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Structural stability for the resonant porous penetrative convection

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We study the structural stability of a problem in a porous medium when the density of saturating liquid is a nonlinear function of temperature and an internal heat source is present. We prove a convergence result for the Forchheimer coefficient. That is to say, when $\lambda \to 0$, the solution of the non-isothermal flow in a porous medium of the Forchheimer type, see (1.1), can converge to the solution of the equivalent Darcy type.

Key words: Structural stability; Forchheimer equations; The Forchheimer coefficient; Darcy equations

1 Introduction

The question of continuous dependence or convergence of solutions of problems in partial differential equations on coefficients in the equations has been extensively studied in recent years for a variety of problems. This is sometimes referred to as the question of structural stability. The concept of structural stability in which the study of continuous dependence (or convergence) is on changes in the model itself rather than the initial data. Many references to work of this nature are given in the monograph of Ames and Straughan [2], which studies the structural stability with respect to changes in the model itself. This means changes in coefficients in the partial differential equations may be reflected physically by changes in constitutive parameters. We believe that the mathematical analysis of these equations will help to reveal their applicability in physics. On the other hand, continuous dependence (or convergence) results are important because of the inevitable error that arises in both numerical computation and physical measurement of data. It is relevant to know the magnitude of the effect of such errors in the solutions.

The model equations (Brinkman–Darcy–Forchheimer equations) describing flow in a porous medium are discussed by Nield and Beijan [14] and Straughan [26, 27]. Several papers in the literature have dealt with the Saint-Venant-type spatial decay results for Brinkman–Darcy–Forchheimer and other equations for porous media (see, e.g. [5, 9, 16–19, 24, 25]). More recent work on stability and continuous dependence questions in porous media problems has been carried out by [1, 3, 4, 6–8, 10–13, 16, 20–23].

In [28], Straughan investigated the continuous dependence on the heat source for the momentum equation for flow in a porous saturated material of the Forchheimer type

$$\begin{cases} u_i + \lambda \mid u \mid u_i = -p_{,i} + g_i T + h_i T^2, \\ \dfrac{\partial u_i}{\partial x_i} = 0, \\ \dfrac{\partial T}{\partial t} + u_i \dfrac{\partial T}{\partial x_i} = \Delta T + Q, \end{cases} \tag{1.1}$$

where u_i is the average fluid velocity in the porous medium, λ is the Forchheimer coefficient, T is the concentration (or the temperature) and p is the pressure. Here $g_i(x)$, $h_i(x)$ are gravity fields, and without loss of generality, we assume g_i, h_i satisfy $\mid g \mid\leqslant 1$, $\mid h \mid\leqslant 1$ and $\mid \nabla g \mid\leqslant 1$, $\mid \nabla h \mid\leqslant 1$. Here also Δ is the Laplacian operator and $Q(x,t)$ is a prescribed heat source (or sink).

Equations (1.1) hold in the region $\Omega \times [0,\tau]$, where Ω is a bounded, simply connected and star-shaped domain with boundary $\partial\Omega$ in R^3, and τ is a given number satisfying $0 \leqslant \tau < \infty$. Associated with (1.1), we impose the boundary conditions

$$u_i n_i = 0, \quad T = l(x,t) \quad (x,t) \in \partial\Omega \times [0,\tau], \tag{1.2}$$

and additionally the concentration is given at $t = 0$, i.e.

$$T(x,0) = T_0(x) \quad x \in \Omega. \tag{1.3}$$

In [28], Straughan obtained the continuous dependence result on the heat source in equations (1.1). We continue his work and study another aspect of structural stability. We will derive the convergence result on the Forchheimer coefficient, λ. We cannot follow the method presented in [28], because the case when $\lambda \to 0$ is more difficult to tackle than the case in [28].

In the present paper, the comma is used to indicate partial differentiation, and the differentiation with respect to the direction x_k is denoted as $,k$, thus $u_{,i}$ denotes $\frac{\partial u}{\partial x_i}$. The usual summation convection is employed with repeated Latin subscripts summed from 1 to 3. Hence, $u_{i,i} = \sum_{i=1}^{3} \frac{\partial u_i}{\partial x_i}$, and $\| \cdot \|$ denotes the norm of L^2.

2 A priori bounds for $\int_0^t \int_\Omega |\nabla u|^2 dx d\eta$, $\int_\Omega |u|^2 dx$ and $\int_0^t \int_\Omega |\nabla T|^2 dx d\eta$

In the course of producing the result of convergence on the coefficient of (1.1), we find it easy if we can derive an *a priori* bound or a maximum principle for the concentration T. In order to get a bound for T, we divide T into $T = T_1 + T_2$, where T_1 and T_2 satisfy the following equations respectively:

$$\begin{cases} \dfrac{\partial T_1}{\partial t} + u_i \dfrac{\partial T_1}{\partial x_i} = \Delta T_1, \\ T_1(x,t) = l(x,t) \quad (x,t) \in \partial\Omega \times [0,\tau], \\ T_1(x,0) = T_0(x) \quad x \in \Omega, \end{cases} \tag{2.1}$$

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and

$$\begin{cases} \dfrac{\partial T_2}{\partial t} + u_i \dfrac{\partial T_2}{\partial x_i} = \Delta T_2 + Q, \\ T_2(x,t) = 0 \quad (x,t) \in \partial\Omega \times [0,\tau], \\ T_2(x,0) = 0 \quad x \in \Omega. \end{cases} \tag{2.2}$$

In [15, pp. 432–433], Payne *et al.* reached the result

$$\sup_{[0,\tau]} \|T_1\|_\infty \leqslant T_M, \tag{2.3}$$

where

$$T_M = \max\left\{\|T_0\|_\infty, \sup_{[0,\tau]} l_\infty\right\},$$

and l_∞ is the maximum of l on $\partial\Omega$.

