the context of annuities and pension plans, much effort has been devoted to the accumulation phase as witnessed by a rich literature. The reader can refer, for example, to Cairns (2000), Haberman and Sung (2002), Haberman and Vigna (2002) also for comprehensive lists of references. The paper by Milevsky and Promislov (2001) specifically deals with the uncertainty in future mortality and the relevant consequences on annuity pricing. In this framework, the present paper aims to stimulate the debate about riskiness inherent in the decumulation phase of annuities, in the context of annuity portfolios and pension plans.

2 Risks and solvency

In this section and in section 3 as well we deal with risks and solvency referring to a context which is more general than the one concerned by immediate annuities. A more general context has some advantages. In particular:

- it paves the way for a more extensive risk and solvency analysis, also involving the accumulation phase;
- it allows for other types of benefit, for example death benefits, which can be provided by a pension scheme (and obviously by a life office);
- a link with the approach to solvency assessment procedures proposed by the Faculty of Actuaries' Solvency Working Party (1986) and adopted in this paper is more easily established.

2.1 Types of risks

In order to state the terminology to be used in the following sections, a description of some 'causes' of risks in life insurance and pension plans follows. Causes of risk can be easily found in items that constitute the scenario in which a life office or a pension plan operates.

The mortality risk is originated by the random lifetimes of insureds and annuitants; it can be split as follows. The risk of random fluctuations of the actual number of deaths around the expected arises from purely random variability of mortality. Mortality trends different from the forecasted trend lead to the risk of systematic deviation from the expected number of deaths. Clearly parameter and model uncertainty produces this type of risk. In particular the longevity risk originates from long-term mortality trends at adult and old ages. Finally, the catastrophe risk arises from the possibility of exceptionally high mortality due to particular events (natural disasters, wars, etc.).

The investment risk originates from facts concerning the financial market in which the insurer or the pension plan invests. The investment risk can be split, for example, as follows. The performance risk concerns insurance covers and annuities with financial guarantees, and arises from the possibility that the asset value is lower than the liability value. The mismatching risk within unit-linked products originates from the possibility that the investment portfolio differs from the set of assets whose performance determines the unit value. The risk of default arises from the possibility that the institution issuing the financial instruments purchased by the insurer or the pension plan does not pay the promised amount at maturity.

Further risks typically concern the life insurance business. In particular, expense risks originate from the behaviour of insurer's expenses compared with the income from expense loadings included in the premiums. Option risks are determined by choices of the policyholders. Actually, many recent products should be regarded as 'options packages', owing to the flexibility allowed by the policy conditions in terms of annual premium amount, annuity options, insurability options, switching options in united linked contracts, etc. Finally, economic risk is any risk not classified in the above categories and arising from the evolution of the economic scenario. Inflation, taxes and exchange rates constitute examples of items of economic risk. Correlations among risks clearly emerge from the above considerations. For example, the policyholders' behaviour also reflects financial and economic issues. So, increasing rates of interest may encourage surrenders should the bonus policy adopted by the insurer be considered not profitable enough.

The 'severity' of the various types of risk is influenced by the size of the portfolio or pension plan and the distribution of the benefits. As far as the role of the size is concerned risks can be classified into (a) pooling risks, whose effect (conveniently assessed) decreases as the size increases, and (b) non-pooling risks, the effect of which is independent of the size and the financial impact increases as the size increases. Within the category of mortality risks, random fluctuations constitute a pooling risk, as a larger size helps compensations. On the contrary, longevity risk is a non-pooling risk, since systematic deviations affect all individuals in the same direction. See Olivieri (2001) for a more comprehensive discussion of longevity risk in life insurance and annuities. Investment risk is, of course, a non-pooling risk.

2.2 Results affected by risks

The effects of risks can be measured in terms of various 'results'. As already mentioned, the topics we are dealing with concern life insurance and annuity portfolios as well as pension schemes. To simplify the text, we will present problems and relevant approaches in terms of a life insurance and annuity portfolio only. The same terminology will be used in section 3. With slight and obvious modifications all arguments can be referred to pension schemes.

From a merely intuitive point of view, portfolio results can be classified as follows. Firstly (following Colotti and Oliveri, 1998), we can distinguish between: (1) cash results simply referring to monetary inflows and outflows; (2) profit results involving reserves and aiming to measure annual and total profits; (3) asset/liability results which explicitly consider the value of assets and the capital invested into the business.

Results can be examined at several detailed levels as far as time is concerned. In particular it is possible to work with: (1) annual results, formally represented by vector-valued functions, for example the sequence of expected annual premiums over a given number of years; (2) single-figure results, aiming to provide appropriate summaries, for example the present value of expected annual premiums.

Time intervals assumed in order to define the size of the sequence of annual results or to produce single-figure results lead to the following classification: (1) short-term results concerning one to two years, say; (2) medium-term results, concerning three to five years; (3) cohort results referring to the time interval within which a given cohort of policies terminates; (4) long-term results, concerning more than five years (Single-figure asymptotic results in principle belong to this category. In practice, however, this latter type of results does not comply with numerical procedures needed in a 'corporate' approach to risk analysis).

Working with a given set of results obviously requires a rigorous definition of the concept of 'portfolio' (or pension scheme) which the results refer to. Actually the portfolio can be thought of as a 'closed' collection of policies already in force, as well as an 'open' collection of policies to which future business can contribute. So, (1) the run-off approach is based on the assumption that the insurer's activities are restricted to the in-force portfolio and hence cease when the portfolio comes to its end; (2) in the going-concern approach it is assumed that the insurer's activities develope with new acquisitions throughout the chosen time span. Adopting the run-off approach rather than the going-concern approach means different effects are produced by some types of risks. For example, random mortality fluctuations (i.e. a pooling risk) normally lead to a higher variability in a run-off context, owing to the decreasing portfolio size.

3 A model for risk investigation and solvency assessment

3.1 Notation

In this section we describe the framework within which risk analyses are performed. More precisely, we make reference to a model which allows us to assess cash flow, profit and patrimonial results (for more details, see Colotti and Olivieri (1998), in particular as far as portfolio valuations are concerned). What follows applies to a generic portfolio of life insurance products. The following notation is adopted:

Z_t random portfolio fund at time t;
A_t random value of assets at time t;
P_t random office premium income at time $t-1$ (for tth year);
E_t random outcome for expenses paid at time $t-1$ (for tth year);
C_t random outcome for death benefits paid at time t;
S_t random outcome for living benefits paid at time t;
$\mathcal{Y}_t$ random value at time t of future obligations;
$\mathcal{V}_t$ random portfolio reserve set up at time t;
I_t random financial incomes in year $(t-1, t)$;
J_t random capital gains or losses on asset value in year $(t-1, t)$;
K_t random payment (withdrawal) by the insurer to (from) the fund at time t ($K_t > 0$ denotes a payment, $K_t < 0$ a withdrawal).

Living benefits paid at the beginning of the year, i.e. at time $t-1$, can be represented by letting P_t be negative (in absolute value equal to the benefit).

Randomness of the above-mentioned quantities comes in general from all the types of risks described in section 2.1. In particular, the portfolio reserve is random because it is the sum of a random number of individual reserves. The individual reserve, usually assessed as the actuarial value of future net payments, can be random in its turn in the case of benefits linked to some financial index.

3.2 Cash flow analysis

Cash flow analysis can be performed either retrospectively, looking at past incomes and outcomes, or prospectively, looking at future payments. In a retrospective approach, the behaviour of the *portfolio fund* describes purely cash incomes and outcomes occurred up to the time which the fund is referred to. Let Z_0 denote the (known) value of the fund at time 0 (the valuation time). The random path of the fund is recursively described (starting from the given value at time 0) as follows:

$$Z_t = Z_{t-1} + P_t - E_t + I_t - C_t - S_t + K_t \tag{3.1}$$

The random variable $Z_t - \mathcal{V}_t$ is usually called the *free portfolio fund*.

In a prospective approach, we define the random value of future obligations as follows

$$\mathcal{Y}_t = \sum_{h=1}^{\infty} [(C_{t+h} + S_{t+h} - K_{t+h})\, v(t, t+h) - (P_{t+h} - E_{t+h})\, v(t, t+h-1)] \tag{3.2}$$

where $v(t, t+h)$ denotes the (random) discount factor for the period $(t, t+h)$, chosen accordingly to the future performance of investments (in this regard see also section 3.7). Note that a broad sense is attached to the term 'obligations', since $\mathcal{Y}_t$ includes payments both to/from policyholders ('industrial' obligations) and to/from shareholders ('corporate' obligations).

3.3 Profit analysis

Annual profit emerging from the management of the portfolio can be assessed as follows

$$U_t = P_t - E_t + I_t + J_t - C_t - S_t + \mathcal{V}_{t-1} - \mathcal{V}_t \tag{3.3}$$

The annual profit U_t can be split in several ways, reflecting the profit sources. However, we do not go deeply into this aspect, since it is not directly required in solvency analysis. We just mention the following relationships between profit and the portfolio fund, which can be easily verified

$$U'_t = \Delta Z_t - \Delta \mathcal{V}_t - K_t \tag{3.4}$$

$$Z_t - \mathcal{V}_t = (Z_0 - \mathcal{V}_0) + \sum_{h=1}^{t} (U'_h + K_h) \tag{3.5}$$

where $U'_t = U_t - J_t$ and, for a given function f_t, $\Delta f_t = f_t - f_{t-1}$.

3.4 Patrimonial analysis

A patrimonial analysis consists in the investigation of both assets and liabilities. We assume that the value of assets can be recursively described as follows (for a given value of A_0)

$$A_t = A_{t-1} + \Delta Z_t + J_t \tag{3.6}$$

where ΔZ_t expresses new financial resources, and hence new investments, emerging in year $(t-1, t)$, whilst J_t represents increases (capital gains) or decreases (capital losses) in asset value in the same year.

