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ABUNDANT REES MATRIX SEMIGROUPS

M. V. LAWSON

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Abstract

The class of abundant semigroups originally arose from 'homological' considerations in the theory of S-systems: they are the semigroup theoretic counterparts of PP-rings. Cancellative monoids, full subsemigroups of regular semigroups as well as the multiplicative semigroups of PP-rings are abundant. In this paper we investigate the properties of Rees matrix semigroups over abundant semigroups. Some of our results generalise McAlister's work on regular Rees matrix semigroups.

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The relations $\mathscr{L}^*$ and $\mathscr{R}^*$ on a semigroup S are generalisations of the familiar Green's relations $\mathscr{L}$ and $\mathscr{R}$. Two elements a and b in S are said to be $\mathscr{L}^*$*-related* if and only if they are related in some oversemigroup of S, the relation $\mathscr{R}^*$ being defined dually. A semigroup is called *left* (*right*) *abundant* if each $\mathscr{R}^*(\mathscr{L}^*)$-class contains an idempotent, and *abundant* if it is both left and right abundant. See Fountain [4] for a discussion of abundant semigroups.

Regular semigroups are abundant, and, in this case, $\mathscr{L}^* = \mathscr{L}$ and $\mathscr{R}^* = \mathscr{R}$. Inverse semigroups belong to the class of abundant semigroups whose idempotents constitute a semilattice, namely the class of *adequate semigroups* introduced by Fountain in [3]. Much work has been devoted to finding analogues of theorems concerning regular semigroups for certain classes of abundant semigroup: witness Armstrong's paper [1] generalising Schein's work on inverse semigroups to a class of adequate semigroups called type A.

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Recently, considerable attention has been paid to investigating Rees matrix semigroups over semigroups with certain desirable properties (consult Meakin's survey article [11] for an excellent discussion of these ideas). The structure of abundant semigroups all of whose idempotents are primitive has been completely described by Fountain [4] in terms of Rees matrix semigroups. McAlister [10] showed that the regular elements of a Rees matrix semigroup over a regular semigroup form a subsemigroup called a *regular Rees matrix semigroup*. It follows readily that a regular Rees matrix semigroup over an inverse semigroup is also locally inverse.

In this paper we show that we can define the abundant part of a Rees matrix semigroup over an abundant semigroup, which is itself a semigroup. In the particular case when the Rees matrix is taken over a type A semigroup, the resulting abundant part, which we call an *abundant Rees matrix semigroup*, is idempotent connected and locally type A with a regular semiband. We discuss the background to abundant semigroups in the preliminary first section.

1. Preliminaries

We begin by recalling some basic results and definitions adapted from Ljapin [9], page 395, which we shall use repeatedly.

LEMMA 1.1. *Let S be a semigroup and let a and b be elements of S. Then $(a,b) \in \mathscr{L}^*$ if and only if, for all x, $y \in S^1$, we have $ax = ay$ if and only if $bx = by$.*

This condition is somewhat simplified when one of the elements concerned is an idempotent, and this is the form in which it is most often applied, namely:

LEMMA 1.2. *Let S be a semigroup and let a be an element of S and e an idempotent of S. Then $(a,e) \in \mathscr{L}^*$ if and only if we have $ae = a$, and for all x, $y \in S^1$, $ax = ay$ implies $ex = ey$.*

It is easy to see that $\mathscr{L}^*$ is a right congruence and $\mathscr{R}^*$ a left congruence. The $\mathscr{L}^*$-class containing the element a of S will be denoted by L_a^*, or by $L_a^*(S)$ if we wish to make the semigroup concerned explicit. We shall often denote by a^* (respectively, a^+) an arbitrary idempotent in L_a^* (respectively, R_a^*). If A is a subset of S, we shall write $E(A)$ to denote the set of idempotents in A. We recall that we may define a partial order ω on $E(S)$ by declaring that, for e, $f \in E(S)$,

$e\omega f$ if and only if $ef = fe = e$. We write $\omega(e) = \{f \in E(S) \mid f\omega e\}$. The subsemigroup $\langle E(S)\rangle$ generated by the idempotents of a semigroups S is called a *semiband*.

If U is a subsemigroup of a semigroup S, then we always have $\mathscr{L}^*(S) \cap (U \times U) \subseteq \mathscr{L}^*(U)$.

