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SOLUTIONS OF p-LAPLACE EQUATIONS WITH INFINITE BOUNDARY VALUES: THE CASE OF NON-AUTONOMOUS AND NON-MONOTONE NONLINEARITIES

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Abstract For a non-negative and non-trivial real-valued continuous function h on $\bar{\Omega} \times [0,\infty)$ such that $h(x,0)=0$ for all $x \in \Omega$, we study the boundary-value problem

$$\left.\begin{aligned} \Delta_p u &= h(x,u) \quad \text{in } \Omega, \\ u &= \infty \qquad\quad \text{on } \partial\Omega, \end{aligned}\right\} \tag{BVP}$$

where $\Omega \subseteq \mathbb{R}^N$, $N \geqslant 2$, is a bounded smooth domain and $\Delta_p := \operatorname{div}(|\mathrm{D}u|^{p-2}\mathrm{D}u)$ is the p-Laplacian. This work investigates growth conditions on $h(x,t)$ that would lead to the existence or non-existence of distributional solutions to (BVP). In a major departure from past works on similar problems, in this paper we do not impose any special structure on the inhomogeneous term $h(x,t)$, nor do we require any monotonicity condition on h in the second variable. Furthermore, $h(x,t)$ is allowed to vanish in either of the variables.

Keywords: boundary blow-up solution; minimality principle; p-Laplacian

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1. Introduction

Let Ω be a bounded smooth domain in $\mathbb{R}^N$ for $N \geqslant 2$, let $f\colon [0,\infty) \to [0,\infty)$ be a continuous function with $f(0)=0$, and let $\omega \in C(\Omega,[0,\infty))$. The boundary-value problem

$$\left.\begin{aligned} \Delta u &= \omega(x) f(u) \quad \text{in } \Omega, \\ u &= \infty \qquad\qquad \text{on } \partial\Omega, \end{aligned}\right\} \tag{1.1}$$

has been investigated extensively. The boundary condition is understood in the sense that $u(x) \to \infty$ as $x \to \partial\Omega$. Solutions of (1.1), when they exist, are referred to as boundary blow-up solutions, explosive solutions or large solutions. It appears that this type of problem was first considered by Bieberbach in 1916 (see [**3**]). He studied (1.1) in the plane ($N=2$) with $f(t)=\exp(t)$ in order to address a question that arose in mathematical physics. Later, a geometric problem prompted Radmacher [**41**] to study the same problem in space ($N = 3$). In their pioneering works, Keller [**23**] and Osserman [**40**] identified a necessary and sufficient condition on a smooth increasing function $f(t)$ for (1.1) to admit a classical solution in Ω when ω is a positive constant. In the literature this condition has come to be known as the Keller–Osserman condition. Since then, numerous existence results for (1.1) have been investigated when f is a smooth increasing function and the potential $\omega \in C(\Omega)$ is non-negative. For instance, in [**24**] Lair shows that the Keller–Osserman condition remains a necessary and sufficient condition for existence of a solution to (1.1) provided that $\omega \in C(\bar{\Omega}, [0, \infty))$ satisfies the so-called circumferentially positive (or c-positive) condition, namely, if $\omega(x_0) = 0$ for some $x_0 \in \Omega$, then there is a domain $\mathcal{O}$ with $x_0 \in \mathcal{O} \subset\subset \Omega$ such that ω is positive on the boundary $\partial\mathcal{O}$ (see [**25**]). In recent years several questions related to (1.1) have been studied. Investigations on existence, asymptotic boundary behaviour and uniqueness have received particular attention. Here we mention a few of the works of Bandle and Marcus [**1**, **2**], Dindoš [**10**], Lazer and McKenna [**26**, **27**], Loewner and Nirenberg [**29**], Cîrstea and Rădulescu [**5**, **6**], García-Milián *et al*. [**14**–**16**], Lair [**24**], López-Gómez [**30**–**34**], Marcus and Véron [**35**, **36**], Matero [**37**, **38**], Véron [**45**, **46**], Zhang [**47**], and Zhang *et al*. [**48**]. We refer the reader to the monograph [**42**] for an extensive list of references.

We should mention that extensive work has also been done when the Laplacian in (1.1) is replaced by other elliptic operators. We cite the works of Du and Guo [**11**], and Gladiali and Porru [**18**] that investigate (1.1) when the Laplace operator is replaced by the p-Laplace operator, and for equations involving the Monge–Ampère operator we refer the reader to [**19**, **38**], while [**22**, **43**] deal with (1.1) with the Laplace operator replaced by the k-Hessian operator.

Some other important investigations related to problem (1.1) for non-monotonic nonlinearities with special structure are [**4**, **15**, **30**]. Partly motivated by this work, in this paper we seek to establish the existence of boundary blow-up weak solutions to equations related to the p-Laplacian. To state the problem, let $h\colon \bar{\Omega} \times [0, \infty) \to [0, \infty)$ be a continuous function. In this work we wish to study some general conditions on h under which the following boundary-value problem admits a local weak solution $u \in W^{1,p}_{\mathrm{loc}}(\Omega) \cap C(\Omega)$:

$$\left.\begin{aligned} \Delta_p u &= h(x, u) \quad \text{in } \Omega, \\ u &= \infty \quad \text{on } \partial\Omega. \end{aligned}\right\} \tag{1.2}$$

Here, $1 < p < \infty$ and $\Delta_p u := \operatorname{div}(|\mathrm{D}u|^{p-2}\mathrm{D}u)$ is the p-Laplacian. We would like to point out that, unlike the work available in the literature of which we are aware, in this paper we consider problem (1.2) for a fairly general inhomogeneous term $h(x, t)$. In particular, we do not require $h(x, t)$ to be monotonic in the second variable, we allow $h(x, t)$ to vanish in $\Omega \times (0, \infty)$, and no special structure is imposed on $h(x, t)$. We introduce an appropriate Keller–Osserman-type condition that reduces to the standard Keller–Osserman condition when the inhomogeneous term has the special form $h(x, t) = \omega(x)f(t)$ for increasing $f(t)$. In the event that $h(x, t) = \omega(x)f(t)$, where $f(t)$ is increasing, the main tool used in the investigation of (1.2) is the comparison principle (see [**7**, **11**, **38**, **39**]). While the comparison principle is not available when the inhomogeneous term $h(x, t)$ is not monotonic, we develop a useful minimality principle that serves as a substitute for the comparison principle. This is motivated by the recent paper [**12**] in which Dumont *et al*. consider the problem of existence of boundary blow-up solutions to problem (1.1) with $\omega(x) \equiv 1$, but without requiring f to be non-decreasing. In [**12**] the authors use a minimality principle as a replacement to the comparison principle.

The paper is organized as follows. In § 2, we will fix some notation, provide some basic definitions and recall some useful results that will be needed in the paper. In this section we also develop a minimality principle that will be used as an important tool that serves as a replacement to the comparison principle in the investigation of the main problem when $h(x,t)$ is not necessarily monotonic. In § 3 we identify a condition, referred to as the Keller–Osserman condition on autonomous, but non-monotonic, nonlinearities $g(t)$, and we use this condition to show existence of boundary blow-up solutions on balls of arbitrarily small radii. This is a refinement of the standard Keller–Osserman condition used when $g(t)$ is increasing. In § 4 we introduce a useful condition, called the lower Keller–Osserman condition, that is useful for studying existence of boundary blow-up solutions of (1.2) with non-autonomous and non-monotonic inhomogeneous term $h(x,t)$. In § 5 the upper Keller–Osserman condition on h will be introduced and this will be shown to be a necessary condition for existence of boundary blow-up solutions to (1.2). In this section we will introduce yet another condition on the inhomogeneous term $h(x,t)$ that if satisfied at a boundary point of Ω, is such that problem (1.2) would fail to admit a positive solution. In § 6 there is a brief discussion of the lower (upper) Keller–Osserman conditions vis-a-vis the special structure inhomogeneous term $h(x,t) = \omega(x)f(t)$. Here f is not necessarily monotonic. In this case, the lower and upper Keller–Osserman conditions become comparable and lead to a necessary and sufficient condition on f for problem (1.2) to admit a positive boundary blow-up solution. Finally, Appendix A provides an example of a specific inhomogeneous term $h(x,t)$ that is used to illustrate the main results of the paper. We also use this section to verify some side comments made in the paper.

2. Preliminaries

We begin by recalling some standard notation. Given $p > 1$, we set

$$p' := \frac{p}{p-1} \quad \text{and} \quad p^* := \frac{pN}{N-p} \text{ if } 1 < p < N \text{ and } p^* = \infty \text{ otherwise.}$$

Let $h\colon \Omega \times \mathbb{R} \to [0,\infty)$ be a Carathéodory function, that is, a function such that $x \mapsto h(x,t)$ is measurable in Ω for each $t \in \mathbb{R}$, and $t \mapsto h(x,t)$ is continuous on $\mathbb{R}$ for almost every (a.e.) $x \in \Omega$. We use $\mathcal{N}_h$ to denote the Nemyitskii operator $\mathcal{N}_h u(x) = h(x,u(x))$ for any given measurable function u.

The focus of our investigation in this paper is the following equation:

$$\Delta_p u = h(x,u) \quad (x \in \Omega). \tag{2.1}$$

Solutions of (2.1) will be understood in the distributional sense. To be precise, we recall the following notion.

Definition 2.1. We say that $u \in W^{1,p}(\Omega) \cap C(\Omega)$ is a weak subsolution of (2.1) in Ω if $\mathcal{N}_h u \in L^{p'}(\Omega)$, and the inequality

$$\int_\Omega |\mathrm{D}u|^{p-2}\mathrm{D}u \cdot \mathrm{D}\varphi \leqslant -\int_\Omega h(x,u)\varphi \tag{2.2}$$

holds for all $\varphi \in W_0^{1,p}(\Omega)$ with $\varphi \geqslant 0$ in Ω. Likewise, $u \in W^{1,p}(\Omega) \cap C(\Omega)$ is called a weak supersolution in Ω if $\mathcal{N}_h u \in L^{p'}(\Omega)$ and the inequality in (2.2) is reversed for all $\varphi \in W_0^{1,p}(\Omega)$ with $\varphi \geqslant 0$ in Ω. Finally, we say that u is a weak solution of (2.1) in Ω if and only if u is both a weak subsolution and a weak supersolution of problem (2.1) in Ω.

Given $\vartheta \in W^{1,p}(\Omega)$, let us consider the boundary-value problem

$$\left.\begin{aligned} \Delta_p u &= h(x,u) && \text{in } \Omega, \\ u &= \vartheta && \text{on } \partial\Omega. \end{aligned}\right\} \tag{2.3}$$

A function $u \in W^{1,p}(\Omega) \cap C(\Omega)$ is said to be a weak subsolution (supersolution) of problem (2.3) in Ω if and only if u is a weak subsolution (supersolution) of problem (2.1) in Ω such that $(u-\vartheta)^+ \in W_0^{1,p}(\Omega)$ $((u-\vartheta)^- \in W_0^{1,p}(\Omega))$. By a weak solution u of (2.3) in Ω we mean a function u that is both a weak subsolution and a weak supersolution of (2.3) in Ω.

