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CONVERGENCE OF TANDEM BROWNIAN QUEUES

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Abstract

It is known that in a stationary Brownian queue with both arrival and service processes equal in law to Brownian motion, the departure process is a Brownian motion, identical in law to the arrival process: this is the analogue of Burke's theorem in this context. In this paper we prove convergence in law to this Brownian motion in a tandem network of Brownian queues: if we have an arbitrary continuous process, satisfying some mild conditions, as an initial arrival process and pass it through an infinite tandem network of queues, the resulting process weakly converges to a Brownian motion. We assume independent and exponential initial workloads for all queues.

Keywords: Brownian queue; tandem queues; Burke's theorem

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1. Introduction

In 1956 Burke [3] obtained one fundamental result for queueing theory. The first part of this result states that given a Poisson arrival process with rate $\lambda < 1$, and an independent service Poisson process with rate 1, which together define an M/M/1 queue, the departure process is Poisson with parameter λ. The second part states a factorization property: the length of the queue at any given time is independent of future arrivals and past departures. Several extensions of this result have followed, see, for example, [4], [5], and [12].

The Brownian queue is a continuous-valued model for a queue, which is indeed the heavy traffic limit of an M/M/1 queue. We define it as follows. Denote by $\mathcal{R}\colon D[0,\infty) \to D[0,\infty)$ the operator in the space of càdlàg functions (i.e. right-continuous functions whose discontinuities, if any, are of jump type) given by

$$\mathcal{R}(f)(t) := f(t) - \inf_{0\le u\le t}\{f(u)\wedge 0\}, \tag{1}$$

called Skorokhod reflection mapping, since it solves the Skorokhod problem; see, for instance, [6, p. 14],). Now, let us denote by $D_0[0,\infty)$ the set of functions $f \in D[0,\infty)$ such that $f(0) = 0$. Given two functions $a, s \in D_0[0,\infty)$ and some nonnegative number $q_0 \ge 0$, we define the departure operator $\mathcal{D}\colon D_0[0,\infty)^2 \times [0,\infty) \to D_0[0,\infty)$ by

$$\mathcal{D}(a,s,q_0)(t) := s(t) + \inf_{0\le u\le t}\{(q_0 + a(u) - s(u))\wedge 0\} = (q_0 + a - \mathcal{R}(q_0 + a - s))(t). \tag{2}$$

See Figure 1 for an illustration of this operator acting on some arbitrary càdlàg functions. Note that $\{\mathcal{D}(a,s,q_0)(t)\}_{t\ge 0}$ can be seen as the reflection of the process s downwards at the upper (time-varying) boundary $q_0 + a$ (see, for example, [9, Appendix A]).

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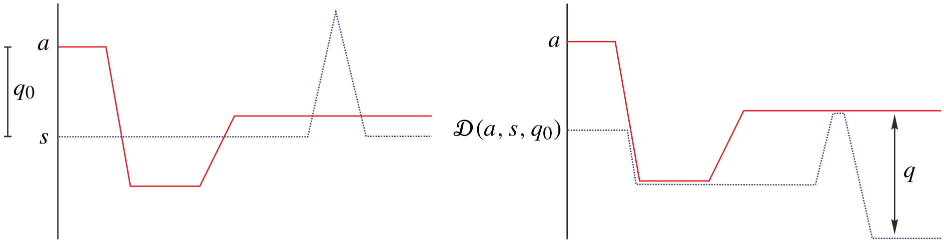

FIGURE 1: The càdlàg queue.

This mapping provides the definition of the departure process from a fluid queue in the context of càdlàg functions according to this interpretation: a represents the arrival process, s the service process, q_0 the initial workload of the queue, and $\mathcal{D}(a, s, q_0)$ the departure process. Other important processes are the queue length process given by $q = \mathcal{R}(q_0 + a - s)$, and the free process defined as $q_0 + a - s$. This is a general way to define a fluid queue; in the case where a and s are nondecreasing functions, and q_0 is some nonnegative number, we have a storage system in the usual sense.

