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FLUCTUATIONS OF THE LOCAL TIMES OF THE SELF-REPELLING RANDOM WALK WITH DIRECTED EDGES

LAURE MARÊCHÉ,* *Institut de Recherche Mathématique Avancée, UMR 7501 Université de Strasbourg et CNRS*

Abstract

In 2008, Tóth and Vető defined the self-repelling random walk with directed edges as a *non-Markovian* random walk on $\mathbb{Z}$: in this model, the probability that the walk moves from a point of $\mathbb{Z}$ to a given neighbor depends on the number of previous crossings of the directed edge from the initial point to the target, called the local time of the edge. Tóth and Vető found that this model exhibited very peculiar behavior, as the process formed by the local times of all the edges, evaluated at a stopping time of a certain type and suitably renormalized, converges to a deterministic process, instead of a random one as in similar models. In this work, we study the fluctuations of the local times process around its deterministic limit, about which nothing was previously known. We prove that these fluctuations converge in the Skorokhod M_1 topology, as well as in the uniform topology away from the discontinuities of the limit, but *not* in the most classical Skorokhod topology. We also prove the convergence of the fluctuations of the aforementioned stopping times.

Keywords: Self-interacting random walks; self-repelling random walk with directed edges; local times; functional limit theorems; fluctuations

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1. Introduction and results

1.1. Self-interacting random walks

The study of self-interacting random walks began in 1983 in an article of Amit *et al.* [1]. Before [1], the expression 'self-avoiding random walk' referred to paths on graphs that do not intersect themselves. However, these are not easy to construct step by step; hence one would consider the set of all possible paths of a given length. Since one does not follow a single path as it grows with time, this is not really a random walk model. In order to work with an actual random walk model with self-avoiding behavior, the authors of [1] introduced the *'true' self-avoiding random walk*. This is a random walk on $\mathbb{Z}^d$ for which, at each step, the position of the process at the next step is chosen randomly from among the neighbors of the current position, depending on the number of previous visits to said neighbors, with lower probabilities for those that have been visited the most. This process is a random walk in the sense that it is constructed

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* Postal address: 7 rue René Descartes, 67000 Strasbourg, France. Email address: laure.mareche@math.unistra.fr

step by step, but unlike most random walks in the literature, it is *non-Markovian*: at each step, the law of the next step depends on the whole past of the process.

It turns out that the 'true' self-avoiding random walk is hard to study. This led to the introduction by Tóth [13, 14, 15] of non-Markovian random walks *with bond repulsion*, for which the probability of going from one site to another, instead of depending of the number of previous visits to the target, depends on the number of previous crossings of the undirected edge between the two sites, which is called the *local time* of the edge, with lower probabilities for the edges that have been crossed the most in the past. These walks are much easier to study, at least on $\mathbb{Z}$, because one can apply the *Ray–Knight approach* to them. This approach was introduced by Ray and Knight in [11, 2], and was used for the first time for non-Markovian random walks by Tóth in [13, 14, 15]. Since then, it has been applied to many other non-Markovian random walks, such as a continuous-time version of the 'true' self-avoiding random walk in [18], *edge-reinforced random walks* (see the corresponding part of the review [9] and references therein), and *excited random walks* (see [4] and references therein). The Ray–Knight approach works as follows: though the random walk itself is not Markovian, if we stop it when the local time at a given edge has reached a certain threshold, then the local times on the edges will form a Markov chain, which enables their analysis. Thanks to this approach, Tóth was able to prove scaling limits for the local times process for many different random walks with bond repulsion in his works [13, 14, 15]. The law of the limit depends on the random walk model, but it is always a random process (the model studied by Tóth in [16] has a deterministic limit, but it is not a random walk with bond repulsion, as it is *self-attracting*: the more an edge has been crossed in the past, the more likely it is to be crossed in the future).

1.2. The self-repelling random walk with directed edges

In 2008, Tóth and Vető [17] introduced a process seemingly very similar to the aforementioned random walks with bond repulsion, in which the probability of going from one site to another depends on the number of crossings of the *directed* edge between them, instead of the crossings of the undirected edge. This process, called *self-repelling random walk with directed edges*, is a nearest-neighbor random walk on $\mathbb{Z}$ defined as follows. For any set A, we denote by $|A|$ the cardinality of A. Let $w : \mathbb{Z} \mapsto (0, +\infty)$ be a non-decreasing and non-constant function. We will denote the walk by $(X_n)_{n\in\mathbb{N}}$. We set $X_0 = 0$, and for any $n \in \mathbb{N}$, $i \in \mathbb{Z}$, we denote by $\ell^{\pm}(n, i) = |\{0 \le m \le n-1 \mid (X_m, X_{m+1}) = (i, i \pm 1)\}|$ the number of crossings of the directed edge $(i, i \pm 1)$ before time n, that is, the *local time* of the directed edge at time n. Then

$$\mathbb{P}(X_{n+1} = X_n \pm 1) = \frac{w(\pm(\ell^-(n, X_n) - \ell^+(n, X_n)))}{w(\ell^+(n, X_n) - \ell^-(n, X_n)) + w(\ell^-(n, X_n) - \ell^+(n, X_n))}.$$

Using the local time of directed edges instead of that of undirected edges may seem like a very small change in the definition of the process, but the behavior of the self-repelling random walk with directed edges is actually very different from that of classical random walks with bond repulsion. Indeed, Tóth and Vető [17] were able to prove that the local times process has a *deterministic* scaling limit, which is in sharp contrast with the random limit processes obtained for the random walks with bond repulsion on undirected edges [13–15] and even for the simple random walk [2].

The result of [17] is as follows. For any $a \in \mathbb{R}$, we let $a_+ = \max(a, 0)$. If for any $n \in \mathbb{N}$, $i \in \mathbb{Z}$, we denote by $T^{\pm}_{n,i}$ the stopping time defined by $T^{\pm}_{n,i} = \min\{m \in \mathbb{N} \mid \ell^{\pm}(m, i) = n\}$, then $T^{\pm}_{n,i}$ is almost surely finite by Proposition 1 of [17], and we have the following.

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Theorem 1. ([17, Theorem 1].) *For any $\theta > 0$, $x \in \mathbb{R}$, we have that*

$$\sup_{y\in\mathbb{R}} \left| \frac{1}{N}\ell^+ \left(T^{\pm}_{\lfloor N\theta\rfloor,\lfloor Nx\rfloor}, \lfloor Ny\rfloor\right) - \left(\frac{|x|-|y|}{2}+\theta\right)_+ \right|$$

converges in probability to 0 when N tends to $+\infty$.

Thus the local times process of the self-repelling random walk with directed edges admits the deterministic scaling limit $y \mapsto \left(\frac{|x|-|y|}{2}+\theta\right)_+$, which has the shape of a triangle. This also implies the following result on convergence to a deterministic limit for the $T^{\pm}_{\lfloor N\theta\rfloor,\lfloor Nx\rfloor}$.

Proposition 1. ([17, Corollary 1].) *For any $\theta > 0$, $x \in \mathbb{R}$, we have that $\frac{1}{N^2}T^{\pm}_{\lfloor N\theta\rfloor,\lfloor Nx\rfloor}$ converges in probability to $(|x|+2\theta)^2$ when N tends to $+\infty$.*

The deterministic character of these limits makes the behavior of the self-repelling random walk with directed edges very unusual, hence worthy of study. In particular, it is natural to consider the possible fluctuations of the local times process and of the $T^{\pm}_{\lfloor N\theta\rfloor,\lfloor Nx\rfloor}$ around their deterministic limits. However, before the present paper, nothing was known about these fluctuations. In this work, we prove convergence in distribution of the fluctuations of the local times process and of the $T^{\pm}_{\lfloor N\theta\rfloor,\lfloor Nx\rfloor}$. It happens that the limit of the fluctuations of the local times process is discontinuous; therefore, before stating the results, we have to be careful about the topology in which it may converge.

1.3. Topologies for the convergence of the local times process

For any interval $I \subset R$, let DI be the space of *càdlàg* functions on I, that is, the set of functions $I \mapsto \mathbb{R}$ that are right-continuous and have left limits everywhere in I. For any function $Z : I \mapsto \mathbb{R}$, we denote by $\|Z\|_\infty = \sup_{y\in I} |Z(y)|$ the uniform norm of Z on I. The uniform norm on I gives a topology on DI, but it is often too strong to deal with discontinuous functions.

For discontinuous càdlàg functions, the most widely used topology is the *Skorokhod J_1 topology*, introduced by Skorokhod in [12] (see [10, Chapter VI] for a course), which is often called 'the' Skorokhod topology. Intuitively, two functions are close in this topology if they are close for the uniform norm after allowing some small perturbation of time. Rigorously, for $a < b$ in $\mathbb{R}$, the Skorokhod J_1 topology on $D[a, b]$ is defined as follows. We denote by $\Lambda_{a,b}$ the set of functions $\lambda : [a, b] \mapsto [a, b]$ that are bijective, strictly increasing, and continuous (they correspond to the possible perturbations of time), and we denote by $\mathrm{Id}_{a,b} : [a, b] \mapsto [a, b]$ the identity map, defined by $\mathrm{Id}_{a,b}(y) = y$ for all $y \in [a, b]$. The Skorokhod J_1 topology on $D[a, b]$ is defined through the following metric: for any $Z_1, Z_2 \in D[a, b]$, we set

$$d_{J_1,a,b}(Z_1, Z_2) = \inf_{\lambda\in\Lambda_{a,b}} \max\left(\|Z_1 \circ \lambda - Z_2\|_\infty, \|\lambda - \mathrm{Id}_{a,b}\|_\infty\right).$$

It can be proven rather easily that this is indeed a metric. We can then define the Skorokhod J_1 topology on $D(-\infty, \infty)$ with the following metric: if for any sets $A_1 \subset A_2$ and A_3 and any function $Z : A_2 \mapsto A_3$, we denote by $Z|_{A_1}$ the restriction of Z to A_1, then for $Z_1, Z_2 \in D(-\infty, \infty)$, we set

$$d_{J_1}(Z_1, Z_2) = \int_0^{+\infty} e^{-a}\left(d_{J_1,-a,a}\left(Z_1|_{[-a,a]}, Z_2|_{[-a,a]}\right) \wedge 1\right)\mathrm{d}a.$$

The Skorokhod J_1 topology is widely used to study the convergence of càdlàg functions. However, when the limit function has a jump, which will be the case here, convergence in the

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Skorokhod J_1 topology requires the converging functions to have a single big jump approximating the jump of the limit process. To account for other cases, like having the jump of the limit functions approximated by several smaller jumps in quick succession or by a very steep continuous slope, one has to use a less restrictive topology, such as the *Skorokhod M_1 topology*.

The Skorokhod M_1 topology was also introduced by Skorokhod in [12] (see [19, Section 3.3] for an overview). For any $a < b$ in $\mathbb{R}$, the Skorokhod M_1 distance on $D[a, b]$ is defined as follows: the distance between two functions will be roughly 'the distance between the completed graphs of the functions'. More rigorously, if $Z \in D[a, b]$, we set $Z(a^-) = Z(a)$, and for any $y \in (a, b]$, we set $Z(y^-) = \lim_{y' \to y, y' < y} Z(y')$. Then the *completed graph of Z* is

$$\Gamma_Z = \{(y, z) \mid y \in [a, b],\ \exists\, \varepsilon \in [0, 1] \text{ such that } z = \varepsilon Z(y^-) + (1 - \varepsilon)Z(y)\}.$$

To express the 'distance between two such completed graphs', we need to define the *parametric representations* of Γ_Z (by abuse of notation, we will often write 'the parametric representations of Z'). We define an order on Γ_Z as follows: for $(y_1, z_1), (y_2, z_2) \in \Gamma_Z$, we have $(y_1, z_1) \leq (y_2, z_2)$ when $y_1 < y_2$ or when $y_1 = y_2$ and $|Z(y_1^-) - z_1| \leq |Z(y_1^-) - z_2|$. A *parametric representation* of Γ_Z is a continuous, surjective function $(u, r) : [0, 1] \mapsto \Gamma_Z$ that is non-decreasing with respect to this order; thus intuitively, when t goes from 0 to 1, $(u(t),r(t))$ 'travels through the completed graph of Z from its beginning to its end'. A parametric representation of Z always exists (see [19, Remark 12.3.3]). For $Z_1, Z_2 \in D[a, b]$, the Skorokhod M_1 distance between Z_1 and Z_2, denoted by $d_{M_1,a,b}(Z_1, Z_2)$, is $\inf\left\{ \max\left(\|u_1 - u_2\|_\infty, \|r_1 - r_2\|_\infty\right)\right\}$, where the infimum is on the parametric representations (u_1, r_1) of Z_1 and (u_2, r_2) of Z_2. It can be proven that this indeed gives a metric (see [19, Theorem 12.3.1]), and this metric defines the Skorokhod M_1 topology on $D[a, b]$. For any $a > 0$, we will denote $d_{M_1,-a,a}$ by $d_{M_1,a}$ for short. We can now define the Skorokhod M_1 topology in $D(-\infty, \infty)$ through the following metric: for $Z_1, Z_2 \in D(-\infty, \infty)$, we set

$$d_{M_1}(Z_1, Z_2) = \int_0^{+\infty} e^{-a}\left(d_{M_1,a}\left(Z_1|_{[-a,a]}, Z_2|_{[-a,a]}\right) \wedge 1\right)\mathrm{d}a.$$

It can be seen that the Skorokhod M_1 topology is weaker than the Skorokhod J_1 topology (see [19, Theorem 12.3.2]), and thus less restrictive. Indeed, since the distance between two functions is roughly 'the distance between the completed graphs of the functions', the Skorokhod M_1 topology will allow a function with a jump to be the limit of functions with steep slopes or with several smaller jumps. For this reason, the Skorokhod M_1 topology is often more suitable when one is considering convergence to a discontinuous function.

1.4. Results

We are now ready to state our results on the convergence of the fluctuations of the local times process. For any $\theta > 0$, $x \in \mathbb{R}$, $\iota \in \{+, -\}$, for any $N \in \mathbb{N}^*$, we define functions Y_N^-, Y_N^+ as follows: for any $y \in \mathbb{R}$, we set

$$Y_N^\pm(y) = \frac{1}{\sqrt{N}}\left(\ell^\pm\left(T^\iota_{\lfloor N\theta\rfloor,\lfloor Nx\rfloor}, \lfloor Ny\rfloor\right) - N\left(\frac{|x| - |y|}{2} + \theta\right)_+\right).$$

The functions $Y_N^\pm$ actually depend on ι, but to make the notation lighter, we do not write this dependency explicitly. Moreover, $\left(B_y^x\right)_{y\in\mathbb{R}}$ will denote a two-sided Brownian motion with $B_x^x = 0$ and variance $\mathrm{Var}(\rho_-)$, where ρ_- is the distribution on $\mathbb{Z}$ defined later in (3). We prove

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the following convergence for the fluctuations of the local times process of the self-repelling random walk with directed edges.

Theorem 2. *For any $\theta > 0$, $x \in \mathbb{R}$, $\iota \in \{+, -\}$, the process $Y_N^{\pm}$ converges in distribution to $\left(B_y^x \mathbb{1}_{\{y\in[-|x|-2\theta,|x|+2\theta)\}}\right)_{y\in\mathbb{R}}$ in the Skorokhod M_1 topology on $D(-\infty, +\infty)$ when N tends to $+\infty$.*

Therefore, the fluctuations of the local times process have a diffusive limit behavior. However, it is necessary to use the Skorokhod M_1 topology here, as the following result states that convergence does not occur in the stronger Skorokhod J_1 topology.

Proposition 2. *For any $\theta > 0$, $x \in \mathbb{R}$, $\iota \in \{+, -\}$, the process $Y_N^{\pm}$ does not converge in distribution in the Skorokhod J_1 topology on $D(-\infty, +\infty)$ when N tends to $+\infty$.*

We stress the fact that the use of the Skorokhod M_1 topology is required only to deal with the discontinuities of the limit process at $-|x| - 2\theta$ and $|x| + 2\theta$. Indeed, if we consider the process on an interval that does not include $-|x| - 2\theta$ or $|x| + 2\theta$, it converges in the much stronger topology given by the uniform norm, as stated in the following result.

Proposition 3. *For any $\theta > 0$, $x \in \mathbb{R}$, $\iota \in \{+, -\}$, for any closed interval $I \in \mathbb{R}$ that does not contain $-|x| - 2\theta$ or $|x| + 2\theta$, the process $\left(Y_N^{\pm}(y)\right)_{y\in I}$ converges in distribution to $\left(B_y^x \mathbb{1}_{\{y\in[-|x|-2\theta,|x|+2\theta)\}}\right)_{y\in I}$ in the topology on DI given by the uniform norm when N tends to $+\infty$.*

Finally, we also prove the convergence of the fluctuations of $T^{\pm}_{\lfloor N\theta\rfloor,\lfloor Nx\rfloor}$. For any $\sigma^2 > 0$, we denote by $\mathcal{N}\left(0, \sigma^2\right)$ the Gaussian distribution with mean 0 and variance σ^2, and we recall that ρ_- will be defined in (3). We then have the following.

Proposition 4. *For any $\theta > 0$, $x \in \mathbb{R}$, $\iota \in \{+, -\}$, we have that*

$$\frac{1}{N^{3/2}}\left(T^{\iota}_{\lfloor N\theta\rfloor,\lfloor Nx\rfloor} - N^2(|x| + 2\theta)^2\right)$$

converges in distribution to $\mathcal{N}(0, \frac{32}{3}\mathrm{Var}(\rho_-)((|x| + \theta)^3 + \theta^3))$ when N tends to $+\infty$.

Remark 1. Instead of studying the fluctuations of $\ell^{\pm}\left(T^{\iota}_{\lfloor N\theta\rfloor,\lfloor Nx\rfloor}, .\right)$, it may seem more natural to consider those of $\ell^{\pm}(N^2, .)$. However, the Ray–Knight arguments that allow one to study $\ell^{\pm}\left(T^{\iota}_{\lfloor N\theta\rfloor,\lfloor Nx\rfloor}, .\right)$ completely break down for $\ell^{\pm}(N^2, .)$, and it is not even clear whether these two processes should have the same behavior.

Remark 2. Apart from the article of Tóth and Vető [17] that introduced the self-repelling random walk with directed edges, there have been a few other works on this model. These works were motivated by another important question, that of the existence of a scaling limit for $(X_n)_{n\in\mathbb{N}}$, which means the convergence in distribution of the process $\left(\frac{1}{N^{\alpha}} X_{\lfloor Nt\rfloor}\right)_{t\geq 0}$ for some α. Obtaining such a scaling limit for the trajectory of the random walk is harder than obtaining scaling limits for the local times. Indeed, for the random walks with bond repulsion with undirected edges introduced by Tóth in [13–15], the scaling limits for the local times have been known since the introduction of the models, but the scaling limits for the trajectories are not known. Some results were proven by Kosygina, Mountford, and Peterson in [3], but they do not cover all models. For the self-repelling random walk with directed edges, the behavior of the scaling limit of the trajectory turns out to be surprising. Indeed, Mountford, Pimentel, and

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Valle proved in [7] that $\frac{1}{\sqrt{N}}X_N$ converges in distribution, but Mountford and the author showed in [6] that $\left(\frac{1}{\sqrt{N}}X_{\lfloor Nt\rfloor}\right)_{t\geq 0}$ does *not* converge in distribution, and that the trajectories of the walk satisfy a more complex limit theorem, of a new kind.

1.5. Proof ideas

We begin by explaining why the limit of the local times process $Y_N^{\pm}$ is the process $\left(B_y^x\mathbb{1}_{\{y\in[-|x|-2\theta,|x|+2\theta)\}}\right)_{y\in\mathbb{R}}$, and we describe the ideas behind the proofs of Theorem 2 and Proposition 3. To show the convergence of the local times process, we use a Ray–Knight argument; that is, we notice that $\left(\ell^-\left(T^\iota_{\lfloor N\theta\rfloor,\lfloor Nx\rfloor}, i\right)\right)_i$ is a Markov chain. Moreover, as long as $\ell^-\left(T^\iota_{\lfloor N\theta\rfloor,\lfloor Nx\rfloor}, i\right)$ is not too low, the quantities

$$\ell^-\left(T^\iota_{\lfloor N\theta\rfloor,\lfloor Nx\rfloor}, i+1\right)-\ell^-\left(T^\iota_{\lfloor N\theta\rfloor,\lfloor Nx\rfloor}, i\right)$$

will roughly be independent and identically distributed (i.i.d.) random variables, in the sense that they can be coupled with i.i.d. random variables with a high probability of being equal to them. This coupling was already used in [17] to prove the convergence of $\frac{1}{N}\ell^+\left(T^{\pm}_{\lfloor N\theta\rfloor,\lfloor Nx\rfloor}, \lfloor Ny\rfloor\right)$ to its deterministic limit (for a given y, the coupling makes this convergence a law of large numbers). However, when $\ell^-\left(T^\iota_{\lfloor N\theta\rfloor,\lfloor Nx\rfloor}, \lfloor Ny\rfloor\right)$ is too low, the coupling fails and the $\ell^-\left(T^\iota_{\lfloor N\theta\rfloor,\lfloor Nx\rfloor}, \lfloor Ny\rfloor+1\right)-\ell^-\left(T^\iota_{\lfloor N\theta\rfloor,\lfloor Nx\rfloor}, \lfloor Ny\rfloor\right)$ are no longer i.i.d. We have to prove that this occurs only around $|x|+2\theta$ and $-|x|-2\theta$, and most of our work is dealing with what happens there. To show that it occurs only around $|x|+2\theta$ and $-|x|-2\theta$, we control the amplitude of the fluctuations to prove that the local times are close to their deterministic limit. This limit is large inside $(-|x|-2\theta, |x|+2\theta)$, so we can use the coupling inside this interval; thus the $\ell^-\left(T^\iota_{\lfloor N\theta\rfloor,\lfloor Nx\rfloor}, \lfloor Ny\rfloor+1\right)-\ell^-\left(T^\iota_{\lfloor N\theta\rfloor,\lfloor Nx\rfloor}, \lfloor Ny\rfloor\right)$ are roughly i.i.d. there, and hence the fluctuations will converge to a Brownian motion by Donsker's invariance principle. When we are close to $|x|+2\theta$ (the same reasoning works for $-|x|-2\theta$), the deterministic limit will be small, and hence the local times will also be small; tools from [17] then allow us to prove that they reach 0 quickly. Once they reach 0, we notice that for $y\geq|x|+2\theta$, if $\ell^-\left(T^\iota_{\lfloor N\theta\rfloor,\lfloor Nx\rfloor}, \lfloor Ny\rfloor\right)=0$, then the walk X did not go from $\lfloor Ny\rfloor$ to $\lfloor Ny\rfloor+1$ before time $T^\iota_{\lfloor N\theta\rfloor,\lfloor Nx\rfloor}$, so it did not go to $\lfloor Ny\rfloor+1$ before time $T^\iota_{\lfloor N\theta\rfloor,\lfloor Nx\rfloor}$; hence $\ell^-\left(T^\iota_{\lfloor N\theta\rfloor,\lfloor Nx\rfloor}, j\right)=0$ for any $j\geq\lfloor Ny\rfloor$. Therefore, once the local times process reaches 0, it stays there. Consequently, we expect $\ell^-\left(T^\iota_{\lfloor N\theta\rfloor,\lfloor Nx\rfloor}, \lfloor Ny\rfloor\right)$ to be 0 when $y>|x|+2\theta$, and thus to have no fluctuations when $y>|x|+2\theta$; similar statements hold when $y<-|x|-2\theta$. This is why our limit is $\left(B_y^x\mathbb{1}_{\{y\in[-|x|-2\theta,|x|+2\theta)\}}\right)_{y\in\mathbb{R}}$.

Since Proposition 3 only describes convergence away from $-|x|-2\theta$ and $|x|+2\theta$, the previous arguments are enough to prove it. To prove the convergence in the Skorokhod M_1 topology on $D(-\infty, +\infty)$ stated in Theorem 2, we need to handle what happens around $-|x|-2\theta$ and $|x|+2\theta$ with more precision. We first have to bound the difference between the local times and the i.i.d. random variables of the coupling even where the coupling fails. Afterwards comes the most important part of the paper: defining parametric representations of $Y_N^{\pm}$ and of the sum of the i.i.d. random variables of the coupling, properly renormalized and set to 0 outside of $[-|x|-2\theta, |x|+2\theta)$, and then proving that they are close to each other. That allows

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us to prove that $Y_N^{\pm}$ is close in the Skorokhod M_1 distance to a process that will converge in distribution to $\left(B_y^x \mathbb{1}_{\{y\in[-|x|-2\theta,|x|+2\theta)\}}\right)_{y\in\mathbb{R}}$ in the Skorokhod M_1 topology, which lets us complete the proof of Theorem 2.