A left ideal I of a semigroup S is called a *left $*$-ideal* if and only if for each $a \in S$, $L_a^* \subseteq S$. Likewise, we have the dual and two sided notions of *right $*$-ideal* and *$*$-ideal*. The intersection of any family of left (right)$*$-ideals is either empty or again a left (right)$*$-ideal. If a is an element of S, then there exists a smallest left (right)$*$-ideal containing a, called the *principal left (right)$*$-ideal generated by a*. We denote this ideal by $L^*(a)$ (respectively $R^*(a)$). See Fountain [4] for more information on $*$-ideals. For our purposes we only need the following definition. For elements x and y of S, we define $R_x^* \leqslant R_y^*$ if and only if $R^*(x) \subseteq R^*(y)$. Then we have the following result from Lawson [8].

LEMMA 1.3. *Let S be an abundant semigroup.*
(a) *For any elements a, $x \in S$, $R_{ax}^* \leqslant R_a^*$.*
(b) *If a, b are regular elements of S, then $R_a^* \leqslant R_b^*$ if and only if $R_a \leqslant R_b$.*

We have already defined an adequate semigroup to be an abundant semigroup whose idempotents form a semilattice. Fountain [3] showed that in an adequate semigroup, the elements a^* and a^+ are uniquely determined. An adequate semigroup is called *type A* if for each element a and idempotent e, we have both $ea = a(ea)^*$ and $ae = (ae)^+a$. Note that we also have notions of *right abundant*, *right adequate* and *right type A semigroups* by taking only the conditions relating to $\mathscr{L}^*$ in each case. If we state a result which holds for the $\mathscr{L}^*$ relation, the dual of that result is simply the left-right dual, with $\mathscr{R}^*$ replacing $\mathscr{L}^*$.

Let S be an abundant semigroup, let $E = E(S)$, and put $B = \langle E\rangle$ and $\langle e\rangle = \langle E(eBe)\rangle$, where e is an idempotent of S. It is easy to see that $\langle e\rangle = \langle \omega(e)\rangle$. El-Qallali and Fountain [2] called an abundant semigroup *idempotent connected* (*I.C.*) if and only if, for each element $a \in S$, and for some $a^+ \in E(R_a^*)$ and some $a^* \in E(L_a^*)$, there exists a bijection α: $\langle a^+\rangle \to \langle a^*\rangle$ satisfying $xa = a(x\alpha)$ for all $x \in \langle a^+\rangle$. In fact α will also be a homomorphism, and it may be verified that the phrase 'for some $a^+ \in E(R_a^*)$ and some $a^* \in L_a^*(s)$' may be replaced by the phrase 'for all $a^+ \in E(R_a^*)$ and all $a^* \in E(L_a^4(s))$'.

Regular semigroups are I.C., for if a is an element of a regular semigroup S, and if a' is an inverse of a, then we have an isomorphism α: $\langle aa'\rangle \to \langle a'a\rangle$ given by $x\alpha = a'xa$ for $x \in \langle aa'\rangle$, and, clearly, $xa = a(x\alpha)$.

From [2] we record the following result.

LEMMA 1.4. *An adequate semigroup is I.C. if and only if it is type* A.

For our purposes it will be more convenient to work with an alternative characterisation of I.C. provided below (see Lawson [8] for a proof).

PROPOSITION 1.5. *Let S be an abundant semigroup. Then the following are equivalent*:

(a) *S is I.C.*

(b) *For each $a \in S$ the following two conditions hold.*

(i) *For each $e \in \omega(a^*)$ there exists $f \in \omega(a^+)$ such that $ae = fa$.*

(ii) *For each $e' \in \omega(a^+)$ there exists $f' \in \omega(a^*)$ such that $e'a = af'$.*

An I.C. abundant semigroup with regular semiband will be called *concordant*. Any full subsemigroup of a regular semigroup is concordant, although the converse does not, in general, hold; for example, not every cancellative monoid can be embedded in a group.

If S is a semigroup, then McAlister [10] calls subsemigroups of the form eSe, where e is an idempotent, *local submonoids*. The proof of the following lemma is left as an exercise.

LEMMA 1.6. *If S is an abundant semigroup and e is an idempotent, then eSe is an abundant subsemigroup of S. In addition, if S is I.C., then eSe is I.C.*

If each local submonoid of a semigroup S is adequate (type A) then we will call *S locally adequate (type A)*. We will be particularly interested in the concordant, locally type A semigroups.