A closely related concept that is more convenient in the context of problem (1.2) is that of a weak local solution. More generally, we say that u is a weak local subsolution (supersolution or solution) of (2.1) in Ω if and only if $u \in W_{\mathrm{loc}}^{1,p}(\Omega) \cap C(\Omega)$ and u is a weak subsolution (supersolution or solution) of (2.1) in $\mathcal{O}$ for every open set $\mathcal{O} \subset\subset \Omega$.

Finally, we say that $u \in W_{\mathrm{loc}}^{1,p}(\Omega) \cap C(\Omega)$ is a weak local solution (subsolution or supersolution) of problem (1.2) in Ω if and only if u is a weak local solution (subsolution or supersolution) of (2.1) in Ω such that $u = \infty$ on $\partial\Omega$.

If u is a subsolution (or a supersolution) of (2.1) in Ω, we indicate this by writing

$$\Delta_p u \geqslant h(x,u) \text{ in } \Omega \quad (\text{or } \Delta_p u \leqslant h(x,u) \text{ in } \Omega).$$

Let us now suppose that h satisfies the following condition.

(G) Given a compact interval $I \subseteq \mathbb{R}$, there is a function $g_I \in L^q(\Omega)$ for some $q > (p^*)'$ such that

$$|h(x,t)| \leqslant g_I(x) \quad \text{for a.e. } x \in \Omega \text{ and for all } t \in I.$$

We recall the following theorem, whose proof can be found in [**28**].

Theorem 2.2. *Let $h\colon \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function that satisfies condition* (G), *and let ϑ be a given constant. Suppose that $u_* \in W^{1,p}(\Omega)$ is a weak subsolution of (2.3) in Ω, and that $u^* \in W^{1,p}(\Omega)$ is a weak supersolution of (2.3) in Ω such that $u_* \leqslant u^*$ in Ω. Then problem (2.3) admits a minimal weak solution $u \in W^{1,p}(\Omega)$ such that $u_* \leqslant u \leqslant u^*$ in Ω, in the sense that if w is a weak solution of (2.3) such that $u_* \leqslant w \leqslant u^*$ in Ω, then $u \leqslant w$ in Ω.*

We now deduce the following minimality principle from the above theorem.

Theorem 2.3 (minimality principle). *Under the assumptions given in Theorem 2.2, there is a weak solution $u \in W^{1,p}(\Omega)$ of (2.3) with the following properties.*

(i) $u_* \leqslant u$ *in* Ω.

(ii) *Given any subdomain* $\mathcal{O} \subseteq \Omega$ *and any weak supersolution* $w \in W^{1,p}(\mathcal{O})$ *of (2.1) in* $\mathcal{O}$ *such that* $u_* \leqslant w$ *in* $\mathcal{O}$ *and* $u \leqslant w$ *on* $\partial\mathcal{O}$*, we have* $u \leqslant w$ *in* $\mathcal{O}$.

(iii) u *is unique.*

Proof. Let $u \in W^{1,p}(\Omega)$ be as given by Theorem 2.2 with $u_* \leqslant u \leqslant u^*$ in Ω. We will show that u has the property stated in part (ii) and that u is the desired unique solution, thus completing the proof. To this end, let $\mathcal{O} \subseteq \Omega$ be a given subdomain and suppose that w is a weak supersolution of (2.1) in $\mathcal{O}$ with the stated properties. Let us set $H(x,t) := \max\{h(x,t), h_0(x,t)\}$, where

$$h_0(x,t) := \begin{cases} h(x,u_*(x)) & \text{if } x \in \mathcal{O} \text{ and } t < u_*(x), \\ h(x,t) & \text{if } x \in \Omega \setminus \mathcal{O}, \text{ or if } x \in \mathcal{O} \text{ and } u_*(x) \leqslant t \leqslant w(x), \\ h(x,w(x)) & \text{if } x \in \mathcal{O} \text{ and } t > w(x). \end{cases}$$

Let us first note that H is a Carathéodory function that satisfies condition (G). Furthermore, we observe that u_* is a weak subsolution and u is a weak supersolution of the Dirichlet problem

$$\left.\begin{aligned} \Delta_p z &= H(x,z) \quad \text{in } \Omega, \\ z &= \vartheta \quad \text{on } \partial\Omega. \end{aligned}\right\} \tag{2.4}$$

By Theorem 2.2, problem (2.4) admits a solution v in Ω such that $u_* \leqslant v \leqslant u$ in Ω, and in particular, $u_* \leqslant v \leqslant u^*$ in Ω. We wish to show that $v \leqslant w$ in $\mathcal{O}$. Note that $0 \leqslant (v-w)^+ \leqslant (u-w)^+$ in $\mathcal{O}$ and since $(u-w)^+ \in W_0^{1,p}(\mathcal{O})$ it follows that $(v-w)^+ \in W_0^{1,p}(\mathcal{O})$ (see [**20**]). Let $\mathcal{Q} := \{x \in \mathcal{O} \colon v(x) > w(x)\}$. Using $(v-w)^+$ as a test function we see that

$$\begin{aligned} \int_{\mathcal{Q}} |\mathrm{D}v|^{p-2}\mathrm{D}v \cdot \mathrm{D}(v-w) &= \int_{\mathcal{O}} |\mathrm{D}v|^{p-2}\mathrm{D}v \cdot \mathrm{D}(v-w)^+ \\ &= -\int_{\mathcal{O}} H(x,v)(v-w)^+ \\ &\leqslant -\int_{\mathcal{O}} h(x,w)(v-w)^+. \end{aligned} \tag{2.5}$$

On the other hand, recalling that $\Delta_p w \leqslant h(x,w)$ in $\mathcal{O}$, we find that

$$\begin{aligned} \int_{\mathcal{Q}} |\mathrm{D}w|^{p-2}\mathrm{D}w \cdot \mathrm{D}(v-w) &= \int_{\mathcal{O}} |\mathrm{D}w|^{p-2}\mathrm{D}w \cdot \mathrm{D}(v-w)^+ \\ &\geqslant -\int_{\mathcal{O}} h(x,w)(v-w)^+. \end{aligned} \tag{2.6}$$

Putting (2.5) and (2.6) together, we find that

$$0 \leqslant \int_{\mathcal{Q}} (|\mathrm{D}v|^{p-2}\mathrm{D}v - |\mathrm{D}w|^{p-2}\mathrm{D}w) \cdot (\mathrm{D}v - \mathrm{D}w) \leqslant 0. \tag{2.7}$$

The lower bound in (2.6) is a consequence of the following, easily verifiable, inequality:

$$(|\eta|^{p-2}\eta - |\zeta|^{p-2}\zeta) \cdot (\eta - \zeta) > 0 \quad \forall \eta, \zeta \in \mathbb{R}^N,\ \eta \neq \zeta. \tag{2.8}$$

Because of the strict inequality in (2.8) for $\eta \neq \zeta$, we conclude that in fact $\mathrm{D}(v-w) = 0$ almost everywhere on $\mathcal{Q}$. Therefore, $\mathrm{D}(v-w)^+ = 0$ almost everywhere in $\mathcal{O}$, so that $(v-w)^+ = c$ for some constant $c \geqslant 0$. Since $(v-w)^+ \in W_0^{1,p}(\mathcal{O})$, it follows that $(v-w)^+ = 0$ in $\mathcal{O}$, and thus $v \leqslant w$ in $\mathcal{O}$. As a consequence $u_* \leqslant v \leqslant w$ in $\mathcal{O}$, and hence we have $H(x,v) = h(x,v)$ in Ω. Therefore, v is actually a solution of problem (2.3) such that $u_* \leqslant v \leqslant u^*$ in Ω. Since u is a minimal solution, Theorem 2.2 shows that $u \leqslant v$ in Ω. In summary, we find that $u = v$ in Ω. In particular, $u = v \leqslant w$ in $\mathcal{O}$, as desired. The uniqueness of u is clear from part (ii) of the theorem. □

Remark 2.4. Suppose that β_1 and β_2 are supersolutions of problem (2.3) such that $u_* \leqslant \beta_i$ in Ω for $i = 1, 2$. Let u_1 and u_2 be the minimal solutions of problem (2.3) such that $u_* \leqslant u_i \leqslant \beta_i$ in Ω for $i = 1, 2$ as given in Theorem 2.2. Since $u_* \leqslant u_2$ in Ω and $u_1 = u_2$ on $\partial\Omega$, an application of Theorem 2.3 (ii) with $\mathcal{O} := \Omega$ shows that $u_1 \leqslant u_2$ in Ω. Interchanging the roles of u_1 and u_2, we also have $u_2 \leqslant u_1$ in Ω. Therefore, the weak solution $u \in W^{1,p}(\Omega)$ given in Theorem 2.3 depends on the weak subsolution u_* and the boundary data ϑ, and not on the choice of the weak supersolution u^* in Theorem 2.2. Consequently, we will refer to the unique weak solution $u \in W^{1,p}(\Omega)$ of problem (2.3) that satisfies properties (i) and (ii) of Theorem 2.3 as the minimal weak solution of (2.3) *relative to the subsolution* u_*.

The next lemma will be useful in establishing the existence of boundary blow-up solutions of (1.2).