To prove Proposition 2, that is, that $Y_N^{\pm}$ does not converge in the J_1 topology, we first notice that since the J_1 topology is stronger than the M_1 topology, if $Y_N^{\pm}$ did converge in the J_1 topology its limit would be $\left(B_y^x \mathbb{1}_{\{y\in[-|x|-2\theta,|x|+2\theta)\}}\right)_{y\in\mathbb{R}}$. However, this is not possible, as $\left(B_y^x \mathbb{1}_{\{y\in[-|x|-2\theta,|x|+2\theta)\}}\right)_{y\in\mathbb{R}}$ has a jump at $|x|+2\theta$, while the jumps of $Y_N^{\pm}$ have typical size of order $\frac{1}{\sqrt{N}}$, so the jump in $\left(B_y^x \mathbb{1}_{\{y\in[-|x|-2\theta,|x|+2\theta)\}}\right)_{y\in\mathbb{R}}$ is approximated in $Y_N^{\pm}$ by either a sequence of small jumps or a continuous slope, which prevents the convergence in the Skorokhod J_1 topology.

Finally, to prove Proposition 4 on the fluctuations of $T^{\iota}_{\lfloor N\theta\rfloor,\lfloor Nx\rfloor}$, we use the fact that we have

$$T^{\iota}_{\lfloor N\theta\rfloor,\lfloor Nx\rfloor} = \sum_{i\in\mathbb{Z}} \left(\ell^{+}\left(T^{\iota}_{\lfloor N\theta\rfloor,\lfloor Nx\rfloor}, i\right) + \ell^{-}\left(T^{\iota}_{\lfloor N\theta\rfloor,\lfloor Nx\rfloor}, i\right)\right).$$

It can be checked that $\left|\ell^{+}\left(T^{\iota}_{\lfloor N\theta\rfloor,\lfloor Nx\rfloor}, i\right) - \ell^{-}\left(T^{\iota}_{\lfloor N\theta\rfloor,\lfloor Nx\rfloor}, i+1\right)\right|$ equals 0 or 1; hence it is enough to control the $\ell^{-}\left(T^{\iota}_{\lfloor N\theta\rfloor,\lfloor Nx\rfloor}, i\right)$. By using the coupling for the

$$\ell^{-}\left(T^{\iota}_{\lfloor N\theta\rfloor,\lfloor Nx\rfloor}, i+1\right) - \ell^{-}\left(T^{\iota}_{\lfloor N\theta\rfloor,\lfloor Nx\rfloor}, i\right)$$

when $\ell^{-}\left(T^{\iota}_{\lfloor N\theta\rfloor,\lfloor Nx\rfloor}, i\right)$ is high enough, and using our estimates on the size of the window in which $\ell^{-}\left(T^{\iota}_{\lfloor N\theta\rfloor,\lfloor Nx\rfloor}, i\right)$ is neither high enough nor 0, we can prove that $T^{\iota}_{\lfloor N\theta\rfloor,\lfloor Nx\rfloor}$ is close to the integral of the sum of the i.i.d. random variables of the coupling, which will yield the convergence.

1.6. Organization of the paper

In Section 2, we define the coupling between the increments of the local time and i.i.d. random variables and prove some of its properties. In Section 3, we control where the local times hit 0, as well as where the local times are too low for the coupling of Section 2 to be useful. In Section 4, we prove a bound on the Skorokhod M_1 distance between $Y_N^{\pm}$ and the renormalized sum of the i.i.d. random variables of the coupling set to 0 outside of $[-|x|-2\theta, |x|+2\theta)$, by writing explicit parametric representations of the two functions. In Section 5, we complete the proof of the convergence of $Y_N^{\pm}$ stated in Theorem 2 and Proposition 3. In Section 6 we prove that, as claimed in Proposition 2, $Y_N^{\pm}$ does not converge in the J_1 topology. Finally, in Section 7 we prove the convergence of the fluctuations of $T^{\pm}_{\lfloor N\theta\rfloor,\lfloor Nx\rfloor}$ stated in Proposition 4.

In what follows, we set $\theta > 0$, $\iota \in \{+, -\}$, and $x > 0$ (the cases $x < 0$ and $x = 0$ can be dealt with in the same way). To simplify the notation, we set $T_N = T^{\iota}_{\lfloor N\theta\rfloor,\lfloor Nx\rfloor}$. Moreover, for any $a, b \in \mathbb{R}$, we set $a \vee b = \max(a, b)$ and $a \wedge b = \min(a, b)$.

2. Coupling of the local times increments with i.i.d. random variables

Our goal in this section will be to couple the $\ell^{\pm}(T_N, i+1) - \ell^{\pm}(T_N, i)$ with i.i.d. random variables and to prove some properties of this coupling. This part of the work is not very different from what was done in [17], but we still recall the concepts and definitions from that paper. If we fix $i \in \mathbb{Z}$ and observe the evolution of $(\ell^{-}(n, i) - \ell^{+}(n, i))_{n\in\mathbb{N}}$, and if we ignore

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the steps at which $\ell^-(n, i) - \ell^+(n, i)$ does not move (i.e. those at which the random walk is not at i), then we obtain a Markov chain ξ_i whose distribution ξ has the following transition probabilities: for all $n \in \mathbb{N}$,

$$\mathbb{P}(\xi(n+1) = \xi(n) \pm 1) = \frac{w(\mp \xi(n))}{w(\xi(n)) + w(-\xi(n))},$$

and $\xi_i(0) = 0$. Now, we set $\tau_{i,\pm}(0) = 0$, and for any $n \in \mathbb{N}$, we define $\tau_{i,\pm}(n+1) = \inf\{m > \tau_{i,\pm}(n) \mid \xi_i(m) = \xi_i(m-1) \pm 1\}$, so that $\tau_{i,+}(n)$ is the time of the nth upward step of ξ_i and $\tau_{i,-}(n)$ is the time of the nth downward step of ξ_i. Then, since the distribution of ξ is symmetric, the processes $(\eta_{i,+}(n))_{n\in\mathbb{N}} = (-\xi_i(\tau_{i,+}(n)))_{n\in\mathbb{N}}$ and $(\eta_{i,-}(n))_{n\in\mathbb{N}} = (\xi_i(\tau_{i,-}(n)))_{n\in\mathbb{N}}$ have the same distribution, called η, and it can be checked that η is a Markov chain.

We are going to give an expression for $\ell^\pm(T_N, i+1) - \ell^\pm(T_N, i)$ depending on the $\eta_{i,-}$, $\eta_{i,+}$. We assume N large enough (so that $\lfloor Nx \rfloor - 1 > 0$). By definition of T_N we have $X_{T_N} = \lfloor Nx \rfloor \iota 1$. If $i \leq 0$ we thus have $X_{T_N} > i$, which means the last step of the walk at i before T_N was going to the right, so the last step of ξ_i was a downward step, and by definition of $\ell^+(T_N, i)$ we have that ξ_i made $\ell^+(T_N, i)$ downward steps; hence

$$\ell^-(T_N, i) - \ell^+(T_N, i) = \xi_i(\tau_{i,-}(\ell^+(T_N, i))) = \eta_{i,-}(\ell^+(T_N, i)),$$

which yields $\ell^-(T_N, i) - \ell^+(T_N, i) = \eta_{i,-}(\ell^+(T_N, i))$. In addition, $\ell^-(T_N, i) = \ell^+(T_N, i-1)$; hence

$$\ell^+(T_N, i-1) = \ell^+(T_N, i) + \eta_{i,-}(\ell^+(T_N, i)).$$

If $0 < i < \lfloor Nx \rfloor$ (for $\iota = -$) or $0 < i \leq \lfloor Nx \rfloor$ (for $\iota = +$), the last step of the walk at i was also going to the right, so we also have $\ell^-(T_N, i) - \ell^+(T_N, i) = \eta_{i,-}(\ell^+(T_N, i))$. However, $\ell^-(T_N, i) = \ell^+(T_N, i-1) - 1$, so $\ell^+(T_N, i-1) = \ell^+(T_N, i) + \eta_{i,-}(\ell^+(T_N, i)) + 1$. Finally, if $i \geq \lfloor Nx \rfloor$ (for $\iota = -$) or $i > \lfloor Nx \rfloor$ (for $\iota = +$), then the last step of the walk at i was going to the left, so the last step of ξ_i was an upward step, and ξ_i made $\ell^-(T_N, i)$ upward steps; therefore

$$\ell^-(T_N, i) - \ell^+(T_N, i) = \xi_i\big(\tau_{i,+}\big(\ell^-\big(T_N, i\big)\big)\big) = -\eta_{i,+}\big(\ell^-\big(T_N, i\big)\big),$$

which yields $\ell^-(T_N, i) - \ell^+(T_N, i) = -\eta_{i,+}\big(\ell^-\big(T_N, i\big)\big)$. Moreover, $\ell^+(T_N, i) = \ell^-(T_N, i+1)$, and hence $\ell^-(T_N, i+1) = \ell^-(T_N, i) + \eta_{i,+}\big(\ell^-\big(T_N, i\big)\big)$.

We are going to use these results to deduce an expression for the $\ell^\pm(T_N, i)$ which will be very useful throughout this work. Defining $\chi(N) = \lfloor Nx \rfloor$ if $\iota = -$ and $\chi(N) = \lfloor Nx \rfloor + 1$ if $\iota = +$, for $i \geq \chi(N)$ we have

$$\ell^-(T_N, i) = \ell^-(T_N, \chi(N)) + \sum_{j=\chi(N)}^{i-1} \eta_{j,+}(\ell^-(T_N, j)),$$

and for $i < \chi(N)$ we have

$$\ell^+(T_N, i) = \ell^+(T_N, \chi(N) - 1) + \sum_{j=i+1}^{\chi(N)-1} \big(\eta_{j,-}(\ell^+(T_N, j)) + \mathbb{1}_{\{j>0\}}\big).$$

Now, we remember that the definition of T_N implies $\ell^\iota(T_N, \lfloor Nx \rfloor) = \lfloor N\theta \rfloor$, so if $\iota = -$ we have $\ell^-(T_N, \chi(N)) = \lfloor N\theta \rfloor$ and $\ell^+(T_N, \chi(N) - 1) = \ell^-(T_N, \chi(N)) = \lfloor N\theta \rfloor$, while if $\iota = +$

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we have $\ell^+(T_N, \chi(N)-1) = \lfloor N\theta \rfloor$ and $\ell^-(T_N, \chi(N)) = \ell^+(T_N, \chi(N)-1) - 1 = \lfloor N\theta \rfloor - 1$. Consequently, we have the following:

$$\begin{aligned} &\text{If } i \geq \chi(N), \quad \ell^-(T_N, i) = \lfloor N\theta \rfloor - \mathbb{1}_{\{\iota=+\}} + \sum_{j=\chi(N)}^{i-1} \eta_{j,+}(\ell^-(T_N, j)). \\ &\text{If } i < \chi(N), \quad \ell^+(T_N, i) = \lfloor N\theta \rfloor + \sum_{j=i+1}^{\chi(N)-1} \big(\eta_{j,-}(\ell^+(T_N, j)) + \mathbb{1}_{\{j>0\}}\big). \end{aligned} \tag{1}$$

We will also need to remember the following:

$$\begin{aligned} &\text{If } i \geq \chi(N), \quad \ell^-(T_N, i) - \ell^+(T_N, i) = -\eta_{i,+}\big(\ell^-\big(T_N, i\big)\big). \\ &\text{If } i < \chi(N), \quad \ell^-(T_N, i) - \ell^+(T_N, i) = \eta_{i,-}(\ell^+(T_N, i)). \end{aligned} \tag{2}$$

To couple the $\ell^\pm(T_N, i+1) - \ell^\pm(T_N, i)$ with i.i.d. random variables, we need to understand the $\eta_{i,+}\big(\ell^-\big(T_N, i\big)\big)$ and the $\eta_{i,-}(\ell^+(T_N, i))$. The paper [17] proved that the following measure ρ_- is the unique invariant probability distribution of the Markov chain η:

$$\forall i \in \mathbb{Z}, \quad \rho_-(i) = \frac{1}{R} \prod_{j=1}^{\lfloor |2i+1|/2 \rfloor} \frac{w(-j)}{w(j)} \quad \text{with} \quad R = \sum_{i\in\mathbb{Z}} \prod_{j=1}^{\lfloor |2i+1|/2 \rfloor} \frac{w(-j)}{w(j)}. \tag{3}$$

We also denote by ρ_0 the measure on $\frac{1}{2} + \mathbb{Z}$ defined by $\rho_0(\,\cdot\,) = \rho_-\big(\,\cdot - \frac{1}{2}\big)$.

We are now in position to construct the coupling of the $\ell^\pm(T_N, i+1) - \ell^\pm(T_N, i)$ with i.i.d. random variables $(\zeta_i)_{i\in\mathbb{Z}}$. The idea is that η can be expected to converge to its invariant distribution ρ_-; hence when $\ell^\pm(T_N, i)$ is large, $\eta_{i,\mp}\big(\ell^\pm(T_N, i)\big)$ will be close to a random variable of law ρ_-. More rigorously, we begin by defining an i.i.d. sequence $(r_i)_{i\in\mathbb{Z}}$ of random variables of distribution ρ_- so that if $i \geq \chi(N)$, then $\mathbb{P}\big(r_i \neq \eta_{i,+}\big(\lfloor N^{1/6} \rfloor\big)\big)$ is minimal, and if $i < \chi(N)$, then $\mathbb{P}\big(r_i \neq \eta_{i,-}\big(\lfloor N^{1/6} \rfloor\big)\big)$ is minimal. We can then define i.i.d. Markov chains $(\bar\eta_{i,+}(n))_{n\geq \lfloor N^{1/6} \rfloor}$ for $i \geq \chi(N)$ and $(\bar\eta_{i,-}(n))_{n\geq \lfloor N^{1/6} \rfloor}$ for $i < \chi(N)$ so that $\bar\eta_{i,\pm}\big(\lfloor N^{1/6} \rfloor\big) = r_i$, $\bar\eta_{i,\pm}$ is a Markov chain of distribution equal to that of η, and if $\bar\eta_{i,\pm}\big(\lfloor N^{1/6} \rfloor\big) = \eta_{i,\pm}\big(\lfloor N^{1/6} \rfloor\big)$, then $\bar\eta_{i,\pm}(n) = \eta_{i,\pm}(n)$ for any $n \geq \lfloor N^{1/6} \rfloor$. Since ρ_- is invariant for η, if $n \geq \lfloor N^{1/6} \rfloor$, then the $\bar\eta_{i,+}(n)$ for $i \geq \chi(N)$ and $\bar\eta_{i,-}(n)$ for $i < \chi(N)$ have distribution ρ_-. We define the random variables $(\zeta_i)_{i\in\mathbb{Z}}$ as follows: for $i \geq \chi(N)$ we set $\zeta_i = \bar\eta_{i,+}\big(\ell^-(T_N, i) \vee \lfloor N^{1/6} \rfloor\big) + \frac{1}{2}$, and for $i < \chi(N)$ we set $\zeta_i = \bar\eta_{i,-}\big(\ell^+(T_N, i) \vee \lfloor N^{1/6} \rfloor\big) + \frac{1}{2}$. For $i \geq \chi(N)$, (1) implies that $\ell^-(T_N, i)$ depends only on the $\eta_{j,+}$, $\chi(N) \leq j \leq i-1$, and hence is independent from $\bar\eta_{i,+}$, which implies that ζ_i has distribution ρ_0 and is independent from the ζ_j, $\chi(N) \leq j \leq i-1$. This together with a similar argument for $i < \chi(N)$ implies that the $(\zeta_i)_{i\in\mathbb{Z}}$ are i.i.d. with distribution ρ_0.

We will prove several properties of $(\zeta_i)_{i\in\mathbb{Z}}$ that we will use in the remainder of the proof. In order to do that, we need the following lemma from [17].

Lemma 1. ([17, Lemma 1].) *There exist two constants $\tilde c = \tilde c(w) > 0$ and $\tilde C = \tilde C(w) < +\infty$ such that for any $n \in \mathbb{N}$,*

$$\mathbb{P}(\eta(n) = i | \eta(0) = 0) \leq \tilde C e^{-\tilde c |i|} \quad \textit{and} \quad \sum_{i\in\mathbb{Z}} |\mathbb{P}(\eta(n) = i | \eta(0) = 0) - \rho_-(i)| \leq \tilde C e^{-\tilde c n}.$$

Firstly, we want to prove that our coupling is actually useful: that the ζ_i are close to the $\ell^\pm(T_N, i+1) - \ell^\pm(T_N, i)$. More precisely, we will show that except on an event of

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probability tending to 0, if $\ell^{\pm}(T_N, i)$ is large then $\zeta_i = \eta_{i,\mp}(\ell^{\pm}(T_N, i)) + 1/2$, which (1) relates to $\ell^{\pm}(T_N, i+1) - \ell^{\pm}(T_N, i)$. We define

$$\begin{aligned} \mathcal{B}_1^- = \{\exists\, i \in \{-\lceil 2(|x| + 2\theta)N\rceil, \dots, \chi(N) - 1\}, \ell^+(T_N, i) \geq \lfloor N^{1/6} \rfloor \\ \text{and } \zeta_i \neq \eta_{i,-}(\ell^+(T_N, i)) + 1/2\}, \\ \mathcal{B}_1^+ = \{\exists\, i \in \{\chi(N), \dots, \lceil 2(|x| + 2\theta)N\rceil\}, \ell^-(T_N, i) \geq \lfloor N^{1/6} \rfloor \\ \text{and } \zeta_i \neq \eta_{i,+}\big(\ell^-\big(T_N, i\big)\big) + 1/2\}. \end{aligned} \tag{4}$$

Lemma 1 will allow us to prove the following.

Lemma 2. $\mathbb{P}\big(\mathcal{B}_1^-\big)$ *and* $\mathbb{P}\big(\mathcal{B}_1^+\big)$ *tend to 0 when* $N \to +\infty$.

Proof. By definition, for any $i \in \{-\lceil 2(|x| + 2\theta)N\rceil, \dots, \chi(N) - 1\}$ we have

$$\zeta_i = \bar{\eta}_{i,-}\big(\ell^+(T_N, i) \vee \lfloor N^{1/6} \rfloor\big) + \frac{1}{2},$$

which is $\bar{\eta}_{i,-}(\ell^+(T_N, i)) + \frac{1}{2}$ when $\ell^+(T_N, i) \geq \lfloor N^{1/6} \rfloor$. Now, $\bar{\eta}_{i,-} = \eta_{i,-}$ if $\bar{\eta}_{i,-}\big(\lfloor N^{1/6} \rfloor\big) = \eta_{i,-}\big(\lfloor N^{1/6} \rfloor\big)$; that is, $r_i = \eta_{i,-}\big(\lfloor N^{1/6} \rfloor\big)$. We deduce that

$$\mathbb{P}\big(\mathcal{B}_1^-\big) \leq \mathbb{P}\big(\exists\, i \in \{-\lceil 2(|x| + 2\theta)N\rceil, \dots, \chi(N) - 1\}, r_i \neq \eta_{i,-}\big(\lfloor N^{1/6} \rfloor\big)\big).$$

Now, for any $i < \chi(N)$, we have $\mathbb{P}\big(r_i \neq \eta_{i,-}\big(\lfloor N^{1/6} \rfloor\big)\big)$ minimal, and thus smaller than $\tilde{C}e^{-\tilde{c}\lfloor N^{1/6} \rfloor}$ by Lemma 1. Consequently, when N is large enough, we have $\mathbb{P}\big(\mathcal{B}_1^-\big) \leq 3(|x| + 2\theta)N\tilde{C}e^{-\tilde{c}\lfloor N^{1/6} \rfloor}$, which tends to 0 when $N \to +\infty$. The proof for $\mathbb{P}\big(\mathcal{B}_1^+\big)$ is the same. □

Unfortunately, the previous lemma does not allow us to control the local times when $\ell^{\pm}(T_N, i)$ is small. In order to do that, we show several additional properties. We have to control the probability of

$$\begin{aligned} \mathcal{B}_2 = &\big\{\exists\, i \in \{-\lceil 2(|x| + 2\theta)N\rceil, \dots, \lceil 2(|x| + 2\theta)N\rceil\}, |\zeta_i| \geq N^{1/16}\big\} \\ &\cup \big\{\exists\, i \in \{-\lceil 2(|x| + 2\theta)N\rceil, \dots, \chi(N) - 1\}, |\eta_{i,-}(\ell^+(T_N, i)) + 1/2| \geq N^{1/16}\big\} \\ &\cup \big\{\exists\, i \in \{\chi(N), \dots, \lceil 2(|x| + 2\theta)N\rceil\}, |\eta_{i,+}\big(\ell^-\big(T_N, i\big)\big) + 1/2| \geq N^{1/16}\big\}. \end{aligned}$$

Lemma 3. $\mathbb{P}(\mathcal{B}_2)$ *tends to 0 when N tends to* $+\infty$.

Proof. It is enough to find some constants $c > 0$ and $C < +\infty$ such that for any $i \in \{-\lceil 2(|x| + 2\theta)N\rceil, \dots, \lceil 2(|x| + 2\theta)N\rceil\}$ we have

$$\mathbb{P}\big(|\zeta_i| \geq N^{1/16}\big) \leq Ce^{-cN^{1/16}},$$

for any $i \in \{-\lceil 2(|x| + 2\theta)N\rceil, \dots, \chi(N) - 1\}$ we have

$$\mathbb{P}\big(|\eta_{i,-}(\ell^+(T_N, i)) + 1/2| \geq N^{1/16}\big) \leq Ce^{-cN^{1/16}},$$

and for all $i \in \{\chi(N), \dots, \lceil 2(|x| + 2\theta)N\rceil\}$ we have

$$\mathbb{P}\big(|\eta_{i,+}\big(\ell^-\big(T_N, i\big)\big) + 1/2| \geq N^{1/16}\big) \leq Ce^{-cN^{1/16}}.$$

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For all $i \in \mathbb{Z}$, ζ_i has distribution ρ_0, which has exponential tails; hence there exist constants $c' = c'(w) > 0$ and $C' = C'(w) < +\infty$ such that for $i \in \{-\lceil 2(|x| + 2\theta)N\rceil, \dots, \lceil 2(|x| + 2\theta)N\rceil\}$ we have $\mathbb{P}\big(|\zeta_i| \geq N^{1/16}\big) \leq C' e^{-c'N^{1/16}}$. We now consider $i \in \{-\lceil 2(|x| + 2\theta)N\rceil, \dots, \chi(N) - 1\}$ and $\mathbb{P}\big(|\eta_{i,-}(\ell^+(T_N, i)) + 1/2| \geq N^{1/16}\big)$ (the $\mathbb{P}\big(|\eta_{i,+}\big(\ell^-\big(T_N, i\big)\big) + 1/2| \geq N^{1/16}\big)$ can be dealt with in the same way). Equation (1) implies that $\ell^+(T_N, i)$ depends only on the $\eta_{j,-}$ for $j > i$, and hence is independent of $\eta_{i,-}$. This implies that

$$\mathbb{P}\big(|\eta_{i,-}(\ell^+(T_N, i)) + 1/2| \geq N^{1/16}\big) = \sum_{k\in\mathbb{N}} \mathbb{P}\big(|\eta_{i,-}(k) + 1/2| \geq N^{1/16}\big)\mathbb{P}\big(\ell^+(T_N, i) = k\big).$$

Therefore the first part of Lemma 1 implies that

$$\begin{aligned}\mathbb{P}\big(|\eta_{i,-}(\ell^+(T_N, i)) + 1/2| \geq N^{1/16}\big) &\leq \sum_{k\in\mathbb{N}} \frac{2\tilde{C}e^{\tilde{c}/2}}{1 - e^{-\tilde{c}}} e^{-\tilde{c}N^{1/16}} \mathbb{P}\big(\ell^+(T_N, i) = k\big) \\ &= \frac{2\tilde{C}e^{\tilde{c}/2}}{1 - e^{-\tilde{c}}} e^{-\tilde{c}N^{1/16}},\end{aligned}$$

which is enough. □

We will also need the following, which is a fairly standard result on large deviations.

Lemma 4. *For any* $\alpha > 0$, $\varepsilon > 0$, $\mathbb{P}\Big(\max_{0\leq i_1\leq i_2\leq\lceil N^\alpha\rceil} \Big|\sum_{i=i_1}^{i_2} \zeta_i\Big| \geq N^{\alpha/2+\varepsilon}\Big)$ *tends to 0 when* $N \to +\infty$.