2. Abundant Rees matrix semigroups

Let S be an arbitrary semigroup, let I and Λ be non-empty sets, and let P be a $\Lambda \times I$-matrix with entries from S. The set of triples $I \times S \times \Lambda$ is a semigroup under the multiplication given by $(i, s, \lambda)(j, t, \mu) = (i, sp_{\lambda j}t, \mu)$, where $s, t \in S$, $i, j \in I$ and $\lambda, \mu \in \Lambda$. We shall denote this semigroup by $\mathscr{M} = \mathscr{M}(S; I, \Lambda; P)$ and call it the *Rees matrix semigroup over S with sandwich matrix P*

The proof of the following lemma is obtained by direct calculation.

LEMMA 2.1. *Let $\mathscr{M} = \mathscr{M}(S; I, \Lambda; P)$ be the Rees matrix semigroup considered above.*

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(a) *The element* (i, s, λ) *is an idempotent if and only if* $s = sp_{\lambda i}s$. *In particular, s is regular.*

(b) *If* (i, s, λ) *and* (j, t, μ) *are idempotents, then* $(i, s, \lambda)\omega(j, t, \mu)$ *in* $\mathscr{M}$ *if and only if* $i = j$, $\lambda = \mu$ *and both* $sp_{\mu j}\omega tp_{\mu j}$ *and* $p_{\mu j}s\omega p_{\mu j}t$ *in S.*

For the sake of clarity we shall write the starred Green's relations on $\mathscr{M}$ simply as $\mathscr{L}^*$ and $\mathscr{R}^*$. The next lemma gives an explicit description of those relations.

LEMMA 2.2. *Let* $\mathscr{M} = \mathscr{M}(S; I, \Lambda; P)$ *be a Rees matrix semigroup. Then for any* $(i, s, \lambda) \in \mathscr{M}$:

(a) *The* $\mathscr{L}^*$*-class of* (i, s, λ) *contains an idempotent if and only if there exists* $t \in S$ *and* $j \in I$ *with* $t = t_{p_{\lambda j}}t$ *and* $s\mathscr{L}^*t$.

(b) *The* $\mathscr{R}^*$*-class of* (i, s, λ) *contains an idempotent if and only if there exists* $m \in S$ *and* $\mu \in \Lambda$ *with* $m = mp_{\mu i}m$ *and* $s\mathscr{R}^*m$.

PROOF. We need only prove (a), since (b) follows by a dual argument. Let (j, t, μ) be an idempotent of $\mathscr{M}$ and suppose that $(i, s, \lambda)\mathscr{L}^*(j, t, \mu)$. In the first instance, this implies that $(i, s, \lambda)(j, t, \mu) = (i, s, \lambda)$. From this we deduce that $\mu = \lambda$ and $s = sp_{\lambda j}t$. Now let (k, x, ν) and (l, y, ζ) be arbitrary elements of $\mathscr{M}$ and suppose that the following equation holds:

$$(i, s, \lambda)(k, x, \nu) = (i, s, \lambda)(l, y, \zeta).$$

Then, from the fact that $(i, s, \lambda)\mathscr{L}^*(j, t, \mu)$, we must have

$$(j, t, \mu)(k, x, \nu) = (j, t, \mu)(l, y, \zeta).$$

Multiplying out these elements and comparing components yields the following: for all $x,\ y \in S$ and for all $k, l \in I$, $s(p_{\lambda k}x) = s(p_{\lambda l}y)$ implies $t(p_{\lambda k}x) = t(p_{\lambda l}y)$. Now let a and b be arbitrary elements of S, and suppose that $sa = sb$. Since $s = sp_{\lambda j}t$, we have that $s(p_{\lambda j}ta) = s(p_{\lambda j}tb)$, but this implies that $t(p_{\lambda j}ta) = t(p_{\lambda j}tb)$ by the above. But $t = tp_{\lambda j}t$, and so $ta = tb$. On the other hand, if $ta = tb$, then it is immediate that $sa = sb$, since $s = sp_{\lambda j}t$. From this reasoning it readily follows that $s\mathscr{L}^*t$ in S.

Conversely, if we are given $s\mathscr{L}^*t$ and $t = tp_{\lambda j}t$, then we have immediately that $s = sp_{\lambda j}t$ and the result follows by direct calculation.

Let S be an arbitrary semigroup and let $\mathscr{L}^*$ and $\mathscr{R}^*$ refer to S. An element a of S is called *right abundant* if and only if $a\mathscr{L}^*e$ for some idempotent e in S. Dually a is called *left abundant* if $a\mathscr{R}^*f$ for some idempotent f in S. An element is called *abundant* if it is both left and right abundant in S. We let $\mathscr{R}\mathscr{A}(S)$ denote the set of right abundant elements of S, $\mathscr{L}\mathscr{A}(S)$ the set of left abundant elements, and $\mathscr{A}(S)$ the set of abundant elements. We shall always assume, where necessary, that these sets are non-empty.