If a and s are independent Poisson processes, and q_0 some nonnegative integer, then we define the queue length process of a classical queue by

$$q(t) = q_0 + a(t) - s\left(\int_0^t \mathbf{1}_{\{q(u)>0\}}\, \mathrm{d}u\right).$$

This is the M/M/1 queue and is known to be a continuous-time Markov chain. Since s is a Poisson process, independent of a and q_0, it follows that $\{q(t), t \geq 0\}$ is identical in law to

$$q(t) = q_0 + a(t) - \int_0^t \mathbf{1}_{\{q(u)>0\}}\, s(\mathrm{d}u); \tag{3}$$

see, for example, [2]. But then

$$q(t) = q_0 + a(t) - s(t) + \ell(t),$$

where $\ell(t) = \int_0^t \mathbf{1}_{\{q(u)=0\}}\, s(\mathrm{d}u)$ has the property that it is a càdlàg nondecreasing function starting from $\ell(0) = 0$ which increases only at points t such that $q(t) = 0$. It follows from Skorokhod's theorem that

$$\ell(t) = -\inf_{0 \leq u \leq t} \{(q(0) + a(u) - s(u)) \wedge 0\},$$

and so the process of (3) is also given by $q = \mathcal{R}(q_0 + a - s)$. In addition, the process $s - \ell$ is nondecreasing. It is the departure process from the M/M/1 queue and it also satisfies $\mathcal{D}(a, s, q_0) = s - \ell$.

Consider now the special case where a is a standard Brownian motion and s a Brownian motion with positive drift c and choose the initial workload q_0 as a random variable having the stationary distribution of this fluid queue (namely an exponential random variable with parameter c; see [6, p. 15]). We assume that a, s, and q_0 are independent. This definition matches the one given by O'Connell and Yor [14] of the stationary version of the Brownian queue (for positive times), as Norros and Salminen [15] pointed out.

For this model (and further generalizations of functionals of Brownian motion) an analogue result to Burke's theorem is presented in [14].

Theorem 1. *Let B_t^1, B_t^2 be standard Brownian motions, and $\mathcal{E}$ an exponential variable of parameter c. Assume that all random elements are independent. Then*

(i) *$\{D_t\}_{t\geq 0}$, defined by $D_t := \mathcal{D}(B_t^1, B_t^2 + c\,t, \mathcal{E})$, has the law of a standard Brownian motion.*

(ii) *Define the process $\{Q_t\}_{t\geq 0}$ by $Q_t = \mathcal{R}(B_t^1 - (B_t^2 + c\,t) + \mathcal{E})$. Then $\{D_s : 0 \leq s < t\}$ and Q_t are independent.*

(Note that her we abuse notation and write $\mathcal{D}(B_t^1, B_t^2 + c\,t, \mathcal{E})$ instead of $\mathcal{D}(\{B_t^1\}_{t\geq 0}, \{B_t^2 + c\,t\}_{t\geq 0}, \mathcal{E})$.) The proof of this result goes back to [7], in the context of multiclass stations, and relies on weak convergence arguments or, alternatively, on path properties of the Brownian motion.

A tandem queueing network is a system of queues where there is an arrival process A^1, and a sequence $\{S^n\}_{n\geq 1}$ of service processes, all independent. The system is defined recursively. The initial queue is fed from the arrival process A^1, and has departures determined by the service process S^1. For $n \geq 2$, the arrival process for the nth queue is defined as the departure process of the $(n-1)$th queue and the departures are determined by the service process S^n.

When the initial arrival process has a Poisson law, Burke's theorem allows us to treat a tandem system of queues at any fixed time as if the queues acted independently. For example, take a two-node system of tandem queues, with Poisson(λ) arrivals, $\lambda < 1$, Poisson(1) service processes, all independent, and sample the initial length of each queue from its stationary measure. Because of the first part of Burke's theorem, the departure process of the first node is a Poisson(λ) process. Moreover, due to the second part of Burke's theorem, the departure process of the first queue prior to time t is independent of Q_t^1, the length of that queue at time t. Then the length of the second queue at time t, Q_t^2, is independent from Q_t^1, and it follows that the invariant measure of the system is a product measure. The factorization property from Burke's theorem has thus enabled the analysis of more complex systems.