For the 'balance condition', at each time the portfolio reserve (i.e. the value of industrial liabilities) and shareholders' fund, M_t, must be equal to the value of assets. Then we have

$$A_t = M_t + \mathscr{V}_t \tag{3.7}$$

whence, trivially, $M_t = A_t - \mathscr{V}_t$. We easily obtain

$$U_t = \Delta A_t - \Delta \mathscr{V}_t - K_t \tag{3.8}$$

$$M_t = M_0 + \sum_{h=1}^{t} (U_h + K_h) \tag{3.9}$$

Further relationships among the three types of portfolio results obviously hold; readers are referred to Colotti and Olivieri (1998).

3.5 Solvency

Albeit solvency is a fundamental concept in insurance theory and practice, it is not clearly and unequivocally defined. What is commonly adopted is a stochastic approach to risk analysis. In this framework, solvency is meant as the ability to meet, with an assigned (high) probability, the random liabilities as described by a realistic (experience based) probabilistic structure.

The concept of stochastic solvency requires some specifications. In particular, choices are needed with respect to:

- the portfolio results which can be used to assess the above ability;
- the time span which the above results must be referred to;

in case the time span is longer than one year (as it is common):

- vector-valued results vs single-figure results;
- run-off vs going-concern approach;

the meaning of the above choices can be easily understood in terms of the classification scheme presented in section 2.

In order to make the above-mentioned choices, the point of view from which solvency is ascertained must be stated. Policyholders, shareholders and the supervisory authority represent possible viewpoints on the insurance business. However, the policyholders' and shareholders' perspectives involve profitability requirements probably higher than those implied by the need to meet liabilities. Such requirements would lead to a concept of insurer's solidity, rather than solvency. So, we will restrict our attention to the supervisory authority's perspective.

The supervisory authority is charged to protect mainly the interests of present and forthcoming policyholders. From this point of view, cash and patrimonial results are mostly important. A long-term perspective should be considered; shorter horizons are consistent with a careful and frequent monitoring of the portfolio as well as with severe solvency requirements.

3.6 Reserve-based solvency requirements

In the traditional approach to solvency, the ability to meet random liabilities (see section 3.5) is meant as the ability to set up the reserve for each policy in the portfolio and hence the portfolio reserve. In terms of patrimonial results, the insurance company is therefore able to meet its liabilities at a given time t if the asset value is greater or equal to the portfolio reserve (see (3.7))

$$M_t \geqslant 0 \tag{3.10}$$

In this framework, M_t is usually referred to as the *solvency margin* assigned to the portfolio.

Let 0 denote the time at which solvency is ascertained and assume $M_0 \geqslant 0$. Given a time horizon of T years, we say that the insurer has a solvency of degree $1-\varepsilon$ if and only if

$$\Pr\left\{ \bigwedge_{t=1}^{T} M_t \geqslant 0 \right\} = 1-\varepsilon \tag{3.11}$$

Specific solvency requirements can be found by choosing proper values for the quantities which affect (3.11), i.e.

- the probability ε (ruin probability);
- the time span T;
- the capital flows K_t;
- a run-off or a going-concern approach.

In particular, if the point of view of the supervisory authority is adopted, it seems natural to choose $K_t = 0$ except for the time of valuation, when a proper solvency margin must be financed. On the contrary, when shareholders' perspective is considered, the choice of the flows K_t could reflect their targets concerning the timing of shareholders' capital investment into the portfolio or dividend distribution.

Note that the assumed time span T implies the consideration of incomes and outcomes over a period of n years, with $n \geqslant T$. Actually, for all t the portfolio reserve $\mathcal{V}_t$ takes into account premiums and benefits falling due after time t; so, for any fixed T, the time interval actually involved in the solvency ascertainment has a term given by T plus the largest residual duration of policies virtually in force at time T.

Our aim is to determine the solvency margin required at time 0 such that, for a given choice of the quantities mentioned above, condition (3.11) is satisfied. Since we adopt the viewpoint of the supervisory authority, we assume $K_t = 0$, $t = 1, 2, \ldots$ and $K_0 \geqslant 0$. We denote the required solvency margin $M_0^{(R)}$ (the superscript 'R' indicates that a reserve-based requirement has been adopted). Note that

$$A_0^{(R)} = M_0^{(R)} + \mathscr{V}_0 \tag{3.12}$$

is the required (total) investment at time 0.

Definition (3.11) is based on a vector-valued portfolio result, since the quantities $M_1, M_2, \ldots, M_T$ are taken into account. A different concept of solvency can be stated referring to a single-figure result; typically

$$\Pr\{M_T \geqslant 0\} = 1 - \varepsilon \tag{3.13}$$

Of course, equation (3.13) leads to a weaker concept of solvency than (3.11).

Solvency requirements can be formulated also with reference to cash results. In this case, the insurance company is solvent at a given time if the free portfolio fund is positive, i.e. if

$$Z_t - \mathscr{V}_t \geqslant 0 \tag{3.14}$$

Note that now the solvency margin linked to the portfolio is given by $Z_t - \mathscr{V}_t$.

Solvency requirements based on the following conditions

$$\Pr\left\{\bigwedge_{t=1}^{T} Z_t - \mathscr{V}_t \geqslant 0\right\} = 1 - \varepsilon \tag{3.15}$$

$$\Pr\{Z_T - \mathscr{V}_T \geqslant 0\} = 1 - \varepsilon \tag{3.16}$$

lead to a required initial free portfolio fund (or solvency margin) $[Z_0^{(R)} - \mathscr{V}_0]$, and a required initial fund $Z_0^{(R)}$, for a given choice of the ruin probability ε, the time horizon T and the run-off or going-concern approach (we still assume $K_0 \geqslant 0$ and $K_t = 0$, $t = 1, 2, \ldots$). The main difference between $[Z_0^{(R)} - \mathscr{V}_0]$ and $M_0^{(R)}$ is due to capital gains or losses. In periods of high investment performance, requirements (3.15) and (3.16) could be more severe than (3.11) and (3.13), since capital gains are disregarded. The reverse obviously holds in periods of investment under performance. In section 4 we will make some hypotheses under which cash and patrimonial results are equivalent.

Further requirements can be given considering short-term cash flows, thus involving in particular liquid assets as well as premiums, expenses and sums assured paid in a year. We do not discuss such requirements (see, for example, Kahane, Tapiero and Laurent, 1989).

3.7 Obligations-based solvency requirements

The disadvantage of the solvency requirements discussed in section 3.6 is that they refer to the notion of reserve, which is usually assessed (at the individual level) as an expected value (on a conservative basis) of future industrial obligations (i.e. benefits less premiums). When lifetime living benefits are concerned, this approach might be lacking. Actually, owing to the uncertainty in mortality trends at adult and old ages and in the future performance of investments, it is difficult to judge the appropriateness of a reserve profile simply based on a deterministic view of future scenarios. For such products, solvency requirements based on the random value of future obligations might be more appropriate than those discussed in section 3.6.

We will therefore consider requirements according to which the insurance company is judged solvent at time t if at that time it is able to meet its future obligations (a similar approach has been adopted in Faculty of Actuaries' Solvency Working Party, 1986). In patrimonial terms, solvency requirements like the following

$$\Pr\left\{\bigwedge_{t=1}^{T} A_t - \mathcal{Y}_t \geqslant 0\right\} = 1 - \varepsilon \tag{3.17}$$

$$\Pr\{A_T - \mathcal{Y}_T \geqslant 0\} = 1 - \varepsilon \tag{3.18}$$

can be considered.

Note that, besides the above mentioned, another advantage of definitions (3.17) and (3.18), when compared with (3.11) and (3.13), is that $\mathcal{Y}_t$ includes strictly industrial obligations (i.e. with respect to policyholders) as well as corporate obligations (i.e. with respect to shareholders). On the contrary $\mathcal{V}_t$, used in (3.11) and (3.13), only considers future payments to policyholders. Hence, definitions (3.17) and (3.18) could be more appropriate when solvency is ascertained from the point of view of shareholders. In the following, we consider the supervisory authority's perspective. Hence, as in section 3.6, we assume $K_t = 0$ except for the time of valuation.

With regard to the value of future obligations, $\mathcal{Y}_t$ should for consistency be calculated with discount factors chosen accordingly to the future behaviour of both financial incomes, I_{t+h}, and capital gains/losses, J_{t+h}, since they both affect the value of assets (see (3.6)). Further, the summation in (3.2) should be extended to the largest residual duration of policies virtually in force at time T. Denoting with n such duration, we have $\mathcal{Y}_n = 0$ and we can verify that, under the information available at time 0, the following equality holds (for whatever choice of the flows K_t)

$$\Pr\left\{\bigwedge_{t=1}^{T} A_t - \mathcal{Y}_t \geqslant 0\right\} = \Pr\{A_n \geqslant 0\} \tag{3.19}$$

Actually, under the hypotheses adopted so far, assets at time t can be expressed as follows

$$A_t = A_0 \frac{1}{v(0,\, t)} + \sum_{h=1}^{t} \left[(P_h - E_h) \frac{1}{v(h-1,\, t)} - (C_h + S_h - K_h) \frac{1}{v(h,\, t)} \right] \tag{3.6$'$}$$

(where $\frac{1}{v(s,\, z)}$ represents the accumulated value of one monetary unit invested at time s for $z - s$ years, which for simplicity is inclusive of financial income and capital

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gains/losses) and liabilities as follows

$$\mathcal{Y}_t = \sum_{h=t+1}^{n} [(C_h + S_h - K_h)\, v(t, h) - (P_h - E_h)\, v(t, h-1)] \qquad (3.2')$$

We can first of all note that under the information available at time 0 (when all future flows and rates of interest are random) the following result holds

$$\begin{aligned} A_t - \mathcal{Y}_t \geqslant 0 &\Leftrightarrow A_t \frac{1}{v(t, n)} - \mathcal{Y}_t \frac{1}{v(t, n)} \geqslant 0 \\ &\Leftrightarrow A_0 \frac{1}{v(0, n)} + \sum_{h=1}^{t} \left[(P_h - E_h) \frac{1}{v(h-1, n)} - (C_h + S_h - K_h) \frac{1}{v(h, n)} \right] \\ &\quad - \sum_{h=t+1}^{n} \left[(C_h + S_h - K_h) \frac{1}{v(h, n)} - (P_h - E_h) \frac{1}{v(h-1, n)} \right] \geqslant 0 \\ &\Leftrightarrow A_n \geqslant 0 \end{aligned}$$

We then have (denoting by $\mathcal{I}_0$ the information available at time 0)

$$\Pr\left\{ \bigwedge_{t=1}^{T} A_t - \mathcal{Y}_t \geqslant 0 \mid \mathcal{I}_0 \right\} = \Pr\left\{ \bigwedge_{t=1}^{T} A_n \geqslant 0 \mid \mathcal{I}_0 \right\} = \Pr\{A_n \geqslant 0 \mid \mathcal{I}_0\} \qquad (3.19')$$

Interpretation is as follows: given that A_t includes all flows relating to $[0, t]$ and $\mathcal{Y}_t$ those relating to $[t, n]$, when the difference $A_t - \mathcal{Y}_t$ is concerned, the overall flows relating to $[0, n]$ are under consideration. Given the information available at time 0, it is then equivalent to refer such flows to time t or to time n (or to some other time).