LEMMA 2.3. *If* $\mathscr{A}(S)$ $(\mathscr{RA}(S), \mathscr{LA}(S))$ *is a subsemigroup of* S, *then it is (left, right) abundant.*

PROOF. For any subsemigroup U of S we always have $\mathscr{L}^*(S) \cap (U \times U) \subseteq \mathscr{L}^*(U)$ and $\mathscr{R}^*(S) \cap (U \times U) \subseteq \mathscr{R}^*(U)$. Because $\mathscr{A}(S)$, $\mathscr{RA}(S)$ and $\mathscr{LA}(S)$ are clearly full in S the result follows.

For a semigroup S and a regular element a of S, we shall denote by $V(a)$ the set of inverses of a in S. If x is an arbitrary element of S, we define the set $R(x)$ by $R(x) = \{a \in S \mid axa = a\}$. Using this notation, we can summarise Lemma 2.2 as follows: $(i, s, \lambda) \in \mathscr{AM}$ if and only if there exists $(\mu, j) \in \Lambda \times I$ such that $R_s^* \cap R(p_{\mu i}) \neq \varnothing$ and $L_s^* \cap R(p_{\lambda j}) \neq \varnothing$, where we write $\mathscr{AM}$ in preference to $\mathscr{A}(\mathscr{M})$.

We now come to a result which will be crucial for our main work.

THEOREM 2.4. *Let* S *be a left abundant semigroup. For* $s \in S$, *if* $R_s^* \cap R(p) \neq \varnothing$ *for some* $p \in S$, *then for all* $e \in E(S)$ *we have* $R_{se}^* \cap R(p) \neq \varnothing$,

PROOF. Let $s \in S$ and suppose that $x \in R_s^* \cap R(p)$, so that $x\mathscr{R}^*s$ and $x = xpx$. Let $e \in E(S)$. By Lemma 1.3 we have $R_{se}^* \leqslant R_s^*$. Since S is left abundant, there exist idempotents $s^+ \in E(R_s^*)$ and $(se)^+ \in E(R_{se}^*)$. Now, $R_{(se)^+}^* = R_{se}^* \leqslant R_s^* = R_{s^+}^*$, and so $R_{(se)^+}^* \leqslant R_{s^+}^*$. But $(se)^+$ and s^+ are regular, so, by Lemma 1.3 $R_{(se)^+} \leqslant R_{s^+}$. Consequently, $(se)^+ \omega^r s^+$, and so $(se)^+ s^+ \mathscr{R} (se)^+$ and $s^+(se)^+ = (se)^+$. Since $x\mathscr{R}^*s^+$, and because $\mathscr{R}^*$ is a left congruence, we have $(se)^+ x \mathscr{R}^* (se)^+ s^+$, whence $(se)^+ x \mathscr{R}^* (se)^+ \mathscr{R}^* se$. Since $x = xpx$, it follows that $pxp \in V(x)$, so $x \cdot pxp \in R_x^* = R_s^*$. Hence $xp \in E(R_{x^+}^*)$, and so $xp \cdot s^+ = s^+$. Now, $(se)^+ x \cdot p(se)^+ x = (se)^+ \cdot xp \cdot (se)^+ x = (se)^+ xps^+(se)^+ x = (se)^+ s^+$, and $(se)^+ x = (se)^+(se)^+ x = (se)^+ x$. It follows that $(se)^+ x \in R_{se}^* \cap R(p)$.

COROLLARY 2.5. *If* S *is left (right) abundant, then* $\mathscr{LAM}(S)$ $(\mathscr{RAM}(S))$ *is a subsemigroup of* $\mathscr{M}(S) = \mathscr{M}(S; I, \Lambda; P)$. *Hence* $\mathscr{LAM}(S)$ $(\mathscr{RAM}(S))$ *is left (right) abundant. Furthermore,* $\mathscr{AM}(S)$ *is an abundant semigroup of* $\mathscr{M}(S)$.