Proof. Let $0 \leq i_1 \leq i_2 \leq \lceil N^\alpha\rceil$, and let us study $\mathbb{P}\Big(\Big|\sum_{i=i_1}^{i_2} \zeta_i\Big| \geq N^{\alpha/2+\varepsilon}\Big)$. We know the ζ_i, $i \in \mathbb{Z}$, are i.i.d. with distribution ρ_0, and it can be checked that ρ_0 is symmetric with respect to 0, so from that and the Markov inequality we get

$$\begin{aligned}\mathbb{P}\left(\left|\sum_{i=i_1}^{i_2} \zeta_i\right| \geq N^{\alpha/2+\varepsilon}\right) &\leq 2\mathbb{P}\left(\sum_{i=i_1}^{i_2} \zeta_i \geq N^{\alpha/2+\varepsilon}\right) \\ = 2\mathbb{P}\left(\exp\left(\frac{1}{N^{\alpha/2}}\sum_{i=i_1}^{i_2}\zeta_i\right) \geq \exp\left(N^\varepsilon\right)\right) &\leq 2e^{-N^\varepsilon}\mathbb{E}\left(\exp\left(\frac{1}{N^{\alpha/2}}\sum_{i=i_1}^{i_2}\zeta_i\right)\right) \\ &\leq 2e^{-N^\varepsilon}\prod_{i=i_1}^{i_2}\mathbb{E}\left(\exp\left(\frac{1}{N^{\alpha/2}}\zeta_i\right)\right).\end{aligned} \tag{5}$$

Now, if ζ has distribution ρ_0, we can write

$$\exp\left(\frac{1}{N^{\alpha/2}}\zeta\right) = 1 + \frac{1}{N^{\alpha/2}}\zeta + \frac{1}{2}\left(\frac{1}{N^{\alpha/2}}\zeta\right)^2 \exp\left(\frac{1}{N^{\alpha/2}}\zeta'\right)$$

with $|\zeta'| \leq |\zeta|$. Since ρ_0 is symmetric with respect to 0, we have $\mathbb{E}(\zeta) = 0$; therefore

$$\begin{aligned}\mathbb{E}\left(\exp\left(\frac{1}{N^{\alpha/2}}\zeta\right)\right) &= 1 + \mathbb{E}\left(\frac{1}{2}\left(\frac{1}{N^{\alpha/2}}\zeta\right)^2\exp\left(\frac{1}{N^{\alpha/2}}\zeta'\right)\right) \\ &\leq 1 + \frac{1}{2N^\alpha}\mathbb{E}\left(\zeta^2\exp\left(\frac{1}{N^{\alpha/2}}|\zeta|\right)\right).\end{aligned}$$

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Moreover, ρ_0 has exponential tails; hence there exist constants $C < +\infty$ and $c > 0$ such that $\mathbb{E}(\zeta^2 e^{c|\zeta|}) \le C$. When N is large enough, $\frac{1}{N^{\alpha/2}} \le c$; therefore

$$\mathbb{E}\Big(\exp\Big(\frac{1}{N^{\alpha/2}}\zeta\Big)\Big) \le 1 + \frac{C}{2N^\alpha} \le \exp\Big(\frac{C}{2N^\alpha}\Big).$$

Together with (5), this yields

$$\mathbb{P}\left(\left|\sum_{i=i_1}^{i_2}\zeta_i\right| \ge N^{\alpha/2+\varepsilon}\right) \le 2e^{-N^\varepsilon} e^{(i_2-i_1+1)\frac{C}{2N^\alpha}} \le 2e^{-N^\varepsilon} e^{\left(\lceil N^\alpha\rceil+1\right)\frac{C}{2N^\alpha}} \le 2e^C e^{-N^\varepsilon}$$

when N is large enough. We deduce that when N is large enough,

$$\mathbb{P}\left(\max_{0\le i_1\le i_2\le\lceil N^\alpha\rceil}\left|\sum_{i=i_1}^{i_2}\zeta_i\right| \ge N^{\alpha/2+\varepsilon}\right) \le (\lceil N^\alpha\rceil + 1)^2 2e^C e^{-N^\varepsilon},$$

which tends to 0 when N tends to $+\infty$. □

We also prove an immediate application of Lemma 4, which we will use several times. If we define

$$\mathcal{B}_3^- = \left\{\max_{-\lfloor(|x|+2\theta)N\rfloor-N^{3/4}\le i_1\le i_2\le-\lfloor(|x|+2\theta)N\rfloor+N^{3/4}}\left|\sum_{i=i_1}^{i_2}\zeta_i\right| \ge N^{19/48}\right\},$$

$$\mathcal{B}_3^+ = \left\{\max_{\lfloor(|x|+2\theta)N\rfloor-N^{3/4}\le i_1\le i_2\le\lfloor(|x|+2\theta)N\rfloor+N^{3/4}}\left|\sum_{i=i_1}^{i_2}\zeta_i\right| \ge N^{19/48}\right\},$$

we have the following lemma.

Lemma 5. *$\mathbb{P}(\mathcal{B}_3^-)$ and $\mathbb{P}(\mathcal{B}_3^+)$ tend to 0 when N tends to $+\infty$.*

Proof. Since the $(\zeta_i)_{i\in\mathbb{Z}}$ are i.i.d.,

$$\mathbb{P}(\mathcal{B}_3^+) = \mathbb{P}(\mathcal{B}_3^-) = \mathbb{P}\left(\max_{0\le i_1\le i_2\le 2\lceil N^{3/4}\rceil}\left|\sum_{i=i_1}^{i_2}\zeta_i\right| \ge N^{19/48}\right),$$

which is smaller than $\mathbb{P}\Big(\max_{0\le i_1\le i_2\le\lceil N^{37/48}\rceil}\left|\sum_{i=i_1}^{i_2}\zeta_i\right| \ge N^{19/48}\Big)$ when N is large enough. Moreover, Lemma 4, used with $\alpha = 37/48$ and $\varepsilon = 1/96$, yields that the latter probability tends to 0 when N tends to $+\infty$. □

3. Where the local times approach 0

The aim of this section is twofold. Firstly, we need to control the place where $\ell^-(T_N, i)$ hits 0 when i is to the right of 0, as well as the place where $\ell^+(T_N, i)$ hits 0 when i is to the left of 0. Secondly, we have to show that even when $\ell^\pm(T_N, i)$ is close to 0, the local times do not stray too far away from the coupling. For any $N \in \mathbb{N}$, we define $I^+ = \inf\{i \ge \chi(N) \mid \ell^-(T_N, i) = 0\}$ and $I^- = \sup\{i < \chi(N) \mid \ell^+(T_N, i) = 0\}$. We notice that $\ell^+(T_N, I^-) = 0$,

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and from the definition of T_N we have $\ell^+(T_N, i) > 0$ for any $0 \le i \le \chi(N) - 1$; hence $I^- < 0$. We first state an elementary result that we will use many times in this work.

Lemma 6. *For any $i \ge I^+$ or $i \le I^-$ we have $\ell^\pm(T_N, i) = 0$.*

Proof. Since $\ell^+(T_N, I^-) = 0$ and the random walk is at $\lfloor Nx \rfloor \iota 1 > 0$ at time T_N, the random walk did not reach I^- before time T_N; thus $\ell^\pm(T_N, i) = 0$ for any $i \le I^-$. Moreover, $\ell^-(T_N, \chi(N)) > 0$ by definition of T_N, so $I^+ > \chi(N)$; thus $X_{T_N} < I^+$, and hence $\ell^-(T_N, I^+) = 0$ implies that the random walk did not reach I^+ before time T_N. Thus $\ell^\pm(T_N, i) = 0$ for any $i \ge I^+$. □

We will also need the auxiliary random variables $\tilde{I}^+ = \inf\{i \ge \chi(N) \mid \ell^-(T_N, i) \le \lfloor N^{1/6} \rfloor\}$ and $\tilde{I}^- = \sup\left\{i < \chi(N) \mid \ell^+(T_N, i) \le \lfloor N^{1/6} \rfloor\right\}$.

3.1. Place where we hit 0

We have the following result on the control of I^+ and I^-.

Lemma 7. *For any $\delta > 0$, $\mathbb{P}\big(|I^- + (|x| + 2\theta)N| \ge N^{\delta+1/2}\big)$ and $\mathbb{P}(|I^+ - (|x| + 2\theta)N| \ge N^{\delta+1/2})$ tend to 0 when N tends to $+\infty$.*

Proof. The idea is to control the fluctuations of the local times around their deterministic limit: as long as $\ell^\pm(T_N, i)$ is large, the $\ell^\pm(T_N, i+1) - \ell^\pm(T_N, i)$ will be close to the i.i.d. random variables of the coupling, so the fluctuations of $\ell^\pm(T_N, i)$ around its deterministic limit are bounded and $\ell^\pm(T_N, i)$ can be small only when the deterministic limit is small, that is, around $-(|x| + 2\theta)N$ and $(|x| + 2\theta)N$. We spell out the proof only for I^-, as the argument for I^+ is similar.

The fact that $\mathbb{P}\big(I^- + (|x| + 2\theta)N \le -N^{\delta+1/2}\big)$ tends to 0 when N tends to $+\infty$ comes from the inequalities (51) and (53) in [17], so we only have to prove that $\mathbb{P}(I^- + (|x| + 2\theta)N \ge N^{\delta+1/2})$ tends to 0 when N tends to $+\infty$. Since $I^- \le \tilde{I}^-$, it is enough to prove that $\mathbb{P}(\tilde{I}^- + (|x| + 2\theta)N \ge N^{\delta+1/2})$ tends to 0 when N tends to $+\infty$. Since by Lemma 2 we have that $\mathbb{P}\big(\mathcal{B}_1^-\big)$ tends to 0 when N tends to $+\infty$, it is enough to prove that $\mathbb{P}(\tilde{I}^- + (|x| + 2\theta)N \ge N^{\delta+1/2}, \big(\mathcal{B}_1^-\big)^c)$ tends to 0 when N tends to $+\infty$.

We now assume N is large enough, $\tilde{I}^- + (|x| + 2\theta)N \ge N^{\delta+1/2}$, and $\big(\mathcal{B}_1^-\big)^c$. Then there exists $i \in \{\lceil -(|x| + 2\theta)N + N^{\delta+1/2} \rceil, \dots, \chi(N) - 1\}$ such that $\ell^+(T_N, i) \le \lfloor N^{1/6} \rfloor$ and $\ell^+(T_N, j) > \lfloor N^{1/6} \rfloor$ for all $j \in \{i+1, \dots, \chi(N) - 1\}$. Thus, by (1) we get

$$\lfloor N\theta \rfloor + \sum_{j=i+1}^{\chi(N)-1} \big(\eta_{j,-}(\ell^+(T_N, j)) + \mathbb{1}_{\{j>0\}}\big) = \ell^+(T_N, i) \le \lfloor N^{1/6} \rfloor.$$

Furthermore, for all $j \in \{i+1, \dots, \chi(N) - 1\}$, since $\big(\mathcal{B}_1^-\big)^c$ occurs and $\ell^+(T_N, j) > \lfloor N^{1/6} \rfloor$, we have $\eta_{j,-}(\ell^+(T_N, j)) + 1/2 = \zeta_j$. We deduce that

$$\lfloor N\theta \rfloor + \sum_{j=i+1}^{\chi(N)-1} \left(\zeta_j + \frac{\mathbb{1}_{\{j>0\}} - \mathbb{1}_{\{j\le 0\}}}{2}\right) \le \lfloor N^{1/6} \rfloor;$$

thus

$$\sum_{j=i+1}^{\chi(N)-1} \zeta_j + \lfloor N\theta \rfloor + \sum_{j=i+1}^{\chi(N)-1} \frac{\mathbb{1}_{\{j>0\}} - \mathbb{1}_{\{j\le 0\}}}{2} \le \lfloor N^{1/6} \rfloor.$$

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Moreover, since $i \in \{\lceil -(|x|+2\theta)N + N^{\delta+1/2}\rceil, \dots, \chi(N)-1\}$, we have

$$\sum_{j=i+1}^{\chi(N)-1} \frac{\mathbb{1}_{\{j>0\}} - \mathbb{1}_{\{j\leq 0\}}}{2} = \frac{1}{2}(\chi(N)-1+i)$$
$$\geq \frac{1}{2}\left(Nx - 2 - (|x|+2\theta)N + N^{\delta+1/2}\right)$$
$$= -\theta N + \frac{1}{2}N^{\delta+1/2} - 1.$$

This yields $\sum_{j=i+1}^{\chi(N)-1} \zeta_j + \lfloor N\theta \rfloor - \theta N + \frac{1}{2}N^{\delta+1/2} - 1 \leq \lfloor N^{1/6} \rfloor$; hence

$$\sum_{j=i+1}^{\chi(N)-1} \zeta_j \leq -\frac{1}{2}N^{\delta+1/2} + \lfloor N^{1/6} \rfloor + 2 \leq -N^{(1+\delta)/2}$$

since N is large enough. Consequently, when N is large enough,

$$\mathbb{P}\left(\tilde{I}^- + (|x|+2\theta)N \geq N^{\delta+1/2}, \left(\mathcal{B}_1^-\right)^c\right)$$
$$\leq \mathbb{P}\left(\exists\, i \in \{\lceil -(|x|+2\theta)N + N^{\delta+1/2}\rceil, \dots, \chi(N)-1\}, \sum_{j=i+1}^{\chi(N)-1} \zeta_j \leq -N^{(1+\delta)/2}\right).$$

Since the ζ_i, $i \in \mathbb{Z}$, are i.i.d., when N is large enough this yields

$$\mathbb{P}\left(\tilde{I}^- + (|x|+2\theta)N \geq N^{\delta+1/2}, \left(\mathcal{B}_1^-\right)^c\right) \leq \mathbb{P}\left(\max_{0\leq i_1\leq i_2\leq \lceil N^{1+\delta/2}\rceil} \left|\sum_{i=i_1}^{i_2} \zeta_i\right| \geq N^{(1+\delta)/2}\right),$$

which tends to 0 when N tends to $+\infty$ by Lemma 4 (applied with $\alpha = 1+\delta/2$ and $\varepsilon = \delta/4$). This shows that $\mathbb{P}\left(I^- + (|x|+2\theta)N \geq N^{\delta+1/2}\right)$ converges to 0 when N tends to $+\infty$, which completes the proof of Lemma 7. □

3.2. Control of low local times

We have to show that even when $\ell^{\pm}(T_N, i)$ is small, the local times are not too far from the random variables of the coupling. In order to do that, we first prove that the window where $\ell^{\pm}(T_N, i)$ is small but not zero—that is, between $\tilde{I}^+$ and I^+ and between I^- and $\tilde{I}^-$—is small. Afterwards, we will give bounds on what happens inside. We begin by showing the following easy result.

Lemma 8. $\mathbb{P}(\tilde{I}^- \geq 0)$ *tends to 0 when* $N \to +\infty$.

Proof. Let N be large enough. If $\tilde{I}^- \geq 0$, there exists $i \in \{0, \dots, \lfloor Nx \rfloor\}$ such that $\ell^+(T_N, i) \leq \lfloor N^{1/6} \rfloor$. Since N is large enough, this implies $\ell^+(T_N, i) \leq N\theta/2$; therefore

$$\sup_{y\in\mathbb{R}} \left|\frac{1}{N}\ell^+\left(T_N, \lfloor Ny \rfloor\right) - \left(\frac{|x|-|y|}{2} + \theta\right)_+\right| \geq \theta/2.$$

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Moreover, by [17, Theorem 1], $\sup_{y\in\mathbb{R}} \left|\frac{1}{N}\ell^+(T_N, \lfloor Ny\rfloor) - \left(\frac{|x|-|y|}{2}+\theta\right)_+\right|$ converges in probability to 0 when N tends to $+\infty$; hence we deduce that

$$\mathbb{P}\left(\sup_{y\in\mathbb{R}}\left|\frac{1}{N}\ell^+(T_N, \lfloor Ny\rfloor) - \left(\frac{|x|-|y|}{2}+\theta\right)_+\right| \geq \theta/2\right)$$

tends to 0 when $N \to +\infty$. Therefore $\mathbb{P}(\tilde{I}^- \geq 0)$ tends to 0 when $N \to +\infty$. □

In order to control I^+, I^-, $\tilde{I}^+$, and $\tilde{I}^-$, we will use the fact that the local times behave as the Markov chain L from [17], defined as follows. We consider i.i.d. copies of the Markov chain η starting at 0, called $(\eta_m)_{m\in\mathbb{N}}$. For any $m \in \mathbb{N}$, we then set $L(m+1) = L(m) + \eta_m(L(m))$. We let $\tau = \inf\{m \in \mathbb{N} \mid L(m) \leq 0\}$. The following was proven in [17].

Lemma 9. ([17, Lemma 2].) *There exists a constant $K < +\infty$ such that for any $k \in \mathbb{N}$ we have $\mathbb{E}(\tau|L(0) = k) \leq 3k + K$.*

Since the local times will behave as L, Lemma 9 implies that if the local time starts out small, then the time at which it reaches 0 has small expectation and hence is not too large. This will help us to prove the following control on the window where $\ell^\pm(T_N, i)$ is small but not zero.

Lemma 10. *$\mathbb{P}\big(I^+ - \tilde{I}^+ \geq N^{1/4}\big)$ and $\mathbb{P}\big(\tilde{I}^- - I^- \geq N^{1/4}\big)$ tend to 0 when $N \to +\infty$.*

Proof. Let N be large enough. We deal only with $\mathbb{P}\big(\tilde{I}^- - I^- \geq N^{1/4}\big)$, since $\mathbb{P}\big(I^+ - \tilde{I}^+ \geq N^{1/4}\big)$ can be dealt with in the same way and with simpler arguments. Thanks to Lemma 8, it is enough to prove that $\mathbb{P}\big(\tilde{I}^- - I^- \geq N^{1/4}, \tilde{I}^- < 0\big)$ tends to 0 when $N \to +\infty$. Moreover, if $\tilde{I}^- < 0$, thanks to (1), for any $i < \tilde{I}^-$ we get $\ell^+(T_N, i) = \ell^+(T_N, \tilde{I}^-) + \sum_{j=i+1}^{\tilde{I}^-} \eta_{j,-}(\ell^+(T_N, j))$, which allows us to prove that $(\ell^+(T_N, \tilde{I}^- - i))_{i\in\mathbb{N}}$ is a Markov chain with the transition probabilities of L. Therefore, recalling the notation just before Lemma 9, we have

$$\begin{aligned}
&\mathbb{P}\left(\tilde{I}^- - I^- \geq N^{1/4}, \tilde{I}^- < 0\right) \\
&= \sum_{k=0}^{\lfloor N^{1/6}\rfloor} \mathbb{P}\left(\tilde{I}^- - I^- \geq N^{1/4}, \tilde{I}^- < 0 \,\middle|\, \ell^+(T_N, \tilde{I}^-) = k\right) \mathbb{P}\left(\ell^+(T_N, \tilde{I}^-) = k\right) \\
&= \sum_{k=0}^{\lfloor N^{1/6}\rfloor} \mathbb{P}\left(\tau \geq N^{1/4} \,\middle|\, L(0) = k\right) \mathbb{P}\left(\ell^+(T_N, \tilde{I}^-) = k\right) \\
&\leq \sum_{k=0}^{\lfloor N^{1/6}\rfloor} \frac{1}{N^{1/4}} \mathbb{E}(\tau|L(0) = k)\mathbb{P}(\ell^+(T_N, \tilde{I}^-) = k).
\end{aligned}$$

By Lemma 9 we deduce that

$$\begin{aligned}
\mathbb{P}\left(\tilde{I}^- - I^- \geq N^{1/4}, \tilde{I}^- < 0\right) &\leq \frac{1}{N^{1/4}} \sum_{k=0}^{\lfloor N^{1/6}\rfloor} (3k + K)\mathbb{P}(\ell^+(T_N, \tilde{I}^-) = k) \\
&\leq \frac{3N^{1/6} + K}{N^{1/4}} \leq 4N^{-1/12},
\end{aligned}$$

since N is large enough; hence $\mathbb{P}(\tilde{I}^- - I^- \geq N^{1/4}, \tilde{I}^- < 0)$ tends to 0 when $N \to +\infty$, which completes the proof. □

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We are now going to prove that even when $\ell^{\pm}(T_N, i)$ is small, the local times are not too far from the random variables of the coupling. More precisely, for any $n \in \mathbb{N}$, we define the following events:

$$\mathcal{B}_4^- = \left\{\exists\, i \in \{I^-, \dots, \chi(N)-1\}, \left|\sum_{j=i+1}^{\chi(N)-1} \left(\eta_{j,-}\big(\ell^+(T_N, j)\big) + 1/2\right) - \sum_{j=i+1}^{\chi(N)-1} \zeta_j\right| \geq N^{1/3}\right\},$$
$$\mathcal{B}_4^+ = \left\{\exists\, i \in \{\chi(N), \dots, I^+\}, \left|\sum_{j=\chi(N)}^{i-1} (\eta_{j,+}(\ell^-(T_N, j)) + 1/2) - \sum_{j=\chi(N)}^{i-1} \zeta_j\right| \geq N^{1/3}\right\}.$$

Lemma 11. *$\mathbb{P}\big(\mathcal{B}_4^-\big)$ and $\mathbb{P}\big(\mathcal{B}_4^+\big)$ tend to 0 when N tend to $+\infty$.*

Proof. The idea of the argument is that when $\ell^{\pm}(T_N, i)$ is large, $\eta_{i,\mp}(\ell^{\pm}(T_N, i)) + 1/2 = \zeta_i$ thanks to Lemma 2; that the window where $\ell^{\pm}(T_N, i)$ is small is bounded by Lemma 10; and that inside this window the $\eta_{i,\mp}(\ell^{\pm}(T_N, i)) + 1/2$, ζ_i are also bounded by Lemma 3. We spell out the proof only for $\mathbb{P}\big(\mathcal{B}_4^-\big)$, since the proof for $\mathbb{P}\big(\mathcal{B}_4^+\big)$ is the same. By Lemma 7, we have that $\mathbb{P}(I^- \leq -2(|x| + \theta)N)$ tends to 0 when N tends to $+\infty$. Furthermore, Lemma 10 implies that $\mathbb{P}\big(\tilde{I}^- - I^- \geq N^{1/4}\big)$ tends to 0 when N tends to $+\infty$. In addition, by Lemmas 2 and 3 we have that $\mathbb{P}\big(\mathcal{B}_1^-\big)$ and $\mathbb{P}(\mathcal{B}_2)$ tend to 0 when N tends to $+\infty$. Consequently, it is enough to prove that for N large enough, if $\big(\mathcal{B}_1^-\big)^c$ and $(\mathcal{B}_2)^c$ occur, if $\tilde{I}^- - I^- < N^{1/4}$, and if $I^- > -2(|x| + \theta)N$, then $\big(\mathcal{B}_4^-\big)^c$ occurs. We assume $\big(\mathcal{B}_1^-\big)^c$, $(\mathcal{B}_2)^c$, $\tilde{I}^- - I^- < N^{1/4}$, and $I^- > -2(|x| + \theta)N$. Since $\big(\mathcal{B}_1^-\big)^c$ occurs and $\tilde{I}^- \geq I^- > -2(|x| + \theta)N$, we get $\zeta_i = \eta_{j,-}(\ell^+(T_N, j)) + 1/2$ for any $i \in \{\tilde{I}^- + 1, \dots, \chi(N) - 1\}$. Therefore, if $i \in \{\tilde{I}^-, \dots, \chi(N) - 1\}$ we get

$$\sum_{j=i+1}^{\chi(N)-1} \left(\eta_{j,-}\big(\ell^+(T_N, j)\big) + 1/2\right) - \sum_{j=i+1}^{\chi(N)-1} \zeta_j = 0,$$

and for $i \in \{I^-, \dots, \tilde{I}^- - 1\}$ we have

$$\left|\sum_{j=i+1}^{\chi(N)-1} \left(\eta_{j,-}\big(\ell^+(T_N, j)\big) + 1/2\right) - \sum_{j=i+1}^{\chi(N)-1} \zeta_j\right| = \left|\sum_{j=i+1}^{\tilde{I}^-} \left(\eta_{j,-}\big(\ell^+(T_N, j)\big) + 1/2\right) - \sum_{j=i+1}^{\tilde{I}^-} \zeta_j\right|$$
$$\leq \sum_{j=i+1}^{\tilde{I}^-} \big(|\eta_{j,-}(\ell^+(T_N, j)) + 1/2| + |\zeta_j|\big) \leq 2(\tilde{I}^- - I^-)N^{1/16},$$

since $\big(\mathcal{B}_2^-\big)^c$ occurs, $i + 1 \geq I^- > -2(|x| + \theta)N$, and by definition $\tilde{I}^- \leq \chi(N) - 1 \leq 2(|x| + \theta)N$. Moreover, we assumed $\tilde{I}^- - I^- < N^{1/4}$, which implies

$$\left|\sum_{j=i+1}^{\chi(N)-1} \left(\eta_{j,-}\big(\ell^+(T_N, j)\big) + 1/2\right) - \sum_{j=i+1}^{\chi(N)-1} \zeta_j\right| \leq 2N^{1/4}N^{1/16} = 2N^{5/16} < N^{1/3}$$

when N is large enough. Consequently, for any $i \in \{I^-, \dots, \chi(N) - 1\}$ we have

$$\left|\sum_{j=i+1}^{\chi(N)-1} \left(\eta_{j,-}\big(\ell^+(T_N, j)\big) + 1/2\right) - \sum_{j=i+1}^{\chi(N)-1} \zeta_j\right| < N^{1/3};$$

therefore $\big(\mathcal{B}_4^-\big)^c$ occurs, which completes the proof. □

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4. Skorokhod M_1 distance

The goal of this section is to prove that when N is large, $Y_N^{\pm}$ is close in the Skorokhod M_1 distance to the function Y_N defined as follows. For any N large enough, for $y \in \mathbb{R}$, we set

$$Y_N(y) = \frac{1}{\sqrt{N}} \sum_{i=\lfloor Ny \rfloor+1}^{\chi(N)-1} \zeta_i$$

if $y \in \left[-|x| - 2\theta, \frac{\chi(N)}{N}\right)$,

$$Y_N(y) = \frac{1}{\sqrt{N}} \sum_{i=\chi(N)}^{\lfloor Ny \rfloor-1} \zeta_i$$

if $y \in \left[\frac{\chi(N)}{N}, |x| + 2\theta\right)$, and $Y_N(y) = 0$ otherwise. We want to prove the following proposition.