Following steps similar to those described above, we can further verify that

$$\Pr\left\{ \bigwedge_{t=1}^{T} A_t - \mathcal{Y}_t \geqslant 0 \right\} = \Pr\{A_T - \mathcal{Y}_T \geqslant 0\} \qquad (3.19'')$$

Relation (3.19) shows in particular that, as for reserve-based requirements, a time span greater than T is involved. However, whilst in the reserve-based approach expected values of future payments are considered (at least at individual level) we are now dealing with random values only (which allows simplifications (3.19) and (3.19″) to be made).

For a given choice of the ruin probability ε, under a run-off or going concern approach, conditions (3.17) and (3.18) (which according to (3.19) and (3.19″) are equivalent and independent of the time span T) lead to a required initial (total) investment $A_0^{(O)}$, to be financed both with premiums and shareholders' fund (the up-letter 'O' indicates that an obligations-based requirement has been adopted). In particular, given the portfolio reserve at time 0 (which should be built with premiums), the required solvency margin according to requirement (3.19) is

$$M_0^{(O)} = A_0^{(O)} - \mathcal{V}_0 \qquad (3.20)$$

In terms of cash results, future obligations are compared with the portfolio fund. Solvency requirements can be formulated as follows

$$\Pr\left\{\bigwedge_{t=1}^{T} Z_t - \mathscr{Y}_t \geqslant 0\right\} = 1-\varepsilon$$
$$\Leftrightarrow \Pr\{Z_T - \mathscr{Y}_T \geqslant 0\} = 1-\varepsilon$$
$$\Leftrightarrow \Pr\{Z_n \geqslant 0\} = 1-\varepsilon \tag{3.21}$$

Note that in (3.21), for consistency, $\mathscr{Y}_t$ should be based on discount factors chosen according only to the future behaviour of financial incomes, I_h (see (3.1)).

For a given choice of the ruin probability ε and the approach (run-off or going concern), condition (3.21) leads to a required initial fund $Z_0^{(O)}$ to be assigned to the portfolio. Given the initial reserve $\mathscr{V}_0$, the required initial free portfolio fund (or solvency margin) is then $[Z_0^{(O)} - \mathscr{V}_0]$.

It is worth noting that the obligation-based requirements approach, to some extent, is a 'fair value' concept of solvency. Clearly, as regards assets the fair valuation leads to a 'market' value, whereas the fair value of liabilities can be meant as a value calculated according to the 'best estimate', namely, in a stochastic framework, the most realistic probability structure describing the random mortality of annuitants. Indeed an appropriate choice of the probability structures underpinning requirements (3.17), (3.18) or (3.21) leads to a realistic representation of future scenarios and hence to a fair valuation of both assets and liabilities.

4 Demographic risk in a portfolio of immediate annuities

4.1 Hypotheses

The meaning and implications, in numerical terms, of the solvency requirements discussed in the previous section are investigated with reference to a portfolio of immediate life annuities. Slight modifications allow us to refer the discussion to a pension scheme. Since these products are in particular affected by longevity and investment risk, we concentrate on these risks only, disregarding expense, option and economic risks.

More precisely, we consider a portfolio of immediate annuities, homogeneous in terms of entry time, age, annual amount, etc. For simplicity, we disregard expenses and consider constant benefits. Let N_t be the random number of annuitants at time t, $t=0, 1, \ldots$ (in particular, N_0 is a known quantity), R the annual amount for each annuity, R_t the total amount paid at time t to annuitants alive at that time and y the age at time 0.

In this section, we adopt a deterministic hypothesis with reference to investment performance, denoting with i_t^* the rate of interest for the tth year (investment risk is dealt with in section 5). Note that in a deterministic framework capital gains/losses are meaningless; hence $A_t = Z_t$ at each time t.

Under our hypotheses, the portfolio fund, future obligations and the portfolio reserve are now described (letting $P=-R$ and $P_t=-R_t$) as follows

$$Z_t = Z_{t-1} - R_t + I_t = (Z_{t-1} - N_{t-1}R)(1+i_t^*) \tag{4.1}$$

$$\mathcal{Y}_t = \sum_{h=1}^{\omega-y-t} R_{t+h}\, v(t,\ t+h-1) = \sum_{h=1}^{\omega-y-t} RN_{t+h-1}\, v(t,\ t+h-1) \tag{4.2}$$

$$\mathcal{V}_t = N_t V_t = N_t R \ddot{a}_{y+t} \tag{4.3}$$

where ω is the maximum age (whence $\omega-y-t$ is the maximum duration of a policy at time t), $v(t, t+h-1)$ is calculated according to the interest rates i_{t+1}^*, i_{t+2}^*, …, i_{t+h-1}^* and $\ddot{a}_{y+t}$ denotes the actuarial value of a unitary immediate annuity for a person aged $y+t$, calculated according to a proper valuation basis. We note that shareholders' capital flows K_t are disregarded because the viewpoint of the supervisory authority is considered.

Requirements (3.15) and (3.21) are adopted. In order to perform the valuation, we still need to assume a mortality model.

4.2 Mortality assumptions

Denoting with q_x and p_x the mortality and survival rates at age x, we model life annuitants' mortality by assuming

$$\frac{q_x}{p_x} = GH^x \tag{4.4}$$

where the q_x/p_x is the so-called 'odds'. Note that the right-hand side of (4.4) is the third term in the well-known Heligman–Pollard law, i.e. the term describing the old-age pattern of mortality (see Heligman and Pollard, 1980). The expression of the mortality rate q_x and the survival function $S(x)$ ($S(x)=\Pr\{T_0>x\}$, with T_0 the lifetime for a newborn), can be easily obtained. Note that the parameter G expresses the level of senescent mortality and H the rate of increase of senescent mortality itself.

In order to represent mortality trends, we will adopt projected survival functions. Mortality trends at adult ages reveal decreasing annual probabilities of death and in particular:

(i) the concentration of deaths around the mode of the probability density function $f_0(x) = -\frac{dS(x)}{dx}$, whose graph is also called 'curve of deaths', increases with time; in terms of the graph of the survival function, this implies the so-called 'rectangularization';
(ii) the mode of the curve of deaths moves towards very old ages, leading to the so-called 'expansion'.

Commonly, mortality projections are extrapolations of (recent) trends as far as these can be perceived from mortality statistics. A different approach leads to models

Table 1. *Mortality laws for annuitants*

	[min]	[med]	[max]
G	0.000042	0.000002	0.0000001
H	1.09803	1.13451	1.17215

expressing the basic characteristics of the evolving scenario in which mortality improvements take place. In this case the projection model should catch the aspects (i) and (ii). To this purpose, analytical mortality laws should be used as, for example, the Heligman–Pollard law. According to this approach, mortality trends are represented assuming that the parameters of the mortality law are functions of the calendar year. Hence, also the mode and the variance of the curve of deaths depend on the calendar year. Then, the adequacy of the projection model can be checked comparing the behaviour of the curve of deaths with the scenario characteristics described above (see (i) and (ii)).

It should be stressed that the increasing concentration of deaths around the mode of the curve of deaths reduces the risk affecting annuity benefits, whatever the location of the mode may be. Hence, any given mortality projection implying this feature reveals a reduction of the mortality risk originating from random fluctuations, with respect to the results produced by a non-projected mortality model. However, the mortality projection itself is affected by uncertainty. Evaluating the degree of uncertainty and incorporating this evaluation in the actuarial model, a higher mortality risk emerges. The additional risk, called the longevity risk, is attributable to systematic deviations of the mortality from the projected mortality assumed in the calculation basis (used in pricing or reserving).

As far as the mortality risk is concerned, we adopt the following approaches:

(1) a 'deterministic' approach, implemented by using a given projected survival function; this approach only allows for the random fluctuation risk;
(2) a 'stochastic' approach, implemented by using a set of projected survival functions, representing the uncertainty inherent in the projection; a 'degree of belief' will be assigned to each function; this approach allows for both the random fluctuation risk and the longevity risk.

More precisely, we will define three projected survival functions, denoted by $S^{[\mathrm{min}]}(x)$, $S^{[\mathrm{med}]}(x)$ and $S^{[\mathrm{max}]}(x)$, expressing, respectively, a little, a medium and a high reduction in mortality. In the deterministic approach only $S^{[\mathrm{med}]}(x)$ is used, whilst the stochastic approach involves the three functions.

As stated at the beginning of this section, a Heligman–Pollard-like model is adopted. The parameters of the three survival functions are shown in table 1 (the maximum age, ω, is set equal to 110). As it emerges from such parameters, the projected functions have been obtained so that they perform the trends of expansion and rectangularization. Figures 1 and 2 show, respectively, the three survival functions and the related probability density functions.