PROOF. Let (i, s, λ), $(j, t, \mu) \in \mathscr{LAM}(S)$, where S is left abundant. Then $R_s^* \cap R(p_{\nu i}) \neq \varnothing$ for some $\nu \in \Lambda$, and $R_t^* \cap R(p_{\zeta j}) \neq \varnothing$ for some $\zeta \in \Lambda$, by Lemma 2.2. Now $(i, j, \lambda)(j, t, \mu) = (i, sp_{\lambda j}t, \mu)$. We need to show that $R_{sp_{\lambda j}t}^* \cap R(p_{\pi i}) \neq \varnothing$ for some $\pi \in \Lambda$. Now S is left abundant, so there exists an idempotent $e \in S$ with $p_{\lambda j}t\mathscr{R}^*e$. Hence $sp_{\lambda j}t\mathscr{R}^*se$, since $\mathscr{R}^*$ is a left congruence. But from Theorem 2.4 and the fact that $R_s^* \cap R(p_{\nu i}) \neq \varnothing$, we have $R_{se}^* \cap R(p_{\nu i}) \neq \varnothing$: in other words, $R_{sp_{\lambda j}t}^* \cap R(p_{\nu i}) \neq \varnothing$, and so we can take $\pi = \nu$.

The fact that $\mathscr{LAM}(S)$ is a subsemigroup, together with Lemma 2.3, gives that $\mathscr{LAM}(S)$ is a left abundant subsemigroup of $\mathscr{M}(S)$.

Let S be a semigroup with a non-empty set E of idempotents. Recall that we can define two quasi-orders on E as follows: for $e, f \in E$, $e\omega^r f$ if and only if $fe = e$ and $e\omega^l f$ if and only if $ef = e$. Let $M(e, f)$ denote the quasi-ordered set $(\omega^l(e) \cap \omega^r(f), <)$, where $<$ is defined by $g < h$ if and only if $eg\omega^r eh$ and $gf\omega^l hf$. Then Nambooripad [12] defines the *sandwich set* of e and f to be $S(e, f) = \{h \in M(e, f) \mid g < h \text{ for all } g \in M(e, f)\}$. Define $S_1(e, f) = \{h \in M(e, f)\colon ehf = ef\}$. Then Theorem 1.1 of [12] shows that if ef is a regular element of S, then $S_1(e, f) = S(e, f) \neq \varnothing$.

The following result is due to Nambooripad [12].

LEMMA 2.6. *Let S be a semigroup in which $\langle E(S)\rangle$ is regular, suppose that a and b are regular elements of S, and let $a' \in V(a)$ and $b' \in V(b)$. Then $b'S(a'a, bb')a' \subseteq V(ab)$.*

As a corollary to this result, we have the following, which should be compared with Corollary 1.3 of McAlister [10].

COROLLARY 2.7. *Let S be a semigroup with $\langle E(S)\rangle$ regular and let $x_i \in S$, $1 \leqslant i \leqslant n$, be regular elements with $x_1' \in V(x_1)$ and $x_n' \in V(x_n)$. Then $V(x_1 \cdots x_n) \cap x_n'Sx_1' \neq \varnothing$.*

For any semigroup S we shall denote the subset of regular elements of S by $\operatorname{Reg}(S)$. The next result should be compared with Proposition 1.3 of Fountain [3] or Result 1 of Hall [5].

PROPOSITION 2.8. *The following are equivalent for a semigroup S.*

(a) *S is abundant and $\langle E(S)\rangle$ is regular.*

(b) *S is abundant and $\operatorname{Reg}(S)$ is a regular subsemigroup.*

(c) *$\operatorname{Reg}(S)$ is a regular subsemigroup and intersects each $\mathscr{L}^*$-class and each $\mathscr{R}^*$-class.*

Since regular semigroups are abundant, we would hope that in their case $\mathscr{AM}(S; I, \Lambda; P)$ would be regular and coincide with the regular Rees matrix semigroup $\mathscr{RM}(S; I, \Lambda; P)$ defined in McAlister [10]. Our hopes are borne out by the following result.

PROPOSITION 2.9. *Let S be a regular semigroup. Then $\mathscr{AM}(S; I, \Lambda; P) = \mathscr{RM}(S; I, \Lambda; P)$.*

PROOF. Clearly $\mathscr{RM}(S; I, \Lambda; P) \subseteq \mathscr{AM}(S; I, \Lambda; P)$. Now let (i, s, λ) be an abundant element. We have $R_s^* \cap R(p_{\mu i}) \neq \varnothing$ and $L_s^* \cap R(p_{\lambda j}) \neq \varnothing$ for some $(j, \mu) \in I \times \Lambda$. Hence there exist elements t and s in S with $s\mathscr{R}^*t$, $t = tp_{\mu i}t$ and $s\mathscr{L}^*m$, where $m = mp_{\lambda j}m$. So $s = tp_{\mu i}s$ and $s = sp_{\lambda j}m$, from which we obtain $s = tp_{\mu i} \cdot s \cdot p_{\lambda j}m$. Now $tp_{\mu i}$ and $p_{\lambda j}m \in E(s)$, so from Corollary 2.7 we have $p_{\lambda j}mStp_{\mu i} \cap V(s) \neq \varnothing$. But by Lemma 2.1 of McAlister [10] this is just the condition for (i, s, λ) to be regular.