Proposition 5. $\mathbb{P}\big(d_{M_1}\big(Y_N^{\pm}, Y_N\big) > 3N^{-1/12}\big)$ *tends to 0 when N tends to* $+\infty$.

If we denote by $\mathcal{B}$ the event

$$\mathcal{B}_1^- \cup \mathcal{B}_1^+ \cup \mathcal{B}_2 \cup \mathcal{B}_3^- \cup \mathcal{B}_3^+ \cup \mathcal{B}_4^- \cup \mathcal{B}_4^+ \cup \left\{|I^- + (|x| + 2\theta)N| \geq N^{3/4}\right\}$$
$$\cup \left\{|I^+ - (|x| + 2\theta)N| \geq N^{3/4}\right\},$$

it will be enough to prove the following proposition.

Proposition 6. *When N is large enough, for all* $a > 0$ *with* $|(|x| + 2\theta) - a| > N^{-1/8}$, *we have that* $\mathcal{B}^c \subset \left\{d_{M_1,a}\big(Y_N^{\pm}|_{[-a,a]}, Y_N|_{[-a,a]}\big) \leq 2N^{-1/12}\right\}$.

Proof of Proposition 5 given Proposition 6. We assume Proposition 6 holds. Then, when N is large enough, if $\mathcal{B}^c$ occurs, for all $a > 0$ with $|(|x| + 2\theta) - a| > N^{-1/8}$ we have $d_{M_1,a}\big(Y_N^{\pm}|_{[-a,a]}, Y_N|_{[-a,a]}\big) \leq 2N^{-1/12}$, which yields that

$$\begin{aligned}
d_{M_1}\big(Y_N^{\pm}, Y_N\big) &= \int_0^{+\infty} e^{-a}\big(d_{M_1,a}\big(Y_N^{\pm}|_{[-a,a]}, Y_N|_{[-a,a]}\big) \wedge 1\big)\mathrm{d}a \\
&\leq \int_0^{+\infty} e^{-a} 2N^{-1/12}\mathrm{d}a + 2N^{-1/8} \\
&= 2N^{-1/12} + 2N^{-1/8} \leq 3N^{-1/12}.
\end{aligned}$$

This implies that $\mathbb{P}\big(d_{M_1}\big(Y_N^{\pm}, Y_N\big) > 3N^{-1/12}\big) \leq \mathbb{P}(\mathcal{B})$ when N is large enough. In addition,

$$\begin{aligned}
\mathbb{P}(\mathcal{B}) \leq\; & \mathbb{P}\big(\mathcal{B}_1^-\big) + \mathbb{P}\big(\mathcal{B}_1^+\big) + \mathbb{P}(\mathcal{B}_2) + \mathbb{P}\big(\mathcal{B}_3^-\big) + \mathbb{P}\big(\mathcal{B}_3^+\big) + \mathbb{P}\big(\mathcal{B}_4^-\big) + \mathbb{P}\big(\mathcal{B}_4^+\big) \\
& + \mathbb{P}\big(|I^- + (|x| + 2\theta)N| \geq N^{3/4}\big) + \mathbb{P}\big(|I^+ - (|x| + 2\theta)N| \geq N^{3/4}\big).
\end{aligned}$$

Applying Lemmas 2, 3, 5, 7, and 11 implies that $\mathbb{P}(\mathcal{B})$ tends to 0 when N tends to $+\infty$; hence $\mathbb{P}\big(d_{M_1}\big(Y_N^{\pm}, Y_N\big) > 3N^{-1/12}\big)$ tends to 0 when N tends to $+\infty$, which is Proposition 5. □

The remainder of this section is devoted to the proof of Proposition 6. The first thing we do is show that between $\frac{(-(|x|+2\theta)N) \vee I^-}{N}$ and $\frac{((|x|+2\theta)N) \wedge I^+}{N}$, the functions $Y_N^{\pm}$ and Y_N are close in uniform distance, which is the following lemma.

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Lemma 12. *When N is large enough, if $(\mathcal{B}_2)^c$, $\left(\mathcal{B}_4^-\right)^c$, and $\left(\mathcal{B}_4^+\right)^c$ occur, we have the following: if $I^+ < (|x| + 2\theta)N$, then for any*

$$y \in \left[\frac{(-(|x|+2\theta)N) \vee I^-}{N}, \frac{((|x|+2\theta)N) \wedge I^+}{N}\right]$$

we have $\left|Y_N^{\pm}(y) - Y_N(y)\right| \le N^{-1/12}$. *If $I^+ \geq (|x| + 2\theta)N$, then we have* $\left|Y_N^{\pm}(y) - Y_N(y)\right| \le N^{-1/12}$ *for* $y \in \left[\frac{(-(|x|+2\theta)N)\vee I^-}{N}, \frac{((|x|+2\theta)N)\wedge I^+}{N}\right)$.

Proof of Lemma 12. Writing down the proof is only a technical matter, as the meaning of $\left(\mathcal{B}_4^{\pm}\right)^c$ is that the local times are close to the process formed from the random variables of the coupling. The event $(\mathcal{B}_2)^c$ is there to ensure that the difference terms that appear will be small. We spell out the proof only for Y^-, as the proof for Y^+ is similar. We assume $(\mathcal{B}_2)^c$, $\left(\mathcal{B}_4^-\right)^c$, and $\left(\mathcal{B}_4^+\right)^c$. Then if

$$y \in \left[\frac{\chi(N)}{N}, \frac{((|x|+2\theta)N) \wedge I^+}{N}\right]$$

$\big($if $I^+ \geq (|x| + 2\theta)N$ we exclude the case $y = \frac{((|x|+2\theta)N)\wedge I^+}{N}\big)$, we have $y \in \left[\frac{\chi(N)}{N}, |x| + 2\theta\right)$, so

$$\left|Y_N^-(y) - Y_N(y)\right| = \frac{1}{\sqrt{N}}\left|\ell^-\left(T_N, \lfloor Ny\rfloor\right) - N\left(\frac{|x|-|y|}{2}+\theta\right)_+ - \sum_{i=\chi(N)}^{\lfloor Ny\rfloor-1}\zeta_i\right|;$$

thus by (1) we obtain that $\left|Y_N^-(y) - Y_N(y)\right|$ is equal to

$$\begin{aligned}
&\frac{1}{\sqrt{N}}\left|\lfloor N\theta\rfloor - \mathbb{1}_{\{\iota=+\}} + \sum_{i=\chi(N)}^{\lfloor Ny\rfloor-1}\eta_{i,+}\left(\ell^-\left(T_N, i\right)\right) - N\left(\frac{|x|-|y|}{2}+\theta\right)_+ - \sum_{i=\chi(N)}^{\lfloor Ny\rfloor-1}\zeta_i\right|\\
&\quad\le \frac{1}{\sqrt{N}}\left|\sum_{i=\chi(N)}^{\lfloor Ny\rfloor-1}\eta_{i,+}\left(\ell^-\left(T_N, i\right)\right) + \frac{\lfloor Ny\rfloor - \chi(N)}{2} - \sum_{i=\chi(N)}^{\lfloor Ny\rfloor-1}\zeta_i\right| + \frac{3}{\sqrt{N}}\\
&\quad= \frac{1}{\sqrt{N}}\left|\sum_{i=\chi(N)}^{\lfloor Ny\rfloor-1}\left(\eta_{i,+}\left(\ell^-\left(T_N, i\right)\right) + 1/2\right) - \sum_{i=\chi(N)}^{\lfloor Ny\rfloor-1}\zeta_i\right| + \frac{3}{\sqrt{N}}.
\end{aligned}$$

Now, $y \in \left[\frac{\chi(N)}{N}, \frac{((|x|+2\theta)N)\wedge I^+}{N}\right]$ implies $\lfloor Ny\rfloor \in \{\chi(N), \dots, I^+\}$; thus $\left(\mathcal{B}_4^+\right)^c$ yields $\left|Y_N^-(y) - Y_N(y)\right| \le \frac{1}{\sqrt{N}}N^{1/3} + \frac{3}{\sqrt{N}} \le N^{-1/12}$ when N is large enough. We now consider the case $y \in \left[\frac{(-(|x|+2\theta)N)\vee I^-}{N}, \frac{\chi(N)}{N}\right)$. Then $y \in \left[-|x| - 2\theta, \frac{\chi(N)}{N}\right)$, and hence

$$\left|Y_N^-(y) - Y_N(y)\right| = \frac{1}{\sqrt{N}}\left|\ell^-\left(T_N, \lfloor Ny\rfloor\right) - N\left(\frac{|x|-|y|}{2}+\theta\right)_+ - \sum_{i=\lfloor Ny\rfloor+1}^{\chi(N)-1}\zeta_i\right|.$$

Now, (2) yields $|\ell^-\left(T_N, \lfloor Ny\rfloor\right) - \ell^+\left(T_N, \lfloor Ny\rfloor\right)| = |\eta_{\lfloor Ny\rfloor,-}\left(\ell^+\left(T_N, \lfloor Ny\rfloor\right)\right)|$, which is smaller than $N^{1/16} + 1/2$ thanks to $(\mathcal{B}_2)^c$. We deduce that

$$\left|Y_N^-(y) - Y_N(y)\right| \le \frac{1}{\sqrt{N}}\left|\ell^+\left(T_N, \lfloor Ny\rfloor\right) - N\left(\frac{|x|-|y|}{2}+\theta\right)_+ - \sum_{i=\lfloor Ny\rfloor+1}^{\chi(N)-1}\zeta_i\right| + \frac{N^{1/16}+1/2}{\sqrt{N}};$$

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thus (1) implies that $\left|Y_N^-(y)-Y_N(y)\right|$ is smaller than

$$\frac{1}{\sqrt{N}}\left|\lfloor N\theta\rfloor+\sum_{i=\lfloor Ny\rfloor+1}^{\chi(N)-1}\left(\eta_{i,-}\left(\ell^+\left(T_N,i\right)\right)+\mathbb{1}_{\{i>0\}}\right)\right.$$

$$\left.-N\left(\frac{|x|-|y|}{2}+\theta\right)_+-\sum_{i=\lfloor Ny\rfloor+1}^{\chi(N)-1}\zeta_i\right|+\frac{N^{1/16}+1/2}{\sqrt{N}}$$

$$\leq\frac{1}{\sqrt{N}}\left|\sum_{i=\lfloor Ny\rfloor+1}^{\chi(N)-1}\left(\eta_{i,-}\left(\ell^+\left(T_N,i\right)\right)+\mathbb{1}_{\{i>0\}}\right)+\frac{\lfloor Ny\rfloor+1-\chi(N)}{2}-\sum_{i=\lfloor Ny\rfloor+1}^{\chi(N)-1}\zeta_i\right|+\frac{N^{1/16}+3}{\sqrt{N}}$$

$$\leq\frac{1}{\sqrt{N}}\left|\sum_{i=\lfloor Ny\rfloor+1}^{\chi(N)-1}\left(\eta_{i,-}\left(\ell^+\left(T_N,i\right)\right)+1/2\right)-\sum_{i=\lfloor Ny\rfloor+1}^{\chi(N)-1}\zeta_i\right|+\frac{N^{1/16}+3}{\sqrt{N}}.$$

Furthermore, $y\in\left[\frac{(-(|x|+2\theta)N)\vee I^-}{N},\frac{\chi(N)}{N}\right)$ implies $\lfloor Ny\rfloor\in\{I^-,\dots,\chi(N)-1\}$; hence $\left(\mathcal{B}_4^-\right)^c$ yields

$$\left|Y_N^-(y)-Y_N(y)\right|\leq\frac{1}{\sqrt{N}}N^{1/3}+\frac{N^{1/16}+3}{\sqrt{N}}\leq N^{-1/12}$$

when N is large enough. Consequently, for any $y\in\left[\frac{(-(|x|+2\theta)N)\vee I^-}{N},\frac{((|x|+2\theta)N)\wedge I^+}{N}\right]$ we have $\left|Y_N^-(y)-Y_N(y)\right|\leq N^{-1/12}$, which completes the proof of Lemma 12. □

We now prove Proposition 6. Let $a>0$ be such that $|(|x|+2\theta)-a|>N^{-1/8}$. We will prove that when N is large enough, $\mathcal{B}^c\subset\left\{d_{M_1,a}\left(Y_N^{\pm}|_{[-a,a]},Y_N|_{[-a,a]}\right)\leq 2N^{-1/12}\right\}$, and the threshold for N given by the proof will not depend on the value of a. There will be two cases depending on whether a is smaller than $|x|+2\theta$ or not.

4.1. Case $\mathbf{a\in\left(0,|x|+2\theta-N^{-1/8}\right)}$

This is the easier case. Indeed, the interval $[-a,a]$ will then be contained in $\left[\frac{(-(|x|+2\theta)N)\vee I^-}{N},\frac{((|x|+2\theta)N)\wedge I^+}{N}\right)$, inside which $Y_N^{\pm}$ and Y_N are close for the uniform norm by Lemma 12. We may then define parametric representations $\left(u_N^{\pm},r_N^{\pm}\right)$ and (u_N,r_N) of $Y_N^-|_{[-a,a]}$ and $Y_N|_{[-a,a]}$ 'following the graphs of $Y_N^{\pm}|_{[-a,a]}$ and $Y_N|_{[-a,a]}$ together' so that $u_N^{\pm}(t)=u_N(t)$ for all $t\in[0,1]$, and

$$\left\|r_N^{\pm}-r_N\right\|_\infty\leq\sup_{y\in[-a,a]}\left|Y_N^{\pm}(y)-Y_N(y)\right|$$

(an explicit construction of these representations can be found in the first arXiv version of this paper [5]). We deduce that

$$d_{M_1,a}\left(Y_N^{\pm}|_{[-a,a]},Y_N|_{[-a,a]}\right)\leq\sup_{y\in[-a,a]}\left|Y_N^{\pm}(y)-Y_N(y)\right|.$$

Moreover, if $\mathcal{B}^c$ occurs, since $a\in\left(0,|x|+2\theta-N^{-1/8}\right)$, for any $y\in[-a,a]$ we have $y\in\left(-|x|-2\theta+N^{-1/8},|x|+2\theta-N^{-1/8}\right)$; thus $-(|x|+2\theta)N+N^{3/4}\leq Ny\leq(|x|+2\theta)N-$

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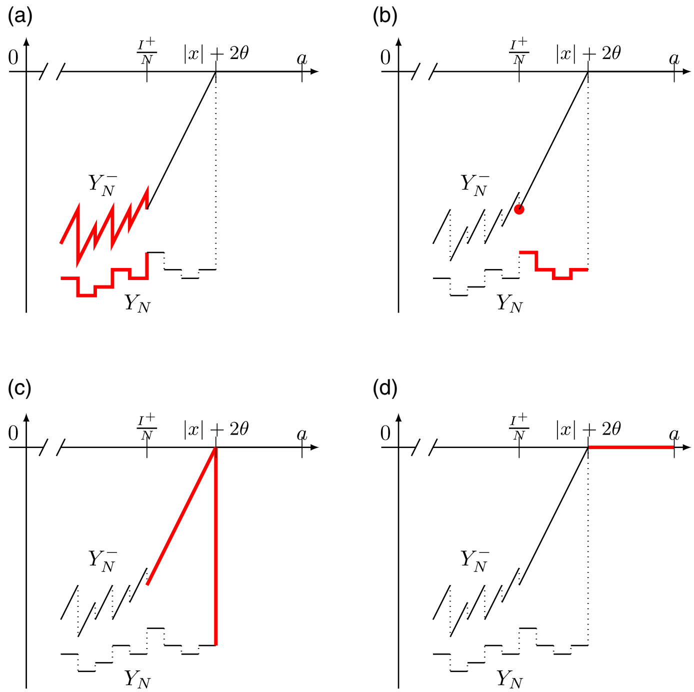

FIGURE 1. The successive steps of the parametric representations of $Y_N^-|_{[-a,a]}$ and $Y_N|_{[-a,a]}$ if $I^+ \leq \lfloor(|x|+2\theta)N\rfloor$. At each step, the parts of the graphs through which the parametric representations travel are thickened.

$N^{3/4}$, which implies $I^- < Ny < I^+$, and hence $y \in \left(\frac{(-(|x|+2\theta)N)\vee I^-}{N}, \frac{((|x|+2\theta)N)\wedge I^+}{N}\right)$. So by Lemma 12 we have $\left|Y_N^{\pm}(y) - Y_N(y)\right| \leq N^{-1/12}$. Consequently, if $\mathcal{B}^c$ occurs, then $d_{M_1,a}\left(Y_N^{\pm}|_{[-a,a]}, Y_N|_{[-a,a]}\right) \leq N^{-1/12}$.

4.2. Case $a > |x| + 2\theta + N^{-1/8}$

This is the harder case, as we have to deal with what happens around $|x|+2\theta$ and $-|x|-2\theta$. We write down only the proof for Y_N^-, since the proof for Y_N^+ is similar (one may remember that (2) allows us to bound the $\ell^-(T_N, i) - \ell^+(T_N, i)$ when $(\mathcal{B}_2)^c$ occurs, and hence when $\mathcal{B}^c$ occurs). Once again, we will define parametric representations $\left(u_N^-, r_N^-\right)$ and (u_N, r_N) of $Y_N^-|_{[-a,a]}$ and $Y_N|_{[-a,a]}$. The definition will depend on whether $I^+ \leq \lfloor(|x|+2\theta)N\rfloor$ or not, and also on whether $I^- \geq -\lfloor(|x|+2\theta)N\rfloor$ or not. We explain it for abscissas in $[0,a]$ depending on whether $I^+ \leq \lfloor(|x|+2\theta)N\rfloor$ or not; the constructions for abscissas in $[-a, 0]$ are similar, depending on whether $I^- \geq -\lfloor(|x|+2\theta)N\rfloor$ or not.

We first assume $I^+ \leq \lfloor(|x|+2\theta)N\rfloor$. Between 0 and $\frac{I^+}{N}$, as in the case $a \in (0, |x|+2\theta - N^{-1/8})$, the parametric representations will follow the completed graphs of Y_N^- and Y_N in parallel (see Figure 1(a)). The next step, once $\left(u_N^-, r_N^-\right)$ has reached $\left(\frac{I^+}{N}, Y_N^-\left(\frac{I^+}{N}\right)\right)$, is to freeze it there while (u_N, r_N) follows the graph of Y_N from $\left(\frac{I^+}{N}, Y_N\left(\frac{I^+}{N}\right)\right)$ to $(|x|+2\theta, Y_N((|x|+2\theta))^-)$ (see Figure 1(b)). For $y \geq \frac{I^+}{N}$ we have $\ell^-\left(T_N, \lfloor Ny\rfloor\right) = 0$ (see Lemma 6); thus $Y_N(y) =$

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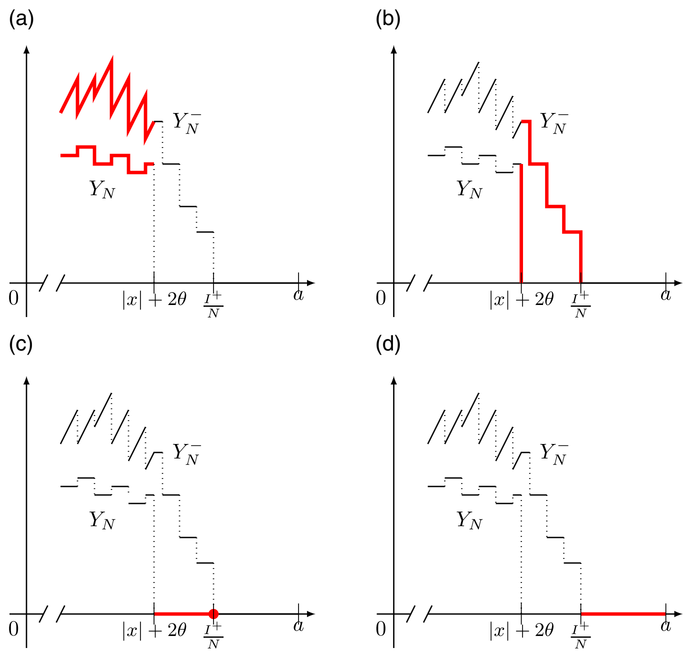

FIGURE 2. The successive steps of the parametric representations of $Y_N^-|_{[-a,a]}$ and $Y_N|_{[-a,a]}$ if $I^+ > \lfloor(|x|+2\theta)N\rfloor$. At each step, the parts of the graphs through which the parametric representations travel are thickened.

$-N\big(\frac{|x|-|y|}{2}+\theta\big)_+$, and hence $Y_N^- : [\frac{I^+}{N}, |x|+2\theta] \mapsto \mathbb{R}$ is affine. Therefore, the following step is to simultaneously move (u_N^-, r_N^-) from $\Big(\frac{I^+}{N}, Y_N^-\Big(\frac{I^+}{N}\Big)\Big)$ to $(|x|+2\theta, Y_N^-(|x|+2\theta)) = (|x|+2\theta, 0)$ and (u_N, r_N) from $(|x|+2\theta, Y_N((|x|+2\theta)^-))$ to $(|x|+2\theta, 0)$ (see Figure 1(c)); here the two parametric representations will remain close. After this step, both parametric representations are at $(|x|+2\theta, 0)$, and they will go together to $(a,0)$ (see Figure 1(d)).

We now assume $I^+ > \lfloor(|x|+2\theta)N\rfloor$. We also assume $\frac{I^+}{N} \le a$ $\big($if $\frac{I^+}{N} > a$, we may choose anything for (u_N^-, r_N^-), (u_N, r_N); this will not happen if $\mathcal{B}^c$ occurs$\big)$. Between 0 and $|x|+2\theta$, the parametric representations will follow the completed graphs of Y_N^- and Y_N in parallel (see Figure 2(a)). Once the abscissa $|x|+2\theta$ is reached, the next step is to move (u_N^-, r_N^-) from $(|x|+2\theta, Y_N^-(|x|+2\theta))$ to $\Big(\frac{I^+}{N}, Y_N^-\Big(\frac{I^+}{N}\Big)\Big)$, which is $\big(\frac{I^+}{N}, 0\big)$, and at the same time to move (u_N, r_N) from $(|x|+2\theta, Y_N(|x|+2\theta))$ to $(|x|+2\theta, 0)$ (see Figure 2(b)). We will prove the two representations are close by controlling the local times. At the next step we freeze (u_N^-, r_N^-) at $\big(\frac{I^+}{N}, 0\big)$ while (u_N, r_N) goes from $(|x|+2\theta, 0)$ to $\big(\frac{I^+}{N}, 0\big)$ (see Figure 2(c)). After this step, both parametric representations are at $\big(\frac{I^+}{N}, 0\big)$, and they will go together from $\big(\frac{I^+}{N}, 0\big)$ to $(a,0)$ (see Figure 2(d)). Again, a more rigorous definition of the parametric representations is available in the first arXiv version of this paper [5].