Lemma 2.5. *Any locally uniformly bounded sequence* $\{u_k\}$ *of solutions of (2.1) in* $W^{1,p}(\Omega)$ *contains a subsequence that converges locally uniformly to a solution* $u \in W^{1,p}_{\mathrm{loc}}(\Omega) \cap C(\Omega)$ *of problem (2.1).*

Proof. Let $\{u_k\}$ be a locally uniformly bounded sequence of solutions of (2.1). Given an open set $\mathcal{O} \subset\subset \Omega$, there is a constant M depending on $\mathcal{O}$ such that

$$\|u_k\|_{L^\infty(\mathcal{O})} \leqslant M, \quad k = 1, 2, \ldots.$$

Consequently, there are positive constants $\alpha > 0$ and $C > 0$, depending on p, N, M only, and $\mathcal{O}$ (see [**8**]) such that

$$|\mathrm{D}u_k(x)| \leqslant C \quad \text{and} \quad |\mathrm{D}u_k(x) - \mathrm{D}u_k(y)| \leqslant C|x-y|^\alpha, \quad x, y \in \mathcal{O},\ k = 1, 2, \ldots.$$

By virtue of the Arzelà–Ascoli theorem, we can extract a subsequence, still denoted by $\{u_k\}$, such that $u_k \to u$ and $\mathrm{D}u_k \to v$ for some $u \in C^{1,\alpha}(\mathcal{O})$ and $v \in (C^\alpha(\mathcal{O}))^N$ uniformly

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on $\mathcal{O}$. In fact, we have $v = \mathrm{D}u$. Therefore, a Cantor diagonalization argument applied to an exhaustion $\{\mathcal{O}_j\}$ of Ω with $\mathcal{O}_j \subset\subset \mathcal{O}_{j+1} \subset\subset \Omega$ shows that there is $u \in C^{1,\alpha}_{\mathrm{loc}}(\Omega)$ such that $u_k \to u$ and $\mathrm{D}u_k \to \mathrm{D}u$ locally uniformly on Ω. To show that u is a solution of problem (2.1), we take $\varphi \in C^\infty_{\mathrm{c}}(\Omega)$ and set $\mathcal{O} := \mathrm{supp}(\varphi) \subset\subset \Omega$. Let us first recall some useful inequalities (see [**9**]) that hold for all $\xi, \zeta \in \mathbb{R}^N$:

$$||\xi|^{p-2}\xi - |\zeta|^{p-2}\zeta| \leqslant \begin{cases} C|\xi-\zeta|(|\xi|+|\zeta|)^{p-2} & \text{if } p \geqslant 2, \\ C|\xi-\zeta|^{p-1} & \text{if } 1 < p \leqslant 2, \end{cases} \tag{2.9}$$

where C is a positive constant independent of ξ and ζ. Using (2.9), it follows that

$$|\mathrm{D}u_k|^{p-2}\mathrm{D}u_k \cdot \mathrm{D}\varphi \to |\mathrm{D}u|^{p-2}\mathrm{D}u \cdot \mathrm{D}\varphi$$

uniformly in $\mathcal{O}$. As a consequence of this, and the dominated convergence theorem, we have

$$\begin{aligned} -\int_{\mathcal{O}} h(x,u)\varphi = -\lim_{k\to\infty}\int_{\mathcal{O}} h(x,u_k)\varphi &= \lim_{k\to\infty}\int_{\mathcal{O}} |\mathrm{D}u_k|^{p-2}\mathrm{D}u_k \cdot \mathrm{D}\varphi \\ &= \int_{\mathcal{O}} |\mathrm{D}u|^{p-2}\mathrm{D}u \cdot \mathrm{D}\varphi. \end{aligned}$$

Thus, $u \in W^{1,p}_{\mathrm{loc}}(\Omega) \cap C^{1,\alpha}_{\mathrm{loc}}(\Omega)$ is indeed a solution of (2.1) as claimed. □

Let us consider the following boundary-value problem in balls B:

$$\left.\begin{aligned} \Delta_p u = g(u) \quad & \text{in } B, \\ u = \vartheta \quad & \text{on } \partial B. \end{aligned}\right\} \tag{2.10}$$

As a first application of the minimality principle, we prove the existence of a radial solution to problem (2.10).

Lemma 2.6. *Let $g\colon [0,\infty) \to [0,\infty)$ be any continuous function such that $g(0) = 0$. Given any ball B and a constant $\vartheta > 0$, the minimal solution u of problem (2.10) with respect to the subsolution $u_* = 0$ is radial and belongs to $C^{1,\alpha}(B)$ for some $0 < \alpha < 1$.*

Proof. Without loss of generality, we assume that B is a ball centred at the origin. Note that $u_* \equiv 0$ and $u^* = \vartheta$ are a subsolution and a supersolution of (2.10), respectively. Let u be the minimal solution of (2.10) with respect to u_* such that $0 \leqslant u \leqslant \vartheta$. Note that, by [**8**], we have $u \in C^{1,\alpha}(B)$. Let A be any orthogonal matrix and define $v(x) = u(Ax)$. Then computation shows that

$$\mathrm{D}v(x) = A^{\mathrm{T}}\mathrm{D}u(Ax) \quad \text{and} \quad v = \vartheta \quad \text{on } \partial B.$$

Given any $\varphi \in W_0^{1,p}(B)$, let $\psi(x) = \varphi(A^{\mathrm{T}}x)$. On using a change of variables, and writing $\langle\cdot,\cdot\rangle$ for the Euclidean inner product, we have

$$\begin{aligned}\int_B |\mathrm{D}v|^{p-2}\langle \mathrm{D}v, \mathrm{D}\varphi\rangle &= \int_B |\mathrm{D}u|^{p-2}\langle A^{\mathrm{T}}\mathrm{D}u, \mathrm{D}\varphi\rangle \\ &= \int_B |\mathrm{D}u|^{p-2}\langle \mathrm{D}u, A\mathrm{D}\varphi\rangle \\ &= \int_B |\mathrm{D}u|^{p-2}\langle \mathrm{D}u, \mathrm{D}\psi\rangle = -\int_B g(u)\psi = -\int_B g(v)\varphi.\end{aligned}$$

Thus, v is a solution of (2.10) such that $u_* = 0 \leqslant v$ in B. Since $v = u$ on ∂B, by the minimality principle, we see that $u(x) \leqslant u(Ax)$. Likewise, we see that $u(x) \leqslant u(A^{\mathrm{T}}x)$ in B. Thus, indeed $u(x) = u(Ax)$ in B, showing that u is radial. □

The minimality principle (Theorem 2.3), and Lemma 2.6 can now be used to establish the following useful result on existence of a blow-up solution to problem (1.2).

Lemma 2.7. *Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain and let $h \in C(\Omega \times \mathbb{R}, [0,\infty))$ satisfy condition* (G). *Let $u_* \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$ be a weak subsolution of (2.1) in Ω and let $u^* \in W^{1,p}_{\mathrm{loc}}(\Omega) \cap C(\Omega)$ be a weak local supersolution of (1.2) in Ω such that $u_* \leqslant u^*$ in Ω. There is then a minimal weak local solution $u \in W^{1,p}_{\mathrm{loc}}(\Omega) \cap C(\Omega)$ of (1.2) such that $u_* \leqslant u \leqslant u^*$ in Ω, in the sense that if $w \in W^{1,p}_{\mathrm{loc}}(\Omega) \cap C(\Omega)$ is any weak local solution of (1.2) such that $u_* \leqslant w$, then $u \leqslant w$ in Ω. Moreover, if Ω is a ball and $h(x,t) := g(t)$ is independent of x, where $g(0) = 0$, then the minimal solution relative to $u_* = 0$ is radial.*

Proof. Let $\ell := \operatorname{ess\,sup}_\Omega u_*$ and for each positive integer j we consider the boundary-value problem

$$\left.\begin{aligned}\Delta_p u &= h(x,u) \quad \text{in } \Omega,\\ u &= \ell + j \qquad \text{on } \partial\Omega.\end{aligned}\right\} \qquad (D_j)$$

For each positive integer j we note that $u_* \leqslant w_j$ in $\bar{\Omega}$, where $w_j := \ell + j$, and that w_j is a supersolution of (D_j). By the minimality principle (Theorem 2.3), we pick the minimal solution $u_j \in W^{1,p}(\Omega)$ of problem (D_j) relative to u_*, so that $u_* \leqslant u_j \leqslant \ell + j$ in $\bar{\Omega}$. Since, for arbitrary and sufficiently small $\delta > 0$, u^* is a supersolution of problem (D_j) in $\{x \in \Omega\colon \operatorname{dist}(x, \partial\Omega) > \delta\}$, by Theorem 2.3 (ii) we conclude that in fact $u_* \leqslant u_j \leqslant u^*$ in Ω for any j. Let us now observe that u_{j+1} is a supersolution of (D_j) such that $u_* \leqslant u_{j+1}$ in Ω, and $u_j \leqslant u_{j+1}$ on $\partial\Omega$. Since u_j is the minimal solution of problem (D_j) relative to u_*, again using Theorem 2.3 (ii), we conclude that $u_j \leqslant u_{j+1}$ in $\bar{\Omega}$. Thus, we have constructed a non-decreasing sequence $\{u_j\}$ of solutions of problem (2.1) in $W^{1,p}(\Omega)$ such that $u_* \leqslant u_j \leqslant u^*$ in Ω for all $j = 1, 2, \dots$. By Lemma 2.5, we conclude that $\{u_j\}$ contains a subsequence that converges locally uniformly to a solution $u \in W^{1,p}_{\mathrm{loc}}(\Omega) \cap C(\Omega)$. Consequently, we have $u_* \leqslant u \leqslant u^*$ in Ω, and $u = \infty$ on $\partial\Omega$. Thus, $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ is a solution of problem (1.2) such that $u_* \leqslant u \leqslant u^*$ in Ω, as asserted. It remains to show that u is minimal relative to u_*. So, suppose that $w \in W^{1,p}_{\mathrm{loc}}(\Omega) \cap C(\Omega)$ is a weak local solution of (1.2) such that $u_* \leqslant w$ in Ω. Then given any integer $j \in \mathbb{N}$, there is $\delta_j > 0$ sufficiently small such that $\ell + j \leqslant w(x)$ provided that $x \in \Omega$ and $\operatorname{dist}(x, \partial\Omega) < \delta_j$. Since u_j is a minimal solution of problem (D_j) such that $u_j \leqslant w$ on $\partial\Omega_{\delta_j}$, we conclude that $u_j \leqslant w$ in Ω. Therefore, $u \leqslant w$ in Ω, as claimed. Finally, suppose that Ω is a ball and that h is independent of x. Then, by Lemma 2.6, each minimal solution u_j of problem (D_j) is radial, and hence the limit u is radial as well. □

Remark 2.8. In what follows Lemma 2.7 will be used when the inhomogeneous term $h(x, t)$ satisfies the condition $h(x, 0) \equiv 0$ in Ω. In this case we take $u_* \equiv 0$ as a subsolution of (2.3) and according to Lemma 2.7 one need only find a non-negative supersolution u^* of problem (1.2) to deduce existence of a non-negative solution to (1.2).

3. The Keller–Osserman condition and existence on balls

Let us now consider $g \in C([0, \infty), [0, \infty))$ such that $g(0) = 0$ and $g(c) > 0$ for some $c > 0$. We define $\Phi\colon (0, \infty) \to (0, \infty]$ by

$$\Phi(t) := \int_t^\infty \frac{\mathrm{d}s}{(p'(G(s) - G(t)))^{1/p}}, \quad \text{where } G(t) := \int_0^t g(\zeta)\,\mathrm{d}\zeta. \tag{3.1}$$

By convention we take $\Phi(t) = \infty$ when the integral diverges or $\{s \in [t, \infty) \colon G(s) = G(t)\}$ is a set of positive measure.