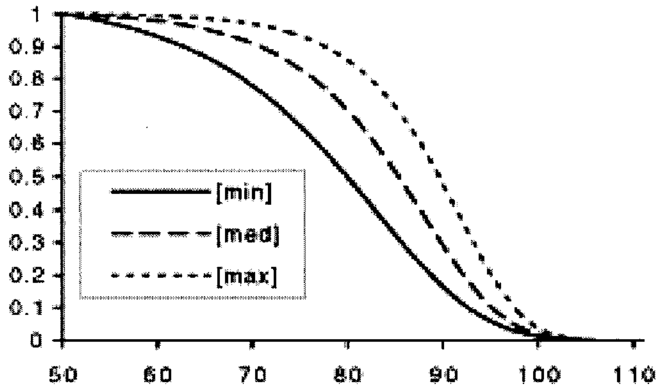

Figure 1. Survival functions.

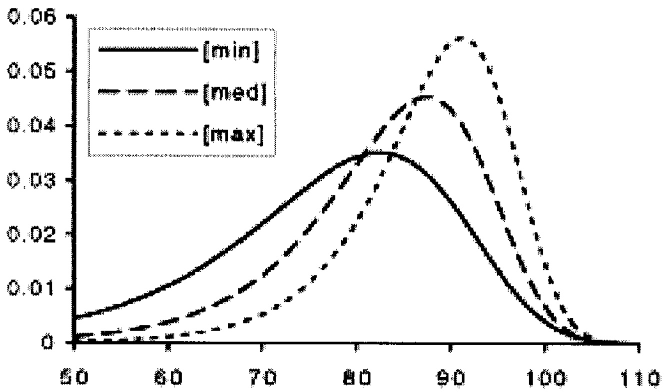

Figure 2. Curves of deaths.

4.3 Solvency in a deterministic approach to mortality

The portfolio we are investigating consists of identical annuities, paid to persons of initial age $y=65$, with annual amount $R=100$. We assume that the future lifetimes of the annuitants have a common distribution and are independent of each other (conditional on any given survival function). The single premium (to be paid at entry) is calculated, for each policy, according to the survival function $S^{[\mathrm{med}]}(x)$ and with a constant annual interest rate $i=0.03$. Further we assume that for each policy in force at time t, $t=0, 1, \ldots$, a reserve must be set up, which is calculated according to such hypotheses.

In the deterministic approach to mortality, the probability distribution of the future lifetime of each insured is known, the only cause of uncertainty consisting in the time of death. The assessment of the solvency margin required according either to a reserve-based or an obligations-based condition is performed through simulation. In order to obtain results easier to interpret, we disregard profit; the actual life duration of the annuitants is thus simulated with the survival function $S^{[\mathrm{med}]}(x)$. Further, we assume $i_t^*=i=0.03$. A run-off approach is adopted (i.e. only the cohort entering at time 0 is considered).

Table 2. *Required solvency margin: $\varepsilon = 0.025$*

	$T=n$		$T=5$			
N_0	$[Z_0^{(R)} - \mathcal{V}_0]$	$\frac{[Z_0^{(R)} - \mathcal{V}_0]}{\mathcal{V}_0}$	$[Z_0^{(R)} - \mathcal{V}_0]$	$\frac{[Z_0^{(R)} - \mathcal{V}_0]}{\mathcal{V}_0}$	$[Z_0^{(O)} - \mathcal{V}_0]$	$\frac{[Z_0^{(O)} - \mathcal{V}_0]}{\mathcal{V}_0}$
1000	32,959	2.180 %	17,242	1.140 %	29,744	1.967 %
2000	44,937	1.486 %	25,253	0.835 %	40,653	1.344 %
3000	54,398	1.199 %	27,771	0.612 %	48,462	1.068 %
4000	65,460	1.082 %	33,340	0.551 %	58,894	0.974 %
5000	73,242	0.969 %	36,621	0.484 %	64,319	0.851 %
6000	79,956	0.881 %	39,597	0.437 %	68,790	0.758 %
7000	87,585	0.828 %	44,861	0.424 %	80,891	0.764 %
8000	90,332	0.747 %	47,302	0.391 %	81,071	0.670 %
9000	100,555	0.739 %	49,744	0.366 %	90,192	0.663 %
10000	103,149	0.682 %	52,032	0.344 %	92,398	0.611 %

In table 2 the solvency margin required at time 0 is quoted according to conditions (3.15) and (3.21). In the former case, the time spans $T=n=\omega-y$ and $T=5$ years are alternatively chosen. When the largest time span (i.e. $T=n$) is chosen, $[Z_0^{(R)} - \mathcal{V}_0]$ and $[Z_0^{(O)} - \mathcal{V}_0]$ are quite similar. However, the reserve-based approach seems to be more severe than the obligations-based one. In order to understand why consider that, since $\mathcal{V}_n = 0$, we have $Z_n - \mathcal{V}_n = Z_n$. According to requirement (3.21), solvency is in practice ascertained only at time n (checking the positivity of Z_n), whilst under (3.15) it is ascertained not only at time n (checking the positivity of $Z_n - \mathcal{V}_n = Z_n$), but also at time t, $t<n$, when a negative free portfolio fund can occur. It can therefore happen that $Z_t - \mathcal{V}_t < 0$ even though $Z_n = 0$, whence the reserve-based approach leads to higher solvency margins than the obligations-based one (note that this situation can emerge both from random interest rates and random fluctuations in mortality; given our current hypotheses, the differences between $[Z_0^{(R)} - \mathcal{V}_0]$ and $[Z_0^{(O)} - \mathcal{V}_0]$ in table 2 are due to the latter). When $T=5$, $[Z_0^{(R)} - \mathcal{V}_0]$ seems to be insufficient when compared to $[Z_0^{(O)} - \mathcal{V}_0]$. This is due to the fact that the risk of random fluctuations of the portfolio fund around the reserve (roughly speaking, of the number of survivors around their expected value) is heavy in the long run, but not within a short horizon. As far as the size of the portfolio is concerned, table 2 shows that the required margin decreases as N increases. This is due to the fact that a deterministic approach to mortality only catches the risk of random fluctuations which, as it is well-known, is a pooling risk (i.e. its effect decreases when larger numbers of similar policies are dealt with). Finally, note that since by definition the initial reserve $\mathcal{V}_0$ is equal to the single premium, the solvency margin at time 0 must be financed with shareholders' funds.

Results in tables 3 and 4 have been obtained by performing the valuation at time h, $h=0, 5, 10, \ldots$, in order to inspect the behaviour of the required margin through time (amendments to formulae in section 3 are straightforward). At each valuation time h we have considered, for a given initial size N_0, a portfolio of $E(N_h)$ insureds, each aged $y+h$, and a portfolio reserve $\mathcal{V}_h = E(N_h)V_h$ (the expected value $E(N_h)$ is calculated according to the assumed mortality distribution, i.e. $S^{[\text{med}]}(x)$). The increase in the

Table 3. *Required solvency margin* $[Z_h^{(R)} - \mathcal{V}_h]/\mathcal{V}_h \times 100$*;* $\varepsilon = 0.025$

	$T=n$			$T=5$		
h	$N_0=1000$	$N_0=5000$	$N_0=10000$	$N_0=1000$	$N_0=5000$	$N_0=10000$
0	2.180	0.969	0.682	1.140	0.484	0.344
5	2.613	1.206	0.808	1.501	0.663	0.450
10	3.124	1.430	0.981	2.079	0.953	0.649
15	4.146	1.761	1.239	2.982	1.321	0.937
20	5.442	2.270	1.675	4.372	1.823	1.371
25	7.489	3.357	2.351	6.222	2.984	2.046
30	14.089	5.866	4.071	12.694	5.313	3.786
35	41.613	15.339	10.492	38.071	14.166	9.850

Table 4. *Required solvency margin* $[Z_h^{(O)} - \mathcal{V}_h]/\mathcal{V}_h \times 100$*;* $\varepsilon = 0.025$

h	$N_0=1000$	$N_0=5000$	$N_0=10000$
0	1.967	0.851	0.611
5	2.345	1.024	0.741
10	2.972	1.243	0.910
15	3.809	1.634	1.154
20	5.022	2.136	1.511
25	7.148	3.103	2.242
30	13.996	5.329	3.807
35	40.934	14.407	9.928

(relative) solvency margin, whatever requirement is adopted, is due to the fact that the size of the portfolio reduces and the age of the annuitants increases with time. Note, however, that the increase is stronger when a reserve-based approach with horizon $T=5$ is chosen. The example shows that, as it is quite intuitive, such an approach requires proper adjustments once the given horizon has been reached. Moreover, it implies a different capital allocation than the other two conditions examined numerically (i.e. $T=n$ and the obligations-based approach), in the sense that the lower initial required fund must be followed, at the subsequent times, by a greater increase of the fund itself. As far as financing of the required fund $Z_h^{(R)}$ ($Z_h^{(O)}$) is concerned, in case the reserve actually cumulated is not equal to its expected value, a shareholders' fund greater than $[Z_h^{(R)} - \mathcal{V}_h]$ ($[Z_h^{(O)} - \mathcal{V}_h]$) could be necessary.

Finally, tables 5–8 show dependence on the ruin probability.