Now we want an *a priori* bound or a maximum principle for T_2. To this end, we form the combination

$$\int_0^t \int_\Omega T_2^{2p-1}(T_{2,\eta} + u_i T_{2,i} - \Delta T_2 - Q)dxd\eta = 0.$$

After some integration by parts, we can then show

$$\int_\Omega T_2^{2p} dx + \frac{2(2p-1)}{p} \int_0^t \int_\Omega T_{2,i}^p T_{2,i}^p dxd\eta = 2p \int_0^t \int_\Omega T_2^{2p-1} Q dxd\eta.$$

Hence, we get

$$\int_\Omega T_2^{2p} dx \leqslant (2p-1) \int_0^t \int_\Omega T_2^{2p} dxd\eta + k(p), \tag{2.4}$$

where $k(p) = \int_0^\tau \int_\Omega Q^{2p} dxd\eta$.

Inequality (2.4) is now integrated and then we take the $\frac{1}{2p}$ power to find

$$\left(\int_0^t \int_\Omega T_2^{2p} dxd\eta\right)^{\frac{1}{2p}} \leqslant \left(\int_0^t \int_\Omega e^{(2p-1)(t-\eta)} k(p) dxd\eta\right)^{\frac{1}{2p}}. \tag{2.5}$$

Let $p \to \infty$ and then (2.5) leads to

$$\sup_{[0,\tau]} \|T_2\|_\infty \leqslant e^\tau Q_M, \tag{2.6}$$

where Q_M is the maximum value of $Q(x,t)$ in $\Omega \times [0,\tau]$.

Combining (2.3) and (2.6), we obtain

$$\sup_{[0,\tau]} \|T\|_\infty \leqslant T^M, \tag{2.7}$$

where $T^M = T_M + e^\tau Q_m$.

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Since our convergence result needs the bounds for $\int_0^t\int_\Omega |\nabla u|^2 dx d\eta$ and $\int_\Omega |u|^2 dx$, we must derive bounds for various norms of u_i, T in terms of given data.

Starting with the identity

$$\int_\Omega u_i(u_i+\lambda|u|u_i+p_{,i}-g_iT-h_iT^2)dx=0,$$

and we have

$$\|u\|^2+2\lambda\int_\Omega |u|^3dx\leqslant 2\int_\Omega T^2dx+2\int_\Omega T^4dx\leqslant 2(T^M)^2|\Omega|(1+(T^M)^2). \tag{2.8}$$

The following argument (2.9)–(2.15) is as in [12]. For completeness, we include it here. We shall also require a bound for the gradient of u_i, and we start with

$$\int_\Omega u_{i,j}u_{i,j}dx=\int_\Omega u_{i,j}(u_{i,j}-u_{j,i})dx+\int_\Omega u_{i,j}u_{j,i}dx. \tag{2.9}$$

Integrating by parts, and using $(1.1)_2$, we obtain

$$\begin{aligned}\int_\Omega u_{i,j}u_{j,i}dx&=\oint_{\partial\Omega} u_{i,j}u_jn_ids-\int_\Omega u_{i,ij}u_jdx=\oint_{\partial\Omega}(u_in_i)_{,j}u_jds-\oint_{\partial\Omega}u_iu_jn_{i,j}ds\\&=-\oint_{\partial\Omega}u_iu_jn_{i,j}ds.\end{aligned} \tag{2.10}$$

If Ω is convex follows that

$$\oint_{\partial\Omega}u_iu_jn_{i,j}ds\geqslant 0.$$

Thus, for convex Ω,

$$\int_\Omega u_{i,j}u_{j,i}dx\leqslant 0. \tag{2.11}$$

For non-convex Ω with boundary of bounded curvature

$$\int_\Omega u_{i,j}u_{j,i}dx\leqslant k_0\oint_{\partial\Omega}|u|^2ds, \tag{2.12}$$

where k_0 depends on the Gaussian curvature of $\partial\Omega$ (see [29]).