We will use the following Keller–Osserman-type condition, hereafter referred to as the Keller–Osserman condition, on g. See [**12**]. A related condition appears in many different contexts, among which we cite the pioneering work of Vazquez [**44**]:

$$\liminf_{t \to \infty} \Phi(t) = 0. \tag{3.2}$$

For $1 < p < \infty$ the Keller–Osserman condition (3.2) is equivalent to the following condition:

$$\Phi(t) < \infty \quad \text{for some } t \geqslant 0. \tag{3.3}$$

This was shown to be the case for $p = 2$ in [**12**]. In Appendix A we show the equivalence for $1 < p < \infty$.

Remark 3.1. We remark that if $g(t) > 0$ for $t > 0$, and g is non-decreasing in $(0, \infty)$, then for any given $1 < p < \infty$, conditions (3.2) and (3.3) are both equivalent to the condition that

$$\int_t^\infty \frac{1}{G(s)^{1/p}}\,\mathrm{d}s < \infty \quad \text{for some } t > 0. \tag{3.4}$$

We refer the reader to Appendix A for a justification of this statement.

Let us now consider the following boundary blow-up problem in balls:

$$\left.\begin{aligned} \Delta_p u &= g(u) \quad \text{in } B, \\ u &= \infty \qquad \text{on } \partial B. \end{aligned}\right\} \tag{3.5}$$

Here, B stands for a ball in $\mathbb{R}^N$.

We begin with the following lemma.

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Lemma 3.2. *Suppose that g satisfies the Keller–Osserman condition (3.3). There is an $R > 0$ such that problem (3.5) admits a non-negative solution $v \in W^{1,p}_{\mathrm{loc}}(B) \cap C^{1,1}(B)$, where $B = B(x_0, R)$ and $x_0 \in \mathbb{R}^N$.*

Proof. Without loss of generality, we take x_0 to be the origin. Since g satisfies the Keller–Osserman condition, let $a > 0$ such that $\Phi(a) < \infty$. Let v be the minimal solution of the following boundary-value problem with respect to the subsolution $u_* = 0$:

$$\left.\begin{aligned} \Delta_p u = g(u) \quad & \text{in } B, \\ u = a \quad & \text{on } \partial B, \end{aligned}\right\} \tag{3.6}$$

where $B = B(0, \rho)$ is a ball of radius ρ centred at the origin. Here $\rho > 0$ is a fixed positive real number chosen such that

$$\frac{\rho(p-1)}{N-p} > (p')^{1/p}\Phi(a) \tag{3.7}$$

if $1 < p < N$, but is arbitrary otherwise. By Lemma 2.6, v is a radial function that belongs to $C^{1,\alpha}(B)$ for some $0 < \alpha < 1$. Let $w(r) := v(x)$, where $r = |x|$. Then w satisfies the initial-value problem

$$(r^{N-1}|w'|^{p-2}w')' = r^{N-1}g(w(r)), \quad w(0) = v(0),\ w'(0) = 0,\ 0 < r < R, \tag{3.8}$$

where this equation is understood in the weak sense. Here $(0, R)$ is the maximal interval of existence of the function w. We note that $R > \rho$ and $w(\rho) = a$. From (3.8) we see that (see [**21**, Theorem 3.1.4])

$$r^{N-1}|w'|^{p-2}w' = \int_0^r t^{N-1}g(w(t))\,\mathrm{d}t. \tag{3.9}$$

It follows that $w' > 0$. Therefore, we have

$$w'(r) = \left(r^{1-N}\int_0^r t^{N-1}g(w(t))\,\mathrm{d}t\right)^{1/(p-1)},$$

and we note that this implies that $v \in C^{1,1}(0, R)$.

As a consequence of (3.8) we see that $r^{N-1}(w')^{p-1}$ is non-decreasing, and hence $r^{(N-1)/(p-1)}w'$ is non-decreasing. Multiplying (3.9) first by $r^{(N-1)/(p-1)}w'(r)$ and using the monotonicity of $r^{(N-1)/(p-1)}w'$ leads to

$$\begin{aligned} r^{p(N-1)/(p-1)}(w'(r))^p &= r^{(N-1)/(p-1)}w'(r)\int_0^r t^{N-1}g(w(t))\,\mathrm{d}t \\ &\geqslant \int_0^r t^{p(N-1)/(p-1)}g(w(t))w'(t)\,\mathrm{d}t \\ &\geqslant \int_\rho^r t^{p(N-1)/(p-1)}g(w(t))w'(t)\,\mathrm{d}t \\ &\geqslant \rho^{p(N-1)/(p-1)}\int_\rho^r g(w(t))w'(t)\,\mathrm{d}t \\ &= \rho^{p(N-1)/(p-1)}[G(w(r)) - G(w(\rho))]. \end{aligned}$$

Therefore, for any $r \in (\rho, R)$ we have the following inequality:

$$\frac{w'(r)}{(p'(G(w(r)) - G(w(\rho))))^{1/p}} \geqslant \frac{1}{(p')^{1/p}} \left(\frac{\rho}{r}\right)^{(N-1)/(p-1)}. \tag{3.10}$$

Integrating (3.10) on (ρ, r), we find that

$$\int_{w(\rho)}^{w(r)} \frac{1}{(p'(G(t) - G(w(\rho))))^{1/p}} \,\mathrm{d}t \geqslant \frac{1}{(p')^{1/p}} \rho^{(N-1)/(p-1)} \int_{\rho}^{r} t^{-(N-1)/(p-1)} \,\mathrm{d}t. \tag{3.11}$$

That is, for any $\rho < r < R$ we have

$$\begin{aligned} \frac{1}{(p')^{1/p}} \rho^{(N-1)/(p-1)} \int_{\rho}^{r} t^{-(N-1)/(p-1)} \,\mathrm{d}t &\leqslant \int_{w(\rho)}^{w(r)} \frac{\mathrm{d}t}{(p'(G(t) - G(w(\rho))))^{1/p}} \\ &\leqslant \int_{w(\rho)}^{\infty} \frac{\mathrm{d}t}{(p'(G(t) - G(w(\rho))))^{1/p}} \\ &< \Phi(w(\rho)) \\ &= \Phi(a). \end{aligned} \tag{3.12}$$

Now, suppose that $R = \infty$. Note that

$$\lim_{r \to \infty} \frac{\rho^{(N-1)/(p-1)}}{(p')^{1/p}} \int_{\rho}^{r} t^{-(N-1)/(p-1)} \,\mathrm{d}t = \begin{cases} \infty & \text{if } p \geqslant N, \\ \dfrac{\rho(p-1)}{(p')^{1/p}(N-p)} & \text{if } 1 < p < N. \end{cases}$$

This, together with (3.7) and (3.12), provides the needed contradiction. Therefore, we conclude that $R < \infty$, and hence $w(R) = \infty$ or $w'(R) = \infty$. In either case, it follows that $w(R) = \infty$. To see this, note that if $w'(R) = \infty$, then we conclude from (3.9) that $w(R) = \infty$, for otherwise $t^{N-1} g(w(t))$ will be bounded in $(0, R)$, and therefore the integral in (3.9) will be finite. □

Lemma 3.3. *Suppose that g satisfies the Keller–Osserman condition (3.3). Then problem (3.5) admits a weak local solution in balls of arbitrarily small radius.*

Proof. Lemma 3.2 shows that problem (3.5) has a solution in some ball. Let

$$R = \inf\{\rho > 0\colon \text{problem (3.5) has a solution in a ball of radius } \rho\}.$$

We claim that $R = 0$. Suppose that $R > 0$. By the Keller–Osserman condition (3.2), we choose $a > 0$ large enough such that

$$\Phi(a) < \frac{1}{(p')^{1/p}} \left(\frac{R}{2}\right)^{(N-1)/(p-1)} \int_{R/2}^{R} t^{-(N-1)/(p-1)} \,\mathrm{d}t.$$

Consider the minimal solution v of problem (3.6) in the ball $B(0, R/2)$ with subsolution $u_* = 0$ and supersolution $u^* = a$. The definition of R allows us to extend the solution v into the ball $B(0, R)$. Let $\phi(r) := v(x)$, where $r = |x|$. Then ϕ is a solution of

$$(r^{N-1} |\phi'|^{p-2} \phi')' = r^{N-1} g(\phi(r)), \quad \phi(0) = v(0),\ \phi'(0) = 0,\ 0 < r < R.$$

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We note that $\phi(R/2) = a$. Proceeding as in the proof of Lemma 3.2, inequality (3.12) shows that

$$\frac{1}{(p')^{1/p}}\left(\frac{R}{2}\right)^{(N-1)/(p-1)}\int_{R/2}^{R} t^{-(N-1)/(p-1)}\,\mathrm{d}t \leqslant \Phi(\phi(R/2)) = \Phi(a).$$

However, this contradicts the choice of a, and hence we must have $R = 0$, as claimed. $\square$

4. The lower Keller–Osserman condition

Remark 4.1. From hereon we will assume, without further mention, that the inhomogeneous term $h(x,t)$ in problem (1.2) satisfies the following conditions.

(1) $h\colon \bar{\Omega} \times [0,\infty) \to [0,\infty)$ is continuous and non-trivial.

(2) $h(x,0) \equiv 0$ in $\bar{\Omega}$.

We note that h satisfies condition (G). With such a function h we associate a non-negative function $h_*\colon [0,\infty) \to [0,\infty)$ whose growth rate at infinity will provide fairly general conditions on h that would ensure the existence of a non-negative solution to problem (1.2). Thus, for each $(x,r) \in \bar{\Omega} \times (0,\infty)$, we define $h_*(t;x,r)\colon [0,\infty) \to [0,\infty)$ by

$$h_*(t;x,r) := \min\{h(z,t)\colon z \in \hat{B}(x,r)\}, \tag{4.1}$$

where $\hat{B}(x,r) := \overline{B(x,r)} \cap \bar{\Omega}$. The following are consequences of the definition (4.1).

(i) $h_*(t;x,r) > 0$ for some $(x,t,r) \in \bar{\Omega} \times (0,\infty) \times (0,\infty)$.

(ii) $h_*(0;x,r) = 0$ for all $(x,r) \in \bar{\Omega} \times (0,\infty)$.

(iii) $h_*(t;x,r)$ is continuous in $t \in \mathbb{R}$.

(iv) $h_*(t;x,r)$ is non-increasing in r.

To proceed further we introduce some notation. For $(x,r) \in \bar{\Omega} \times (0,\infty)$ we set

$$H_*(t;x,r) := \int_0^t h_*(\zeta;x,r)\,\mathrm{d}\zeta \quad \forall t > 0.$$

In analogy with (3.1), for a given $x \in \bar{\Omega}$ such that $h_*(\tau;x,r) > 0$ for some $(\tau,r) \in (0,\infty) \times (0,\infty)$ we set

$$\Phi_*(t;x,r) := \int_t^\infty \frac{\mathrm{d}s}{(p'(H_*(s;x,r) - H_*(t;x,r)))^{1/p}} \quad \forall t > 0.$$

Remark 4.2. Since $h_*(t;x,r)$ is non-increasing in r, it should be noted that $\Phi_*(t;x,r)$ is non-decreasing in r. We direct the reader to Appendix A for a proof.