To conclude the examples relating to the deterministic approach, we just mention that investigations performed with either the survival function $S^{[\min]}(x)$ or $S^{[\max]}(x)$ suggest comments similar to those above discussed. However, due to the phenomenon of rectangularization, the solvency margin required according to $S^{[\max]}(x)$ ($S^{[\min]}(x)$) is lower (higher) than that assessed using $S^{[\mathrm{med}]}(x)$. Finally, it must be mentioned that the relatively low levels of $\frac{Z_h - \mathcal{V}_h}{\mathcal{V}_h}$ in all the examples presented is

Table 5. *Required solvency margin* $[Z_0^{(R)} - \mathcal{V}_0]/\mathcal{V}_0 \times 100$

	$T=n$			$T=5$		
N_0	$\varepsilon=0.01$	$\varepsilon=0.025$	$\varepsilon=0.05$	$\varepsilon=0.01$	$\varepsilon=0.025$	$\varepsilon=0.05$
1000	2.503	2.180	1.887	1.254	1.140	1.024
2000	1.736	1.486	1.272	0.949	0.835	0.696
3000	1.475	1.199	1.080	0.686	0.612	0.543
4000	1.226	1.082	1.082	0.636	0.551	0.488
5000	1.147	0.969	0.844	0.581	0.484	0.420
6000	0.969	0.881	0.773	0.511	0.437	0.390
7000	0.946	0.828	0.715	0.473	0.424	0.376
8000	0.868	0.747	0.676	0.454	0.391	0.324
9000	0.861	0.739	0.642	0.471	0.366	0.308
10000	0.775	0.682	0.588	0.380	0.344	0.295

Table 6. *Required solvency margin* $[Z_0^{(O)} - \mathcal{V}_0]/\mathcal{V}_0 \times 100$

N_0	$\varepsilon=0.01$	$\varepsilon=0.025$	$\varepsilon=0.05$
1000	2.314	1.967	1.644
2000	1.589	1.344	1.107
3000	1.347	1.068	0.927
4000	1.116	0.974	0.974
5000	1.053	0.851	0.725
6000	0.927	0.758	0.645
7000	0.845	0.764	0.647
8000	0.808	0.670	0.589
9000	0.762	0.663	0.576
10000	0.764	0.611	0.523

due to the fact that, because of the deterministic approach to mortality (and to investment as well), only the risk of random fluctuations has been accounted for.

4.4 Solvency in a stochastic approach to mortality

The assessment of the solvency margin is now obtained considering explicitly uncertainty in future mortality trends. To this aim, we first consider the three survival functions $S^{[\min]}(x)$, $S^{[\mathrm{med}]}(x)$ and $S^{[\max]}(x)$, assuming that each of them is meant as a possible distribution of the future lifetime. The weights $\rho^{[\min]}$, $\rho^{[\mathrm{med}]}$ and $\rho^{[\max]}$ represent, respectively, the 'degree of belief' of such functions.

The single premium for each policy and the individual reserve are still calculated with the survival function $S^{[\mathrm{med}]}(x)$ and the interest rate $i=0.03$; we let $i_t^*=i=0.03$. Unless otherwise stated, we assume $\rho^{[\min]}=0.2$, $\rho^{[\mathrm{med}]}=0.6$, $\rho^{[\max]}=0.2$ (reflecting the fact that $S^{[\mathrm{med}]}(x)$, which is used for pricing and reserving, is assumed to give the most reliable mortality description).

The investigation is carried out through simulation. We now deal with two causes of uncertainty: the actual distribution of the future lifetimes and the time of death

Table 7. *Required solvency margin* $[Z_h^{(R)} - \mathcal{V}_h]/\mathcal{V}_h \times 100$*;* $N_0 = 5000$

	$T=n$			$T=5$		
h	$\varepsilon=0.01$	$\varepsilon=0.025$	$\varepsilon=0.05$	$\varepsilon=0.01$	$\varepsilon=0.025$	$\varepsilon=0.05$
0	1.147	0.969	0.844	0.581	0.484	0.420
5	1.377	1.206	1.006	0.804	0.663	0.578
10	1.569	1.430	1.224	1.041	0.953	0.798
15	2.129	1.761	1.556	1.638	1.321	1.157
20	2.633	2.270	1.958	2.307	1.823	1.598
25	3.749	3.357	2.846	3.369	2.984	2.532
30	6.829	5.866	4.717	6.391	5.313	4.421
35	17.530	15.339	13.104	17.265	14.166	12.395

Table 8. *Required solvency margin* $[Z_h^{(O)} - \mathcal{V}_h]/\mathcal{V}_h \times 100$*;* $N_0 = 5000$

h	$\varepsilon=0.01$	$\varepsilon=0.025$	$\varepsilon=0.05$
0	1.053	0.851	0.725
5	1.283	1.024	0.914
10	1.517	1.243	1.090
15	1.930	1.634	1.346
20	2.507	2.136	1.795
25	3.575	3.103	2.657
30	6.543	5.329	4.389
35	16.205	14.407	12.193

of each insured. Firstly, the survival function must be chosen (through simulation) and then, assuming that under a given lifetime distribution the annuitants are independent risks, the actual duration of life of each person is simulated.

Tables 9 to 15 show the results of valuations of the same type as those performed in the deterministic framework. The following aspects must be stressed.

For a given choice of parameters (i.e. size of the portfolio, rate of interest, ruin probability etc.), the comparison between the deterministic and the stochastic case shows a heavy increase of the required solvency margin in the latter. This is due to the fact that a stochastic framework allows us to analyse not only the risk of random fluctuations in the number of survivors, but also that of systematic deviations (i.e. the longevity risk), which is a non-pooling risk. Considering for example table 9, the decreasing behaviour of the relative required solvency margin with respect to N_0 is due to the pooling effect of random fluctuations; however, its magnitude is rather stable and its value seems to tend to a large positive amount (as could be checked considering larger portfolios); hence, the non-pooling effect of longevity risk is witnessed.

Requirement (3.15) with time span $T=5$ leads to a solvency margin significantly lower than either the case $T=n$ or requirement (3.21). This shows first of all that the longevity risk reveals itself in the long run. Secondly, the choice of assessing the

Table 9. *Required solvency margin; $\varepsilon = 0.025$*

	$T=n$		$T=5$			
N_0	$[Z_0^{(R)} - \mathcal{V}_0]$	$\frac{[Z_0^{(R)} - \mathcal{V}_0]}{\mathcal{V}_0}$	$[Z_0^{(R)} - \mathcal{V}_0]$	$\frac{[Z_0^{(R)} - \mathcal{V}_0]}{\mathcal{V}_0}$	$[Z_0^{(O)} - \mathcal{V}_0]$	$\frac{[Z_0^{(O)} - \mathcal{V}_0]}{\mathcal{V}_0}$
1000	218,172	14.431 %	43,869	2.902 %	217,543	14.389 %
2000	425,758	14.081 %	83,771	2.770 %	425,222	14.063 %
3000	632,324	13.941 %	123,672	2.727 %	631,324	13.919 %
4000	840,797	13.903 %	162,592	2.689 %	839,595	13.883 %
5000	1,046,066	13.838 %	202,827	2.683 %	1,045,173	13.826 %
6000	1,253,963	13.824 %	241,871	2.666 %	1,252,419	13.807 %
7000	1,457,367	13.771 %	279,694	2.643 %	1,455,255	13.751 %
8000	1,660,461	13.729 %	318,527	2.634 %	1,658,925	13.716 %
9000	1,867,065	13.722 %	357,704	2.629 %	1,865,099	13.707 %
10000	2,072,830	13.710 %	395,813	2.618 %	2,071,272	13.700 %

Table 10. *Required solvency margin $[Z_h^{(R)} - \mathcal{V}_h]/\mathcal{V}_h \times 100$; $\varepsilon = 0.025$*

	$T=n$			$T=5$		
h	$N_0=1000$	$N_0=5000$	$N_0=10000$	$N_0=1000$	$N_0=5000$	$N_0=10000$
0	14.431	13.838	13.710	2.902	2.683	2.618
5	16.525	15.830	15.651	4.658	4.322	4.249
10	18.236	17.320	17.073	7.140	6.664	6.543
15	18.979	17.585	17.272	10.146	9.322	9.175
20	17.766	16.008	15.562	12.690	11.379	11.098
25	15.047	12.060	11.498	13.034	10.769	10.249
30	16.758	11.035	10.072	14.563	9.082	7.948
35	45.498	24.129	21.122	40.183	22.325	19.357

Table 11. *Required solvency margin $[Z_h^{(O)} - \mathcal{V}_h]/\mathcal{V}_h \times 100$; $\varepsilon = 0.025$*

h	$N_0=1000$	$N_0=5000$	$N_0=10000$
0	14.389	13.826	13.700
5	16.479	15.815	15.640
10	18.158	17.292	17.036
15	18.845	17.536	17.218
20	17.538	15.880	15.481
25	14.731	11.814	11.291
30	16.250	10.968	10.061
35	45.258	23.992	21.098

required solvency margin according to (3.15) with $T=5$ implies a strong postponement of the solvency margin building up (as witnessed by tables 10, 11, 14 and 15) and the need to monitor carefully the portfolio in order to adjust the solvency margin in case it is insufficient.

Table 12. *Required solvency margin* $[Z_0^{(R)} - \mathcal{V}_0]/\mathcal{V}_0 \times 100$

	$T=n$			$T=5$		
N_0	$\varepsilon=0.01$	$\varepsilon=0.025$	$\varepsilon=0.05$	$\varepsilon=0.01$	$\varepsilon=0.025$	$\varepsilon=0.05$
1000	14.897	14.431	13.999	3.033	2.902	2.734
2000	14.372	14.081	13.812	2.874	2.770	2.664
3000	14.157	13.941	13.729	2.823	2.727	2.635
4000	14.079	13.903	13.688	2.756	2.689	2.619
5000	13.988	13.838	13.658	2.768	2.683	2.610
6000	13.962	13.824	13.644	2.732	2.666	2.599
7000	13.898	13.771	13.610	2.700	2.643	2.585
8000	13.879	13.729	13.590	2.700	2.634	2.580
9000	13.851	13.722	13.603	2.686	2.629	2.580
10000	13.843	13.710	13.584	2.685	2.618	2.568

Table 13. *Required solvency margin* $[Z_0^{(O)} - \mathcal{V}_0]/\mathcal{V}_0 \times 100$

N_0	$\varepsilon=0.01$	$\varepsilon=0.025$	$\varepsilon=0.05$
1000	14.851	14.389	13.968
2000	14.353	14.063	13.789
3000	14.140	13.919	13.712
4000	14.063	13.883	13.665
5000	13.971	13.826	13.639
6000	13.939	13.807	13.634
7000	13.877	13.751	13.596
8000	13.870	13.716	13.574
9000	13.838	13.707	13.593
10000	13.829	13.700	13.571

It is quite clear that the figures quoted so far strongly depend on the choices assumed, in particular with regard to the mortality model (viz. survival functions and the relevant weights). However, to a large extent what discussed above gives a general perspective of the severity of longevity risk. In table 16 the required solvency margin has been investigated assuming different weights ρ, i.e. $\rho^{[\min]}=0.1$, $\rho^{[\mathrm{med}]}=0.8$, $\rho^{[\max]}=0.1$. Similar results have been obtained under other choices of the weights ρ (considering anyway reasonable that $S^{[\mathrm{med}]}(x)$ is the most realistic scenario).