In case Ω is non-convex, we may use the Poincaré inequality

$$\oint_{\partial\Omega}|u|^2ds\leqslant k_1\int_\Omega|u|^2dx+k_2\int_\Omega|\nabla u|^2dx, \tag{2.13}$$

where the constant k_2 may be small. For instance, if we introduce a vector field $q_i(x)$ satisfying

$$|q_i|,\quad |q_{i,j}|\leqslant M\quad x\in\Omega,\quad q_in_i\geqslant q_0>0\quad x\in\partial\Omega, \tag{2.14}$$

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we have

$$q_0 \oint_{\partial\Omega} |u|^2 ds \leqslant \oint_{\partial\Omega} q_j n_j |u|^2 ds = \int_\Omega q_{j,j}|u|^2 dx + 2\int_\Omega q_j u_i u_{i,j} dx$$
$$\leqslant M\left\{\left(1+\frac{1}{\varepsilon_1}\right)\int_\Omega |u|^2 + \varepsilon_1 \int_\Omega |\nabla u|^2 dx\right\},$$

where we have used the Schwarz inequality in the final step and ε_1 is an arbitrary positive constant.

Combining (2.9) and (2.11)–(2.13), no matter whether Ω is convex or non-convex, for k_2 sufficiently small, we have

$$\int_\Omega u_{i,j}u_{i,j}dx \leqslant \int_\Omega u_{i,j}(u_{i,j}-u_{j,i})dx + k_3\int_\Omega |u|^2 dx. \tag{2.15}$$

Since equation $(1.1)_1$ does not contain the Laplacian, we need a preliminary estimate for the quantity J which is defined in [23] by

$$J(t) = \int_\Omega u_{i,j}(u_{i,j}-u_{j,i})dx. \tag{2.16}$$

Using (1.1)–(1.3), we get

$$\begin{aligned} J &= \int_\Omega (u_{i,j}-u_{j,i})[-\lambda(|u|u_i)_{,j} - p_{,ij} + (g_iT)_{,j} - (h_iT^2)_{,j}]dx \\ &= -\lambda\int_\Omega u_{i,j}|u|u_{i,j}dx - \lambda\int_\Omega u_{i,j}u_i\frac{u_k u_{k,j}}{|u|}dx + \lambda\oint_{\partial\Omega} u_{j,i}|u|u_i n_j ds - \int_\Omega (u_{i,j}-u_{j,i})g_{i,j}T dx \\ &\quad - \int_\Omega (u_{i,j}-u_{j,i})g_iT_{,j}dx - \int_\Omega (u_{i,j}-u_{j,i})h_{i,j}T^2 dx - 2\int_\Omega (u_{i,j}-u_{j,i})h_iTT_{,j}dx. \end{aligned} \tag{2.17}$$

Using the Schwarz inequality, we get for arbitrary ε_2

$$\begin{aligned} J \leqslant& -\lambda\int_\Omega u_{i,j}|u|u_{i,j}dx - \lambda\int_\Omega u_{i,j}u_i\frac{u_k u_{k,j}}{|u|}dx + \lambda\oint_{\partial\Omega} u_{j,i}|u|u_i n_j ds \\ &+ 4\varepsilon_2\int_\Omega (u_{i,j}-u_{j,i})(u_{i,j}-u_{j,i})dx + \frac{1}{4\varepsilon_2}\int_\Omega T_{,j}T_{,j}dx + \frac{1}{4\varepsilon_2}\int_\Omega T^2dx + \frac{1}{4\varepsilon_2}\int_\Omega T^4 dx \\ &+ \frac{(T^M)^2}{\varepsilon_2}\int_\Omega T_{,j}T_{,j}dx. \end{aligned} \tag{2.18}$$

Note that

$$\int_\Omega (u_{i,j}-u_{j,i})(u_{i,j}-u_{j,i})dx = 2\int_\Omega (u_{i,j}-u_{j,i})u_{i,j}dx. \tag{2.19}$$

Thus, we choose $\varepsilon_2 = \frac{1}{16}$ in (2.18), and obtain

$$\begin{aligned} J \leqslant& -2\lambda\int_\Omega u_{i,j}|u|u_{i,j}dx - 2\lambda\int_\Omega u_{i,j}u_i\frac{u_k u_{k,j}}{|u|}dx + 2\lambda\oint_{\partial\Omega} u_{j,i}|u|u_i n_j ds + (8+(32T^M)^2) \\ &\times \int_\Omega T_{,j}T_{,j}dx + 8|\Omega|(T^M)^2(1+(T^M)^2). \end{aligned} \tag{2.20}$$

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Now, we need to bound $\oint_{\partial\Omega} u_{j,i}|u|u_i n_j ds$. We have

$$\oint_{\partial\Omega} u_{j,i}|u|u_i n_j ds = \oint_{\partial\Omega} (u_j n_j)_{,i}|u|u_i ds - \oint_{\partial\Omega} u_j n_{j,i}|u|u_i ds$$
$$= -\oint_{\partial\Omega} u_j n_{j,i}|u|u_i ds. \tag{2.21}$$

If Ω is convex, we can get

$$\oint_{\partial\Omega} u_j|u|u_i n_{j,i} ds \geqslant 0,$$

and thus

$$\oint_{\partial\Omega} u_{j,i}|u|u_i n_j ds \leqslant 0. \tag{2.22}$$

If Ω is non-convex, we write on $\partial\Omega$ that

$$u_{j,i} = \frac{\partial u_j}{\partial n} n_i + a^{\alpha\beta} x^i_{;\alpha} \frac{\partial u_j}{\partial \theta^\beta},$$

where θ^β are the surface coordinates, $a^{\alpha\beta}$ is determined from the surface metric tensor and $x^i_{;\alpha}$ are tangent vectors. Then

$$\oint_{\partial\Omega} |u|u_i u_{j,i} n_j ds = \oint_{\partial\Omega} |u|u_i n_i \frac{\partial u_j}{\partial n} n_j ds + \oint_{\partial\Omega} |u|u_i n_j a^{\alpha\beta} x^i_{;\alpha} \frac{\partial u_j}{\partial \theta^\beta} ds.$$