We now give the following definition.

Definition 4.3. We say that $h(x,t)$ satisfies a lower Keller–Osserman condition at $x_0 \in \bar{\Omega}$ if there is $r \in (0,\infty)$ such that

$$\liminf_{t\to\infty} \Phi_*(t; x_0, r) = 0. \tag{4.2}$$

Recalling the statements just before Remark 3.1, we see that (4.2) is equivalent to

$$\Phi_*(t; x_0, r) < \infty \quad \text{for some } t > 0. \tag{4.3}$$

If h satisfies a lower Keller–Osserman condition at x_0, then in view of Remark 4.2 we see that (4.2) holds for all sufficiently small $r > 0$.

As an example, we consider $h(x,t) := \omega(x)t^{b(x)}(1 + \cos\lambda t)$, where $\omega \in C(\bar{\Omega})$ is non-negative and non-trivial, $b \in C(\bar{\Omega})$ is positive and λ is a constant. Given $x_0 \in \Omega$ such that $\omega(x_0) > 0$ and $b(x_0) > p - 1$, it can be shown that h satisfies a lower Keller–Osserman condition at x_0. The reader is referred to Appendix A for a detailed discussion of this example.

Remark 4.4. Suppose that $h(x,t) > 0$ for all $(x,t) \in \bar{\Omega} \times (0,\infty)$ and $h(x,t)$ is non-decreasing in t for each $x \in \bar{\Omega}$. The requirement that h satisfies a lower Keller–Osserman condition at $x_0 \in \bar{\Omega}$ is equivalent to

$$\int_t^\infty \frac{\mathrm{d}s}{H_*(s; x_0, r)^{1/p}} < \infty$$

for some $r > 0$ and some $t > 0$. See Remark 3.1.

Theorem 4.5. *Suppose that $\mathcal{O} \subset\subset \Omega$ and assume that h satisfies a lower Keller–Osserman condition at each $x \in \partial\mathcal{O}$. Then there is a constant $C > 0$, independent of ϑ, such that*

$$0 \leqslant u_\vartheta \leqslant C \quad \text{in } \mathcal{O}$$

for any minimal solution $u_\vartheta \in W^{1,p}(\Omega)$ of (2.3) relative to the subsolution $u_* \equiv 0$ and any constant $\vartheta > 0$.

Proof. Let $\vartheta > 0$ be a given constant, and suppose that u_ϑ is the minimal solution of (2.3) relative to $u_* \equiv 0$. By Lemma 3.3, given $z \in \partial\mathcal{O}$ there is a ball $B(z, r_z) \subseteq \Omega$ with $0 < r_z \leqslant \rho_z$ for some sufficiently small $\rho_z > 0$ such that the following problem admits a non-negative solution $v_z \in W^{1,p}_{\mathrm{loc}}(B(z, r_z)) \cap C(B(z, r_z))$:

$$\begin{aligned} \Delta_p w &= h_*(w; z, \rho_z) && \text{in } B(z, r_z), \\ w &= \infty && \text{on } \partial B(z, r_z). \end{aligned}$$

Since $\Delta_p v_z = h_*(v_z(x); z, \rho_z) \leqslant h(x, v_z(x))$ for $x \in B(z, r_z)$, by the minimality principle (Theorem 2.3), we see that $u_\vartheta \leqslant v_z$ on $B(z, r_z)$. In particular, $u_\vartheta \leqslant M_z$ in the ball $B(z, r_z/2)$, where

$$M_z := \max\{v_z(x) \colon x \in B(z, r_z/2)\}.$$

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From the open cover $\mathcal{U} := \{B(z, r_z/2) \colon z \in \partial\mathcal{O}\}$ of $\partial\mathcal{O}$ we pick a finite subcover

$$\{B(z_j, r_{z_j}/2) \colon j = 1, 2, \dots, m\}.$$

Therefore,

$$u_\vartheta \leqslant C \quad \text{on } \partial\mathcal{O},$$

where $C := \max\{M_{z_j} : j = 1, \dots, m\}$. Since $C > 0$ is a supersolution of $\Delta_p u = h(x, u)$ such that $u_\vartheta \leqslant C$ on $\partial\mathcal{O}$, again by the minimality principle we see that $u_* \leqslant u_\vartheta \leqslant C$ in $\mathcal{O}$, as claimed. □

In preparation for the statement of our main theorem, we introduce the following definition.

Definition 4.6. We say that h satisfies a circumferential lower Keller–Osserman condition at $x_0 \in \Omega$ if and only if there is an open set $\mathcal{O}$ with $x_0 \in \mathcal{O} \subset\subset \Omega$ such that h satisfies a lower Keller–Osserman condition at every $x \in \partial\mathcal{O}$.

Remark 4.7. We remark that if h satisfies a lower Keller–Osserman condition at $x_0 \in \Omega$, then h satisfies a circumferential lower Keller–Osserman condition at x_0 as well.

We are now ready to state and prove our main result on existence of solutions to problem (1.2). It will be convenient to use the following notation in the theorem and its proof. Given $\delta > 0$ we will write

$$\Omega_\delta := \{x \in \Omega \colon \operatorname{dist}(x, \partial\Omega) < \delta\} \quad \text{and} \quad \Omega^\delta := \{x \in \Omega \colon \operatorname{dist}(x, \partial\Omega) > \delta\}.$$

Theorem 4.8. *Suppose that h satisfies a circumferential lower Keller–Osserman condition at every point of Ω_δ for some $\delta > 0$. Then problem (1.2) admits a non-negative solution $u \in W^{1,p}_{\mathrm{loc}}(\Omega) \cap C(\Omega)$.*

Proof. For each positive integer k, consider the Dirichlet problem

$$\left.\begin{aligned} \Delta_p u &= h(x, u) \quad \text{in } \Omega, \\ u &= k \qquad\quad\ \text{on } \partial\Omega. \end{aligned}\right\} \tag{4.4}$$

For each k, note that $u_* \equiv 0$ is a subsolution and $u^* \equiv k$ is a supersolution of (4.4). Let $u_k \in W^{1,p}(\Omega)$ be the minimal solution to (4.4) with respect to u_*, so that $0 \leqslant u_k \leqslant k$ for all k. By the minimality principle (Theorem 2.3), we see that $u_k \leqslant u_{k+1}$ on $\bar{\Omega}$. We wish to show that the sequence $\{u_k\}$ is locally uniformly convergent in Ω. For this it suffices, according to Lemma 2.5, to show that the sequence $\{u_k\}$ is uniformly bounded in Ω^ε for each $0 < \varepsilon < \delta$. To this end, first we show that given $z \in \partial\Omega^\varepsilon$ there are positive constants $r_{z,\varepsilon} > 0$ and $M_{z,\varepsilon} > 0$ such that

$$0 \leqslant u_k \leqslant M_z \quad \text{in } B(z, r_{z,\varepsilon}) \text{ for all } k = 1, 2, \dots.$$

Obviously, we have $\partial\Omega^\varepsilon \subseteq \Omega_\delta$. Let $z \in \partial\Omega^\varepsilon$. By hypothesis there is an open set $\mathcal{O} \subset\subset \Omega$ containing z such that h satisfies a lower Keller–Osserman condition at each $x \in \partial\mathcal{O}$. But

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then, by Theorem 4.5, there is a constant $M_{z,\varepsilon}$, independent of k, such that $0 \leqslant u_k \leqslant M_{z,\varepsilon}$ in $\mathcal{O}$. In particular, there is a ball $B(z, r_{z,\varepsilon}) \subseteq \Omega$ such that $0 \leqslant u_k \leqslant M_{z,\varepsilon}$ in $B(z, r_{z,\varepsilon})$ for all k, as claimed.

Therefore, we have shown that given $z \in \partial\Omega^\varepsilon$ there is a ball $B(z, r_{z,\varepsilon}) \subseteq \Omega$ and a positive constant $M_{z,\varepsilon}$, independent of k, such that

$$0 \leqslant u_k \leqslant M_{z,\varepsilon} \quad \text{in } B(z, r_{z,\varepsilon}),\ k = 1, 2, \dots .$$

By compactness it follows that there is a constant $M_\varepsilon > 0$, independent of k, such that

$$0 \leqslant u_k \leqslant M_\varepsilon \quad \text{on } \partial\Omega^\varepsilon,\ k = 1, 2, \dots .$$

By the minimality principle (Theorem 2.3), we see that

$$0 \leqslant u_k \leqslant M_\varepsilon \quad \text{in } \Omega^\varepsilon,\ k = 1, 2, \dots .$$

By Lemma 2.5 it follows that the sequence $\{u_k\}$ has a subsequence that converges locally uniformly to $u \in W^{1,p}_{\mathrm{loc}}(\Omega) \cap C(\Omega)$ such that $\Delta_p u = h(x, u)$ in Ω, in the weak sense. Since $u_k \leqslant u$ in Ω, it follows that $u(x) \to \infty$ as $x \to \partial\Omega$, as desired. □

The next result shows that under appropriate conditions on the inhomogeneous term h, problem (1.2) admits infinitely many non-negative solutions. For this, we assume that there is $\varpi > 0$ such that

$$h(x, t) \leqslant h(x, t + \varpi) \quad \forall (x, t) \in \Omega \times \mathbb{R}. \tag{4.5}$$

Proposition 4.9. *Suppose that $h(x, \tau) \equiv 0$ in Ω for all $\tau \in \mathcal{D}$, where $\mathcal{D}$ is some unbounded discrete set of positive real numbers, and assume that condition (4.5) holds for some $\varpi > 0$. If problem (1.2) has a non-negative solution in $W^{1,p}_{\mathrm{loc}}(\Omega) \cap C(\Omega)$, then problem (1.2) admits infinitely many non-negative solutions.*