It could be argued that when just three scenarios are dealt with, the dramatic increase in the solvency margin when considering also the longevity risk is due to the difference between the reserve calculated with the 'worst' scenario and the reserve based on the central scenario. Denoting by $V_t^{[\cdot]}$ the individual reserve calculated with the survival function $S^{[\cdot]}(x)$, we find

$$\frac{V_0^{[\max]}}{V_0^{[\mathrm{med}]}} - 1 = 13.383\%$$

Table 14. *Required solvency margin* $[Z_h^{(R)} - \mathcal{V}_h]/\mathcal{V}_h \times 100\,;\ N_0 = 5000$

	$T=n$			$T=5$		
h	$\varepsilon=0.01$	$\varepsilon=0.025$	$\varepsilon=0.05$	$\varepsilon=0.01$	$\varepsilon=0.025$	$\varepsilon=0.05$
0	13.988	13.838	13.658	2.768	2.683	2.610
5	16.077	15.830	15.603	4.429	4.322	4.225
10	17.600	17.320	17.037	6.829	6.664	6.492
15	17.900	17.585	17.212	9.569	9.322	9.092
20	16.609	16.008	15.485	11.753	11.379	10.949
25	12.824	12.060	11.235	11.404	10.769	10.097
30	12.630	11.035	9.725	10.201	9.082	8.053
35	27.794	24.129	20.302	25.656	22.325	18.827

Table 15. *Required solvency margin* $[Z_h^{(O)} - \mathcal{V}_h]/\mathcal{V}_h \times 100\,;\ N_0 = 5000$

h	$\varepsilon=0.01$	$\varepsilon=0.025$	$\varepsilon=0.05$
0	13.971	13.826	13.639
5	16.051	15.815	15.581
10	17.589	17.292	17.011
15	17.849	17.536	17.154
20	16.539	15.880	15.394
25	12.593	11.814	11.060
30	12.503	10.968	9.681
35	27.660	23.992	20.114

So, in the example the magnitude of the difference between such reserves is actually similar to the magnitude of the required solvency margin.

However this example does not lead to any general conclusion, given that the individual reserve is an expected value of liabilities, whilst the solvency reserve is related to the right tail of the distribution of assets. In order to have a better understanding, let us consider a wider set of scenarios, as depicted in table 17.

Note that the scenarios differ one from the other in terms of the way they represent the phenomena of rectangularization and expansion (in both cases, levels increase moving from scenario [1] to [7]).

In table 18 the solvency margins obtained according to the different requirements discussed previously and for some values of the ruin probability are quoted. Table 19 shows similar results, but with a different choice for the weights ρ. For the sake of brevity, in both cases just one portfolio size has been considered, i.e. $N_0 = 1000$ (as well as only one valuation time, i.e. time 0). The individual reserve is still calculated with $S^{[\mathrm{med}]}(x)$.

In order to reach some conclusions, first of all it is worth noting that the (individual) reserve calculated according to the worst scenario among the seven currently under investigation (i.e. with $S^{[7]}(x)$) is 20.631 % higher than that calculate according to $S^{[\mathrm{med}]}(x)$. Setting aside a solvency margin simply based on the comparison of

Table 16. *Required solvency margin; $\varepsilon=0.025$. (1) deterministic approach; (2) stochastic approach: $\rho^{[min]}=0.2$, $\rho^{[med]}=0.6$, $\rho^{[max]}=0.2$; (3) stochastic approach: $\rho^{[min]}=0.1$, $\rho^{[med]}=0.8$, $\rho^{[max]}=0.1$*

	$\frac{[Z_0^{(R)}-\mathcal{V}_0]}{\mathcal{V}_0}\times 100, T=n$			$\frac{[Z_0^{(O)}-\mathcal{V}_0]}{\mathcal{V}_0}\times 100$		
N_0	(1)	(2)	(3)	(1)	(2)	(3)
1000	2.180	14.431	14.024	1.967	14.389	13.984
2000	1.486	14.081	13.816	1.344	14.063	13.793
3000	1.199	13.941	13.733	1.068	13.919	13.713
4000	1.082	13.903	13.691	0.974	13.883	13.678
5000	0.969	13.838	13.643	0.851	13.826	13.629
6000	0.881	13.824	13.665	0.758	13.807	13.653
7000	0.828	13.771	13.602	0.764	13.751	13.583
8000	0.747	13.729	13.610	0.670	13.716	13.592
9000	0.739	13.722	13.608	0.663	13.707	13.599
10000	0.682	13.710	13.581	0.611	13.700	13.566

Table 17. *Mortality laws for annuitants*

	[1]	[2](=[min])	[3]	[4](=[med])	[5]	[6](=[max])	[7]
G	0.000178	0.000042	0.000009	0.000002	0.0000004	0.0000001	0.00000001
H	1.07968	1.09803	1.11670	1.13450	1.15379	1.17215	1.19208

Table 18. *Required solvency margin; $\rho^{[1]}=\rho^{[7]}=0.05$, $\rho^{[2]}=\rho^{[6]}=0.10$, $\rho^{[3]}=\rho^{[5]}=0.15$, $\rho^{[4]}=0.40$*

	$T=n$		$T=5$			
ε	$[Z_0^{(R)}-\mathcal{V}_0]$	$\frac{[Z_0^{(R)}-\mathcal{V}_0]}{\mathcal{V}_0}$	$[Z_0^{(R)}-\mathcal{V}_0]$	$\frac{[Z_0^{(R)}-\mathcal{V}_0]}{\mathcal{V}_0}$	$[Z_0^{(O)}-\mathcal{V}_0]$	$\frac{[Z_0^{(O)}-\mathcal{V}_0]}{\mathcal{V}_0}$
0.025	311,832	20.626 %	48,571	3.213 %	311,525	20.605 %
0.05	253,906	16.794 %	44,625	2.952 %	253,460	16.765 %
0.1	203,123	13.435 %	38,134	2.522 %	202,538	13.396 %

reserves calculated with different survival functions (as some practice suggests) on the one hand would disregard the risk of random fluctuations (which obviously can be considered separately) and on the other would disregard a valuation of the probability of ruin, possibly leading to unsound capital allocation.

5 Allowing for financial risk

5.1 Investment hypotheses

The examples in the previous section show the importance of the mortality risk, namely of the longevity risk, in life annuity portfolios. Another important source of

Table 19. *Required solvency margin*; $\rho^{[1]}=\rho^{[7]}=0.01$, $\rho^{[2]}=\rho^{[6]}=0.04$, $\rho^{[3]}=\rho^{[5]}=0.20$, $\rho^{[4]}=0.50$

	$T=n$		$T=5$			
ε	$[Z_0^{(R)}-\mathcal{V}_0]$	$\frac{[Z_0^{(R)}-\mathcal{V}_0]}{\mathcal{V}_0}$	$[Z_0^{(R)}-\mathcal{V}_0]$	$\frac{[Z_0^{(R)}-\mathcal{V}_0]}{\mathcal{V}_0}$	$[Z_0^{(O)}-\mathcal{V}_0]$	$\frac{[Z_0^{(O)}-\mathcal{V}_0]}{\mathcal{V}_0}$
0.025	206,852	13.682 %	39,592	2.619 %	206,293	13.645 %
0.05	156,250	10.335 %	33,531	2.218 %	155,152	10.262 %
0.1	108,272	7.161 %	26,871	1.777 %	107,111	7.085 %

risk comes from investment. In this section we intend to analyse the global riskiness arising from both mortality and investment.

As far as financial modelling is concerned, we do not deal with the problem of assessing the market value of the different types of financial instruments in which the insurance company can invest, nor with the problem of choosing a proper investment strategy. We simply adopt a model for the short interest rate, whose (random) behaviour should reflect interest incomes as well as capital gains/losses arising from the performance of the assets linked to the portfolio.

Let r_t be the short interest rate, expressing the instantaneous total rate of return on assets. We assume that its behaviour can be described with the Vasicek model (i.e. with an Ornstein-Uhlenbeck process). Thus

$$dr_t=\alpha(\gamma-r_t)dt+\sigma dW_t \tag{5.1}$$

where $\{W_t\}$ is a standard Wiener process and α, γ and σ are positive constants. Note that γ represents the long-term mean of the short rate, α a friction force bringing the process back towards γ and σ the diffusion coefficient. We do not discuss further the features of model (5.1) (see Vasicek, 1977). We just mention that the Vasicek model is often used in actuarial applications since it represents quite satisfactorily the long-term development of the rate of return (see, for example, Parker (1997) and the list of references therein). Note that (5.1) could lead to a negative value for the short rate; under our hypotheses this is anyway acceptable, since capital losses are admitted.

Having assumed that the behaviour of r_t includes both financial incomes and capital gains/losses, solvency is investigated in patrimonial terms. Assets are now described as follows

$$A_t=A_{t-1}-N_{t-1}R+I_t+J_t \tag{5.2}$$

whilst the value of future obligations and the portfolio reserve are still given by (4.2) and (4.3), respectively (considering the perspective of the supervisory authority, the flows K_t are disregarded). Requirements (3.11) and (3.19) are considered.