The first term on the right is zero due to the boundary conditions, and the second term may be integrated by parts to find that

$$\oint_{\partial\Omega} |u|u_i u_{j,i} n_j ds = \oint_{\partial\Omega} |u|u_i a^{\alpha\beta} x^i_{;\alpha} (n_j u_j)_{;\beta} ds - \oint_{\partial\Omega} |u|u_i a^{\alpha\beta} x^i_{;\alpha} u_j n^j_{;\beta} ds$$
$$= -\oint_{\partial\Omega} |u|u_i a^{\alpha\beta} x^i_{;\alpha} u_j n^j_{;\beta} ds.$$

Next, by the Gauss–Weingarten relationship $n^j_{;\beta} = -b^\xi_\beta x^j_{;\xi}$, where b^ξ_β is determined from the second fundamental form of the surface, we find that

$$\oint_{\partial\Omega} |u|u_i u_{j,i} n_j ds = \oint_{\partial\Omega} |u|u_i x^i_{;\alpha} u_j x^j_{;\xi} b^{\xi\alpha} ds. \tag{2.23}$$

From (2.23), we have

$$\left|\oint_{\partial\Omega}\right| u|u_i u_{j,i} n_j ds| \leqslant k_4 \oint_{\partial\Omega} |u|^3 ds, \tag{2.24}$$

where $k_4 = \max_{\partial\Omega}\{|x^i_{;\alpha}|^2, |b^{\xi\alpha}|\}$.
Because Ω is star-shaped, we define

$$m = \min_{\partial\Omega} |x_k n_k| > 0. \tag{2.25}$$

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We note that from the divergence theorem

$$\oint_{\partial\Omega} x_k n_k |u|^3 ds = \int_\Omega (x_k|u|^3)_{,k} dx = 3\int_\Omega |u|^3 dx + 3\int_\Omega |u| u_i u_{i,j} x_j dx.$$

Thus, by using the arithmetic–geometric mean inequality and (2.25), for arbitrary positive ε_3, we have

$$m\oint_{\partial\Omega} |u|^3 ds \leqslant \left(3+\frac{3}{2\varepsilon_3}\right)\int_\Omega |u|^3 dx + \frac{3R^2\varepsilon_3}{2}\int_\Omega |u| u_{i,j} u_{i,j} dx, \tag{2.26}$$

where R denotes the diameter of the bounded domain Ω.

So

$$0 = \oint_{\partial\Omega} |u| x_j u_j u_i n_i ds = \int_\Omega |u|^3 dx + \int_\Omega |u| x_j u_{j,i} u_i dx + \int_\Omega \frac{u_k u_{k,i} u_i u_j x_j}{|u|} dx.$$

Then, by using the arithmetic–geometric mean inequality on this equation, for positive constants ξ and μ, with $\xi+\mu<2$,

$$[2-(\xi+\mu)]\int_\Omega |u|^3 dx \leqslant \frac{R^2}{\xi}\int_\Omega |u| u_{i,j} u_{i,j} dx + \frac{R^2}{\mu}\int_\Omega \frac{u_k u_{k,j} u_i u_{i,j}}{|u|} dx. \tag{2.27}$$

Hence, employing (2.27) in (2.26),

$$\oint_{\partial\Omega} |u|^3 ds \leqslant \frac{k_5 R^2}{m}\int_\Omega |u| u_{i,j} u_{i,j} dx + k_6 R^2 \int_\Omega \frac{u_k u_{k,j} u_i u_{i,j}}{|u|} dx, \tag{2.28}$$

where $k_5 = \frac{3+\frac{3}{2\varepsilon_3}}{\xi[2-(\xi+\mu)]} + \frac{3\varepsilon_3}{2}$, $k_6 = \frac{3+\frac{3}{2\varepsilon_3}}{\mu[2-(\xi+\mu)]}$.

Combining (2.20), (2.24) and (2.28), we have

$$\begin{aligned} J \leqslant\ & 2\lambda\left(\frac{k_4 k_5 R^2}{m} - 1\right)\int_\Omega |u| u_{i,j} u_{i,j} dx + 2\lambda(k_4 k_6 R^2 - 1)\int_\Omega \frac{u_k u_{k,j} u_i u_{i,j}}{|u|} dx \\ & + (8+(32T^M)^2)\int_\Omega T_{,j} T_{,j} dx + 8|\Omega|(T^M)^2(1+(T^M)^2). \end{aligned} \tag{2.29}$$

To make (2.29) useful requires that the geometry of Ω be such that

$$k_4 k_5 R^2 \leqslant m, \quad K_4 k_6 R^2 \leqslant 1.$$

Thus, from (2.29), we get

$$J \leqslant (8+(32T^M)^2)\int_\Omega T_{,j} T_{,j} dx + 8|\Omega|(T^M)^2(1+(T^M)^2). \tag{2.30}$$

This holds whether Ω is convex or non-convex.