Proof. Without loss of generality we assume that $\varpi \geqslant 1$, and let us fix $\tau_1 \in \mathcal{D}$ and $x_0 \in \Omega$. Suppose that $v \in W^{1,p}_{\mathrm{loc}}(\Omega) \cap C(\Omega)$ is a non-negative blow-up solution of (1.2). Let $w_1 := \tau_1$ and let $v_1 := v + j_1\varpi$, where j_1 is the smallest integer greater than or equal to τ_1. We note that w_1 is a subsolution of (2.1) and, in view of (4.5), v_1 is a supersolution of (1.2) such that $w_1 \leqslant v_1$ in Ω. By Lemma 2.7, we find a solution $u_1 \in W^{1,p}_{\mathrm{loc}}(\Omega) \cap C(\Omega)$ of (1.2) such that $w_1 \leqslant u_1 \leqslant v_1$ in Ω. Let $\tau_2 \in \mathcal{D}$ such that $\tau_2 > u_1(x_0)$ and set $w_2 := \tau_2$ and $v_2 := v + j_2\varpi$, where j_2 is the smallest positive integer greater than or equal to τ_2. Then w_2 is a subsolution of (2.1) and v_2 is a supersolution of (1.2) such that $w_2 \leqslant v_2$. Again, by Lemma 2.7, we find a solution $u_2 \in W^{1,p}_{\mathrm{loc}}(\Omega) \cap C(\Omega)$ of (1.2) with $w_2 \leqslant u_2 \leqslant v_2$, and we note that $u_1(x_0) < u_2(x_0)$. We inductively continue in this manner to produce an infinite number of blow-up solutions $u_j \in W^{1,p}_{\mathrm{loc}}(\Omega) \cap C(\Omega)$ of problem (1.2) such that $0 < u_j(x_0) < u_{j+1}(x_0)$ for $j = 1, 2, \dots$. □

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To provide an example, let $b \in C(\bar{\Omega})$ be positive, and let $\omega \in C(\bar{\Omega})$ be a non-negative and non-trivial function. Suppose that $b(x) > p-1$ and $\omega(x) > 0$ for all $x \in \partial\Omega$. Then, given any $\lambda \in \mathbb{R}$, the following problem has a non-negative solution $u \in C(\Omega)$:

$$\left.\begin{aligned} \Delta_p u &= \omega(x)u^{b(x)}(1+\cos(\lambda u(x))) && \text{in } \Omega, \\ u &= \infty && \text{on } \partial\Omega. \end{aligned}\right\} \tag{4.6}$$

This follows from Theorem 4.8, as $h(x,t) = \omega(x)t^{b(x)}(1+\cos\lambda t)$ satisfies the circumferential lower Keller–Osserman condition in a neighbourhood, relative to $\bar{\Omega}$, of $\partial\Omega$. See Appendix A for details. Moreover, Proposition 4.9 shows that problem (4.6) admits infinitely many non-negative solutions provided that $\lambda \neq 0$.

5. The upper Keller–Osserman condition

In this section we study some conditions on h that are necessary for problem (1.2) to admit non-negative solutions. For this we introduce another function associated with h by a slight modification of the definition of h_* defined in the previous section.

For each $(x,r) \in \bar{\Omega} \times (0,\infty)$, define $h^*(t;x,r)\colon [0,\infty) \to [0,\infty)$ by

$$h^*(t;x,r) := \max\{h(z,t)\colon z \in \hat{B}(x,r)\}.$$

Again, it follows easily from Remark 4.1 that $h^*(t;x,r) > 0$ in $(0,\infty)$ for some $(x,r) \in \bar{\Omega} \times (0,\infty)$, $h^*(0;x,r) = 0$ and $h^*(t;x,r)$ is non-decreasing in $t > 0$, as well as in $r > 0$. One can also see that $h^*(t;x,r)$ is continuous in $t \in \mathbb{R}$.

For $t > 0$ we set

$$H^*(t;x,r) := \int_0^t h^*(\zeta;x,r)\,\mathrm{d}\zeta.$$

In analogy with (3.1) we let

$$\Phi^*(t;x,r) := \int_t^\infty \frac{1}{(p'(H^*(s;x,r) - H^*(t;x,r)))^{1/p}}\,\mathrm{d}s.$$

Remark 5.1. Since $h^*(t;x,r)$ is non-decreasing in r, it follows that $\Phi^*(t;x,r)$ is non-increasing in r.

We now give the following definition.

Definition 5.2. We say that $h(x,t)$ satisfies an upper Keller–Osserman condition at $x_0 \in \bar{\Omega}$ if there is $r \in (0,\infty)$ such that

$$\liminf_{t\to\infty} \Phi^*(t;x_0,r) = 0. \tag{5.1}$$

As noted before, this is equivalent to the following condition:

$$\Phi^*(\alpha;x_0,r) < \infty \quad \text{for some } \alpha > 0. \tag{5.2}$$

If h satisfies an upper Keller–Osserman condition at x_0, then in view of Remark 5.1 we note that condition (5.1) holds for all sufficiently large $r > 0$.

It is clear that $\Phi^*(t;x,r) \leqslant \Phi_*(t;x,r)$ for all $(t,x,r) \in (0,\infty) \times \bar{\Omega} \times (0,\infty)$, and that equality holds if $h(x,t)$ is independent of x, and is non-decreasing in t. In the latter case we see that a lower Keller–Osserman and an upper Keller–Osserman condition are equivalent to the standard Keller–Osserman condition.

The next result shows the necessity of condition (5.2) for problem (1.2) to have non-negative solutions.

Theorem 5.3. *Let $B(z,R) \subseteq \Omega$ and suppose that $h(x,t) > 0$ for some $(x,t) \in B(z,R) \times (0,\infty)$. If problem (1.2) admits a non-negative weak local supersolution in $B(z,R)$, then there is an $\alpha > 0$ such that $\Phi^*(\alpha;z,R) < \infty$. Thus, h satisfies an upper Keller–Osserman condition (5.2) at z.*

Proof. Let $u \in W^{1,p}_{\mathrm{loc}}(B) \cap C(B)$ be a weak local supersolution of (1.2) for some $B := B(z,R) \subseteq \Omega$. Note that $h^*(t;z,R) > 0$ for some $t > 0$. Then we see that u is a supersolution of

$$\left.\begin{aligned} \Delta_p v &= h^*(v;z,R) && \text{in } B,\\ v &= \infty && \text{on } \partial B. \end{aligned}\right\} \tag{5.3}$$

Lemma 2.7 shows that problem (5.3) admits a minimal radial solution v relative to $u_* = 0$. If we write $v(x) := \phi(|z-x|)$, then $\phi \in C^1([0,R))$ is a distributional solution of

$$\left.\begin{aligned} (r^{N-1}|\phi'(r)|^{p-2}\phi'(r))' &= r^{N-1}h^*(\phi(r);z,R), \quad 0<r<R,\\ \phi(0) = v(z), &\quad \phi'(0) = 0, \end{aligned}\right\} \tag{5.4}$$

in $(0,R)$. Note that $\phi' > 0$ for $0<r<R$. Multiplying both sides of the equation in (5.4) by $r^{(N-1)/(p-1)}\phi'(r)$ and integrating on $(0,r)$ for $0<r<R$ leads to

$$\int_0^r (t^{N-1}(\phi'(t))^{p-1})' t^{(N-1)/(p-1)}\phi'(t)\,\mathrm{d}t = \int_0^r t^{p(N-1)/(p-1)} h^*(\phi(t);z,R)\phi'(t)\,\mathrm{d}t.$$

Thus,

$$\begin{aligned} \frac{p-1}{p}(r^{(N-1)/(p-1)}\phi'(r))^p &\leqslant r^{p(N-1)/(p-1)} \int_0^r h^*(\phi(t);z,R)\phi'(t)\,\mathrm{d}t\\ &= r^{p(N-1)/(p-1)} \int_{\phi(0)}^{\phi(r)} h^*(s;z,R)\,\mathrm{d}s\\ &= r^{p(N-1)/(p-1)}(H^*(\phi(t);z,R) - H^*(\phi(0);z,R)). \end{aligned}$$

Consequently, we obtain

$$\frac{\phi'(r)}{(p'(H^*(\phi(t);z,R) - H^*(\phi(0);z,R)))^{1/p}} \leqslant 1, \quad 0<r<R.$$

Integrating this last inequality shows that, for $0<r<R$,

$$\int_{v(z)}^{\phi(r)} \frac{\mathrm{d}r}{(p'(H^*(t;z,R) - H^*(v(z);z,R)))^{1/p}} \leqslant r.$$

In particular, $\Phi^*(v(z);z,R) \leqslant R$. Therefore, h does indeed satisfy an upper Keller–Osserman condition at z. ☐

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We record the following immediate corollary.

Corollary 5.4. *Suppose that problem (1.2) admits a weak local supersolution in every ball $B \subseteq \Omega$. Then h satisfies an upper Keller–Osserman condition at every $x \in \Omega$ for which $h(x,t) > 0$ for some $t > 0$.*

Our next result shows that if $h(x,t)$ grows slower than t^{p-1} at infinity for some $x \in \partial\Omega$, then problem (1.2) does not admit weak local solutions in Ω. To state this more precisely, let

$$h^{\circ}(t;x,r) := \max\{h(z,s)\colon (z,s) \in \hat{B}(x,r) \times [0,t]\}. \tag{5.5}$$

Then h° has similar properties to those of h^*, but in addition $h^{\circ}(t;x,r)$ is now non-decreasing in t. Moreover, it is easily seen that if $h(x,t)$ is non-decreasing in t for each $x \in \bar{\Omega}$, then $h^*(t;x,r) = h^{\circ}(t;x,r)$ for all $(x,t,r) \in \bar{\Omega} \times (0,\infty) \times (0,\infty)$.

The following assumption will be needed to state and prove a non-existence result.

(H) There is a pair $(x,r) \in \bar{\Omega} \times (0,\infty)$ such that

$$\int_1^{\infty} \frac{1}{(h^{\circ}(t;x,r))^{1/(p-1)}}\,\mathrm{d}t = \infty.$$

We will say that h satisfies condition (H) at $x \in \bar{\Omega}$ if $h^0(t;x,r)$ satisfies (H) for some $r > 0$. For instance, if $h(x,t) = O(t^{p-1})$ as $t \to \infty$ for some $x \in \bar{\Omega}$, then h satisfies condition (H) at (x,r) for some $r > 0$.

Once again, it is easy to see that if condition (H) holds at $x \in \bar{\Omega}$ for some $r > 0$, then it holds at (x,s) for all $0 < s < r$.

We introduce the following function:

$$\beta(t) = \frac{1}{t}\int_t^{2t} h^{\circ}(s;x,r)\,\mathrm{d}s, \quad t > 0.$$

One can verify that β is a C^1 function on $(0,\infty)$ with $\beta(0+) = 0$, and $\beta(t) > 0$ for $t > 0$. Note that since $h^{\circ}(t) := h^{\circ}(t;x,r)$ is non-decreasing, we have

$$h^{\circ}(t) \leqslant \beta(t) \leqslant h^{\circ}(2t), \quad t > 0. \tag{5.6}$$

By virtue of the second inequality in (5.6), condition (H) holds for β whenever it holds for h°. This remark will be useful in proving the following non-existence result.