We can now bound the Skorokhod M_1 distance between $Y_N^-|_{[-a,a]}$ and $Y_N|_{[-a,a]}$. From its definition, we have

$$d_{M_1,a}\big(Y_N^-|_{[-a,a]}, Y_N|_{[-a,a]}\big) \le \max\big(\|u_N^- - u_N\|_\infty, \|r_N^- - r_N\|_\infty\big);$$

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hence we only have to prove that $\mathcal{B}^c \subset \{\max\left(\|u_N^- - u_N\|_\infty, \|r_N^- - r_N\|_\infty\right) \le 2N^{-1/12}\}$ when N is large enough. We are going to break down $\left\{\max\left(\|u_N^- - u_N\|_\infty, \|r_N^- - r_N\|_\infty\right) \le 2N^{-1/12}\right\}$ into three events. We may write

$$\left\{\max\left(\|u_N^- - u_N\|_\infty, \|r_N^- - r_N\|_\infty\right) \le 2N^{-1/12}\right\}$$

$$= \left\{\text{between } \frac{(-(|x|+2\theta)N)\vee I^-}{N} \text{ and } \frac{((|x|+2\theta)N)\wedge I^+}{N}, \right.$$
$$\left. \|u_N^- - u_N\|_\infty, \|r_N^- - r_N\|_\infty \le 2N^{-1/12}\right\}$$
$$\cap \left\{\text{between } \frac{((|x|+2\theta)N)\wedge I^+}{N} \text{ and } a, \|u_N^- - u_N\|_\infty, \|r_N^- - r_N\|_\infty \le 2N^{-1/12}\right\}$$
$$\cap \left\{\text{between } -a \text{ and } \frac{(-(|x|+2\theta)N)\vee I^-}{N}, \|u_N^- - u_N\|_\infty, \|r_N^- - r_N\|_\infty \le 2N^{-1/12}\right\}.$$

Consequently, to prove that $\mathcal{B}^c \subset \left\{\max\left(\|u_N^- - u_N\|_\infty, \|r_N^- - r_N\|_\infty\right) \le 2N^{-1/12}\right\}$ when N is large enough and thus complete the proof of Proposition 6, we only have to prove the following lemmas.

Lemma 13. *We have* $\mathcal{B}^c \subset \left\{\text{between } \frac{(-(|x|+2\theta)N)\vee I^-}{N} \text{ and } \frac{((|x|+2\theta)N)\wedge I^+}{N}, \|u_N^- - u_N\|_\infty, \|r_N^- - r_N\|_\infty \le 2N^{-1/12}\right\}$ *when N is large enough.*

Lemma 14. *We have* $\mathcal{B}^c \cap \{I^+ \le \lfloor(|x|+2\theta)N\rfloor\} \subset \left\{\text{between } \frac{((|x|+2\theta)N)\wedge I^+}{N} \text{ and } a, \|u_N^- - u_N\|_\infty, \|r_N^- - r_N\|_\infty \le 2N^{-1/12}\right\}$ *and* $\mathcal{B}^c \cap \{I^- \ge -\lfloor(|x|+2\theta)N\rfloor\} \subset \left\{\text{between } -a \text{ and } \frac{(-(|x|+2\theta)N)\vee I^-}{N}, \|u_N^- - u_N\|_\infty, \|r_N^- - r_N\|_\infty \le 2N^{-1/12}\right\})$ *when N is large enough.*

Lemma 15. *We have* $\mathcal{B}^c \cap \{I^+ > \lfloor(|x|+2\theta)N\rfloor\} \subset \left\{\text{between } \frac{((|x|+2\theta)N)\wedge I^+}{N} \text{ and } a, \|u_N^- - u_N\|_\infty, \|r_N^- - r_N\|_\infty \le 2N^{-1/12}\right\}$ *and* $\mathcal{B}^c \cap \{I^- < -\lfloor(|x|+2\theta)N\rfloor\} \subset \left\{\text{between } -a \text{ and } \frac{(-(|x|+2\theta)N)\vee I^-}{N}, \|u_N^- - u_N\|_\infty, \|r_N^- - r_N\|_\infty \le 2N^{-1/12}\right\})$ *when N is large enough.*

We now prove Lemmas 13, 14, and 15.

Proof of Lemma 13. We assume $\mathcal{B}^c$ occurs. In the part of the parametric representations between $\frac{(-(|x|+2\theta)N)\vee I^-}{N}$ and $\frac{((|x|+2\theta)N)\wedge I^+}{N}$, corresponding to Figures 1(a) and 2(a), we follow the completed graphs of Y_N^- and Y_N in parallel. Therefore $u_N^-(t) = u_N(t)$ and

$$\left|r_N^-(t) - r_N(t)\right| \le \sup\left\{\left|Y_N^-(y) - Y_N(y)\right| : y \in \left[\frac{(-(|x|+2\theta)N)\vee I^-}{N}, \frac{((|x|+2\theta)N)\wedge I^+}{N}\right]\right\}.$$

If $(|x|+2\theta)N$ is not an integer or $I^+ < (|x|+2\theta)N$, then by Lemma 12, this is smaller than N^{-12} when N is large enough, and we are done. If $(|x|+2\theta)N$ is an integer and $I^+ \ge (|x|+2\theta)N$, there is a small complication, since the parametric representations follow the graph of Y_N until $Y_N((|x|+2\theta)^-)$, but should follow the graph of Y_N^- until $Y_N^-(|x|+2\theta)$. The solution is to freeze the representation of Y_N at $Y_N((|x|+2\theta)^-)$ while that of Y_N^- goes from $Y_N^-((|x|+2\theta)^-)$ to $Y_N^-(|x|+2\theta)$. Then, between $\frac{(-(|x|+2\theta)N)\vee I^-}{N}$ and $(|x|+2\theta)^-$, we have

$$\left|r_N^-(t)-r_N(t)\right| \le \sup\left\{\left|Y_N^-(y)-Y_N(y)\right| : y\in\left[\frac{(-(|x|+2\theta)N)\vee I^-}{N},(|x|+2\theta)N\right)\right\} \le N^{-1/12}$$

by Lemma 12 when N is large enough. Furthermore, when going from $Y_N^-((|x|+2\theta)^-)$ to $Y_N^-(|x|+2\theta)$, we have

$$\begin{aligned}\left|r_N^-(t)-r_N(t)\right| &\le \left|Y_N^-((|x|+2\theta)^-)-Y_N((|x|+2\theta)^-)\right|+\left|Y_N^-((|x|+2\theta)^-)-Y_N^-(|x|+2\theta)\right|\\ &\le N^{-1/12}+\left|Y_N^-((|x|+2\theta)^-)-Y_N^-(|x|+2\theta)\right|\end{aligned}$$

when N is large enough. In addition, when N is large enough, (1) yields

$$\begin{aligned}\left|Y_N^-(|x|+2\theta)-Y_N^-((|x|+2\theta)^-)\right| &= \frac{1}{\sqrt{N}}\left|\ell^-\big(T_N,(|x|+2\theta)N\big)-\ell^-\big(T_N,(|x|+2\theta)N-1\big)\right|\\ &= \frac{1}{\sqrt{N}}\left|\eta_{(|x|+2\theta)N-1,+}\big(\ell^-\big(T_N,(|x|+2\theta)N-1\big)\big)\right|\\ &\le \frac{N^{1/16}+1/2}{\sqrt{N}}\end{aligned}$$

since $(\mathcal{B}_2)^c$ occurs. This yields $\left|r_N^-(t)-r_N(t)\right|\le N^{-1/12}+\frac{N^{1/16}+1/2}{\sqrt{N}}\le 2N^{-1/12}$ when N is large enough, which completes the proof. □

Proof of Lemma 14. This lemma deals with the 'right part' of the parametric representations in the case $I^+\le\lfloor(|x|+2\theta)N\rfloor$, and with the 'left part' in the case $I^-\ge-\lfloor(|x|+2\theta)N\rfloor$, corresponding to Panels (b), (c), and (d) of Figure 1. The idea of the argument is that in the step corresponding to Figure 1(b), the representation of Y_N does not move much horizontally, as $\frac{I^+}{N}$ is close to $|x|+2\theta$ by Lemma 7, so it does not have time to move too much vertically. In the step corresponding to Figure 1(c), the representations of Y_N^- and Y_N will thus start from points that are close and go to the same point, which means they stay close to each other.

We now give the rigorous argument. We spell out the proof only for $\mathcal{B}^c\cap\{I^+\le\lfloor(|x|+2\theta)N\rfloor\}$, as the other case is similar. Let us assume $\mathcal{B}^c$ occurs and $I^+\le\lfloor(|x|+2\theta)N\rfloor$. Firstly, we notice that in the part of the parametric representations corresponding to Figure 1(d) we have $\big(u_N^-(t),r_N^-(t)\big)=\big(u_N(t),r_N(t)\big)$, so we consider only the parts corresponding to Figures 1(b) and 1(c). We first consider the case in which $(|x|+2\theta)N$ is not an integer or $I^+<\lfloor(|x|+2\theta)N\rfloor$. We begin by dealing with $\left|u_N^-(t)-u_N(t)\right|$. By the definition of our parametric representations, $\left|u_N^-(t)-u_N(t)\right|\le\left||x|+2\theta-\frac{I^+}{N}\right|$. Furthermore, $\mathcal{B}^c$ occurs; thus we have $|I^+-(|x|+2\theta)N|<N^{3/4}$, and hence $\left|u_N^-(t)-u_N(t)\right|\le N^{-1/4}$.

We now deal with $\left|r_N^-(t)-r_N(t)\right|$. Remembering the definition of our parametric representations, we notice that in the part corresponding to Figure 1(c), r_N^- and r_N are affine functions, so the maximum value of $\left|r_N^-(t)-r_N(t)\right|$ on this part is reached either at the beginning or at the end of the part. Moreover, at the end of the part we have $r_N^-(t)=r_N(t)=0$, so the maximum is reached at the beginning. Therefore, if $\left|r_N^-(t)-r_N(t)\right|\le 2N^{-1/12}$ in the part corresponding to Figure 1(b), then $\left|r_N^-(t)-r_N(t)\right|\le 2N^{-1/12}$ in the part corresponding to Figure 1(c), and this completes the proof when $(|x|+2\theta)N$ is not an integer or $I^+<\lfloor(|x|+2\theta)N\rfloor$.

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We thus have to study the part corresponding to Figure 1(b). By the definition of our parametric representations,

$$\left|r_N^-(t) - r_N(t)\right| \le \sup\left\{\left|Y_N^-\left(\frac{I^+}{N}\right) - Y_N(y)\right| : y \in \left[\frac{I^+}{N}, |x| + 2\theta\right)\right\},$$

so it is enough to prove that when N is large enough, $\sup\left\{\left|Y_N^-\left(\frac{I^+}{N}\right) - Y_N(y)\right| : y \in \left[\frac{I^+}{N}, |x| + 2\theta\right)\right\} \le 2N^{-1/12}$. Moreover, for any $y \in \left[\frac{I^+}{N}, |x| + 2\theta\right)$, we have

$$\left|Y_N^-\left(\frac{I^+}{N}\right) - Y_N(y)\right| \le \left|Y_N^-\left(\frac{I^+}{N}\right) - Y_N\left(\frac{I^+}{N}\right)\right| + \left|Y_N\left(\frac{I^+}{N}\right) - Y_N(y)\right|.$$

Since $\mathcal{B}^c$ occurs, we have that $(\mathcal{B}_2)^c$, $\left(\mathcal{B}_4^-\right)^c$, and $\left(\mathcal{B}_4^+\right)^c$ occur; hence Lemma 12 implies $\left|Y_N^-\left(\frac{I^+}{N}\right) - Y_N\left(\frac{I^+}{N}\right)\right| \le N^{-1/12}$ when N is large enough, and so

$$\left|Y_N^-\left(\frac{I^+}{N}\right) - Y_N(y)\right| \le \left|Y_N\left(\frac{I^+}{N}\right) - Y_N(y)\right| + N^{-1/12} \le \frac{1}{\sqrt{N}}\left|\sum_{i=I^+}^{\lfloor Ny\rfloor - 1} \zeta_i\right| + N^{-1/12}.$$

We deduce that

$$\sup\left\{\left|Y_N^-\left(\frac{I^+}{N}\right) - Y_N(y)\right| : y \in \left[\frac{I^+}{N}, |x| + 2\theta\right)\right\} \le \sup\left\{\frac{1}{\sqrt{N}}\left|\sum_{i=I^+}^{\lfloor Ny\rfloor - 1} \zeta_i\right| : y \in \left[\frac{I^+}{N}, |x| + 2\theta\right)\right\} + N^{-1/12}.$$

Furthermore, $\mathcal{B}^c$ occurs; hence $|I^+ - (|x| + 2\theta)N| < N^{3/4}$, and thus

$$\sup\left\{\left|Y_N^-\left(\frac{I^+}{N}\right) - Y_N(y)\right| : y \in \left[\frac{I^+}{N}, |x| + 2\theta\right)\right\} \le \frac{1}{\sqrt{N}} \max_{\lfloor(|x|+2\theta)N\rfloor - N^{3/4} \le i_1 \le i_2 \le \lfloor(|x|+2\theta)N\rfloor} \frac{1}{\sqrt{N}}\left|\sum_{i=i_1}^{i_2} \zeta_i\right| + N^{-1/12}.$$

Since $\mathcal{B}^c$ occurs, $\left(\mathcal{B}_3^+\right)^c$ occurs; hence

$$\sup\left\{\left|Y_N^-\left(\frac{I^+}{N}\right) - Y_N(y)\right| : y \in \left[\frac{I^+}{N}, |x| + 2\theta\right)\right\} \le \frac{N^{19/48}}{\sqrt{N}} + N^{-1/12} = N^{-5/48} + N^{-1/12} \le 2N^{-1/12},$$

which is enough.

We now consider the case in which $(|x| + 2\theta)N$ is an integer and $I^+ = \lfloor(|x| + 2\theta)N\rfloor$. Then the step corresponding to Figure 1(b) does not exist; we only have to deal with that of Figure 1(c), which comes mostly from Lemma 12, as this lemma ensures $Y_N^-((|x| + 2\theta)^-)$ and $Y_N((|x| + 2\theta)^-)$ are close (we will actually prove they are both close to 0). Since $I^+ = \lfloor(|x| + 2\theta)N\rfloor$, we have $u_N^-(t) = u_N(t)$. Moreover, $\ell^-(T_N, \lfloor N(|x| + 2\theta)\rfloor) = \ell^-(T_N, I^+) = 0$, so

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$Y_N^-(|x|+2\theta)=0$, and hence $r_N^-(t)=0$. Furthermore, $|r_N(t)|\leq|Y_N((|x|+2\theta)^-)|$. Therefore we only have to prove that $|Y_N((|x|+2\theta)^-)|\leq 2N^{-1/12}$ when N is large enough. In addition, $\mathcal{B}^c$ occurs; thus by Lemma 12 we have $|Y_N((|x|+2\theta)^-)-Y_N^-((|x|+2\theta)^-)|\leq N^{-1/12}$ when N is large enough. Moreover, by the definition of Y_N^- and by (1), we have

$$Y_N^-(|x|+2\theta)=Y_N^-((|x|+2\theta)^-)+\frac{1}{\sqrt{N}}\eta_{(|x|+2\theta)N-1,+}\big(\ell^-\big(T_N,(|x|+2\theta)N-1\big)\big),$$

and since $\mathcal{B}$ occurs, $(\mathcal{B}_2)^c$ occurs, which means we get

$$\big|Y_N^-(|x|+2\theta)-Y_N^-((|x|+2\theta)^-)\big|\leq\frac{1}{\sqrt{N}}\big(N^{1/16}+1/2\big)\leq N^{-1/4}.$$

Since $Y_N^-(|x|+2\theta)=0$, this yields $\big|Y_N^-((|x|+2\theta)^-)\big|\leq N^{-1/4}$, which implies $|Y_N((|x|+2\theta)^-)|\leq N^{-1/12}+N^{-1/4}<2N^{-1/12}$. This is enough to complete the proof of Lemma 14. □

Proof of Lemma 15. This lemma deals with the 'right part' of the parametric representations in the case $I^+>\lfloor(|x|+2\theta)N\rfloor$, and with the 'left part' in the case $I^-<-\lfloor(|x|+2\theta)N\rfloor$, corresponding to Panels (b), (c), and (d) of Figure 2. We first give an idea of the argument. The most important part of the proof is to deal with the step corresponding to Figure 2(b). In this step, the function $Y_N^-(y)=\frac{1}{\sqrt{N}}\ell^-\big(T_N,\lfloor Ny\rfloor\big)$ evolves as a sum of $\frac{1}{\sqrt{N}}\eta_{j,+}(\ell^-(T_N,j))$ by (1), which is close to the sum of $\frac{1}{\sqrt{N}}\big(\zeta_j-\frac{1}{2}\big)$ as $\big(\mathcal{B}_4^+\big)^c$ occurs. Since the ζ_j are i.i.d. with mean 0, the sum of $\frac{1}{\sqrt{N}}\zeta_j$ will be small, and the evolution of Y_N^- will be close to that of a deterministic sum of $-\frac{1}{2\sqrt{N}}$. Thus it reaches 0 at constant speed, which is also what our parametric representation of Y_N does.

We now give the proof, beginning with the details of the argument to deal with $\mathcal{B}^c\cap\{I^-<-\lfloor(|x|+2\theta)N\rfloor\}$. Let us assume $\mathcal{B}^c$ occurs and $I^-<-\lfloor(|x|+2\theta)N\rfloor$. We first see that $\frac{I^-}{N}\geq-a$, as since $\mathcal{B}^c$ occurs we have $|I^-+(|x|+2\theta)N|<N^{3/4}$, which implies $\frac{I^-}{N}>-|x|-2\theta-N^{-1/4}$, and by assumption $a>|x|+2\theta+N^{-1/8}$, so $-a<-|x|-2\theta-N^{-1/8}<\frac{I^-}{N}$, which implies $\frac{I^-}{N}\geq-a$. Moreover, in the part of the parametric representations corresponding to Figure 2(d), we have $\big(u_N^-(t),r_N^-(t)\big)=(u_N(t),r_N(t))$. We now consider the equivalent of Figure 2(c). Then $r_N^-(t)=r_N(t)=0$, and $\big|u_N^-(t)-u_N(t)\big|\leq\big|\frac{I^-}{N}+(|x|+2\theta)\big|$, which is strictly smaller than $2N^{-1/12}$ since $|I^-+(|x|+2\theta)N|<N^{3/4}$. It remains to consider the equivalent of Figure 2(b). Then $\big|u_N^-(t)-u_N(t)\big|\leq\big|\frac{I^-}{N}+(|x|+2\theta)\big|$, which is strictly smaller than $2N^{-1/12}$, so we only have to prove $\big|r_N^-(t)-r_N(t)\big|\leq 2N^{-1/12}$.

We are going to study

$$\sup_{y\in\left[\frac{I^-}{N},-|x|-2\theta\right]}\left|Y_N^-(y)-Y_N^-(-|x|-2\theta)+\frac{\lfloor(|x|+2\theta)N\rfloor-\lfloor Ny\rfloor}{2\sqrt{N}}\right|.$$

Let $y\in\big[\frac{I^-}{N},-|x|-2\theta\big]$. By the definition of Y_N^- we have

$$Y_N^-(y)-Y_N^-(-|x|-2\theta)=\frac{1}{\sqrt{N}}\big(\ell^-\big(T_N,\lfloor Ny\rfloor\big)-\ell^-\big(T_N,\lfloor-(|x|+2\theta)N\rfloor\big)\big).$$

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By (2) and since $(\mathcal{B}_2)^c$ occurs (remembering that $\lfloor Ny\rfloor \geq I^- \geq -(|x|+2\theta)N - N^{3/4} \geq -\lceil 2(|x|+2\theta)N\rceil$), we deduce that

$$\left|Y_N^-(y) - Y_N^-(-|x|-2\theta) - \frac{1}{\sqrt{N}}\big(\ell^+\big(T_N, \lfloor Ny\rfloor\big) - \ell^+\big(T_N, \lfloor -(|x|+2\theta)N\rfloor\big)\big)\right|$$

$$= \left|\frac{\eta_{\lfloor Ny\rfloor,-}\big(\ell^+\big(T_N, \lfloor Ny\rfloor\big)\big) - \eta_{\lfloor -(|x|+2\theta)N\rfloor,-}\big(\ell^+\big(T_N, \lfloor -(|x|+2\theta)N\rfloor\big)\big)}{\sqrt{N}}\right| \leq \frac{2N^{1/16}}{\sqrt{N}}.$$

In addition, (1) yields the following:

$$\ell^+\big(T_N, \lfloor Ny\rfloor\big) - \ell^+\big(T_N, \lfloor -(|x|+2\theta)N\rfloor\big)$$

$$= \sum_{i=\lfloor Ny\rfloor+1}^{\chi(N)-1}\big(\eta_{i,-}(\ell^+(T_N,i)) + \mathbb{1}_{\{i>0\}}\big) - \sum_{i=\lfloor -(|x|+2\theta)N\rfloor+1}^{\chi(N)-1}\big(\eta_{i,-}(\ell^+(T_N,i)) + \mathbb{1}_{\{i>0\}}\big)$$

$$= \sum_{i=\lfloor Ny\rfloor+1}^{\chi(N)-1}\big(\eta_{i,-}\big(\ell^+\big(T_N,i\big)\big) + 1/2\big)$$

$$- \sum_{i=\lfloor -(|x|+2\theta)N\rfloor+1}^{\chi(N)-1}\big(\eta_{i,-}\big(\ell^+\big(T_N,i\big)\big) + 1/2\big) - \frac{\lfloor -(|x|+2\theta)N\rfloor - \lfloor Ny\rfloor}{2}.$$

Since $\big(\mathcal{B}_4^-\big)^c$ occurs, this yields

$$\left|\ell^+\big(T_N, \lfloor Ny\rfloor\big) - \ell^+\big(T_N, \lfloor -(|x|+2\theta)N\rfloor\big) + \frac{\lfloor -(|x|+2\theta)N\rfloor - \lfloor Ny\rfloor}{2}\right|$$

$$\leq \left|\sum_{i=\lfloor Ny\rfloor+1}^{\chi(N)-1}\zeta_i - \sum_{i=\lfloor -(|x|+2\theta)N\rfloor+1}^{\chi(N)-1}\zeta_i\right| + 2N^{1/3}$$

$$= \left|\sum_{i=\lfloor Ny\rfloor+1}^{\lfloor -(|x|+2\theta)N\rfloor}\zeta_i\right| + 2N^{1/3}.$$

As we also have

$$\left|Y_N^-(y) - Y_N^-(-|x|-2\theta) - \frac{1}{\sqrt{N}}\big(\ell^+\big(T_N, \lfloor Ny\rfloor\big) - \ell^+\big(T_N, \lfloor -(|x|+2\theta)N\rfloor\big)\big)\right| \leq \frac{2N^{1/16}}{\sqrt{N}},$$

this implies

$$\sup_{y\in\left[\frac{I^-}{N}, -|x|-2\theta\right]}\left|Y_N^-(y) - Y_N^-(-|x|-2\theta) + \frac{\lfloor(|x|+2\theta)N\rfloor - \lfloor Ny\rfloor}{2\sqrt{N}}\right|$$

$$\leq \max_{I^-+1\leq i\leq\lfloor -(|x|+2\theta)N\rfloor}\frac{1}{\sqrt{N}}\left|\sum_{j=i}^{\lfloor -(|x|+2\theta)N\rfloor}\zeta_j\right| + \frac{2N^{1/16}}{\sqrt{N}} + \frac{2N^{1/3}}{\sqrt{N}}.$$

Moreover, $\mathcal{B}^c$ occurs, and hence $|I^- + (|x| + 2\theta)N| < N^{3/4}$ and $\left(\mathcal{B}_3^-\right)^c$ occurs; therefore we obtain that $\sup_{y\in\left[\frac{I^-}{N},-|x|-2\theta\right]}\left|Y_N^-(y) - Y_N^-(-|x|-2\theta) + \frac{\lfloor(|x|+2\theta)N\rfloor-\lfloor Ny\rfloor}{2\sqrt{N}}\right|$ is smaller than the quantity

$$\max_{-\lfloor(|x|+2\theta)N\rfloor-N^{3/4}\le i\le\lfloor-(|x|+2\theta)N\rfloor}\frac{1}{\sqrt{N}}\left|\sum_{j=i}^{\lfloor-(|x|+2\theta)N\rfloor}\zeta_j\right| + \frac{2N^{1/16}}{\sqrt{N}} + \frac{2N^{1/3}}{\sqrt{N}}$$
$$\le \frac{N^{19/48}}{\sqrt{N}} + \frac{2N^{1/16}}{\sqrt{N}} + \frac{2N^{1/3}}{\sqrt{N}} \le 2N^{-5/48}$$

when N is large enough. This yields $\sup_{y\in\left[\frac{I^-}{N},-|x|-2\theta\right]}\left|Y_N^-(y) - Y_N^-(-|x|-2\theta) + \frac{\lfloor(|x|+2\theta)N\rfloor-\lfloor Ny\rfloor}{2\sqrt{N}}\right| \le 2N^{-5/48}$ when N is large enough.