5.2 *Numerical investigations*

We refer to the same portfolio considered in section 4; hence, in particular, we adopt a run-off approach. As far as mortality is concerned, we adopt the three scenarios

Table 20. *Parameters for the short rate*

r_0	0.05	α	0.1
γ	0.04	σ	0.01

Table 21. *Required solvency margin; deterministic approach to mortality, $\varepsilon=0.025$*

	$T=n$		$T=5$			
N_0	$M_0^{(R)}$	$\frac{M_0^{(R)}}{\mathcal{V}_0}$	$M_0^{(R)}$	$\frac{M_0^{(R)}}{\mathcal{V}_0}$	$M_0^{(O)}$	$\frac{M_0^{(O)}}{\mathcal{V}_0}$
1000	89,226	5.902 %	69,633	4.606 %	27,737	1.835 %
2000	177,550	5.872 %	140,135	4.634 %	55,639	1.840 %
3000	265,559	5.855 %	211,177	4.656 %	82,428	1.817 %
4000	352,875	5.835 %	281,198	4.650 %	108,223	1.790 %
5000	441,260	5.837 %	351,858	4.655 %	135,558	1.793 %
6000	529,288	5.835 %	422,707	4.660 %	162,775	1.794 %
7000	616,138	5.822 %	492,382	4.653 %	188,919	1.785 %
8000	703,182	5.814 %	563,850	4.662 %	216,284	1.788 %
9000	790,749	5.811 %	634,518	4.663 %	243,852	1.792 %
10000	879,061	5.814 %	705,166	4.664 %	271,403	1.795 %

described in section 4.1 (see table 1 and figures 1 and 2); both a deterministic and a stochastic framework are dealt with. In any case, we still assume that the future lifetimes of the annuitants have a common distribution and that the single premium and the individual reserve are calculated according to $S^{[\mathrm{med}]}(x)$ and $i=0.03$.

Within a deterministic approach to mortality, the causes of uncertainty are two: the time of death of each annuitant and the level of the short interest rate at any time t. We assume, as it is quite common, independence between demographic and financial variables and we proceed to the assessment of the solvency margin through simulation (assuming independence among the insured risks, as far as mortality is concerned, conditional on a given survival function). Parameters for the financial simulation are given in table 20. They are (arbitrarily) chosen so that they reflect the overall performance of assets, resulting from both market behaviour and the investment strategy of the insurance company. Once the short rate has been simulated, values for I_t+J_t and $v(t, t+h)$ can be easily obtained. For example

$$v(0, h)=\exp\left(-\int_0^h r_u\, du\right) \tag{5.3}$$

Note that under the parameters of table 20, an expected investment profit derives (the average simulated annual interest rate is slightly greater than 0.04). Hence, results quoted in this section are not comparable with those of section 4.

Table 21 shows the solvency margin required at time 0 according to condition (3.11) ($M_0^{(R)}$), with both $T=n$ and $T=5$, as well as to (3.19) ($M_0^{(O)}$). The greater severity of requirement (3.11) (both in the case $T=n$ and $T=5$) is due to the fact that

Table 22. *Required solvency margin; stochastic approach to mortality, $\varepsilon=0.025$*

	$T=n$		$T=5$			
N_0	$M_0^{(R)}$	$\frac{M_0^{(R)}}{\mathcal{V}_0}$	$M_0^{(R)}$	$\frac{M_0^{(R)}}{\mathcal{V}_0}$	$M_0^{(O)}$	$\frac{M_0^{(O)}}{\mathcal{V}_0}$
1000	234,809	15.531 %	105,820	6.999 %	233,627	15.453 %
2000	465,021	15.379 %	211,859	7.007 %	463,350	15.324 %
3000	696,735	15.361 %	318,642	7.025 %	694,796	15.319 %
4000	927,784	15.358 %	424,604	7.021 %	926,516	15.321 %
5000	1,160,254	15.349 %	531,348	7.029 %	1,157,321	15.310 %
6000	1,394,717	15.375 %	638,309	7.037 %	1,391,178	15.336 %
7000	1,622,656	15.333 %	743,647	7.027 %	1,618,809	15.296 %
8000	1,857,986	15.362 %	851,135	7.037 %	1,853,379	15.324 %
9000	2,090,759	15.366 %	957,496	7.037 %	2,086,065	15.331 %
10000	2,317,505	15.329 %	1,063,397	7.034 %	2,313,219	15.300 %

assets, cumulating at a random rate, are compared with the reserve, calculated according to a financial hypotheses which could be significantly different from the actual investment performance. On the contrary, under condition (3.19) assets are compared with the present value of future obligations, calculated under the same investment scenario (actually, (3.19) reduces to the inspection of A_n); hence, a sort of offsetting effect arises for the investment risk when the obligations-based approach is considered, i.e. when the ability to meet liabilities is ascertained on realistic grounds. It must be stressed, in particular, that contrary to the deterministic financial setting (section 4), we find $M_0^{(O)}<M_0^{(R)}$ also in the case of $T=5$, witnessing the fact that investment risk reveals itself already in the short run. As far as the size of the portfolio is concerned, in any case the entity of the relative required solvency margin $\left(\frac{M_0}{\mathcal{V}_0}\right)$ is almost constant; actually, the investment risk is non-pooling and its effect, in relative terms, is independent of the size of the portfolio. Finally, it should be kept in mind that results in table 21 (and 22 as well) have been obtained in the presence of an expected (positive) investment profit. In order to understand its effect, we mention that adopting a deterministic financial scenario and assuming $i_t^*=0.04$ (hence including an investment profit whose magnitude is comparable with that obtained in the stochastic financial framework) it turns out that no solvency margin is required at time 0, the mortality risk being completely covered by the expected financial profit.

In a stochastic approach to mortality, the causes of uncertainty are three: the distribution of future lifetimes, the time of death of each insured and the investment performances. As before, we assume independence between demographic and financial variables; further, we assume that, once the mortality distribution has been assigned, annuitants have independent durations of life. Results in table 22 have been obtained through simulation (assuming $\rho^{[\min]}=0.2$, $\rho^{[\mathrm{med}]}=0.6$ and $\rho^{[\max]}=0.2$), with valuation time 0. Note the sharp increase in the margins $M_0^{(R)}$, under $T=n$, and $M_0^{(O)}$ with respect to table 21, obviously due to longevity risk. In particular, since longevity risk emerges in the long run, $M_0^{(O)}$ is now greater than $M_0^{(R)}$ when $T=5$. Note also that when comparing table 22 with 21 it turns out that $M_0^{(O)}$ is more seriously affected

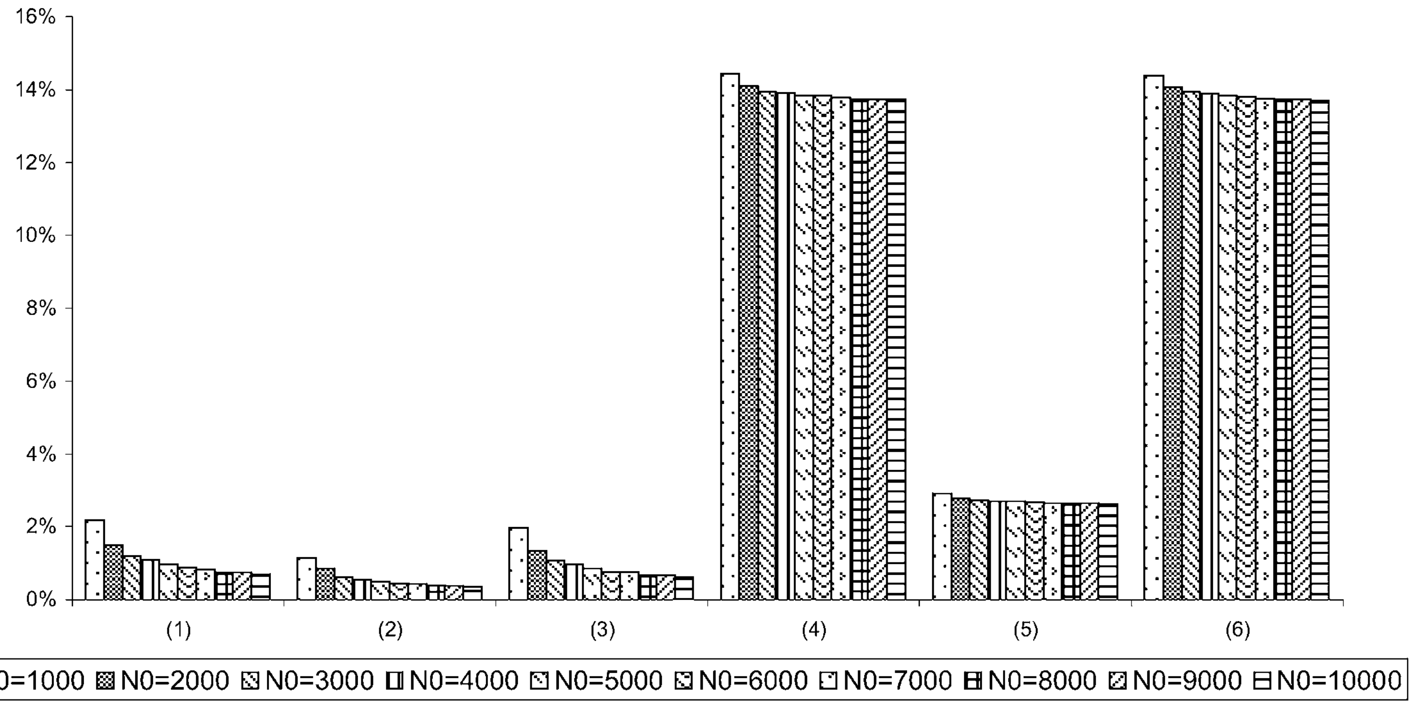

Figure 3. Required solvency margin $\frac{[Z_0 - \mathscr{V}_0]}{\mathscr{V}_0}$; deterministic financial approach, $\varepsilon = 0.025$. (1) deterministic approach to mortality; reserve-based, $T = n$; (2) deterministic approach to mortality; reserve-based, $T = 5$; (3) deterministic approach to mortality; obligations-based; (4) stochastic approach to mortality; reserve-based, $T = n$; (5) stochastic approach to mortality; reserve-based, $T = 5$; (6) stochastic approach to mortality; obligations-based.