In the case when the law of the initial arrival process is not Poisson, a natural question is whether it is possible to prove convergence to the stationary distribution in a tandem system where the number of queues tends to ∞. Assuming an existence result, Anantharam [1] proved the uniqueness of a stationary ergodic fixed point for the ·/M/K queue. Next, Mountford and Prabhakar [13] proved the attractiveness of the Poisson distribution in the class of ergodic stationary point processes on the line. To obtain this result, they used a coloring coupling technique based on an argument of Ekhaus and Gray (unpublished, cited by [13]).

For the Brownian case of tandem systems, some advances have been made. Lieshout and Mandjes [10] calculated the joint distribution of the workload processes in a two-node system and obtained asymptotic results for the same system in the case of Lévy-driven queues [11]. In this paper we present an analogue of the Mountford–Prabhakar theorem for the following Brownian queue.

Theorem 2. *Let A^0 be a process with continuous paths $A^0(\cdot, \omega) : [0, \infty) \to \mathbb{R}$ that do not explode in finite time almost surely (a.s.), and $A^0(0, \omega) \equiv 0$. Let $\{W^n\}_{n\in\mathbb{N}}$ be a family of standard Brownian motions and $\{\mathcal{E}^n\}_{n\in\mathbb{N}}$ a family of exponential random variables with common parameter $c > 0$, all independent. We define recursively the sequence of processes*

$$A^n = \mathcal{D}(A^{n-1}, W^n + ct, \mathcal{E}^n), \qquad n \geq 1,$$

where $\mathcal{D}$ is the departure operator defined in (2). Then A^n weakly converges to a Brownian motion.

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In other words, the departure process of a infinite-node tandem system of Brownian queues is weakly convergent to a Brownian motion for an initial arrival process belonging to a wide class of continuous-valued processes, and a particular set of initial conditions for the tandem queues: all having independent workloads, distributed as the stationary distribution of the Brownian queue. The coupling used in the ·/M/1 case [13] is no longer suitable and we introduce an ad hoc coupling technique that takes advantage of simple path properties of the Brownian motion. This procedure, however, strongly depends on the particular choice of initial workloads for the queues. We are currently working on a version of Theorem 2 where each queue is stationary, using a different approach.

For completeness, we present an elementary proof of Theorem 1, using the heavy-traffic weak limit of the M/M/1 queue, as done in [6], but avoiding the more complex context of multiclass stations.

2. Burke's theorem for Brownian queues

Before proving Theorem 1 we state a corollary of Donsker's theorem.

Lemma 1. *Let $\{P^n\}_{n\in\mathbb{N}}$ be a sequence of Poisson processes with rate $r_n > 0$. Assume that $r_n \to r \in (0, \infty)$. Then*

$$\frac{P^n(nt) - r_n nt}{\sqrt{n}} \xrightarrow{\mathrm{W}} \sqrt{r}B(t),$$

where $\{B(t)\}_{t\geq 0}$ is a standard Brownian motion, and '$\xrightarrow{\mathrm{W}}$' denotes weak convergence of processes.

Proof. Let $P(t)$ be a Poisson process with intensity 1. Since $\{P^n(t) : t \geq 0\} \overset{\mathrm{D}}{=} \{P(r_n t) : t \geq 0\}$, where $\overset{\mathrm{D}}{=}$ denotes equality in law, we have

$$\frac{P^n(nt) - nr_n t}{\sqrt{n}} \overset{\mathrm{D}}{=} \frac{P(nr_n t) - nr_n t}{\sqrt{n}} = \left(\frac{P((nr_n)t) - (nr_n)t}{\sqrt{nr_n}}\right)\sqrt{r_n}.$$

The result follows by the functional central limit theorem; see, for example, [16]. □

Proof of Theorem 1. (i) Define $\lambda_n := 1 - c/\sqrt{n}$. For $n \in \mathbb{N}$ large enough such that $0 < \lambda_n < 1$, we let $\{A^n(t)\}$ be a Poisson process with parameter λ_n, $\{S(t)\}$ a Poisson process with parameter 1, and G^n a geometric random variable with $\mathbb{P}(G^n = x) = \lambda_n^x(1-\lambda_n)$ for $x \in \mathbb{Z}_+$, all independent. Consider an M/M/1 queue with arrival process A^n, service process S, and initial queue length G^n. According to (3), the queue length process is given by $Q^n = \mathcal{R}(G^n + A^n - S)$ and is a stationary process, i.e. for all $t_0 > 0$, the law of $\{Q^n(t_0 + t)\}_{t\geq 0}$ is the law of Q^n. Next, consider the scaled processes

$$\tilde{A}^n(t) := \frac{A^n(nt) - nt\lambda_n}{\sqrt{n}} \quad \text{and} \quad \tilde{S}^n(t) := \frac{S(nt) - nt\lambda_n}{\sqrt{n}} = \frac{S(nt) - nt}{\sqrt{n}} + ct.$$