We also need an explicit expression for the parametric representations. Assume the part of [0,1] devoted to the equivalent of Figure 2(b) in the parametric representations is $[a_N, a'_N]$. We set ϕ equal to the affine function mapping a_N to $-\frac{2I^-}{N}$ and a'_N to $-(|x|+2\theta)N$. Then, if $\phi(t)$ belongs to some $\left[\frac{2i}{N}, \frac{2i+1}{N}\right)$ with $i\in\{I^-,\dots,-\lfloor(|x|+2\theta)N\rfloor-1\}$, we set $\left(u_N^-(t), r_N^-(t)\right) = \left(\phi(t)-\frac{i}{N}, Y_N^-\left(\phi(t)-\frac{i}{N}\right)\right)$, while if $\phi(t)$ belongs to some $\left[\frac{2i+1}{N}, \frac{2i+2}{N}\right]$ for $i\in\{I^-,\dots,-\lfloor(|x|+2\theta)N\rfloor-1\}$, we set

$$\left(u_N^-(t), r_N^-(t)\right) = \left(\frac{i+1}{N}, (-N\phi(t)+2i+2)Y_N^-\left(\left(\frac{i+1}{N}\right)^-\right) + (N\phi(t)-2i-1)Y_N^-\left(\frac{i+1}{N}\right)\right).$$

In addition, we set $(u_N(t), r_N(t)) = (-|x|-2\theta, \hat{\phi}(\phi(t)))$, where $\hat{\phi}$ is the affine function mapping $-|x|-2\theta-\frac{\lfloor(|x|+2\theta)N\rfloor+1}{N}$ to $Y_N(-|x|-2\theta)$ and $\frac{2I^-}{N}$ to 0.

We recall that it is enough to prove $\left|r_N^-(t) - r_N(t)\right| < 2N^{-1/12}$. We are going to study $\left|r_N^-(t) - Y_N^-(-|x|-2\theta) + \frac{\sqrt{N}}{4}\phi(t) - \frac{\lfloor-(|x|+2\theta)N\rfloor}{2\sqrt{N}}\right|$. We first suppose that $\phi(t)\in\left[\frac{2i}{N}, \frac{2i+1}{N}\right)$ with $i\in\{I^-,\dots,-\lfloor(|x|+2\theta)N\rfloor-1\}$. In this case, $r_N^-(t) = Y_N^-\left(\phi(t)-\frac{i}{N}\right)$ and $\left|\frac{\phi(t)}{2} - \frac{1}{N}\lfloor N\left(\phi(t)-\frac{i}{N}\right)\rfloor\right| = \left|\frac{\phi(t)}{2} - \frac{i}{N}\right| \le \frac{1}{2N}$; hence

$$\left|r_N^-(t) - Y_N^-(-|x|-2\theta) + \frac{\sqrt{N}}{4}\phi(t) - \frac{\lfloor-(|x|+2\theta)N\rfloor}{2\sqrt{N}}\right|$$
$$= \left|Y_N^-\left(\phi(t)-\frac{i}{N}\right) - Y_N^-(-|x|-2\theta) + \frac{\lfloor N\left(\phi(t)-\frac{i}{N}\right)\rfloor - \lfloor-(|x|+2\theta)N\rfloor}{2\sqrt{N}} + \frac{\frac{N}{2}\phi(t) - \lfloor N\left(\phi(t)-\frac{i}{N}\right)\rfloor}{2\sqrt{N}}\right|$$
$$\le \left|Y_N^-\left(\phi(t)-\frac{i}{N}\right) - Y_N^-(-|x|-2\theta) + \frac{\lfloor N\left(\phi(t)-\frac{i}{N}\right)\rfloor - \lfloor-(|x|+2\theta)N\rfloor}{2\sqrt{N}}\right| + \frac{\sqrt{N}}{2}\left|\frac{\phi(t)}{2} - \frac{1}{N}\left\lfloor N\left(\phi(t)-\frac{i}{N}\right)\right\rfloor\right|$$

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is smaller than $2N^{-5/48}+\frac{1}{4\sqrt{N}}$, which implies

$$\left|r_N^-(t)-Y_N^-(-|x|-2\theta)+\frac{\sqrt{N}}{4}\phi(t)-\frac{\lfloor-(|x|+2\theta)N\rfloor}{2\sqrt{N}}\right|\leq 2N^{-5/48}+\frac{1}{4\sqrt{N}}.$$

We now consider the case $\phi(t)\in\left[\frac{2i+1}{N},\frac{2i+2}{N}\right]$ with $i\in\{I^-,\dots,-\lfloor(|x|+2\theta)N\rfloor-1\}$. We temporarily denote $N\phi(t)-2i-1$ by ε for short, with $\varepsilon\in[0,1]$. Then we have $r_N^-(t)=(1-\varepsilon)Y_N^-\left(\left(\frac{i+1}{N}\right)^-\right)+\varepsilon Y_N^-\left(\frac{i+1}{N}\right)$, $\left|\frac{\phi(t)}{2}-\frac{i}{N}\right|\leq\frac{1}{N}$, and $\left|\frac{\phi(t)}{2}-\frac{i+1}{N}\right|\leq\frac{1}{2N}$. Therefore,

$$\begin{aligned}
&\left|r_N^-(t)-Y_N^-(-|x|-2\theta)+\frac{\sqrt{N}}{4}\phi(t)-\frac{\lfloor-(|x|+2\theta)N\rfloor}{2\sqrt{N}}\right|\\
&=\Bigg|(1-\varepsilon)\left(Y_N^-\left(\left(\frac{i+1}{N}\right)^-\right)-Y_N^-(-|x|-2\theta)+\frac{\sqrt{N}}{4}\phi(t)-\frac{\lfloor-(|x|+2\theta)N\rfloor}{2\sqrt{N}}\right)\\
&\quad+\varepsilon\left(Y_N^-\left(\frac{i+1}{N}\right)-Y_N^-(-|x|-2\theta)+\frac{\sqrt{N}}{4}\phi(t)-\frac{\lfloor-(|x|+2\theta)N\rfloor}{2\sqrt{N}}\right)\Bigg|\\
&\leq(1-\varepsilon)\left|Y_N^-\left(\left(\frac{i+1}{N}\right)^-\right)-Y_N^-(-|x|-2\theta)+\frac{\sqrt{N}}{4}\phi(t)-\frac{\lfloor-(|x|+2\theta)N\rfloor}{2\sqrt{N}}\right|\\
&\quad+\varepsilon\left|Y_N^-\left(\frac{i+1}{N}\right)-Y_N^-(-|x|-2\theta)+\frac{\sqrt{N}}{4}\phi(t)-\frac{\lfloor-(|x|+2\theta)N\rfloor}{2\sqrt{N}}\right|\\
&\leq(1-\varepsilon)\left|Y_N^-\left(\left(\frac{i+1}{N}\right)^-\right)-Y_N^-(-|x|-2\theta)+\frac{i-\lfloor-(|x|+2\theta)N\rfloor}{2\sqrt{N}}\right|\\
&\quad+\varepsilon\left|Y_N^-\left(\frac{i+1}{N}\right)-Y_N^-(-|x|-2\theta)+\frac{i+1-\lfloor-(|x|+2\theta)N\rfloor}{2\sqrt{N}}\right|\\
&\quad+(1-\varepsilon)\left|\frac{\sqrt{N}}{4}\phi(t)-\frac{i}{2\sqrt{N}}\right|+\varepsilon\left|\frac{\sqrt{N}}{4}\phi(t)-\frac{i+1}{2\sqrt{N}}\right|\\
&\leq(1-\varepsilon)\sup_{y\in\left[\frac{I^-}{N},-|x|-2\theta\right]}\left|Y_N^-(y)-Y_N^-(-|x|-2\theta)+\frac{\lfloor Ny\rfloor-\lfloor-(|x|+2\theta)N\rfloor}{2\sqrt{N}}\right|\\
&\quad+\varepsilon\sup_{y\in\left[\frac{I^-}{N},-|x|-2\theta\right]}\left|Y_N^-(y)-Y_N^-(-|x|-2\theta)+\frac{\lfloor Ny\rfloor-\lfloor-(|x|+2\theta)N\rfloor}{2\sqrt{N}}\right|\\
&\quad+(1-\varepsilon)\frac{\sqrt{N}}{2}\left|\frac{\phi(t)}{2}-\frac{i}{N}\right|+\varepsilon\frac{\sqrt{N}}{2}\left|\frac{\phi(t)}{2}-\frac{i+1}{N}\right|\\
&\leq\sup_{y\in\left[\frac{I^-}{N},-|x|-2\theta\right]}\left|Y_N^-(y)-Y_N^-(-|x|-2\theta)+\frac{\lfloor Ny\rfloor-\lfloor-(|x|+2\theta)N\rfloor}{2\sqrt{N}}\right|+\frac{1}{2\sqrt{N}}\\
&\leq 2N^{-5/48}+\frac{1}{2\sqrt{N}}
\end{aligned}$$

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thanks to our bound on the supremum. Since this was also true for $\phi(t) \in \left[\frac{2i}{N}, \frac{2i+1}{N}\right)$ with $i \in \{I^-, \ldots, -\lfloor(|x|+2\theta)N\rfloor - 1\}$, we have

$$\left| r_N^-(t) - Y_N^-(-|x|-2\theta) + \frac{\sqrt{N}}{4}\phi(t) - \frac{\lfloor -(|x|+2\theta)N \rfloor}{2\sqrt{N}} \right| \le 2N^{-5/48} + \frac{1}{2\sqrt{N}}.$$

The latter expression yields

$$\begin{aligned} \left| r_N^-(t) - r_N(t) \right| &\le \left| r_N(t) - Y_N^-(-|x|-2\theta) + \frac{\sqrt{N}}{4}\phi(t) - \frac{\lfloor -(|x|+2\theta)N \rfloor}{2\sqrt{N}} \right| + 2N^{-5/48} + \frac{1}{2\sqrt{N}} \\ &= \left| \hat{\phi}(\phi(t)) - Y_N^-(-|x|-2\theta) + \frac{\sqrt{N}}{4}\phi(t) - \frac{\lfloor -(|x|+2\theta)N \rfloor}{2\sqrt{N}} \right| + 2N^{-5/48} + \frac{1}{2\sqrt{N}}, \end{aligned}$$

where $\hat{\phi}$ is the affine function mapping $-|x| - 2\theta - \frac{\lfloor(|x|+2\theta)N\rfloor+1}{N}$ to $Y_N(-|x|-2\theta)$ and $\frac{2I^-}{N}$ to 0. Therefore it is enough to prove

$$\left| \hat{\phi}(\phi(t)) - Y_N^-(-|x|-2\theta) + \frac{\sqrt{N}}{4}\phi(t) - \frac{\lfloor -(|x|+2\theta)N \rfloor}{2\sqrt{N}} \right| \le N^{-1/12} + \frac{1}{2\sqrt{N}}$$

to complete the proof. Now, $\hat{\phi}(\phi(t)) - Y_N^-(-|x|-2\theta) + \frac{\sqrt{N}}{4}\phi(t) - \frac{\lfloor -(|x|+2\theta)N \rfloor}{2\sqrt{N}}$ is an affine function of $\phi(t)$, so it is enough to prove the bound for $\phi(t) = -|x| - 2\theta - \frac{\lfloor(|x|+2\theta)N\rfloor+1}{N}$ and for $\phi(t) = \frac{2I^-}{N}$. We first consider $\phi(t) = \frac{2I^-}{N}$. By Lemma 6, $\ell^-(T_N, I^-) = 0$. Moreover, $I^- < -\lfloor(|x|+2\theta)N\rfloor$, and hence $Y_N^-\left(\frac{I^-}{N}\right) = 0$. We deduce that

$$\begin{aligned} &\left| \hat{\phi}(\phi(t)) - Y_N^-(-|x|-2\theta) + \frac{\sqrt{N}}{4}\phi(t) - \frac{\lfloor -(|x|+2\theta)N \rfloor}{2\sqrt{N}} \right| \\ &= \left| -Y_N^-(-|x|-2\theta) + \frac{\sqrt{N}}{4}\frac{2I^-}{N} - \frac{\lfloor -(|x|+2\theta)N \rfloor}{2\sqrt{N}} \right| \\ &= \left| Y_N^-\left(\frac{I^-}{N}\right) - Y_N^-(-|x|-2\theta) + \frac{I^- - \lfloor -(|x|+2\theta)N \rfloor}{2\sqrt{N}} \right| \\ &\le \sup_{y \in \left[\frac{I^-}{N}, -|x|-2\theta\right]} \left| Y_N^-(y) - Y_N^-(-|x|-2\theta) + \frac{\lfloor Ny \rfloor - \lfloor -(|x|+2\theta)N \rfloor}{2\sqrt{N}} \right| \le 2N^{-5/48}, \end{aligned}$$

which is enough. We now consider $\phi(t) = -|x| - 2\theta - \frac{\lfloor(|x|+2\theta)N\rfloor+1}{N}$. Then $\left| \hat{\phi}(\phi(t)) - Y_N^-(-|x|-2\theta) + \frac{\sqrt{N}}{4}\phi(t) - \frac{\lfloor -(|x|+2\theta)N \rfloor}{2\sqrt{N}} \right|$ is equal to

$$\begin{aligned} &\Big| Y_N(-|x|-2\theta) - Y_N^-(-|x|-2\theta) \\ &\qquad + \frac{\sqrt{N}}{4}\left(-|x| - 2\theta - \frac{\lfloor(|x|+2\theta)N\rfloor+1}{N}\right) - \frac{\lfloor -(|x|+2\theta)N \rfloor}{2\sqrt{N}} \Big| \end{aligned}$$

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$$\leq |Y_N(-|x|-2\theta) - Y_N^-(-|x|-2\theta)| + \left|\frac{1}{4\sqrt{N}}(-(|x|+2\theta)N - \lfloor(|x|+2\theta)N\rfloor - 1 - 2\lfloor-(|x|+2\theta)N\rfloor\right|$$

$$\leq |Y_N(-|x|-2\theta) - Y_N^-(-|x|-2\theta)| + \frac{1}{2\sqrt{N}} \leq N^{-1/12} + \frac{1}{2\sqrt{N}}$$

by Lemma 12, which completes the proof for $\mathcal{B}^c \cap \{I^- < -\lfloor(|x|+2\theta)N\rfloor\}$.

The argument to show that $\mathcal{B}^c \cap \{I^+ > \lfloor(|x|+2\theta)N\rfloor\} \subset \{\text{between } \frac{((|x|+2\theta)N)\wedge I^+}{N} \text{ and } a, \|u_N^- - u_N\|_\infty, \|r_N^- - r_N\|_\infty \leq 2N^{-1/12}\}$ is similar and simpler, except for the end of the argument, which we give here. In a similar way as in the previous case, we must bound

$$\left|Y_N((|x|+2\theta)^-) - Y_N^-(|x|+2\theta) + \frac{\sqrt{N}}{4}\left(|x|+2\theta+\frac{\lfloor(|x|+2\theta)N\rfloor}{N}\right) - \frac{\lfloor(|x|+2\theta)N\rfloor}{2\sqrt{N}}\right|$$

$$\leq \left|Y_N((|x|+2\theta)^-) - Y_N^-(|x|+2\theta)\right| + \left|\frac{(|x|+2\theta)N - \lfloor(|x|+2\theta)N\rfloor}{4\sqrt{N}}\right|$$

$$\leq \left|Y_N((|x|+2\theta)^-) - Y_N^-((|x|+2\theta)^-)\right| + \left|Y_N^-((|x|+2\theta)^-) - Y_N^-(|x|+2\theta)\right| + \frac{1}{4\sqrt{N}};$$

hence Lemma 12 yields

$$\left|Y_N((|x|+2\theta)^-) - Y_N^-(|x|+2\theta) + \frac{\sqrt{N}}{4}\left(|x|+2\theta+\frac{\lfloor(|x|+2\theta)N\rfloor}{N}\right) - \frac{\lfloor(|x|+2\theta)N\rfloor}{2\sqrt{N}}\right|$$

$$\leq N^{-1/12} + \left|Y_N^-((|x|+2\theta)^-) - Y_N^-(|x|+2\theta)\right| + \frac{1}{4\sqrt{N}}.$$

In addition, the definition of Y_N^- and (1) yield that if $(|x|+2\theta)N$ is not an integer, then $Y_N^-((|x|+2\theta)^-) = Y_N^-(|x|+2\theta)$, while if $(|x|+2\theta)N$ is an integer, then

$$\left|Y_N^-((|x|+2\theta)^-) - Y_N^-(|x|+2\theta)\right| = \frac{1}{\sqrt{N}}\left|\ell^-\big(T_N, (|x|+2\theta)N-1\big) - \ell^-\big(T_N, (|x|+2\theta)N\big)\right|$$

$$= \frac{1}{\sqrt{N}}\left|\eta_{(|x|+2\theta)N-1,+}\big(\ell^-\big(T_N, (|x|+2\theta)N-1\big)\big)\right|$$

$$\leq \frac{N^{1/16}+1/2}{\sqrt{N}},$$

since $(\mathcal{B}_2)^c$ occurs. In all cases we obtain $\left|Y_N^-((|x|+2\theta)^-) - Y_N^-(|x|+2\theta)\right| \leq \frac{N^{1/16}+1/2}{\sqrt{N}}$; therefore,

$$\left|Y_N((|x|+2\theta)^-) - Y_N^-(|x|+2\theta) + \frac{\sqrt{N}}{4}\left(|x|+2\theta+\frac{\lfloor(|x|+2\theta)N\rfloor}{N}\right) - \frac{\lfloor(|x|+2\theta)N\rfloor}{2\sqrt{N}}\right|$$

$$\leq N^{-1/12} + \frac{N^{1/16}+1/2}{\sqrt{N}} + \frac{1}{4\sqrt{N}},$$

which is a bound small enough to complete the proof of the lemma. □

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5. Convergence of the local times process: proof of Theorem 2 and Proposition 3

5.1. Proof of Theorem 2

Our aim is to prove that $Y_N^{\pm}$ converges in distribution to $\big(B_y^x \mathbb{1}_{\{y\in[-|x|-2\theta,|x|+2\theta)\}}\big)_{y\in\mathbb{R}}$ in the Skorokhod M_1 topology on $D(-\infty,+\infty)$ when N tends to $+\infty$. Proposition 5 yields that $Y_N^{\pm}$ is close to the function Y_N defined by $Y_N(y)=\frac{1}{\sqrt{N}}\sum_{i=\lfloor Ny\rfloor+1}^{\chi(N)-1}\zeta_i$ if $y\in\big[-|x|-2\theta,\frac{\chi(N)}{N}\big)$, $Y_N(y)=\frac{1}{\sqrt{N}}\sum_{i=\chi(N)}^{\lfloor Ny\rfloor-1}\zeta_i$ if $y\in\big[\frac{\chi(N)}{N},|x|+2\theta\big)$, and $Y_N(y)=0$ otherwise. One has the feeling that by Donsker's invariance principle, Y_N should converge to $\big(B_y^x\mathbb{1}_{\{y\in[-|x|-2\theta,|x|+2\theta)\}}\big)_{y\in\mathbb{R}}$, and so we should be able to conclude quickly, but rigorously proving the convergence in the Skorokhod M_1 topology on $D(-\infty,+\infty)$ is harder than it looks. We are instead going to use a similar argument with a new process Y''_N which will be 'like Y_N, but continuous in $[-|x|-2\theta,|x|+2\theta)$'. We define it as follows. We first set a process Y'_N thus: if $Ny\in\mathbb{Z}$, then $Y'_N(y)=\frac{1}{\sqrt{N}}\sum_{i=Ny+1}^{\chi(N)-1}\zeta_i$ if $y\in\big(-\infty,\frac{\chi(N)}{N}\big)$ and $Y'_N(y)=\frac{1}{\sqrt{N}}\sum_{i=\chi(N)}^{Ny-1}\zeta_i$ if $y\in\big[\frac{\chi(N)}{N},+\infty\big)$; in between, Y'_N is linearly interpolated. We then define Y''_N by $Y''_N(y)=Y'_N(y)\mathbb{1}_{\{y\in[-|x|-2\theta,|x|+2\theta)\}}$ for any $y\in\mathbb{R}$. Then Y''_N will converge to $\big(B_y^x\mathbb{1}_{\{y\in[-|x|-2\theta,|x|+2\theta)\}}\big)_{y\in\mathbb{R}}$ and be close to Y_N, as stated in the following two lemmas.

Lemma 16. *Y''_N converges to $\big(B_y^x\mathbb{1}_{\{y\in[-|x|-2\theta,|x|+2\theta)\}}\big)_{y\in\mathbb{R}}$ in distribution when N tends to $+\infty$ for the Skorokhod M_1 topology in $D(-\infty,\infty)$.*

Lemma 17. $\mathbb{P}\big(d_{M_1}\big(Y_N^{\pm},Y''_N\big)>N^{-7/16}\big)$ *tends to 0 when N tends to $+\infty$.*

Given these two lemmas, the proof of Theorem 2 is fairly standard. One may for example look at the end of the proof of the Donsker invariance principle in [8] (here Y''_N converges to the desired distribution instead of having it outright, but this convergence yields that the probability that Y''_N is in a closed set has the right limit). Thus we only have to prove Lemmas 16 and 17. In order to do this, we first need two easy lemmas which will also be used later in this work. If we denote by $C[-|x|-2\theta,|x|+2\theta]$ the space of continuous functions $[-|x|-2\theta,|x|+2\theta]\mapsto\mathbb{R}$, then since the $(\zeta_i)_{i\in\mathbb{Z}}$ are i.i.d. with law ρ_0, which is symmetric and so has zero mean, Donsker's invariance principle yields the following.

Lemma 18. *$Y'_N|_{[-|x|-2\theta,|x|+2\theta]}$ converges in distribution to $B^x|_{[-|x|-2\theta,|x|+2\theta]}$ when N tends to $+\infty$ for the topology defined on $C[-|x|-2\theta,|x|+2\theta]$ by the uniform norm.*

The following lemma is also easy to prove.

Lemma 19. *If $(\mathcal{B}_2)^c$ occurs, then* $\sup\{|Y_N(y)-Y''_N(y)|:y\in[-|x|-2\theta,|x|+2\theta)\}\le N^{-7/16}$.

Proof. By the definition of Y_N and Y''_N, we have

$$\begin{aligned}
&\sup\left\{\left|Y_N(y)-Y''_N(y)\right|:y\in[-|x|-2\theta,|x|+2\theta)\right\}\\
&\quad\le\frac{1}{\sqrt{N}}\sup\{|\zeta_i|:-(|x|+2\theta)N\le i\le(|x|+2\theta)N\},
\end{aligned}$$

which is smaller than $\frac{N^{1/16}}{\sqrt{N}}=N^{-7/16}$ if $(\mathcal{B}_2)^c$ occurs. □

We also need the following technical lemma in order to deduce results on the Skorokhod M_1 topology from Lemmas 18 and 19.

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Lemma 20. *Let $N > 0$, and let $Z_1, Z_2 \in D(-\infty, +\infty)$ be functions whose possible discontinuities belong to $\frac{1}{N}\mathbb{Z}$. Then*

$$d_{M_1}\big(\big(Z_1(y)\mathbb{1}_{\{y\in[-|x|-2\theta,|x|+2\theta)\}}\big)_{y\in\mathbb{R}}, \big(Z_2(y)\mathbb{1}_{\{y\in[-|x|-2\theta,|x|+2\theta)\}}\big)_{y\in\mathbb{R}}\big) \leq \sup\{|Z_1(y) - Z_2(y)| : y \in [-|x| - 2\theta, |x| + 2\theta)\}.$$

Proof. Lemma 20 can be proved by writing, for each $a \neq |x| + 2\theta$, parametric representations of the two processes on $[-a, a]$ 'following their completed graphs together' (one can find an explicit construction of such representations in the first arXiv version of this paper [5]). □

Lemma 20 will allow us to deduce Lemma 16 from Lemma 18, and Lemma 17 from Lemma 19 and Proposition 5, which will complete the proof of Theorem 2.

Proof of Lemma 16. Let $f : D(-\infty, +\infty) \mapsto \mathbb{R}$ be bounded and continuous with respect to the Skorokhod M_1 topology on $D(-\infty, +\infty)$. We need to prove that $\mathbb{E}\big(f\big(Y_N''\big)\big)$ converges to $\mathbb{E}\big(f\big(\big(B_y^x\mathbb{1}_{\{y\in[-|x|-2\theta,|x|+2\theta)\}}\big)_{y\in\mathbb{R}}\big)\big)$ when N tends to $+\infty$. We define $g : C[-|x| - 2\theta, |x| + 2\theta] \mapsto \mathbb{R}$ by $g(Z) = f\big(\big(Z(y)\mathbb{1}_{\{y\in[-|x|-2\theta,|x|+2\theta)\}}\big)_{y\in\mathbb{R}}\big)$ for any $Z \in C[-|x| - 2\theta, |x| + 2\theta]$. We then have $\mathbb{E}\big(f\big(Y_N''\big)\big) = \mathbb{E}\big(g\big(Y_N'|_{[-|x|-2\theta,|x|+2\theta]}\big)\big)$ and $\mathbb{E}\big(f\big(\big(B_y^x\mathbb{1}_{\{y\in[-|x|-2\theta,|x|+2\theta)\}}\big)_{y\in\mathbb{R}}\big)\big) = \mathbb{E}\big(g\big(B^x|_{[-|x|-2\theta,|x|+2\theta]}\big)\big)$; hence it is enough to prove that $\mathbb{E}\big(g\big(Y_N'|_{[-|x|-2\theta,|x|+2\theta]}\big)\big)$ converges to $\mathbb{E}\big(g\big(B^x|_{[-|x|-2\theta,|x|+2\theta]}\big)\big)$ when N tends to $+\infty$. Furthermore, Lemma 18 yields that $Y_N'|_{[-|x|-2\theta,|x|+2\theta]}$ converges in distribution to $B^x|_{[-|x|-2\theta,|x|+2\theta]}$ when N tends to $+\infty$ for the topology defined on $C[-|x| - 2\theta, |x| + 2\theta]$ by the uniform norm. Consequently, we only have to prove that g is continuous for this topology.