A combination of (2.8), (2.15), (2.16) and (2.30) leads to

$$\int_0^t \int_\Omega |\nabla u|^2 dx d\eta \leqslant (8+(32T^M)^2)\int_0^t \int_\Omega T_{,j} T_{,j} dx d\eta + k_1(t), \tag{2.31}$$

where $k_1(t) = 2k_3(T^M)^2|\Omega|(1+(T^M)^2) + 8|\Omega|(T^M)^2(1+(T^M)^2)$.

If we want to bound $\int_0^t \int_\Omega |\nabla u|^2 dx d\eta$, we need an *a priori* bound for $\int_0^t \int_\Omega T_{,j} T_{,j} dx d\eta$. To this end, we introduce the harmonic function H, which adopts the same boundary values as T, so we define H by

$$\Delta H = 0 \quad (x,t) \in \Omega \times [0,\tau], \tag{2.32}$$

$$H(x,t) = l(x,t) \quad (x,t) \in \partial\Omega \times [0,\tau]. \tag{2.33}$$

We then form the identity

$$\int_0^t \int_\Omega (T-H)(T_{,\eta} + u_i T_{,i} - \Delta T - Q) dx d\eta = 0. \tag{2.34}$$

Next, we perform several integrations on (2.34) and use the boundary values and properties of H to see that

$$\begin{aligned} 0 = &\frac{1}{2}\|T(t)\|^2 - \frac{1}{2}\|T(0)\|^2 - \int_\Omega HT dx|_{\eta=t} + \int_\Omega H_0 T_0 dx|_{\eta=t} + \int_0^t \int_\Omega H_{,\eta} T dx d\eta \\ &- \int_0^t \int_\Omega H u_i T_{,i} dx d\eta + \int_0^t \int_\Omega T_{,i} T_{,i} dx d\eta - \int_0^t \oint_{\partial\Omega} l \frac{\partial H}{\partial n} dA d\eta - \int_0^t \int_\Omega (T-H) Q dx d\eta, \end{aligned} \tag{2.35}$$

where T_0, H_0 denote $T(t)|_{t=0}$ and $H(t)|_{t=0}$.

To handle the cubic term in (2.35), we let l_m be the maximum value of $l(x,t)$ on $\partial\Omega \times [0,\tau]$ (l_m is taken to be positive) and then since H is harmonic, we know by the maximum principle that $H \leqslant l_m$. Upon employing the Cauchy–Schwarz and arithmetic–geometric mean inequalities, we derive

$$\begin{aligned} \int_0^t \int_\Omega H u_i T_{,i} dx d\eta \leqslant l_m \left(\int_0^t \|u\|^2 d\eta \right)^{\frac{1}{2}} \left(\int_0^t \|\nabla T\|^2 d\eta \right)^{\frac{1}{2}} \leqslant \frac{1}{2} \int_0^t \|\nabla T\|^2 d\eta \\ + \frac{1}{2} l_m^2 \int_0^t \|u\|^2 d\eta. \end{aligned} \tag{2.36}$$

Therefore, (2.36) can be rewritten as

$$\int_0^t \int_\Omega H u_i T_{,i} dx d\eta \leqslant \frac{1}{2} \int_0^t \|\nabla T\|^2 d\eta + \frac{1}{2} l_m^2 h_1(t), \tag{2.37}$$

where $h_1(t) = 2(T^M)^2 |\Omega| (1 + (T^M)^2) t$.

From the arithmetic–geometric mean inequality it follows that

$$\begin{aligned} \int_\Omega HT dx &\leqslant \int_\Omega H^2 dx + \frac{1}{4} \int_\Omega T^2 dx, \\ - \int_\Omega H_0 T_0 dx &\leqslant \frac{1}{2} \int_\Omega H_0^2 dx + \frac{1}{2} \int_\Omega T_0^2 dx, \\ \int_0^t \int_\Omega H_{,\eta} T dx d\eta &\leqslant \int_0^t \int_\Omega H_{,\eta}^2 dx d\eta + \frac{1}{4} \int_0^t \int_\Omega T^2 dx d\eta, \end{aligned}$$

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$$\int_0^t \int_\Omega HQdxd\eta \leqslant \frac{1}{2}\int_0^t \int_\Omega H^2dxd\eta + \frac{1}{2}\int_0^t \int_\Omega Q^2dxd\eta,$$
$$\int_0^t \int_\Omega TQdxd\eta \leqslant \int_0^t \int_\Omega Q^2dxd\eta + \frac{1}{4}\int_0^t \int_\Omega T^2dxd\eta,$$

and by the use of the Cauchy–Schwarz inequality, one finds

$$\int_0^t \oint_{\partial\Omega} l\frac{\partial H}{\partial n}dAd\eta \leqslant \left(\int_0^t \oint_{\partial\Omega} l^2dAd\eta\right)^{\frac{1}{2}} \times \left(\int_0^t \oint_{\partial\Omega} \left(\frac{\partial H}{\partial n}\right)^2 dAd\eta\right)^{\frac{1}{2}}.$$