By Lemma 1, $\{\tilde{A}^n(t)\}$ converges weakly to a Brownian motion B^1 as $n \to \infty$. Also, $\{\tilde{S}^n(t)\}$ converges to $B^2(t)+ct$, where B^2 is a Brownian motion. Finally, let $\tilde{G}^n := G^n/\sqrt{n}$, so that $\tilde{G}^n$ converges to an exponential random variable $\mathcal{E}$ with parameter c. We may choose B^1, B^2, and $\mathcal{E}$ independently. Since A^n, S, G^n are independent, it follows that $(\tilde{A}_n, \tilde{S}_n, \tilde{G}_n)$ converges weakly to $(B^1, \{B^2(t)+ct\}_{t\geq 0}, \mathcal{E})$. Since $\mathcal{R}$ is continuous in the Skorokhod topology (see [16, p. 439]), the departure operator $\mathcal{D}$ is also continuous, as follows by (2). By the continuous mapping theorem, it follows that $\mathcal{D}(\tilde{A}^n(t), \tilde{S}^n(t), \tilde{G}^n)$ converges weakly to $\mathcal{D}(B^1(t), B^2(t) + ct, \mathcal{E})$.

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On the other hand, $D^n := \mathcal{D}(A^n, S, G^n)$ is the departure process of the nth M/M/1 queue. By Burke's theorem, D^n is a rate λ_n Poisson process. So, if we let

$$\tilde{D}^n(t) := \frac{D^n(nt) - nt\lambda_n}{\sqrt{n}},$$

we obtain, again, by Lemma 1, that $\tilde{D}^n$ converges weakly to a standard Brownian motion. One can check directly from (1) and (2) that $\tilde{D}^n = \mathcal{D}(\tilde{A}^n, \tilde{S}^n, \tilde{G}^n)$. Therefore, the weak limit of $\tilde{D}^n$ is simultaneously $\mathcal{D}(B^1(t), B^2(t)+ct, \mathcal{E})$ and a standard Brownian motion. It follows that the law of $\mathcal{D}(B^1(t), B^2(t) + ct, \mathcal{E})$ is a standard Brownian motion.

(ii) Let $\tilde{Q}^n(t) := Q^n(nt)/\sqrt{n}$. From $Q^n = \mathcal{R}(G^n + A^n - S)$, it follows that $\tilde{Q}^n = \mathcal{R}(\tilde{G}^n + \tilde{A}^n - \tilde{S}^n)$. By the same arguments as above, it follows that Q^n converges weakly to $Q(t) = \mathcal{R}(\mathcal{E} + B^1(t) - (B^2(t) + ct))$. In fact, the pair $(\tilde{Q}^n, \tilde{D}^n)$ converges weakly to the pair (Q, D). By the second half of Burke's theorem, $\{D^n(s) : 0 \le s < t\}$ is independent of $Q^n(t)$ and so $\{\tilde{D}^n(s) : 0 \le s < t\}$ is independent of $\tilde{Q}^n(t)$. Since independence is of course preserved in the limit, it follows that $\{D(s) : 0 \le s < t\}$ is independent of $Q(t)$ as asserted. This completes the proof. □

3. Convergence of the tandem Brownian queueing network

We will need the following version of the Borel–Cantelli lemma. Its proof can be found in, for example, [8, p. 131].