Let $(Z_k)_{k\in\mathbb{N}}$ be a sequence in $C[-|x| - 2\theta, |x| + 2\theta]$ converging uniformly to $Z \in C[-|x| - 2\theta, |x| + 2\theta]$ when k tends to $+\infty$. Then Lemma 20 states that for all $k \in \mathbb{N}$,

$$d_{M_1}\big(\big(Z_k(y)\mathbb{1}_{\{y\in[-|x|-2\theta,|x|+2\theta)\}}\big)_{y\in\mathbb{R}}, \big(Z(y)\mathbb{1}_{\{y\in[-|x|-2\theta,|x|+2\theta)\}}\big)_{y\in\mathbb{R}}\big) \leq \sup\{|Z_k(y) - Z(y)| : y \in [-|x| - 2\theta, |x| + 2\theta)\} \leq \|Z_k - Z\|_\infty.$$

Since the latter tends to 0 when k tends to $+\infty$, $(Z_k(y)\mathbb{1}_{\{y\in[-|x|-2\theta,|x|+2\theta)\}})_{y\in\mathbb{R}}$ converges to $\big(Z(y)\mathbb{1}_{\{y\in[-|x|-2\theta,|x|+2\theta)\}}\big)_{y\in\mathbb{R}}$ when k tends to $+\infty$ with respect to the Skorokhod M_1 topology on $D(-\infty, +\infty)$. Since f is continuous with respect to this topology, $(g(Z_k))_{k\in\mathbb{N}}$ converges to $g(Z)$ when k tends to $+\infty$. Consequently g is continuous for the topology defined on $C[-|x| - 2\theta, |x| + 2\theta]$ by the uniform norm, which completes the proof. □

Proof of Lemma 17. We have

$$\mathbb{P}\big(d_{M_1}\big(Y_N^\pm, Y_N''\big) > 4N^{-1/12}\big) \leq \mathbb{P}\big(d_{M_1}\big(Y_N^\pm, Y_N\big) > 3N^{-1/12}\big) + \mathbb{P}\big(d_{M_1}\big(Y_N, Y_N''\big) > N^{-7/16}\big)$$

when N is large enough. By Lemmas 19 and 20 we have $\mathbb{P}\big(d_{M_1}\big(Y_N, Y_N''\big) > N^{-7/16}\big) \leq \mathbb{P}(\mathcal{B}_2)$. Therefore $\mathbb{P}\big(d_{M_1}\big(Y_N^\pm, Y_N''\big) > 4N^{-1/12}\big) \leq \mathbb{P}\big(d_{M_1}\big(Y_N^\pm, Y_N\big) > 3N^{-1/12}\big) + \mathbb{P}(\mathcal{B}_2)$, which tends to 0 when N tends to $+\infty$ by Proposition 5 and Lemma 3. □

5.2. Proof of Proposition 3

Our goal is to prove that for any closed interval $I \in \mathbb{R}$ that does not contain $-|x| - 2\theta$ or $|x| + 2\theta$, the process $\big(Y_N^\pm(y)\big)_{y\in I}$ converges in distribution to $\big(B_y^x\mathbb{1}_{\{y\in[-|x|-2\theta,|x|+2\theta)\}}\big)_{y\in I}$ in the

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topology on $D\,I$ given by the uniform norm when N tends to $+\infty$. We first assume $I=[a,b]$ or $I=[a,+\infty)$ with $a>|x|+2\theta$ (the case $I=[a,b]$ or $I=(-\infty,b]$ with $b<-|x|-2\theta$ can be dealt with in the same way). We are going to prove that outside an event of small probability, $\big(Y_N^{\pm}(y)\big)_{y\in I}=0=\big(B_y^x\mathbb{1}_{\{y\in[-|x|-2\theta,|x|+2\theta)\}}\big)_{y\in I}$. For any $y\geq(|x|+2\theta)\vee\frac{I^+}{N}$, by Lemma 6 we have $\ell^{\pm}\big(T_N,\lfloor Ny\rfloor\big)=0$, and thus $Y_N^{\pm}(y)=0$. We deduce that as soon as $\frac{I^+}{N}\leq a$, we have $\big(Y_N^{\pm}(y)\big)_{y\in I}=0=\big(B_y^x\mathbb{1}_{\{y\in[-|x|-2\theta,|x|+2\theta)\}}\big)_{y\in I}$. In addition, when N is large enough we have $a\geq|x|+2\theta+N^{-1/4}$. Therefore, when N is large enough,

$$\mathbb{P}\Big(\big(Y_N^{\pm}(y)\big)_{y\in I}\neq\big(B_y^x\mathbb{1}_{\{y\in[-|x|-2\theta,|x|+2\theta)\}}\big)_{y\in I}\Big)\leq\mathbb{P}\big(|I^+-(|x|+2\theta)N|\geq N^{3/4}\big),$$

which tends to 0 when N tends to $+\infty$ by Lemma 7. This yields that $\big(Y_N^{\pm}(y)\big)_{y\in I}$ converges in distribution to $\big(B_y^x\mathbb{1}_{\{y\in[-|x|-2\theta,|x|+2\theta)\}}\big)_{y\in I}\big)$ in the topology on $D\,I$ given by the uniform norm.

We now deal with the case $I=[a,b]$ with $-|x|-2\theta<a<b<|x|+2\theta$. The idea is that we will be far from the problems at $-|x|-2\theta$ and $|x|+2\theta$, so that $Y_N^{\pm}$ will be close to Y_N' in all I, and Y_N' converges to the right limit, which means $Y_N^{\pm}$ does as well. We first prove the following lemma.

Lemma 21. *For any $-|x|-2\theta<a<b<|x|+2\theta$, we have that $\mathbb{P}\big(\|Y_N^{\pm}|_{[a,b]}-Y_N'|_{[a,b]}\|_\infty>2N^{-1/12}\big)$ tends to 0 when N tends to $+\infty$.*

Proof. We assume $(\mathcal{B}_2)^c$, $\big(\mathcal{B}_4^-\big)^c$, and $\big(\mathcal{B}_4^+\big)^c$ occur, as well as $|I^-+(|x|+2\theta)N|<N^{3/4}$, $|I^+-(|x|+2\theta)N|<N^{3/4}$. When N is large enough, we have $a\geq-|x|-2\theta+N^{-1/4}>\frac{I^-}{N}$ and $b\leq|x|+2\theta-N^{-1/4}<\frac{I^+}{N}$, so $[a,b]\subset\big(\frac{I^-}{N},\frac{I^+}{N}\big)$. Therefore, for any $y\in[a,b]$, Lemma 12 yields $\big|Y_N^{\pm}(y)-Y_N(y)\big|\leq N^{-1/12}$, and Lemma 19 gives $\big|Y_N(y)-Y_N'(y)\big|\leq N^{-7/16}$. Hence we get $\big|Y_N^{\pm}(y)-Y_N'(y)\big|\leq 2N^{-1/12}$, and we deduce $\big\|Y_N^{\pm}|_{[a,b]}-Y_N'|_{[a,b]}\big\|_\infty\leq 2N^{-1/12}$. This implies

$$\begin{aligned}&\mathbb{P}\Big(\big\|Y_N^{\pm}|_{[a,b]}-Y_N'|_{[a,b]}\big\|_\infty>2N^{-1/12}\Big)\\&\quad\leq\mathbb{P}\Big(\mathcal{B}_2\cup\mathcal{B}_4^-\cup\mathcal{B}_4^+\cup\big\{|I^-+(|x|+2\theta)N|\geq N^{3/4}\big\}\cup\big\{|I^+-(|x|+2\theta)N|\geq N^{3/4}\big\}\Big),\end{aligned}$$

which tends to 0 when N tends to $+\infty$ thanks to Lemmas 3, 7, and 11. □

Moreover, for any $-|x|-2\theta<a<b<|x|+2\theta$, by Donsker's invariance principle, $Y_N'|_{[a,b]}$ converges in distribution to $B^x|_{[a,b]}$ when N tends to $+\infty$ for the topology defined on $D[a,b]$ by the uniform norm. The proof of Proposition 3 from this is standard, as was the proof of Theorem 2 from Lemmas 16 and 17.

6. No convergence in the Skorokhod J_1 topology: proof of Proposition 2

In this section, our aim is to prove that $Y_N^{\pm}$ does not converge in distribution in the Skorokhod J_1 topology on $D(-\infty,+\infty)$ when N tends to $+\infty$. We will first prove that if $Y_N^{\pm}$ converges in the Skorokhod J_1 topology, then the limit has to be the same as in the Skorokhod M_1 topology, that is, $\big(B_y^x\mathbb{1}_{\{y\in[-|x|-2\theta,|x|+2\theta)\}}\big)_{y\in\mathbb{R}}$, by Theorem 2 (this will be Lemma 22). Afterwards, we will prove that $Y_N^{\pm}$ does not converge in distribution in the Skorokhod J_1 topology to $\big(B_y^x\mathbb{1}_{\{y\in[-|x|-2\theta,|x|+2\theta)\}}\big)_{y\in\mathbb{R}}$, by finding some closed set Ξ such that $\limsup_{N\to+\infty}\mathbb{P}\big(Y_N^{\pm}\in\Xi\big)>\mathbb{P}\Big(\big(B_y^x\mathbb{1}_{\{y\in[-|x|-2\theta,|x|+2\theta)\}}\big)_{y\in\mathbb{R}}\in\Xi\Big)$, which is enough by the portmanteau theorem.

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Lemma 22. *If $Y_N^{\pm}$ converges in distribution in the Skorokhod J_1 topology on $D(-\infty, +\infty)$ when N tends to $+\infty$, the limit is $\big(B_y^x \mathbb{1}_{\{y\in[-|x|-2\theta,|x|+2\theta)\}}\big)_{y\in\mathbb{R}}$.*

Proof. The idea is that the Skorokhod J_1 topology is stronger than the Skorokhod M_1 topology. We assume $Y_N^{\pm}$ converges in distribution to some Z in the Skorokhod J_1 topology on $D(-\infty, +\infty)$ when N tends to $+\infty$. It can be proven that for any $a > 0$ we have $d_{M_1,a} \le d_{J_1,-a,a}$. Indeed, this is Theorem 12.3.2 of [19], whose proof is in the internet supplement of that book (just replace the points of discontinuity of x_1 with their image by λ^{-1}). This implies $d_{M_1} \le d_{J_1}$. Therefore, a function $g : D(-\infty, +\infty) \mapsto \mathbb{R}$ that is bounded and continuous for the Skorokhod M_1 topology is also continuous for the Skorokhod J_1 topology. We deduce that $\mathbb{E}\big(g\big(Y_N^{\pm}\big)\big)$ converges to $\mathbb{E}(g(Z))$ when N tends to $+\infty$; thus $Y_N^{\pm}$ converges in distribution to Z in the Skorokhod M_1 topology when N tends to $+\infty$. By Theorem 2, the limit has to be $\big(B_y^x \mathbb{1}_{\{y\in[-|x|-2\theta,|x|+2\theta)\}}\big)_{y\in\mathbb{R}}$.

We now define our closed set Ξ. The idea behind this definition is that with high probability, $B^x_{|x|+2\theta}$ is at some distance from 0; hence, at some point around $|x| + 2\theta$, $Y_N^{\pm}$ will be close to $B^x_{|x|+2\theta}$, and therefore at some distance from 0. Furthermore, at $|x| + 2\theta$ the process $\big(B_y^x \mathbb{1}_{\{y\in[-|x|-2\theta,|x|+2\theta)\}}\big)_{y\in\mathbb{R}}$ will jump directly from $B^x_{|x|+2\theta}$ to 0, while $Y_N^{\pm}$, which can make only jumps of order $\frac{1}{\sqrt{N}}$, will have to cross the distance separating $B^x_{|x|+2\theta}$ from 0 without any big jumps. Therefore, if $\delta_1 > 0$ is much smaller than $B^x_{|x|+2\theta}$, then $Y_N^{\pm}(y)$ will enter the interval $[\delta_1, 2\delta_1]$ for y near $|x| + 2\theta$, while $\big(B_y^x \mathbb{1}_{\{y\in[-|x|-2\theta,|x|+2\theta)\}}\big)_{y\in\mathbb{R}}$ will not. We thus set Ξ to be roughly 'the function enters $[\delta_1, 2\delta_1]$ around $|x| + 2\theta$'. More rigorously, by the definition of B^x, the random variable $B^x_{|x|+2\theta}$ has distribution $\mathcal{N}(0, 2\theta)$, which means there exists $\delta_1 > 0$ such that $\mathbb{P}\big(\big|B^x_{|x|+2\theta}\big| \le 4\delta_1\big) \le 1/8$. Moreover, B^x is continuous; hence there exists $0 < \delta_2 < \theta$ such that $\mathbb{P}\big(\exists\, y \in \big[|x| + 2\theta - \delta_2, |x| + 2\theta\big], \big|B_y^x\big| \le 3\delta_1\big) \le 1/4$. We then define

$$\begin{aligned}\Xi = \{Z \in D(-\infty, +\infty) \mid \exists y \in [|x| + 2\theta - \delta_2, |x| + 2\theta + \delta_2],\\ |Z(y)| \in [\delta_1, 2\delta_1] \text{ or } |Z(y^-)| \in [\delta_1, 2\delta_1]\}\end{aligned}$$

(the inclusion of $Z(y^-)$ is necessary for Ξ to be closed). Then $\mathbb{P}\big(\big(B_y^x \mathbb{1}_{\{y\in[-|x|-2\theta,|x|+2\theta)\}}\big)_{y\in\mathbb{R}} \in \Xi\big) \le 1/4$. We will prove the following two lemmas.

Lemma 23. *When N is large enough, $\mathbb{P}\big(Y_N^{\pm} \in \Xi\big) \ge 1/2$.*

Lemma 24. *Ξ is closed in the Skorokhod J_1 topology on $D(-\infty, +\infty)$.*

With these two lemmas, the proof of Proposition 2 becomes easy.

Proof of Proposition 2. Lemma 23 yields $\limsup_{N\to+\infty} \mathbb{P}\big(Y_N^{\pm} \in \Xi\big) \ge 1/2$, and the definition of Ξ ensures that $\mathbb{P}\big(\big(B_y^x \mathbb{1}_{\{y\in[-|x|-2\theta,|x|+2\theta)\}}\big)_{y\in\mathbb{R}} \in \Xi\big) \le 1/4$; hence $\limsup_{N\to+\infty} \mathbb{P}\big(Y_N^{\pm} \in \Xi\big) > \mathbb{P}\big(\big(B_y^x \mathbb{1}_{\{y\in[-|x|-2\theta,|x|+2\theta)\}}\big)_{y\in\mathbb{R}} \in \Xi\big)$. Since Lemma 24 yields that Ξ is closed in the Skorokhod J_1 topology on $D(-\infty, +\infty)$, the portmanteau theorem implies that $Y_N^{\pm}$ does not converge in distribution in the Skorokhod J_1 topology on $D(-\infty, +\infty)$ to $\big(B_y^x \mathbb{1}_{\{y\in[-|x|-2\theta,|x|+2\theta)\}}\big)_{y\in\mathbb{R}}$ when N tends to $+\infty$. Hence Lemma 22 yields that $Y_N^{\pm}$ does not converge in distribution in the Skorokhod J_1 topology on $D(-\infty, +\infty)$ when N tends to $+\infty$, which is Proposition 2. □

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Thus it remains only to prove Lemmas 23 and 24.

Proof of Lemma 23. The idea is that with good probability, when y is a bit smaller than $|x|+2\theta$, we have that $Y_N^{\pm}(y)$ is of the same order as $B^x_{|x|+2\theta}$, and is thus away from 0, while when y is a bit larger than $|x|+2\theta$, we have $Y_N^{\pm}(y)=0$, so, since Y_N can only make jumps of order $\frac{1}{\sqrt{N}}$, it will enter $[\delta_1, 2\delta_1]$.

We now give the rigorous argument. We begin by assuming that $\left|Y_N^{\pm}(|x|+2\theta-\delta_2)\right|>3\delta_1$ (that is, $Y_N(y)$ is indeed away from 0 when y is a bit smaller than $|x|+2\theta$), that $(\mathcal{B}_2)^c$ occurs, and that $|I^+-(|x|+2\theta)N|<N^{3/4}$; and we prove that when N is large enough, $Y_N^{\pm}\in\Xi$. We first show that $Y_N^{\pm}(|x|+2\theta+\delta_2)=0$. When N is large enough, $\frac{I^+}{N}\leq|x|+2\theta+N^{-1/4}\leq|x|+2\theta+\delta_2$. Moreover, Lemma 6 implies $\ell^{\pm}\big(T_N, \lfloor Ny\rfloor\big)=0$ for any $y\geq\frac{I^+}{N}$, and hence for $y=|x|+2\theta+\delta_2$. This yields $Y_N^{\pm}(|x|+2\theta+\delta_2)=0$. Moreover, we assumed $\left|Y_N^{\pm}(|x|+2\theta-\delta_2)\right|>3\delta_1$. Equations (1) and (2) yield that the jumps of $Y_N^{\pm}$ in $[|x|+2\theta-\delta_2, |x|+2\theta+\delta_2]$ are either $\frac{1}{\sqrt{N}}\eta_{i,+}\big(\ell^-\big(T_N, i\big)\big)$ (if we are dealing with Y_N^-) or $\frac{1}{\sqrt{N}}\eta_{i+1,+}(\ell^-(T_N, i+1))$ (if we are dealing with Y_N^+), with $i\in\{\lfloor(|x|+2\theta-\delta_2)N\rfloor, \dots, \lfloor(|x|+2\theta+\delta_2)N\rfloor-1\}$. Since $(\mathcal{B}_2)^c$ occurs, the jumps of $Y_N^{\pm}$ in $[|x|+2\theta-\delta_2, |x|+2\theta+\delta_2]$ have size at most $\frac{1}{\sqrt{N}}\big(N^{1/16}+1/2\big)$, which tends to 0 when N tends to $+\infty$. Therefore, when N is large enough, there exists $y\in[|x|+2\theta-\delta_2, |x|+2\theta+\delta_2]$ such that $|Y_N^{\pm}(y)|\in[\delta_1, 2\delta_1]$; hence $Y_N^{\pm}\in\Xi$. Consequently, when N is large enough, if $\left|Y_N^{\pm}(|x|+2\theta-\delta_2)\right|>3\delta_1$, $(\mathcal{B}_2)^c$, and $|I^+-(|x|+2\theta)N|<N^{3/4}$ then $Y_N^{\pm}\in\Xi$. This implies that

$$\mathbb{P}\big(Y_N^{\pm}\notin\Xi\big)\leq\mathbb{P}\big(\left|Y_N^{\pm}(|x|+2\theta-\delta_2)\right|\leq 3\delta_1\big)+\mathbb{P}(\mathcal{B}_2)+\mathbb{P}\big(|I^+-(|x|+2\theta)N|\geq N^{3/4}\big).$$

In addition, Lemma 3 and Lemma 7 yield respectively that $\mathbb{P}(\mathcal{B}_2)$ and $\mathbb{P}\big(|I^+-(|x|+2\theta)N|\geq N^{3/4}\big)$ tend to 0 when N tends to $+\infty$. Therefore, it is enough to prove that $\mathbb{P}\big(\left|Y_N^{\pm}(|x|+2\theta-\delta_2)\right|\leq 3\delta_1\big)\leq 3/8$ when N is large enough to deduce that $\mathbb{P}(Y_N^{\pm}\notin\Xi)\leq 1/2$ when N is large enough and complete the proof of Lemma 23.

We now prove that $\mathbb{P}\big(\left|Y_N^{\pm}(|x|+2\theta-\delta_2)\right|\leq 3\delta_1\big)\leq 3/8$ when N is large enough, by noticing that $Y_N^{\pm}(|x|+2\theta-\delta_2)$ is close to $Y'_N(|x|+2\theta-\delta_2)$, which will converge in distribution to $B^x_{|x|+2\theta-\delta_2}$ when N tends to $+\infty$. Lemma 21 implies that $\mathbb{P}\big(\|Y_N^{\pm}|_{[0,|x|+2\theta-\delta_2]}-Y'_N|_{[0,|x|+2\theta-\delta_2]}\|_\infty>2N^{-1/12}\big)$ tends to 0 when N tends to $+\infty$; hence $\mathbb{P}\big(\left|Y_N^{\pm}(|x|+2\theta-\delta_2)-Y'_N(|x|+2\theta-\delta_2)\right|>2N^{-1/12}\big)$ tends to 0 when N tends to $+\infty$, which implies $Y_N^{\pm}(|x|+2\theta-\delta_2)-Y'_N(|x|+2\theta-\delta_2)$ converges in probability to 0 when N tends to $+\infty$. In addition, Lemma 18 states that $Y'_N|_{[-|x|-2\theta,|x|+2\theta]}$ converges in distribution to $B^x|_{[-|x|-2\theta,|x|+2\theta]}$ when N tends to $+\infty$ for the topology defined on $C[-|x|-2\theta, |x|+2\theta]$ by the uniform norm; hence $Y'_N(|x|+2\theta-\delta_2)$ converges in distribution to $B^x_{|x|+2\theta-\delta_2}$ when N tends to $+\infty$. Therefore, Slutsky's theorem yields that $Y_N^{\pm}(|x|+2\theta-\delta_2)$ converges in distribution to $B^x_{|x|+2\theta-\delta_2}$ when N tends to $+\infty$. Moreover, we defined Ξ so that $\mathbb{P}(\exists\, y\in[|x|+2\theta-\delta_2, |x|+2\theta], |B^x_y|\leq 3\delta_1)\leq 1/4$; hence $\mathbb{P}\big(\left|B^x_{|x|+2\theta-\delta_2}\right|\leq 3\delta_1\big)\leq 1/4$. This implies that when N is large enough, $\mathbb{P}\big(\left|Y_N^{\pm}(|x|+2\theta-\delta_2)\right|\leq 3\delta_1\big)\leq 3/8$. □

Proof of Lemma 24. Let $(Z_N)_{N\in\mathbb{N}}$ be a sequence of elements of Ξ converging to Z in the Skorokhod J_1 topology on $D(-\infty, +\infty)$; we will prove that $Z\in\Xi$. By taking a subsequence, we may assume $d_{J_1}(Z, Z_N)<e^{-|x|-2\theta-\delta_2-1}/N$ for any $N\in\mathbb{N}^*$. Then for any $N\in\mathbb{N}^*$, some $a_N>|x|+2\theta+\delta_2+1$ such that $d_{J_1,-a_N,a_N}\big(Z|_{[-a_N,a_N]}, Z_N|_{[-a_N,a_N]}\big)\leq 1/N$ will exist. Indeed,

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if this were not the case, for some N we would have

$$d_{J_1}(Z,Z_N)=\int_0^{+\infty}e^{-a}\big(d_{J_1,-a,a}\big(Z|_{[-a,a]},Z_N|_{[-a,a]}\big)\wedge 1\big)\mathrm{d}a \geq \int_{|x|+2\theta+\delta_2+1}^{+\infty}e^{-a}\frac{1}{N}\mathrm{d}a=e^{-|x|-2\theta-\delta_2-1}/N,$$

which does not happen. For all $N\in\mathbb{N}^*$, since $d_{J_1,-a_N,a_N}\big(Z|_{[-a_N,a_N]},Z_N|_{[-a_N,a_N]}\big)\leq 1/N$, there exists $\lambda_N\in\Lambda_{-a_N,a_N}$ with $\|Z|_{[-a_N,a_N]}\circ\lambda_N-Z_N|_{[-a_N,a_N]}\|_\infty\leq 2/N$ and $\|\lambda_N-\mathrm{Id}_{-a_N,a_N}\|_\infty\leq 2/N$. Moreover, $Z_N\in\Xi$; hence there exists $y_N\in[|x|+2\theta-\delta_2,|x|+2\theta+\delta_2]$ such that $|Z_N(y_N)|\in[\delta_1,2\delta_1]$ or $|Z_N\big(y_N^-\big)|\in[\delta_1,2\delta_1]$. We now define y_N' as follows: if $|Z_N(y_N)|\in[\delta_1,2\delta_1]$ we set $y_N'=y_N$. Otherwise, since $|Z_N\big(y_N^-\big)|\in[\delta_1,2\delta_1]$, we can take some y_N' in $\big[y_N-\frac{1}{N},y_N\big]$ such that $\big|Z_N\big(y_N'\big)\big|\in[\delta_1-1/N,2\delta_1+1/N]$. In both cases, we have $y_N'\in[|x|+2\theta-\delta_2-1/N,|x|+2\theta+\delta_2]$ and $\big|Z_N\big(y_N'\big)\big|\in[\delta_1-1/N,2\delta_1+1/N]$. Furthermore, $\|\lambda_N-\mathrm{Id}_{-a_N,a_N}\|_\infty\leq 2/N$; hence $|\lambda_N\big(y_N'\big)-y_N'|\leq 2/N$, and thus $\lambda_N\big(y_N'\big)\in[|x|+2\theta-\delta_2-3/N,|x|+2\theta+\delta_2+2/N]$. In addition, $\big\|Z\big(\lambda_N\big(y_N'\big)\big)-Z_N\big(y_N'\big)\big\|_\infty\leq 2/N$; hence $\big|Z\big(\lambda_N\big(y_N'\big)\big)\big|\in[\delta_1-3/N,2\delta_1+3/N]$. By taking a subsequence, we may assume that $\lambda_N\big(y_N'\big)$ converges to some $y_\infty\in[|x|+2\theta-\delta_2,|x|+2\theta+\delta_2]$. In addition, Z is càdlàg, so there is a subsequence of $\big(Z\big(\lambda_N\big(y_N'\big)\big)\big)_{N\in\mathbb{N}^*}$ that converges to either $Z(y_\infty)$ or $Z\big(y_\infty^-\big)$. Since $\big|Z\big(\lambda_N\big(y_N'\big)\big)\big|\in[\delta_1-3/N,2\delta_1+3/N]$, we have $|Z(y_\infty)|$ or $\big|Z\big(y_\infty^-\big)\big|$ in $[\delta_1,2\delta_1]$. Therefore $Z\in\Xi$, which completes the proof. □

7. Convergence of the stopping time: proof of Proposition 4

We want to prove Proposition 4, that is, the convergence in distribution of $\frac{1}{N^{3/2}}\big(T_N-N^2(|x|+2\theta)^2\big)$ to the law $\mathcal{N}\big(0,\frac{32}{3}\mathrm{Var}(\rho_-)\big((|x|+\theta)^3+\theta^3\big)\big)$ when N tends to $+\infty$. In order to do that, we will prove that $\frac{1}{N^{3/2}}\big(T_N-N^2(|x|+2\theta)^2\big)$ is close to $2\int_{-|x|-2\theta}^{|x|+2\theta}Y_N'(y)\mathrm{d}y$ (where Y_N' was defined at the beginning of Section 5.1), then that $2\int_{-|x|-2\theta}^{|x|+2\theta}Y_N'(y)\mathrm{d}y$ converges to the desired distribution.