We next employ these estimates together with (2.37) in equation (2.35) to arrive at

$$\begin{aligned}\frac{1}{4}\|T(t)\|^2 + \frac{1}{2}\int_0^t \|\nabla T\|^2 d\eta \leqslant \|T_0\|^2 + \|H\|^2 + \frac{1}{2}\|H_0\|^2 + \int_0^t \|H_{,\eta}\|^2 d\eta + \int_0^t \|H\|^2 d\eta\\ +\frac{3}{2}\int_0^t \|Q\|^2 d\eta + \left(\int_0^t \oint_{\partial\Omega} l^2 dAd\eta\right)^{\frac{1}{2}} \times \left(\int_0^t \oint_{\partial\Omega}\left(\frac{\partial H}{\partial n}\right)^2 dAd\eta\right)^{\frac{1}{2}} + \frac{1}{2}\int_0^t \|T\|^2 d\eta\\ +\frac{1}{2}l_m^2 h_1(t). \qquad (2.38)\end{aligned}$$

In order to obtain an *a priori* estimate, we need to demonstrate that the terms involving H are bounded by the given data. We will use the following Rellich identity

$$0 = \int_\Omega x^i H_{,i}\Delta H dx.$$

Using integration by parts, we can find

$$\begin{aligned}0 = \int_\Omega x^i H_{,i}\Delta H dx &= -\int_\Omega |\nabla H|^2 dx - \int_\Omega x^i H_{,ij}H_{,j}dx + \oint_{\partial\Omega} x^i H_{,i}H_{,j}n_j dA\\ &= -\int_\Omega |\nabla H|^2 dx + \frac{3}{2}\int_\Omega H_{,j}H_{,j}dx - \frac{1}{2}\oint_{\partial\Omega} x^i H_{,j}H_{,j}n_i dA + \oint_{\partial\Omega} x^i H_{,i}H_{,j}n_j dA.\end{aligned}$$

We can get that

$$\frac{1}{2}\int_\Omega |\nabla H|^2 dx - \frac{1}{2}\oint_{\partial\Omega} x^i H_{,j}H_{,j}n_i dA + \oint_{\partial\Omega} x^i H_{,i}H_{,j}n_j dA = 0. \qquad (2.39)$$

Since

$$H_{,i} = \frac{\partial H}{\partial n}n_i + s_i\nabla_s H,$$

where the normal and tangential vectors to $\partial\Omega$ are n and s, respectively, and $\nabla_s H$ is the tangential derivative, we have

$$\frac{1}{2}\oint_{\partial\Omega} x^i H_{,j}H_{,j}n_i dA = \frac{1}{2}\oint_{\partial\Omega} x^i\left(\frac{\partial H}{\partial n}\right)^2 n_i dA + \frac{1}{2}\oint_{\partial\Omega} x^i|\nabla_s H|^2 n_i dA,$$
$$\oint_{\partial\Omega} x^i H_{,i}H_{,j}n_j dA = \oint_{\partial\Omega} x^i\left(\frac{\partial H}{\partial n}\right)^2 n_i dA + \oint_{\partial\Omega} x^i s^i\nabla_s H\frac{\partial H}{\partial n}dA.$$

Thus, (2.39) can be rewritten as

$$\frac{1}{2}\int_{\Omega}|\nabla H|^2dx+\frac{1}{2}\oint_{\partial\Omega}x^i\left(\frac{\partial H}{\partial n}\right)^2n_idA=\frac{1}{2}\oint_{\partial\Omega}x^i|\nabla_sH|^2n_idA-\oint_{\partial\Omega}x^is^i\nabla_sH\frac{\partial H}{\partial n}dA.$$

If we assume Ω is star-shaped with respect to the region and set $\min_{\partial\Omega}|x^in_i|=m$, then there exist positive constants c_1 and c_2 such that

$$\int_{\Omega}|\nabla H|^2dx+c_1\oint_{\partial\Omega}\left(\frac{\partial H}{\partial n}\right)^2dA\leqslant c_2\oint_{\partial\Omega}|\nabla_sH|^2dA. \tag{2.40}$$

Moreover, we have, for any positive eigenvalue of the membrane problem

$$\Delta H+\lambda H=0\quad x\in\Omega,$$
$$H=h\quad x\in\partial\Omega.$$

We have

$$\lambda\int_{\Omega}H^2dx=-\oint_{\partial\Omega}H\frac{\partial H}{\partial n}dA+\int_{\Omega}H_{,i}H_{,i}dx,$$

which results in the inequality

$$\int_{\Omega}H^2dx\leqslant\frac{1}{2\lambda}\oint_{\partial\Omega}H^2dA+\frac{1}{2\lambda}\oint_{\partial\Omega}\left(\frac{\partial H}{\partial n}\right)^2dA+\frac{1}{\lambda}\int_{\Omega}H_{,i}H_{,i}dx.$$

In view of (2.40), we obtain

$$\int_{\Omega}H^2dx\leqslant c_3\oint_{\partial\Omega}H^2dA+c_4\oint_{\partial\Omega}|\nabla_sH|^2dA, \tag{2.41}$$

where c_3 and c_4 are computable constants. Recall that $H=l$ on the boundary of Ω, so (2.41) shows that $\int_{\Omega}H^2dx$ is bounded by the data.