Lemma 2. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, $\{\mathcal{F}_n\}_{n\in\mathbb{N}}$ a filtration such that $\mathcal{F} = \bigcup_{n\in\mathbb{N}} \mathcal{F}_n$, and $O_n \in \mathcal{F}_n$ for each $n \in \mathbb{N}$. Then*

$$\{O_n \text{ infinitely often}\} = \Big\{\sum_{n\in\mathbb{N}} \mathbb{P}(O_{n+1} \mid \mathcal{F}_n) = \infty\Big\} \quad a.s.$$

We also need a contraction property for $\mathcal{D}$. Denote by $\|\cdot\|_{[0,T]}$ the supremum norm on $[0, T]$.

Lemma 3. *Denote the space of continuous real functions that vanish at* 0 *by $C_0[0, \infty)$. Let $a^1, a^2, s \in C_0[0, \infty)$, $q_0 \ge 0$, and let $\mathcal{D}(a^1, s, q_0)$, $\mathcal{D}(a^2, s, q_0)$, be defined as in (2). Then, for any $T > 0$,*

$$\|\mathcal{D}(a^1, s, q_0) - \mathcal{D}(a^2, s, q_0)\|_{[0,T]} \le \|a^1 - a^2\|_{[0,T]}.$$

Proof. We have

$$\begin{aligned}
&\|\mathcal{D}(a^1, s, q_0) - \mathcal{D}(a^2, s, q_0)\|_{[0,T]} \\
&\quad = \sup_{0\le t\le T}\Big|\Big[s_t + \inf_{0\le u\le t}\{(q_0 + a_u^1 - s_u)\wedge 0\}\Big] - \Big[s_t + \inf_{0\le u\le t}\{(q_0 + a_u^2 - s_u)\wedge 0\}\Big]\Big| \\
&\quad = \sup_{0\le t\le T}\Big|\sup_{0\le u\le t}\{(q_0 + a_u^1 - s_u)\vee 0\} - \sup_{0\le u\le t}\{(q_0 + a_u^1 - s_u)\vee 0\}\Big| \\
&\quad \le \sup_{0\le t\le T}|(q_0 + a_t^1 - s_t)\vee 0 - (q_0 + a_t^2 - s_t)\vee 0| \\
&\quad \le \|a^1 - a^2\|_{[0,T]}.
\end{aligned}$$

The first inequality follows from the Lipschitz continuity of the supremum mapping with Lipschitz constant equal to 1 (see, for example, [16, p. 436]), and the second inequality holds since $\|f^+ - g^+\|_{[0,T]} \le \|f - g\|_{[0,T]}$ for every pair f, g of continuous real functions. □

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Proof of Theorem 2. The proof relies on a coupling argument: we show that if different arrival processes are run through the same services, the resulting trajectories are eventually locally coupled. Since we know that there exists a stationary distribution for the system of the tandem queues, given by Theorem 1, we conclude the result.

Starting with an initial arrival process A^0, define the processes A^n, $n \geq 1$, as in the statement of the theorem. In addition, letting B^0 be an independent Brownian motion, define

$$B^n = \mathcal{D}(B^{n-1}, W^n + ct, \mathcal{E}^n) \quad \text{for all } n \geq 1.$$

That is, we consider a second system of tandem queues where we replaced A^0 by B^0 but left all service processes and initial states intact. This second system is now in a steady state. By Theorem 1, each B^n is a standard Brownian motion. So, in order to obtain the announced convergence of the A^n towards a Brownian motion, it is enough to prove that the trajectories of A^n and B^n eventually couple on $[0, T]$ for all $T > 0$.

Fix $T > 0$. The heart of the proof is this: beginning with two different arrival process, there will be a queue indexed, say n, that will have positive workload during $[0, T]$. Then the departure process from this queue will coincide with the service process on $[0, T]$. Since we are using the same service processes for both systems, the departures of both systems will also coincide. This coupling persists on all queues following the queue indexed n. This is made more precise as follows. If

$$W_t^{n+1} + ct \leq \mathcal{E}^{n+1} + A_t^n \quad \text{for all } t \in [0, T],$$

then $\mathcal{R}(\mathcal{E}^{n+1} + A_t^n - (W_t^{n+1} + ct)) = \mathcal{E}^{n+1} + A_t^n - (W_t^{n+1} + ct)$ for all $t \in [0, T]$, and, hence,

$$\begin{aligned} A_t^{n+1} &= \mathcal{D}(A_t^n, W_t^{n+1} + ct, \mathcal{E}^{n+1}) \\ &= \mathcal{E}^{n+1} + A_t^n - \mathcal{R}(\mathcal{E}^{n+1} + A_t^n - (W_t^{n+1} + ct)) \\ &= W_t^{n+1} + ct \quad \text{for all } t \in [0, T]. \end{aligned}$$