LEMMA 2.10. *If S is a semigroup such that $\mathscr{M} = \mathscr{M}(S; I, \Lambda; P)$ is abundant, then S is abundant.*

PROOF. From Lemma 2.2 we note that any element s of S is $\mathscr{L}^*$-related and $\mathscr{R}^*$-related to regular elements.

We now obtain a sufficient condition for a semigroup $\mathscr{M} = \mathscr{M}(S; I, \Lambda; P)$ to be abundant.

PROPOSITION 2.11. *Let S be an abundant monoid and let P be a $\Lambda \times I$-sandwich matrix over S in which each row contains a right unit of S and each column contains a left unit of S. Then $\mathscr{M} = \mathscr{M}(S; I, \Lambda; P)$ is abundant.*

PROOF. Let $(i, s, \lambda) \in \mathscr{M}$. Since S is abundant, there exists an idempotent $e \in E(s)$ such that $s\mathscr{L}^*e$. By hypothesis, for a given λ, there exists an element j of I such that $p_{\lambda j}$ is a right unit of S, so that $p_{\lambda j}p_{\lambda j}^{-1} = 1$. Hence $ep_{\lambda j} \cdot p_{\lambda j}^{-1}e = e$, and so $(p_{\lambda j}^{-1}e)p_{\lambda j}(p_{\lambda j}^{-1}e) = p_{\lambda j}^{-1}e$, or, in other words, $p_{\lambda j}^{-1}e \in R(p_{\lambda j})$. Also $p_{\lambda j}^{-1}e\mathscr{L}e$, and thus $s\mathscr{L}^*p_{\lambda j}^{-1}e$. Consequently, $(j, p_{\lambda j}^{-1}e, \lambda)$ is an idempotent which is $\mathscr{L}^*$-related to (i, s, λ) by Lemma 2.2. This fact, together with its dual, yields the result.

We shall now consider an example which demonstrates that, in general, the idempotents of an abundant Rees matrix semigroup do not generate a regular semiband.

EXAMPLE 2.12. Let S be the cancellative monoid $\mathbb{N}$ under multiplication, let $I = \Lambda = \{1, 2\}$, and let $P = \left(\begin{smallmatrix}1 & 2\\ 2 & 1\end{smallmatrix}\right)$. By Proposition 2.11, $\mathscr{M} = \mathscr{M}(S; I, \Lambda; P)$ is abundant, with exactly two idempotents $(1, 1, 1)$ and $(2, 1, 2)$. The $\mathscr{L}^*$-classes of $\mathscr{M}$ are $\{(1, n, 1), (2, m, 1) \mid n, m \in \mathbb{N}\}$ and $\{(1, n, 2), (2, m, 2) \mid n, m \in \mathbb{N}\}$, and the $\mathscr{R}^*$-classes are $\{(1, n, 1), (1, m, 2) \mid n, m \in \mathbb{N}\}$ and $\{(2, n, 1), (2, m, 2) \mid n, m \in \mathbb{N}\}$. It is easy to see that the product of the two idempotents $(1, 1, 1)(2, 1, 2) = (1, 2, 2)$ is not regular. Consequently $\mathscr{M}$ does not have a regular semiband, even though $\mathbb{N}$

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does. It can be verified that the semiband of $\mathscr{M}$ consists of all elements of the form $(1, 2^n, 1)$, $(1, 2^m, 2)$, $(2, 2^n, 2)$ and $(2, 2^m, 1)$, where $n \geqslant 0$ and $m \geqslant 1$.

We now give sufficient conditions which ensure that an abundant Rees matrix semigroup has a regular semiband.