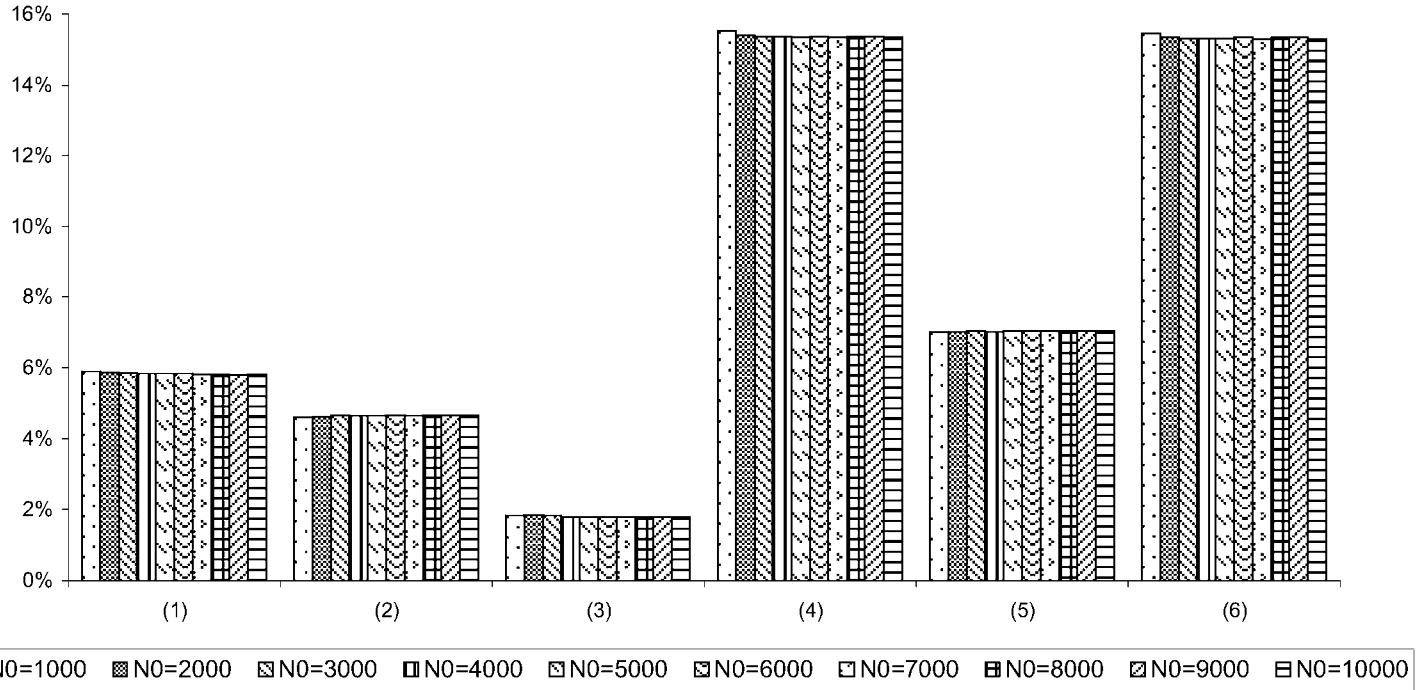

Figure 4. Required solvency margin $\frac{M_0}{\mathcal{V}_0}$; stochastic financial approach, $\varepsilon = 0.025$. (1) deterministic approach to mortality; reserve-based, $T = n$; (2) deterministic approach to mortality; reserve-based, $T = 5$; (3) deterministic approach to mortality; obligations-based; (4) stochastic approach to mortality; reserve-based, $T = n$; (5) stochastic approach to mortality; reserve-based, $T = 5$; (6) stochastic approach to mortality; obligations-based.

by longevity than investment risk. Actually it must be stressed that, contrarily to table 21, $M_0^{(R)}$ under $T=n$ and $M_0^{(O)}$ have the same magnitude. This shows that for longevity risk neither a pooling effect (as for the random fluctuation risk) nor an offsetting effect (as for the investment risk) can be obtained, albeit that the investigation is performed on realistic grounds. In any case, as in table 21, the relative required solvency margin $\frac{M_0}{\mathscr{V}_0}$, plotted against N_0, is almost constant, due to the non-pooling effect of both longevity and investment risk. In order to catch the effect of investment profit, we point out that numerical evaluations performed within a deterministic financial setting lead to a required solvency margin of nearly 3 % of the initial reserve in the reserve-based approach with $T=n$ and in the obligations-based approach and to no margin required in the reserve-based approach with $T=5$.

Finally, with regard to the dependence of the results obtained on the mortality model, we mention that the wider set of scenarios considered at the end of section 4 lead to similar conclusions. For the sake of brevity, the relevant results are omitted.

Figures 3 and 4 offer an overall comparison among some of the results discussed in sections 4 and 5 (relating to the three mortality scenarios represented by $S^{[\min]}(x)$, $S^{[\mathrm{med}]}(x)$, $S^{[\max]}(x)$).

6 Concluding remarks

Solvency requirements for immediate annuities have been analyzed. Particular emphasis has been placed on the longevity risk, i.e. the mortality risk originated by possible systematic deviations from the assumed projected mortality, viz. arising from parameter and model uncertainty.

Solvency is traditionally defined in terms of comparisons between asset and reserve values. In order to avoid problems inherent in the choice of the valuation basis used in reserving, solvency has been defined also in terms of random values of future obligations.

Several numerical examples illustrate solvency requirements produced by the two different approaches. In particular, the results obtained taking into account the mortality as well as the financial risk provide an interesting description of the riskiness inherent in a portfolio of immediate annuities and in a pension plan as well.

Numerical results only provide an illustration, since they obviously depend on the specific assumptions concerning the randomness in mortality as well as on the model used to describe the investment performance. Nevertheless, the results presented in this paper underline the dramatic importance of a sound evaluation of the solvency requirements for life annuities (and, more generally, for insurance products providing lifetime living benefits, for example long-term care covers, post-retirement sickness benefits, etc.).

Further research work should concern various aspects of the risks' inherent in pension annuities, focussing in particular on the relevant solvency requirements. Following a rather traditional approach to solvency, appropriate short-cut formulae, expressing the required amount of assets in terms of quantities which properly reflect important risk drivers (investment policy, number of the annuitants, average age of the annuitants, etc.), should be constructed and proposed in order to make easier a quick assessment of solvency. In this context the solvency requirements should be related to some 'objective' quantity as, for example, the total amount paid yearly to the annuitants, rather than the reserve which is usually considered in traditional short-cut formulae but whose amount heavily depends on the chosen valuation basis.

Aggregate risk models can suggest an innovative approach to solvency requirements for pension annuities. This context seems appropriate for a joint analysis of risks related to the accumulation phase and the decumulation phase, providing tools for quantifying the overall risk exposure of pension schemes and annuity portfolios.

Finally, more emphasis should be placed on the 'value' of pension liabilities. As noted in section 3, obligation-based solvency requirements allow for a fair value-oriented approach to solvency. Focussing on the concept of fair value of pension liabilities is particularly important as a market-based valuation seems difficult, because markets for these liabilities are far from perfect or complete.

References

Benjamin, B. and Soliman, A. S. (1993) *Mortality on the Move*. Oxford: The Institute of Actuaries.

Cairns, A. (2000) Some notes on the dynamics and optimal control of stochastic pension fund models in continuous time. *ASTIN Bulletin*, **30**: 19–55.

Colotti, R. and Olivieri, A. (1998) Profit testing and life portfolio valuation: a threefold approach. *Transactions of the 26th International Congress of Actuaries*, Birmingham, **3**: 357–82.

Faculty of Actuaries' Solvency Working Party (1986) The solvency of life assurance companies. *Transactions of the Faculty of Actuaries*, **39**: 251–340.

Haberman, S. and Sung, J. H. (2002) Dynamic programming approach to pension funding: the case of incomplete state information. *ASTIN Bulletin*, **3**: 129–42.

Haberman, S. and Vigna, E. (2002) Optimal investment strategies and risk measures in defined contribution pension schemes. *Insurance: Mathematics and Economics*, **31**: 35–69.

Heligman, L. and Pollard, J. H. (1980) The age pattern of mortality. *Journal of the Institute of Actuaries*, **107**: 49–80.

Kahane, Y., Tapiero, C. S. and Laurent, J. (1989) Concepts and trends in the study of insurer's solvency. In Cummins, J. D. and Derrig, R. A. (eds), *Financial Models of Insurance Solvency*. Boston: Kluwer Academic Publishers, pp. 219–41.

MacDonald, A. S. (1997) *The Second Actuarial Study of Mortality in Europe*. Oxford: Groupe Consultatif des Associations d'Actuaires des Pays des Communautes Europennes.

MacDonald, A. S., Cairns, A. J. G., Gwilt, P. L. and Miller, K. A. (1998) An international comparison of recent trends in population mortality. *British Actuarial Journal*, **4**(1): 3–141.

Marocco, P. and Pitacco, E. (1998) Longevity risk and life annuity reinsurance. *Transactions of the 26th International Congress of Actuaries*, Birmingham, **6**: 453–79.

Milevsky, M. A. and Promislov, S. D. (2001) Mortality derivatives and the option to annuitise. *Insurance: Mathematics and Economics*, **29**: 299–318.

Olivieri, A. (2001) Uncertainty in mortality projections: an actuarial perspective. *Insurance: Mathematics and Economics*, **29**: 231–45.

Olivieri, A. and Pitacco, E. (2002) Premium systems for post-retirement sickness covers. *Belgian Actuarial Bulletin*, **2**: 15–25.

Parker, G. (1997) Stochastic analysis of the interaction between investment and insurance risk. *North American Actuarial Journal*, **1**(2): 55–71.

Pitacco, E. (2002) Longevity risk in living benefits. to appear in: In Fornero, E. and Luciano, E. (eds), *Developing an annuity market in Europe*. Edward Elgar Publising Ltd.

Rüttermann, M. (1999) Mortality trends worldwide. *Risk Insights, General and Cologne Re*, **3**: 18–20.

Vasicek, O. A. (1977) An equilibrium characterization of the term structure. *Journal of Financial Economics*, **5**: 177–88.