By integrating (2.41) with respect to t, we also obtain

$$\int_0^t\int_{\Omega}H^2dxd\eta\leqslant c_3\int_0^t\oint_{\partial\Omega}H^2dAd\eta+c_4\int_0^t\oint_{\partial\Omega}|\nabla_sH|^2dAd\eta, \tag{2.42}$$

and from (2.40), an integration in t gives

$$c_1\int_0^t\oint_{\partial\Omega}\left(\frac{\partial H}{\partial n}\right)^2dAd\eta\leqslant c_2\int_0^t\oint_{\partial\Omega}|\nabla_sH|^2dA. \tag{2.43}$$

Differentiating (2.32) and (2.33) with respect to t, we see that

$$\Delta H_{,t}=0\quad x\in\Omega\times[0,\tau],$$
$$H_{,t}=h_{,t}\quad x\in\partial\Omega\times[0,\tau].$$

Thus, we can obtain a bound for $\int_0^t \int_\Omega H_{,\eta} H_{,\eta} dx d\eta$ in a similar fashion, leading to the result

$$\int_0^t \int_\Omega H_{,\eta}^2 dx d\eta \leqslant c_5 \int_0^t \oint_{\partial\Omega} H_{,\eta}^2 dA d\eta + c_6 \int_0^t \oint_{\partial\Omega} |\nabla_s H_{,\eta}|^2 dA d\eta. \tag{2.44}$$

From (2.38), we can see

$$\|T(t)\|^2 + 2\int_0^t \|\nabla T\|^2 d\eta \leqslant D_1(t) + 2\int_0^t \int_\Omega T^2 dx d\eta, \tag{2.45}$$

where $D_1(t)$ denotes the data items in (2.38).

Thus,

$$\int_0^t \|\nabla T\|^2 d\eta \leqslant \frac{D_1(t)}{2} + (T^M)^2 |\Omega| t = h_3(t). \tag{2.46}$$

Inserting (2.46) into (2.31), we get

$$\int_0^t \int_\Omega |\nabla u|^2 dx ds \leqslant h_4(t), \tag{2.47}$$

where $h_4(t) = (8 + (32T^M)^2) h_3(t) + k_1(t)$.

3 Convergence as the Forchheimer coefficient tends to zero

Now, let (u_i, p, T) be a solution to the boundary initial-value problem for the Forchheimer equations

$$\begin{cases} u_i + \lambda \mid u \mid u_i = -p_{,i} + g_i T + h_i T^2 \quad (x,t) \in \Omega \times [0,\tau], \\ \dfrac{\partial u_i}{\partial x_i} = 0 \quad (x,t) \in \Omega \times [0,\tau], \\ \dfrac{\partial T}{\partial t} + u_i \dfrac{\partial T}{\partial x_i} = \Delta T + Q \quad (x,t) \in \Omega \times [0,\tau], \end{cases} \tag{3.1}$$

$$u_i n_i = 0, \quad T = l(x,t) \quad (x,t) \in \partial\Omega \times [0,\tau], \tag{3.2}$$

$$T(x,0) = T_0(x) \quad x \in \Omega. \tag{3.3}$$

Furthermore, let (v_i, q, S) be a solution to the corresponding Darcy problem

$$\begin{cases} v_i = -q_{,i} + g_i S + h_i S^2 \qquad (x,t) \in \Omega \times [0,\tau], \\ \dfrac{\partial v_i}{\partial x_i} = 0 \quad (x,t) \in \Omega \times [0,\tau], \\ \dfrac{\partial S}{\partial t} + v_i \dfrac{\partial S}{\partial x_i} = \Delta S + Q \quad (x,t) \in \Omega \times [0,\tau], \end{cases} \tag{3.4}$$

$$v_i n_i = 0, \quad S = l(x,t) \quad (x,t) \in \partial\Omega \times [0,\tau], \tag{3.5}$$

$$S(x,0) = T_0(x) \quad x \in \Omega. \tag{3.6}$$

The object of this section is to demonstrate solutions of (3.1) converge to solutions of (3.4) as $\lambda \to 0$.

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We define the difference variables w_i, π and θ by

$$w_i = u_i - v_i, \pi = p - q, \theta = T - S \tag{3.7}$$

and then (w_i, π, θ) satisfies the boundary initial-value problem

$$\begin{cases} w_i + \lambda \mid u \mid u_i = -\pi_{,i} + g_i\theta + h_i(S+T)\theta & (x,t) \in \Omega \times [0,\tau], \\ \dfrac{\partial w_i}{\partial x_i} = 0 \quad (x,t) \in \Omega \times [0,\tau], \\ \dfrac{\partial \theta}{\partial t} + w_i \dfrac{\partial T}{\partial x_i} + v_i \dfrac{\partial \theta}{\partial x_i} = \Delta\theta \quad (x,t) \in \Omega \times [0,\tau], \end{cases} \tag{3.8}$$

$$w_i n_i = \theta = 0 \quad (x,t) \in \partial\Omega \times [0,\tau], \tag{3.9}$$

$$w_i(x,0) = \theta(x,0) = 0 \quad x \in \Omega. \tag{3.10}$$

We then multiply $(3.8)_1$ by w_i and integrate over Ω to find

$$\|w\|^2 + \lambda \int_\Omega |u|u_i w_i dx = \int_\Omega (g_i\theta + h_i(S+T)\theta) w_i dx. \tag{3.11}$$