PROPOSITION 2.13. *Let S be an abundant semigroup with a regular semiband of idempotents. Let $\mathscr{AM} = \mathscr{AM}(S; I, \Lambda; P)$ be an abundant Rees matrix semigroup over S for which the entries of the sandwich matrix P are all regular. Then $\mathscr{AM}$ has a regular semiband.*

PROOF. Let (i, s, λ) and (j, t, μ) be two idempotents. Then $s = sp_{\lambda i}s$ and $t = tp_{\mu j}t$, whence both s and t are regular elements of S. We need to show that $(i, s, \lambda)(j, t, \mu) = (i, sp_{\lambda j}t, \mu)$ is regular in $\mathscr{AM}$. By Lemma 2.1 of McAlister [10], we need to have $V(sp_{\lambda j}t) \cap p_{\mu k}Sp_{\nu i} \neq \varnothing$ for some $\nu \in \Lambda$ and $k \in I$. But $p_{\lambda i}sp_{\lambda i} \in V(s)$ and $p_{\mu j}tp_{\mu j} \in V(t)$, so by Corollary 2.7 and the fact that $p_{\lambda j}$ is regular, we have $V(sp_{\lambda j}t) \cap p_{\mu j}tp_{\mu j}Sp_{\lambda i}sp_{\lambda i} \neq \varnothing$, and the result follows.

From Proposition 2.8, it follows that, under the conditions of Proposition 2.13, $\text{Reg}(\mathscr{AM}) = \mathscr{RM}$ is a regular subsemigroup of $\mathscr{AM}$. We shall not, in general, assume that P consists entirely of regular elements, unless otherwise stated.

PROPOSITION 2.14. *Let S be an I.C. abundant semigroup and let $\mathscr{M} = \mathscr{M}(S; I, \Lambda; P)$. Then $\mathscr{AM}$ is an I.C. abundant semigroup.*

PROOF. Let $(i, s, \lambda) \in \mathscr{AM}$. Then there exists an idempotent (j, t, λ) with $t = tp_{\lambda j}t$, $s\mathscr{L}^*t$ and $(i, s, \lambda)\mathscr{L}^*(j, t, \lambda)$, and there exists an idempotent (i, m, μ) with $m = mp_{\mu i}m$, $s\mathscr{R}^*m$, and $(i, s, \lambda)\mathscr{R}^*(i, m, \mu)$. Now let $(j, x, \lambda)\omega(j, t, \lambda)$, so that $x = xp_{\lambda j}x$, $xp_{\lambda j}\omega tp_{\lambda j}$ and $p_{\lambda j}x\omega p_{\lambda j}t$. We have that $(i, s, \lambda)(j, x, \lambda) = (i, sp_{\lambda j}x, \lambda)$. Note that $p_{\lambda j}x\omega p_{\lambda t}t\mathscr{L}t\mathscr{L}^*s$. Hence, $p_{\lambda j}x \in \omega(p_{\lambda j}t)$ and $p_{\lambda j}t \in E(l_s^*)$. Since S is I.C., there exists an idempotent e of S such that $sp_{\lambda j}x = es$ and $e \in \omega(mp_{\mu i})$. Now $sp_{\lambda j}x = es = emp_{\mu i}s$; hence $(i, em, \mu)(i, s, \lambda) = (i, emp_{\mu i}s, \lambda) = (i, sp_{\lambda j}x, \lambda) = (i, s, \lambda)(j, x, \lambda)$. Also, $(i, em, \mu)(i, em, \mu) = (i, emp_{\mu i}em, \mu) = (i, em, \mu)$, so (i, em, μ) is an idempotent. Now, $emp_{\mu i} = e \in \omega(mp_{\mu i})$, and both $p_{\mu i}em \cdot p_{\mu i}m = p_{\mu i}em$ and $p_{\mu i}m \cdot p_{\mu i}em = p_{\mu i}em$. Consequently, $(i, em, \mu) \in \omega((i, m, \mu))$. This fact together with its dual yields the result by Proposition 1.5.

We shall now look at the idempotent structure of the local submonoids of $\mathscr{AM}$.

LEMMA 2.15. *Let S be a semigroup and suppose that for each idempotent e of S, $\omega(e)$ is a semilattice. Put $\mathscr{M} = \mathscr{M}(S; I, \Lambda; P)$. Then for each idempotent (j, t, μ) of $\mathscr{M}$, $\omega((j, t, \mu))$ is band.*