Theorem 5.5. *Suppose that h satisfies condition (H) for some $(x_0,r) \in \partial\Omega \times (0,\infty)$. Then problem (1.2) has no weak local supersolution in Ω.*

Proof. Assume to the contrary that problem (1.2) has a weak local supersolution $0 \leqslant u \in W^{1,p}_{\mathrm{loc}}(\Omega) \cap C(\Omega)$. Let $(x_0,r) \in \partial\Omega \times (0,\infty)$ be as in the hypothesis of the theorem. We assume that r is sufficiently small that $u \geqslant \varepsilon > 0$ in $\Omega_B := B(x_0,r) \cap \Omega$ for some ε. In view of (5.6), we note that

$$\beta(u(x)) \geqslant h^{\circ}(u(x);x_0,r) \geqslant h(x,u(x)) \quad (x \in \Omega_B). \tag{5.7}$$

We consider the following smooth and increasing function $\eta\colon [\varepsilon,\infty) \to [0,\infty)$:

$$\eta(t) := \int_{\varepsilon}^{t} \frac{1}{(\beta(s))^{1/(p-1)}}\,\mathrm{d}s, \quad t \geqslant \varepsilon. \tag{5.8}$$

We set

$$w(x) := \eta(u(x)), \quad x \in \Omega_B.$$

Note that $w \in W^{1,p}_{\mathrm{loc}}(\Omega_B) \cap C(\Omega_B)$. We now claim that $\Delta_p w \leqslant 1$ in Ω_B.

To prove the assertion, let $\varphi \in C_c^{\infty}(\Omega_B)$ and $\varphi \geqslant 0$ in Ω_B. Let us observe that $t \to (\eta'(t))^{p-1}$ is smooth and bounded on $[\varepsilon,\infty)$. Therefore, $(\eta'(u))^{p-1} \in W^{1,p}_{\mathrm{loc}}(\Omega_B)$ (see [**17**, Theorem 7.8]). We now use $\zeta := (\eta'(u))^{p-1}\varphi$ as a test function and, on noting that $\eta'' < 0$, we have

$$\begin{aligned}
\int_{\Omega_B} |\mathrm{D}w|^{p-2}\mathrm{D}w \cdot \mathrm{D}\varphi &= \int_{\Omega_B} |\mathrm{D}u|^{p-2}\mathrm{D}u \cdot \mathrm{D}\zeta - (p-1)\int_{\Omega_B} |\mathrm{D}u|^{p}\eta''(u)\eta'(u)^{p-2}\varphi \\
&\geqslant \int_{\Omega_B} |\mathrm{D}u|^{p-2}\mathrm{D}u \cdot \mathrm{D}\zeta \\
&\geqslant -\int_{\Omega_B} h(x,u)(\eta'(u))^{p-1}\varphi \\
&= -\int_{\Omega_B} \frac{h(x,u)}{\beta(u)}\varphi \\
&\geqslant -\int_{\Omega_B} \varphi, \quad \text{by (5.7)}.
\end{aligned}$$

Thus,

$$\Delta_p w \leqslant 1 \quad \text{in } \Omega_B, \tag{5.9}$$

as claimed. Recalling that β satisfies condition (H) and $u = \infty$ on $\partial\Omega$, we see that $w = \infty$ on $B(x_0,r) \cap \partial\Omega$.

We now proceed to show that the conclusion in (5.9) leads to a contradiction.

Let $\chi \in C_c(B(x_0,r))$ such that $\chi \equiv 1$ in $B(x_0,r/3)$ and $\chi \equiv 0$ on $B(x_0,r) \setminus B(x_0,r/2)$. Now let v be the solution of

$$\begin{aligned}
\Delta_p v &= 1 \quad \text{in } \Omega_B, \\
v &= \chi \quad \text{on } \partial\Omega_B.
\end{aligned}$$

Recalling (5.9), and employing a standard comparison principle, we note that

$$kv \leqslant w \quad \text{in } \Omega_B\ \forall k \geqslant 1.$$

By choosing $x_1 \in \Omega_B$ such that $w(x_1) > 0$, we see in particular that

$$kv(x_1) \leqslant w(x_1)$$

for all k. This is an obvious contradiction. □

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We illustrate the above theorem with a simple example. Let $h(x,t) = t^{b(x)}$, where $b \in C(\bar{\Omega}, (0,\infty))$, and consider the problem

$$\left.\begin{aligned} \Delta_p u = h(x,u) \quad & \text{in } \Omega, \\ u = \infty \quad & \text{on } \partial\Omega. \end{aligned}\right\} \tag{5.10}$$

For $t \geqslant 1$ we observe that

$$h^{\circ}(t; x, r) = t^{\max\{b(z):\, z \in \hat{B}(x,r)\}}.$$

Suppose that $b(x_0) < p$ at some point $x_0 \in \partial\Omega$. Since $b \in C(\bar{\Omega}, (0,\infty))$, there is an $r_0 > 0$ such that $b(x) \leqslant \gamma < p - 1$ on $B(x_0, r_0) \cap \bar{\Omega}$. Then $h^{\circ}(t; x_0, r_0) \leqslant t^{\gamma}$ for all $t \geqslant 1$, and hence condition (H) holds. Therefore, Theorem 5.5 shows that problem (5.10) has no weak local supersolution. Hence, for problem (5.10) to admit a supersolution it is necessary that $b(x) \geqslant p$ on $\partial\Omega$.

6. The $h(x,t) = \omega(x)f(t)$ case

Following Lair [**24**], we make the following definition.

Definition 6.1. A function $\omega\colon \Omega \to [0,\infty)$ is said to be circumferentially positive (or just c-positive) if and only if given $x_0 \in \Omega$ with $\omega(x_0) = 0$ there is $\mathcal{O} \subset\subset \Omega$ such that $x_0 \in \mathcal{O}$ and $\omega(x) > 0$ for $x \in \partial\mathcal{O}$.

Let us now consider the following singular boundary-value problem:

$$\left.\begin{aligned} \Delta_p u = \omega(x)f(u) \quad & \text{in } \Omega, \\ u = \infty \quad & \text{on } \partial\Omega. \end{aligned}\right\} \tag{6.1}$$

Suppose that $f(0) = 0$ and $f(c) > 0$ for some $c > 0$. A simple computation shows that if $x_0 \in \bar{\Omega}$ such that $\omega(x_0) > 0$, then for a sufficiently small $r > 0$,

$$\Phi_*(t; x_0, r) = \frac{1}{(\min_{\hat{B}(x_0,r)} \omega)^{1/p}} \Phi(t). \tag{6.2}$$

Likewise, we have

$$\Phi^*(t; x_0, r) = \frac{1}{(\max_{\hat{B}(x_0,r)} \omega)^{1/p}} \Phi(t). \tag{6.3}$$

In both (6.2) and (6.3), Φ is defined as in (3.1) but with g replaced by f. Therefore, in this situation, the lower and upper Keller–Osserman conditions coincide with the Keller–Osserman condition (3.2).

The next result is a consequence of Theorems 4.8 and 5.3. The analogue of the following theorem, in the context of the classical Laplacian operator, was proved in [**24**] under the additional assumptions that $f(t) > 0$ for $t > 0$, f increasing.

Theorem 6.2. *Suppose that* $\omega \in C(\bar{\Omega}, [0,\infty))$ *and* $f\colon [0,\infty) \to [0,\infty)$ *is a non-trivial and continuous function such that* $f(0) = 0$.

(i) *If* ω *is c-positive in* Ω *and* f *satisfies the Keller–Osserman condition (3.3), then problem (6.1) admits a non-negative solution* $u \in W^{1,p}_{\mathrm{loc}}(\Omega) \cap C(\Omega)$.

(ii) *If* ω *is non-trivial on the ball* $B(z,R) \subseteq \Omega$ *and problem (6.1) admits a non-negative solution* u *in a ball* $B(z,R)$*, then* f *satisfies the Keller–Osserman condition (3.3).*

Proof. (i) Suppose first that f satisfies the Keller–Osserman condition (3.3). We note that if $\omega(x_0) > 0$ at some $x_0 \in \Omega$, then by virtue of (6.2), $h(x,t) := \omega(x)f(t)$ satisfies a lower Keller–Osserman condition at x_0. Therefore, it suffices to note that the c-positivity of ω together with the assumption that f satisfies the Keller–Osserman condition (3.3) imply that $h(x,t) := \omega(x)f(t)$ satisfies the circumferential lower Keller–Osserman condition at every point of Ω_δ for any $\delta > 0$. Therefore, part (i) of the theorem follows from Theorem 4.8.

(ii) Let us assume that problem (6.1) has a positive solution in a ball $B(z,R) \subseteq \Omega$ for some $z \in \Omega$ with $\omega(z) > 0$. By Theorem 5.3, $h(x,t) = \omega(x)f(t)$ satisfies an upper Keller–Osserman condition at $z \in \Omega$ with $\omega(z) > 0$. In fact, we have $\Phi^*(a; z, r) < \infty$ for some $a > 0$ and $r > 0$. But then, according to (6.3), we find that $\Phi(a) < \infty$, where Φ is as in (3.1) with G replaced by $F(t) = \int_0^t f(s)\,\mathrm{d}s$. That is, f satisfies the Keller–Osserman condition (3.3). This concludes the proof of the theorem. □

Appendix A.

A.1. An example

Here we look at an example to illustrate some of the main results of the paper. To this end, let us consider the nonlinearity

$$h(x,t) = \omega(x)t^{b(x)}(1 + \cos\lambda t), \tag{A 1}$$

where $b \in C(\bar{\Omega})$ is positive, $\omega \in C(\bar{\Omega})$ is a non-negative and non-trivial function, and λ is a constant.