The last argument also works when we replace A_t^n and A_t^{n+1} by B_t^n and B_t^{n+1}. Define the events

$$O_n := \{\omega \in \Omega \colon W_t^n + ct - \mathcal{E}^n \leq \min(A_t^{n-1}, B_t^{n-1}) \text{ for all } t \in [0, T]\}, \qquad n \in \mathbb{N},$$

and note that O_n belongs to the σ-algebra $\mathcal{F}_n := \sigma(\{A^0, B^0, W^i, \mathcal{E}^i : i \leq n\})$ and that O_n is a coupling event, that is, if O_n occurs then $A^n = B^n$ occurs as well. Since $A^n = B^n$ on $[0, T]$ implies that $A^k = B^k$ on $[0, T]$ for all $k > n$, to prove that the paths A^n and B^n a.s. eventually couple, by Lemma 2, it is enough to prove that

$$\sum_{n=1}^{\infty} \mathbb{E}(\mathbf{1}_{\{O_{n+1}\}} \mid \mathcal{F}_n) = \infty.$$

Define for $n \geq 0$, $\delta_n := \|A^n - B^n\|_{[0,T]}$. Since process A_0 does not explode in finite time a.s., we have $\delta_0 < \infty$ a.s. and, hence,

$$\begin{aligned} \sum_{n=1}^{\infty} \mathbb{E}(\mathbf{1}_{\{O_{n+1}\}} \mid \mathcal{F}_n) &= \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} \mathbf{1}_{\{k-1 \leq \delta_0 < k\}} \mathbb{E}(\mathbf{1}_{\{O_{n+1}\}} \mid \mathcal{F}_n) \\ &= \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} \mathbb{E}(\mathbf{1}_{\{O_{n+1}\}} \mathbf{1}_{\{k-1 \leq \delta_0 < k\}} \mid \mathcal{F}_n) \end{aligned}$$

because $\{k-1 \leq \delta_0 < k\} \in \mathcal{F}_0 \subseteq \mathcal{F}_n$. Since both systems have the same services and initial states, Lemma 3 implies that $\delta_n(\omega) \leq \delta_0(\omega)$ for every $\omega \in \Omega$. Therefore,

$$\sum_{n=1}^{\infty} \mathbb{E}(\mathbf{1}_{\{O_{n+1}\}} \mid \mathcal{F}_n) = \sum_{k=1}^{\infty}\sum_{n=1}^{\infty} \mathbb{E}(\mathbf{1}_{\{O_{n+1}\}} \mathbf{1}_{\{k-1\leq\delta_0<k\}} \mathbf{1}_{\{\delta_n<k\}} \mid \mathcal{F}_n).$$

Note now that $\{W_t^{n+1} + ct - \mathcal{E}^{n+1} \leq B_t^n - \delta_n \text{ for all } t \in [0,T]\} \subseteq O_{n+1}$. Hence,

$$\begin{aligned}
&\sum_{n=1}^{\infty} \mathbb{E}(\mathbf{1}_{\{O_{n+1}\}} \mid \mathcal{F}_n) \\
&\qquad \geq \sum_{k=1}^{\infty}\sum_{n=1}^{\infty} \mathbb{E}(\mathbf{1}_{\{W_t^{n+1}+ct-\mathcal{E}^{n+1}\leq B_t^n-\delta_n \text{ for all } t\in[0,T]\}} \mathbf{1}_{\{k-1\leq\delta_0<k\}} \mathbf{1}_{\{\delta_n<k\}} \mid \mathcal{F}_n) \\
&\qquad \geq \sum_{k=1}^{\infty}\sum_{n=1}^{\infty} \mathbb{E}(\mathbf{1}_{\{W_t^{n+1}+ct-\mathcal{E}^{n+1}\leq B_t^n-k \text{ for all } t\in[0,T]\}} \mathbf{1}_{\{k-1\leq\delta_0<k\}} \mid \mathcal{F}_n) \\
&\qquad = \sum_{k=1}^{\infty}\bigg[\mathbf{1}_{\{k-1\leq\delta_0<k\}} \sum_{n=1}^{\infty} \mathbb{E}(\mathbf{1}_{\{W_t^{n+1}+ct-\mathcal{E}^{n+1}\leq B_t^n-k \text{ for all } t\in[0,T]\}} \mid \mathcal{F}_n)\bigg].
\end{aligned}$$