PROOF. Let $(j, t, \mu) \in E(\mathscr{M})$ and let $(j, x, \mu), (j, y, \mu) \in \omega((j, t, \mu))$. Then $xp_{\mu j} \omega tp_{\mu j}$, $p_{\mu j} x \omega p_{\mu j} t$, $yp_{\mu j} \omega tp_{\mu j}$, and $p_{\mu j} y \omega p_{\mu j} t$. Now, $(j, x, \mu)(j, y, \mu) = (j, xp_{\mu j} y, \mu)$. Consider $xp_{\mu j} y \cdot p_{\mu j} \cdot xp_{\mu j} y = xp_{\mu j} \cdot yp_{\mu j} \cdot xp_{\mu j} y = xp_{\mu j} \cdot xp_{\mu j} \cdot yp_{\mu j} \cdot y = xp_{\mu j} \cdot yp_{\mu j} \cdot y = xp_{\mu j} \cdot y$, since $y \in R(p_{\mu j})$ and $xp_{\mu j}$, $yp_{\mu j} \in \omega(tp_{\mu j})$, which is a semilattice. It follows that $(j, xp_{\mu j} y, \mu)$ is an idempotent. Furthermore, $xp_{\mu j} y \cdot p_{\mu j} \cdot tp_{\mu j} = xp_{\mu j} \cdot yp_{\mu j} \cdot tp_{\mu j} = xp_{\mu j} yp_{\mu j}$, since $yp_{\mu j} \in (tp_{\mu j})$, and $tp_{\mu j} \cdot xp_{\mu j} y \cdot p_{\mu j} = tp_{\mu j} \cdot xp_{\mu j} \cdot yp_{\mu j} = xp_{\mu j} \cdot yp_{\mu j}$. Hence $xp_{\mu j} y \cdot p_{\mu j} \omega tp_{\mu j}$, and similarly $p_{\mu j} \cdot xp_{\mu j} y \omega p_{\mu j} t$, so that $(j, xp_{\mu j} y, \mu) \omega (j, t, \mu)$.

LEMMA 2.16. *Let S be a semigroup with $\omega(e)$ a semilattice for each $e \in E(S)$, and suppose furthermore that the relation $\leqslant_r$ defined on S by $a \leqslant_r b$ if and only if there exists an $e \in E(S)$ such that $a = be$ is a partial order. Then, for each idempotent (j, t, μ) in $E(\mathscr{M})$, $\omega((j, t, \mu))$ is a semilattice.*

PROOF. We need to show, in the notation of Lemma 2.15, that $xp_{\mu j} y = yp_{\mu j} x$. Now, $xp_{\mu j} y = xp_{\mu j} \cdot yp_{\mu j} y = xp_{\mu j} yp_{\mu j} y = yp_{\mu j} \cdot xp_{\mu j} \cdot y = yp_{\mu j} x \cdot p_{\mu j} y$, and, since $p_{\mu j} y \in E(s)$, we have $xp_{\mu j} y \leqslant_r yp_{\mu j} x$. Similarly $yp_{\mu j} x \leqslant_r xp_{\mu j} y$. Hence $xp_{\mu j} y = yp_{\mu j} x$.

We now summarize our results in the following theorem. We remark first that on a right adequate semigroup the relation $\leqslant_r$ given by $a \leqslant_r b$ if and only if $a = be$ for some idempotent e is a partial order. Note also that a (right) abundant semigroup whose idempotents form a band is called *(right) quasi-adequate*.

THEOREM 2.17.

(a) *If S is right abundant and locally right adequate, then $\mathscr{RAM}(S)$ is right abundant and locally right quasi-adequate.*

(b) *If S is right adequate, then $\mathscr{RAM}(S)$ is right abundant and locally right adequate.*

(c) *If S is adequate, then $\mathscr{AM}(S)$ is abundant and locally adequate.*

(d) *If S is type* A, *then $\mathscr{AM}(S)$ is abundant, I.C. and locally type* A.

(e) *If S is type* A, *and if the entries of the sandwich matrix are all regular, then $\mathscr{AM}(S)$ is concordant and locally type* A.

PROOF. By Lemma 1.6, (a) follows from Corollary 2.5 and Lemma 2.15; (b) follows from (a) and Lemma 2.16; (c) is a consequence of (b), and its dual (d) follows from (c) and from Proposition 2.14; and finally (e) derives from (d), from Proposition 2.13, and from Lemma 1.6.

If we look at a particular case of (e), namely where S is a cancellative monoid and the sandwich matrix P consists of invertible elements, then, by Proposition 2.11, $\mathscr{AM}(S, I, \Lambda; P) = \mathscr{M}(S, I, \Lambda; P)$ and is concordant. Note that these are precisely the *completely* $\mathscr{J}$**-simple* semigroups of Fountain [4]: concordant semigroups (without zero) every idempotent of which is primitive.

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Department of Mathematics
University of York
Heslington
York
United Kingdom