Proposition 7. *We have that*

$$\mathbb{P}\left(\left|\frac{1}{N^{3/2}}\big(T_N-N^2(|x|+2\theta)^2\big)-2\int_{-|x|-2\theta}^{|x|+2\theta}Y_N'(y)\mathrm{d}y\right|>5(|x|+2\theta)N^{-1/12}\right)$$

tends to 0 when N tends to $+\infty$.

Proof. The result will come from the fact that T_N can be written as the sum of the local times, which is itself related to the integrals of Y_N^- and Y_N^+, which are close to Y_N by Lemma 12 and hence to Y_N' by Lemma 19. It is enough to prove that if $(\mathcal{B}_2)^c$, $\big(\mathcal{B}_4^-\big)^c$, and $\big(\mathcal{B}_4^+\big)^c$ occur and if $|I^-+(|x|+2\theta)N|<N^{5/8}$, $|I^+-(|x|+2\theta)N|<N^{5/8}$, then

$$\left|\frac{1}{N^{3/2}}\big(T_N-N^2(|x|+2\theta)^2\big)-2\int_{-|x|-2\theta}^{|x|+2\theta}Y_N'(y)\mathrm{d}y\right|\leq 5(|x|+2\theta)N^{-1/12},$$

since Lemma 3 implies that $\mathbb{P}(\mathcal{B}_2)$ tends to 0 when N tends to $+\infty$, Lemma 11 implies that $\mathbb{P}\big(\mathcal{B}_4^-\big)$ and $\mathbb{P}\big(\mathcal{B}_4^+\big)$ tend to 0 when N tends to $+\infty$, and Lemma 7 implies that $\mathbb{P}\big(|I^-+(|x|+2\theta)N|\geq N^{5/8}\big)$ and $\mathbb{P}\big(|I^+-(|x|+2\theta)N|\geq N^{5/8}\big)$ tend to 0 when N tends to $+\infty$. We assume that $(\mathcal{B}_2)^c$, $\left(\mathcal{B}_4^-\right)^c$, and $\left(\mathcal{B}_4^+\right)^c$ occur and that $|I^- + (|x| + 2\theta)N| < N^{5/8}$, $|I^+ - (|x| + 2\theta)N| < N^{5/8}$. Let us study T_N.

In order to do this, we first need to prove an auxiliary result—more precisely, that the following holds when N is large enough:

$$\begin{gathered}\text{if } |i - I^+| \leq N^{5/8} + 1 \text{ or } |i - I^-| \leq N^{5/8} + 1,\\ \text{then } \ell^+(T_N, i) \leq 4N^{11/16} \text{ and } \ell^-(T_N, i) \leq 4N^{11/16}.\end{gathered} \tag{6}$$

We prove (6) for the case $|i - I^-| \leq N^{5/8} + 1$, since the other case is similar. Let $i \in \mathbb{Z}$ so that $|i - I^-| < N^{5/8} + 1$. We notice that since $|I^- + (|x| + 2\theta)N| < N^{5/8}$ we have I^-, $i < 0$ when N is large enough, so (1) yields

$$|\ell^+(T_N, i) - \ell^+(T_N, I^-)| \leq \sum_{|j-I^-|<N^{5/8}+1} \left|\eta_{j,-}\left(\ell^+\left(T_N, j\right)\right)\right|;$$

thus, since $\ell^+(T_N, I^-) = 0$, we have

$$\ell^+(T_N, i) \leq \sum_{|j-I^-|<N^{5/8}+1} \left|\eta_{j,-}\left(\ell^+\left(T_N, j\right)\right)\right|.$$

In addition, we assumed $(\mathcal{B}_2)^c$; hence

$$\ell^+(T_N, i) \leq \sum_{|j-I^-|<N^{5/8}+1} \left(N^{1/16} + 1/2\right) \leq 3N^{5/8}N^{1/16} = 3N^{11/16}$$

when N is large enough. Furthermore, (2) implies

$$\left|\ell^-(T_N, i) - \ell^+(T_N, i)\right| = \left|\eta_{i,-}\left(\ell^+(T_N, i)\right)\right| \leq N^{1/16} + 1/2$$

thanks to $(\mathcal{B}_2)^c$, and hence $\ell^-(T_N, i) \leq 3N^{11/16} + N^{1/16} + 1/2 \leq 4N^{11/16}$ when N is large enough, which completes the proof of (6).

We now write T_N as the sum of the local times and relate $\frac{1}{N^{3/2}}\left(T_N - N^2(|x| + 2\theta)^2\right)$ to the integral of Y^+ and Y^-. We have $T_N = \sum_{i\in\mathbb{Z}} (\ell^+(T_N, i) + \ell^-(T_N, i))$. Moreover, Lemma 6 implies that for all $i \geq I^+$ and $i \leq I^-$ we have $\ell^+(T_N, i) = \ell^-(T_N, i) = 0$. Consequently,

$$T_N = \sum_{i=I^-\wedge(-\lfloor(|x|+2\theta)N\rfloor)}^{I^+\vee\lfloor(|x|+2\theta)N\rfloor} \left(\ell^+(T_N, i) + \ell^-(T_N, i)\right).$$

We thus have

$$\begin{aligned}&\left|\frac{1}{N^{3/2}}\left(T_N - N^2(|x| + 2\theta)^2\right) - \int_{(I^-\wedge(-(|x|+2\theta)N))/N}^{(I^+\vee(|x|+2\theta)N)/N} \left(Y_N^+(y) + Y_N^-(y)\right)\mathrm{d}y\right|\\ &\quad\leq \frac{1}{N^{3/2}}\Big(\ell^+\left(T_N, I^+ \vee \lfloor(|x| + 2\theta)N\rfloor\right) + \ell^-\left(T_N, I^+ \vee \lfloor(|x| + 2\theta)N\rfloor\right)\\ &\qquad + \ell^+\left(T_N, -\lfloor(|x| + 2\theta)N\rfloor - 1\right) + \ell^-\left(T_N, -\lfloor(|x| + 2\theta)N\rfloor - 1\right)\Big).\end{aligned}$$

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Since we assumed $|I^- + (|x| + 2\theta)N| < N^{5/8}$ and $|I^+ - (|x| + 2\theta)N| < N^{5/8}$, Equation (6) yields

$$\left|\frac{1}{N^{3/2}}\big(T_N - N^2(|x|+2\theta)^2\big) - \int_{(I^-\wedge(-(|x|+2\theta)N))/N}^{(I^+\vee(|x|+2\theta)N)/N}\big(Y_N^+(y)+Y_N^-(y)\big)\mathrm{d}y\right| \\ \leq \frac{1}{N^{3/2}}16N^{11/16} = 16N^{-13/16}. \tag{7}$$

We now prove that

$$\int_{(I^-\wedge(-(|x|+2\theta)N))/N}^{(I^+\vee(|x|+2\theta)N)/N}\big(Y_N^+(y)+Y_N^-(y)\big)\mathrm{d}y$$

is close to $2\int_{-(|x|+2\theta)}^{|x|+2\theta} Y_N(y)\mathrm{d}y$. We begin by considering

$$\int_{\chi(N)/N}^{(I^+\vee(|x|+2\theta)N)/N}\big(Y_N^+(y)+Y_N^-(y)\big)\mathrm{d}y.$$

We first assume $I^+ \geq (|x|+2\theta)N$. Since we assumed that $(\mathcal{B}_2)^c$, $\left(\mathcal{B}_4^-\right)^c$, and $\left(\mathcal{B}_4^+\right)^c$ occur, Lemma 12 yields

$$\left|\int_{\chi(N)/N}^{(I^+\vee(|x|+2\theta)N)/N}\big(Y_N^+(y)+Y_N^-(y)\big)\mathrm{d}y - 2\int_{\chi(N)/N}^{|x|+2\theta} Y_N(y)\mathrm{d}y\right| \\ \leq 2\left(|x|+2\theta-\frac{\chi(N)}{N}\right)N^{-1/12} + \int_{|x|+2\theta}^{I^+/N}|Y_N^+(y)+Y_N^-(y))|\mathrm{d}y.$$

In addition, we know $I^+ - (|x|+2\theta)N \leq N^{5/8}$ and (6); hence

$$\int_{|x|+2\theta}^{I^+/N}|Y_N^+(y)+Y_N^-(y))|\mathrm{d}y \\ \leq N^{-3/8}\frac{1}{\sqrt{N}}\left(\max_{\lfloor(|x|+2\theta)N\rfloor\leq i\leq I^+}\ell^+(T_N,i) + \max_{\lfloor(|x|+2\theta)N\rfloor\leq i\leq I^+}\ell^-(T_N,i)\right) \\ \leq N^{-3/8}N^{-1/2}8N^{11/16} = 8N^{-3/16}.$$

We deduce that

$$\left|\int_{\chi(N)/N}^{(I^+\vee(|x|+2\theta)N)/N}\big(Y_N^+(y)+Y_N^-(y)\big)\mathrm{d}y - 2\int_{\chi(N)/N}^{|x|+2\theta} Y_N(y)\mathrm{d}y\right| \\ \leq 2\left(|x|+2\theta-\frac{\chi(N)}{N}\right)N^{-1/12} + 8N^{-3/16}.$$

We now assume $I^+ < (|x|+2\theta)N$. In this case, we have

$$\left|\int_{\chi(N)/N}^{(I^+\vee(|x|+2\theta)N)/N}\big(Y_N^+(y)+Y_N^-(y)\big)\mathrm{d}y - 2\int_{\chi(N)/N}^{|x|+2\theta} Y_N(y)\mathrm{d}y\right| \\ \leq \int_{\chi(N)/N}^{I^+/N}\left|Y_N^+(y)+Y_N^-(y)-2Y_N(y)\right|\mathrm{d}y + \int_{I^+/N}^{|x|+2\theta}\left|Y_N^+(y)+Y_N^-(y)\right|\mathrm{d}y + \int_{I^+/N}^{|x|+2\theta}|2Y_N(y)|\mathrm{d}y.$$

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Moreover, Lemma 12 yields

$$\int_{\chi(N)/N}^{I^+/N} \left|Y_N^+(y)+Y_N^-(y)-2Y_N(y)\right|\mathrm{d}y \leq 2\left(|x|+2\theta-\frac{\chi(N)}{N}\right)N^{-1/12}.$$

Furthermore, for $y \geq \frac{I^+}{N}$ we have $\ell^{\pm}(T_N, \lfloor Ny \rfloor)=0$. Since $|I^+-(|x|+2\theta)N|<N^{5/8}$, this yields $\left|Y_N^{\pm}(y)\right| \leq \frac{1}{2}N^{1/8}$. Thus

$$\int_{I^+/N}^{|x|+2\theta} \left|Y_N^+(y)+Y_N^-(y)\right|\mathrm{d}y \leq \int_{I^+/N}^{|x|+2\theta} N^{1/8}\mathrm{d}y \leq N^{-3/8}N^{1/8}=N^{-1/4}.$$

We deduce that

$$\begin{aligned}&\left|\int_{\chi(N)/N}^{(I^+\vee(|x|+2\theta)N)/N} \left(Y_N^+(y)+Y_N^-(y)\right)\mathrm{d}y - 2\int_{\chi(N)/N}^{|x|+2\theta} Y_N(y)\mathrm{d}y\right|\\ &\quad\leq 2\left(|x|+2\theta-\frac{\chi(N)}{N}\right)N^{-1/12}+N^{-1/4}+\int_{I^+/N}^{|x|+2\theta}|2Y_N(y)|\mathrm{d}y.\end{aligned}$$

In addition, for any $y\in[\frac{I^+}{N}, |x|+2\theta]$, we have

$$|Y_N(y)| \leq \left|Y_N(y)-Y_N\left(\frac{I^+}{N}\right)\right|+\left|Y_N\left(\frac{I^+}{N}\right)-Y_N^-\left(\frac{I^+}{N}\right)\right|+\left|Y_N^-\left(\frac{I^+}{N}\right)\right|.$$

Lemma 12 yields that $\left|Y_N\left(\frac{I^+}{N}\right)-Y_N^-\left(\frac{I^+}{N}\right)\right| \leq N^{-1/12}$, and since $|I^+-(|x|+2\theta)N|<N^{5/8}$ we have

$$\left|Y_N^-\left(\frac{I^+}{N}\right)\right|=\left|\frac{1}{\sqrt{N}}\left(\ell^{\pm}(T_N, I^+)-N\left(\frac{|x|-|I^+/N|}{2}+\theta\right)_+\right)\right| \leq \frac{1}{2}N^{1/8};$$

hence

$$\begin{aligned}|Y_N(y)| &\leq \left|Y_N(y)-Y_N\left(\frac{I^+}{N}\right)\right|+N^{-1/12}+\frac{1}{2}N^{1/8}=\frac{1}{\sqrt{N}}\left|\sum_{i=I^+}^{\lfloor Ny\rfloor-1}\zeta_i\right|+N^{-1/12}+\frac{1}{2}N^{1/8}\\ &\leq \frac{1}{\sqrt{N}}\sum_{i=I^+}^{\lfloor(|x|+2\theta)N\rfloor-1}|\zeta_i|+N^{-1/12}+\frac{1}{2}N^{1/8} \leq \frac{1}{\sqrt{N}}N^{5/8}N^{1/16}+N^{-1/12}+\frac{1}{2}N^{1/8} \leq 2N^{3/16}\end{aligned}$$

since $(\mathcal{B}_2)^c$ occurs. Thus

$$\int_{I^+/N}^{|x|+2\theta}|2Y_N(y)|\mathrm{d}y \leq \int_{I^+/N}^{|x|+2\theta} 4N^{3/16}\mathrm{d}y = N^{-3/8}4N^{3/16}=4N^{-3/16}.$$

We deduce that

$$\begin{aligned}&\left|\int_{\chi(N)/N}^{(I^+\vee(|x|+2\theta)N)/N} \left(Y_N^+(y)+Y_N^-(y)\right)\mathrm{d}y - 2\int_{\chi(N)/N}^{|x|+2\theta} Y_N(y)\mathrm{d}y\right|\\ &\quad\leq 2\left(|x|+2\theta-\frac{\chi(N)}{N}\right)N^{-1/12}+5N^{-3/16}.\end{aligned}$$

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Consequently, in all cases,

$$\left|\int_{\chi(N)/N}^{(I^+\vee(|x|+2\theta)N)/N}\big(Y_N^+(y)+Y_N^-(y)\big)\mathrm{d}y-2\int_{\chi(N)/N}^{|x|+2\theta}Y_N(y)\mathrm{d}y\right|$$
$$\leq 2\left(|x|+2\theta-\frac{\chi(N)}{N}\right)N^{-1/12}+8N^{-3/16}.$$

One can prove similarly that

$$\left|\int_{(I^-\wedge(-(|x|+2\theta)N))/N}^{\chi(N)/N}\big(Y_N^+(y)+Y_N^-(y)\big)\mathrm{d}y-2\int_{-(|x|+2\theta)}^{\chi(N)/N}Y_N(y)\mathrm{d}y\right|$$
$$\leq 2\left(|x|+2\theta+\frac{\chi(N)}{N}\right)N^{-1/12}+8N^{-3/16}.$$

We conclude that

$$\left|\int_{(I^-\wedge(-(|x|+2\theta)N))/N}^{(I^+\vee(|x|+2\theta)N)/N}\big(Y_N^+(y)+Y_N^-(y)\big)\mathrm{d}y-2\int_{-(|x|+2\theta)}^{|x|+2\theta}Y_N(y)\mathrm{d}y\right|$$
$$\leq 4(|x|+2\theta)N^{-1/12}+16N^{-3/16}.$$

We are now in position to conclude. Indeed, the previous result and (7) imply that when N is large enough, $|\frac{1}{N^{3/2}}\big(T_N-N^2(|x|+2\theta)^2\big)-2\int_{-(|x|+2\theta)}^{|x|+2\theta}Y_N(y)\mathrm{d}y|\leq 16N^{-13/16}+4(|x|+2\theta)N^{-1/12}+16N^{-3/16}$. Moreover, $(\mathcal{B}_2)^c$ occurs, so Lemma 19 yields $\sup\{\big|Y_N(y)-Y_N'(y)\big|:y\in[-|x|-2\theta,\,|x|+2\theta)\}\leq N^{-7/16}$; therefore

$$\left|\int_{-(|x|+2\theta)}^{|x|+2\theta}Y_N(y)\mathrm{d}y-\int_{-(|x|+2\theta)}^{|x|+2\theta}Y_N'(y)\mathrm{d}y\right|\leq 2(|x|+2\theta)N^{-7/16}.$$

We deduce that when N is large enough,

$$\left|\frac{1}{N^{3/2}}\big(T_N-N^2(|x|+2\theta)^2\big)-2\int_{-(|x|+2\theta)}^{|x|+2\theta}Y_N'(y)\mathrm{d}y\right|$$
$$\leq 16N^{-13/16}+4(|x|+2\theta)N^{-1/12}+16N^{-3/16}+4(|x|+2\theta)N^{-7/16}$$
$$\leq 5(|x|+2\theta)N^{-1/12},$$

which completes the proof. □

Now that we know $\frac{1}{N^{3/2}}\big(T_N-N^2(|x|+2\theta)^2\big)$ is close to $2\int_{-|x|-2\theta}^{|x|+2\theta}Y_N'(y)\mathrm{d}y$, we need to prove that $2\int_{-|x|-2\theta}^{|x|+2\theta}Y_N'(y)\mathrm{d}y$ converges to the desired distribution. In order to do so, we will use the convergence of Y_N' to a Brownian motion as stated in Lemma 18; thus $2\int_{-|x|-2\theta}^{|x|+2\theta}Y_N'(y)\mathrm{d}y$ will converge to the integral of a Brownian motion. The law of the latter is characterized by the following lemma, where we denote by $(B_t)_{t\in\mathbb{R}^+}$ a standard Brownian motion with $B_0=0$. This lemma is quite standard (the interested reader can find a proof in the first arXiv version of this paper [5]).

Lemma 25. *For any $y>0$, the integral $\int_0^y B_z\mathrm{d}z$ has distribution $\mathcal{N}\left(0,\,\frac{y^3}{3}\right)$.*

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We are now able to prove Proposition 4.

Proof of Proposition 4. Proposition 7 implies that the quantity $\frac{1}{N^{3/2}}\left(T_N - N^2(|x|+2\theta)^2\right) - 2\int_{-|x|-2\theta}^{|x|+2\theta} Y_N'(y)\mathrm{d}y$ converges in probability to 0 when N tends to $+\infty$. Hence, by Slutsky's theorem, to prove Proposition 4, it is enough to prove that $2\int_{-|x|-2\theta}^{|x|+2\theta} Y_N'(y)\mathrm{d}y$ converges in distribution to $\mathcal{N}\left(0,\ \mathrm{Var}(\rho_-)\frac{32}{3}\left((|x|+\theta)^3+\theta^3\right)\right)$ when N tends to $+\infty$. In addition, by Lemma 18, $Y_N'|_{[-|x|-2\theta,|x|+2\theta]}$ converges in distribution to $B^x|_{[-|x|-2\theta,|x|+2\theta]}$ when N tends to $+\infty$, for the topology defined on $C[-|x|-2\theta, |x|+2\theta]$ by the uniform norm. The integral between $-|x|-2\theta$ and $|x|+2\theta$ is continuous for this topology, so $\int_{-|x|-2\theta}^{|x|+2\theta} Y_N'(y)\mathrm{d}y$ converges in distribution to $\int_{-|x|-2\theta}^{|x|+2\theta} B_y^x\mathrm{d}y$ when N tends to $+\infty$. Furthermore, B^x is a two-sided Brownian motion with $B_x^x=0$ and variance $\mathrm{Var}(\rho_-)$, which means we can write

$$\int_{-|x|-2\theta}^{|x|+2\theta} B_y^x\mathrm{d}y = \int_{-|x|-2\theta}^{x} B_y^x\mathrm{d}y + \int_{x}^{|x|+2\theta} B_y^x\mathrm{d}y$$

where $\int_{-|x|-2\theta}^{x} B_y^x\mathrm{d}y$ and $\int_{x}^{|x|+2\theta} B_y^x\mathrm{d}y$ are independent. In addition, $\int_{x}^{|x|+2\theta} B_y^x\mathrm{d}y$ has the distribution of $\sqrt{\mathrm{Var}(\rho_-)}\int_0^{2\theta} B_y\mathrm{d}y$, which we know is $\mathcal{N}\left(0,\ \mathrm{Var}(\rho_-)\frac{(2\theta)^3}{3}\right)$ by Lemma 25, and $\int_{-|x|-2\theta}^{x} B_y^x\mathrm{d}y$ has the distribution of $\sqrt{\mathrm{Var}(\rho_-)}\int_0^{2|x|+2\theta} B_y\mathrm{d}y$, which is $\mathcal{N}\left(0,\ \mathrm{Var}(\rho_-)\frac{(2|x|+2\theta)^3}{3}\right)$ by Lemma 25. We obtain that $\int_{-|x|-2\theta}^{|x|+2\theta} B_y^x\mathrm{d}y$ has the distribution

$$\mathcal{N}\left(0,\ \mathrm{Var}(\rho_-)\frac{(2|x|+2\theta)^3}{3} + \mathrm{Var}(\rho_-)\frac{(2\theta)^3}{3}\right) = \mathcal{N}\left(0,\ \mathrm{Var}(\rho_-)\frac{8}{3}\left((|x|+\theta)^3+\theta^3\right)\right).$$

Consequently, $\int_{-|x|-2\theta}^{|x|+2\theta} Y_N'(y)\mathrm{d}y$ converges in distribution to $\mathcal{N}\left(0,\ \mathrm{Var}(\rho_-)\frac{8}{3}\left((|x|+\theta)^3+\theta^3\right)\right)$ when N tends to $+\infty$, which completes the proof of Proposition 4.

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