Using the Schwarz inequality, we obtain

$$\|w\|^2 \leqslant (1 + X_m)\|\theta\|^2 + 2\lambda \left(\int_\Omega |u|^4 dx \right)^{\frac{1}{2}} \left(\int_\Omega w_i w_i dx \right)^{\frac{1}{2}}, \tag{3.12}$$

where $X_m = 2T^M$.

We can get the same result as (2.8)

$$\|v\|^2 \leqslant 2\int_\Omega S^2 dx + 2\int_\Omega S^4 dx \leqslant 2(T^M)^2|\Omega|(1 + (T^M)^2). \tag{3.13}$$

Combining (2.8) and (3.13), we get

$$\|w\|^2 \leqslant 2(\|u\|^2 + \|v\|^2) \leqslant k^2, \tag{3.14}$$

where $k = (8(T^M)^2|\Omega|(1 + (T^M)^2))^{\frac{1}{2}}$.

Combining (3.12) and (3.14) , we obtain

$$\|w\|^2 \leqslant (1 + X_m)\|\theta\|^2 + 2k\lambda \left(\int_\Omega |u|^4 dx \right)^{\frac{1}{2}}. \tag{3.15}$$

Next, we multiply $(3.8)_3$ by θ and integrate over Ω to find

$$\frac{1}{2}\frac{d}{dt}\|\theta\|^2 + \int_\Omega w_i \frac{\partial T}{\partial x_i}\theta dx + \int_\Omega \theta v_i \frac{\partial \theta}{\partial x_i} dx = \int_\Omega \theta\Delta\theta dx,$$

thus, we have

$$\frac{1}{2}\frac{d}{dt}\|\theta\|^2 = -\int_\Omega |\nabla\theta|^2 dx + \int_\Omega w_i T \theta_{,i} dx. \tag{3.16}$$

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Since we have $\sup_{[0,\tau]} |T| \leqslant T^M$, we obtain

$$\frac{d}{dt}\|\theta\|^2 \leqslant \frac{(T^M)^2}{2}\int_\Omega w^2 dx. \tag{3.17}$$

Combining (3.15) and (3.17), we get

$$\frac{d}{dt}\|\theta\|^2 \leqslant Y_m\|\theta\|^2 + (T^M)^2 k\lambda \left(\int_\Omega |u|^4 dx\right)^{\frac{1}{2}}, \tag{3.18}$$

where $Y_m = \frac{1}{2}(1+X_m)(T^M)^2$.

An integration of (3.18) leads to

$$\|\theta\|^2 \leqslant (T^M)^2 k\lambda e^{Y_m t}\int_0^t \left(\int_\Omega |u|^4 dx\right)^{\frac{1}{2}} d\eta. \tag{3.19}$$

We must give a bound for $\int_0^t \left(\int_\Omega |u|^4 dx\right)^{\frac{1}{2}} d\eta$.

Using the Ladyzenskaya inequality, or using the result (B.17) in [12] by choosing $\delta = 1$, we can get

$$\left(\int_\Omega |u|^4 dx\right)^{\frac{1}{2}} \leqslant M\left\{\frac{5}{4}\|u\|^2 + \frac{3}{4}\|\nabla u\|^2\right\}. \tag{3.20}$$

A combination of (2.8), (2.47) and (3.20) leads to

$$\int_0^t \left(\int_\Omega |u|^4 dx\right)^{\frac{1}{2}} d\eta \leqslant k_2(t), \tag{3.21}$$

where $k_2(t) = \frac{5}{2}M(T^M)^2|\Omega|(1+(T^M)^2) + \frac{3}{4}Mh_4(t)$.

We can obtain

$$\|\theta\|^2 \leqslant (T^M)^2 k\lambda e^{Y_m t} k_2(t). \tag{3.22}$$

Inserting (3.21) and (3.22) into (3.15), we have

$$\int_0^t \|w\|^2 d\eta \leqslant (1+X_m)(T^M)^2 k\lambda e^{Y_m t}\int_0^t k_2(\eta)d\eta + k\lambda k_2(t). \tag{3.23}$$

Inequalities (3.22) and (3.23) demonstrate the convergence of u_i to v_i, T to S as $\lambda \to 0$ in the indicated measure.

4 Conclusions

In this paper, we only study the structural stability for the resonant porous penetrative convection in a bounded domain Ω, and we get the result of convergence of solutions for the Fochheimer coefficient. Following the method proposed in this paper, we can easily get the result of continuous dependence for the Fochheimer coefficient. For the case of the unbounded domain, the method used would be absolutely new, we will consider this case in another paper. The study of the structural stability of these equations in an unbounded domain would be interesting.

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