Let

$$X_n^k := \mathbb{E}(\mathbf{1}_{\{W_t^{n+1}+ct-\mathcal{E}^{n+1}\leq B_t^n-k \text{ for all } t\in[0,T]\}} \mid \mathcal{F}_n).$$

By Theorem 1, it follows that B^n is a Brownian motion for all n and so the random variables $\{X_n^k\}_{n\in\mathbb{N}}$ are identically distributed. Moreover, it holds that the dynamics with respect to the nth step in the tandem queue are Markovian, in particular, given B^n, the process $\{B^n + \mathcal{E}^{n+1} - (W_t^{n+1} + ct)\colon t \in [0,T]\}$ is independent of the processes $\{B^k\colon k < n, W^k\colon k \leq n\}$. Then it follows that the variables $\{X_n^k\}_{n\in\mathbb{N}}$ are independent. Using elementary properties of a Brownian motion, it follows that the X_n^k are nonidentically 0 random variables. Therefore, the sum $\sum_{n=1}^{\infty} X_n^k$ diverges a.s. for all k and this completes the proof. □

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References

[1] Anantharam, V. (1993). Uniqueness of stationary ergodic fixed point for a $\cdot/M/K$ node. *Ann. Appl. Prob.* **3,** 154–172. (Correction: **4** (1994), 607.)
[2] Brémaud, P. (1981). *Point Processes and Queues: Martingale Dynamics*. Springer, New York.
[3] Burke, P. J. (1956). The output of a queuing system. *Operat. Res.* **4,** 699–704.
[4] Draief, M., Mairesse, J. and O'Connell, N. (2003). Joint Burke's theorem and RSK representation for a queue and a store. In *Discrete Random Walks*, Association of Discrete Mathematics and Theoretical Computer Science, Nancy, pp. 69–82.
[5] Ferrari, P. A. and Fontes, L. R. G. (1994). The net output process of a system with infinitely many queues. *Ann. Appl. Prob.* **4,** 1129–1144.
[6] Harrison, J. M. (1985). *Brownian Motion and Stochastic Flow Systems*. John Wiley, New York.
[7] Harrison, J. M. and Williams, R. J. (1990). On the quasireversibility of a multiclass Brownian service station. *Ann. Prob.* **18,** 1249–1268.

[8] KALLENBERG, O. (2002). *Foundations of Modern Probability*, 2nd edn. Springer, New York.
[9] KONSTANTOPOULOS, T. AND ANANTHARAM, V. (1995). Optimal flow control schemes that regulate the burstiness of traffic. *IEEE/ACM Trans. Networking* **3,** 423–432.
[10] LIESHOUT, P. AND MANDJES, M. (2007). Tandem Brownian queues. *Math. Meth. Operat. Res.* **66,** 275–298.
[11] LIESHOUT, P. AND MANDJES, M. (2008). Asymptotic analysis of Lévy-driven tandem queues. *Queueing Systems* **60,** 203–226.
[12] MARTIN, J. B. AND PRABHAKAR, B. (2010). Fixed points for multi-class queues. Preprint. Available at http://arxiv.org/abs/1003.3024v1.
[13] MOUNTFORD, T. AND PRABHAKAR, B. (1995). On the weak convergence of departures from an infinite series of $\cdot/M/1$ queues. *Ann. Appl. Prob.* **5,** 121–127.
[14] O'CONNELL, N. AND YOR, M. (2001). Brownian analogues of Burke's theorem. *Stoch. Process. Appl.* **96,** 285–304.
[15] SALMINEN, P. AND NORROS, I. (2001). On busy periods of the unbounded Brownian storage. *Queueing Systems* **39,** 317–333.
[16] WHITT, W. (2002). *Stochastic-Process Limits*. Springer, New York.

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