A.1.1. *On existence*

Let $x_0 \in \Omega$ such that $\omega(x_0) > 0$ and $b(x_0) > p - 1$. We show below that h satisfies a lower Keller–Osserman condition at x_0. We will assume that $\lambda \neq 0$, as the case $\lambda = 0$ is much easier. Let us set

$$\omega_*(x_0,r) := \min\{\omega(x)\colon x \in \hat{B}(x_0,r)\} \quad \text{and} \quad b_*(x_0,r) := \min\{b(x)\colon x \in \hat{B}(x_0,r)\}.$$

For $t \geqslant 1$ we see that

$$h_*(t; x_0, r) \geqslant \omega_*(x_0,r)t^{b_*(x_0,r)}(1 + \cos\lambda t). \tag{A 2}$$

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Without loss of generality, we suppose that $\lambda > 0$. We fix $r > 0$ sufficiently small such that $b_*(x_0, r) > p - 1$. Let $\beta \geqslant \lambda$ be an arbitrary positive constant such that $\cos\beta = 1$, and set $\alpha := \beta/\lambda$ and $\gamma := (\beta + \pi/2)/\lambda$. We observe that

$$\int_\alpha^\gamma \cos(\lambda s)\,\mathrm{d}s = \frac{1}{\lambda} \quad \text{and} \quad \gamma - \alpha = \frac{\pi}{2\lambda}.$$

If $\alpha \leqslant t \leqslant \gamma$, we have

$$\int_\alpha^t (1 + \cos\lambda s)\,\mathrm{d}s \geqslant t - \alpha.$$

On the other hand, if $t > \gamma$, we have

$$\int_\alpha^t (1 + \cos\lambda s)\,\mathrm{d}s = t - \alpha + \frac{\sin\lambda t}{\lambda} \geqslant t - \alpha - \frac{1}{\lambda} = t - \alpha - \frac{2}{\pi}(\gamma - \alpha) \geqslant \left(1 - \frac{2}{\pi}\right)(t - \alpha).$$

Thus, in any case we have

$$\int_\alpha^t (1 + \cos\lambda s)\,\mathrm{d}s \geqslant \mu(t - \alpha), \quad t \geqslant \alpha, \text{ where } \mu := 1 - \frac{2}{\pi}. \tag{A 3}$$

Let θ be a fixed positive real number θ and set

$$\gamma_\theta = \alpha + \frac{4(\theta + 1)}{\lambda}. \tag{A 4}$$

For $t \geqslant \gamma_\theta$ we have the following chain of inequalities:

$$\begin{aligned}
\int_\alpha^t s^\theta (1 + \cos\lambda s)\,\mathrm{d}s &\geqslant \int_\alpha^t (s - \alpha)^\theta (1 + \cos\lambda s)\,\mathrm{d}s \\
&= \frac{(t - \alpha)^{\theta+1}}{\theta + 1} + \int_\alpha^t (s - \alpha)^\theta \cos\lambda s\,\mathrm{d}s \\
&= \frac{(t - \alpha)^{\theta+1}}{\theta + 1} + \frac{1}{\lambda}(t - \alpha)^\theta \sin\lambda t - \frac{\theta}{\lambda}\int_\alpha^t (s - \alpha)^{\theta-1} \sin\lambda s\,\mathrm{d}s \\
&\geqslant \frac{(t - \alpha)^{\theta+1}}{\theta + 1} + \frac{1}{\lambda}(t - \alpha)^\theta \sin\lambda t - \frac{1}{\lambda}(t - \alpha)^\theta \\
&\geqslant \frac{(t - \alpha)^{\theta+1}}{\theta + 1} - \frac{2}{\lambda}(t - \alpha)^\theta \\
&= \frac{(t - \alpha)^{\theta+1}}{\theta + 1} - \frac{(\gamma_\theta - \alpha)}{2(\theta + 1)}(t - \alpha)^\theta.
\end{aligned}$$

Therefore, using (A 4) in the last inequality above we get the following conclusion. Given a positive real number θ, there exist positive real numbers γ_θ and C_θ such that

$$\int_\alpha^t s^\theta (1 + \cos\lambda s)\,\mathrm{d}s \geqslant C_\theta (t - \alpha)^{\theta+1} \quad \forall t \geqslant \gamma_\theta. \tag{A 5}$$

In fact, γ_θ and C_θ are given by

$$\gamma_\theta - \alpha = \frac{4(\theta + 1)}{\lambda}, \qquad C_\theta := \frac{1}{2(\theta + 1)}.$$

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On using (A 2) and (A 3) we find that for any $s \geqslant \alpha$ (for simplicity we use b_* for $b_*(x_0, r)$ and ω_* for $\omega_*(x_0, r)$)

$$
\begin{aligned}
H_*(s; x_0, r) - H_*(\alpha; x_0, r) &= \int_\alpha^s h_*(z; x_0, r)\,\mathrm{d}z \\
&\geqslant \omega_* \alpha^{b_*} \int_\alpha^s (1 + \cos(\lambda z))\,\mathrm{d}z \\
&\geqslant \mu \omega_* \alpha^{b_*}(s - \alpha).
\end{aligned}
$$

That is,

$$
H_*(s; x_0, r) - H_*(\alpha; x_0, r) \geqslant \mu \omega_* \alpha^{b_*}(s - \alpha) \quad \text{for } s \geqslant \alpha. \tag{A 6}
$$

Fix $\varepsilon > 0$ small enough such that $b_* - \varepsilon > p - 1$. On using (A 2) and (A 5) we find that for any $s \geqslant \gamma_* = \alpha + 4(b_* - \varepsilon + 1)/\lambda$,

$$
\begin{aligned}
H_*(s; x_0, r) - H_*(\alpha; x_0, r) &= \int_\alpha^s h_*(z; x_0, r)\,\mathrm{d}z \\
&\geqslant \omega_* \alpha^{\varepsilon} \int_\alpha^s z^{b_* - \varepsilon}(1 + \cos \lambda z)\,\mathrm{d}z \\
&\geqslant C \omega_* \alpha^{\varepsilon}(s - \alpha)^{b_* + 1 - \varepsilon}.
\end{aligned} \tag{A 7}
$$

From (A 6) and (A 7) we see that

$$
\begin{aligned}
\Phi_*(\alpha; x_0, r) &= \int_\alpha^\infty \frac{1}{(H_*(s; x_0, r) - H_*(\alpha; x_0, r))^{1/p}}\,\mathrm{d}s \\
&= \int_\alpha^{\gamma_*} \frac{1}{(H_*(s; x_0, r) - H_*(\alpha; x_0, r))^{1/p}}\,\mathrm{d}s \\
&\quad + \int_{\gamma_*}^\infty \frac{1}{(H_*(s; x_0, r) - H_*(\alpha; x_0, r))^{1/p}}\,\mathrm{d}s \\
&\leqslant (\mu \omega_* \alpha^{b_*})^{-1/p} \int_\alpha^{\gamma_*} (s - \alpha)^{-1/p}\,\mathrm{d}s \\
&\quad + C(\omega_* \alpha^{\varepsilon})^{-1/p} \int_{\gamma_*}^\infty (s - \alpha)^{(-b_* - 1 + \varepsilon)/p}\,\mathrm{d}s \\
&\leqslant C(\alpha^{-b_*/p} + \alpha^{-\varepsilon/p}).
\end{aligned}
$$

Here C is a constant independent of α. In conclusion, we have shown that given any positive real number $\beta \geqslant \lambda$ such that $\cos \beta = 1$ and $\alpha := \beta/\lambda$ we have

$$
\Phi_*(\alpha; x_0, r) \leqslant C(\alpha^{-b_*/p} + \alpha^{-\varepsilon/p})
$$

with a positive constant C independent of α. Therefore, we see that $\liminf_{t \to \infty} \Phi(t) = 0$. That is, condition (4.2) holds at x_0. Therefore, if $b(x) > p-1$ and $\omega(x) > 0$ on Ω_δ for some $\delta > 0$, then Theorem 4.8 shows that problem (1.2), with $h(x, t)$ given as in (A 1), admits a non-negative solution $u \in W^{1,p}_{\mathrm{loc}}(\Omega) \cap C(\Omega)$. Now let $\lambda \neq 0$. Then $h(x, t)$ satisfies condition (4.5) with $\varpi = 2\pi/|\lambda|$. Since $h(x, t_j) \equiv 0$ in Ω for all j, where $t_j := (2j + 1)\pi/(2|\lambda|)$, Proposition 4.9 shows that problem (1.2) actually admits infinitely many non-negative solutions in $W^{1,p}_{\mathrm{loc}}(\Omega) \cap C(\Omega)$.

A.1.2. *On non-existence*

Let h be as in (A 1). Then there is a positive constant C such that for $t \geqslant 1$,

$$h^*(t;x,r) = \max\{\omega(z)t^{b(z)}(1+\cos(\lambda t))\colon z \in \hat{B}(x,r)\} \leqslant Ct^{\max_{\hat{B}(x,r)} b(z)}. \tag{A 8}$$

Suppose that $b(x_0) < p-1$ and $\omega(x_0) > 0$ for some $x_0 \in \Omega$. Then there is a ball $B := B(x_0, r) \subseteq \Omega$ such that problem (1.2) has no non-negative weak local supersolution u in B. To see this we observe that, by continuity, $0 < b(x) \leqslant \gamma < p-1$ for all $x \in \hat{B}(x_0, r)$ and a sufficiently small $r > 0$. In view of (A 8) we have

$$H^*(s;x_0,r) - H^*(t;x_0,r) = \int_t^s h^*(\tau;x_0,r)\,\mathrm{d}\tau \leqslant Cs^{\gamma+1}, \quad s \geqslant t \geqslant 1,$$

for some $C > 0$.

Thus, $\Phi^*(t;x_0,r) = \infty$ for all $t > 1$. Therefore, Theorem 5.3 shows that problem (1.2) has no non-negative weak local supersolution in $W^{1,p}_{\mathrm{loc}}(B) \cap C(B)$.

Now let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain, and let us assume that $0 < b(x_0) < p-1$ and $\omega(x_0) > 0$ for some $x_0 \in \partial\Omega$. Thus, $h(x_0, t) = O(t^{p-1})$ as $t \to \infty$. Moreover, since $\omega(x_0) > 0$, we see that $h(x_0, \tau) > 0$ for some $\tau > 0$. Consequently, h satisfies condition (H) and thus, by Theorem 5.5, we conclude that problem (1.2) has no solution in $W^{1,p}_{\mathrm{loc}}(\Omega) \cap C(\Omega)$.

A.2. Equivalence of conditions (3.2) and (3.3) for $1 < p < \infty$

Our proof of the equivalence is an adaptation of the argument in [**12**]. Obviously, condition (3.3) is implied by (3.2). So, let us suppose that condition (3.3) holds for some $t > 0$, and show that (3.2) holds as well. We start by noting that (3.3) implies that

$$\int_t^\infty \frac{1}{G(s)^{1/p}}\,\mathrm{d}s < \infty. \tag{A 9}$$

We start with the change of variable $\zeta = G(s)$ to rewrite

$$\int_t^\infty \frac{\mathrm{d}s}{(G(s) - G(t))^{1/p}} = \int_{G(t)}^\infty \frac{\eta(\zeta)}{(\zeta - G(t))^{1/p}}\,\mathrm{d}\zeta,$$

where $\eta(\zeta) = (G^{-1})'(\zeta)$. Therefore, we wish to show that

$$\liminf_{\tau\to\infty} \int_\tau^\infty \frac{\eta(\zeta)}{(\zeta-\tau)^{1/p}}\,\mathrm{d}\zeta = 0. \tag{A 10}$$

Since $\zeta - \tau \leqslant \zeta \leqslant 2(\zeta - \tau)$ for all $\zeta \geqslant 2\tau > 0$, we see that

$$\lim_{\tau\to\infty} \int_{2\tau}^\infty \frac{\eta(\zeta)}{(\zeta-\tau)^{1/p}}\,\mathrm{d}\tau = 0.$$

Consequently, in order to establish (A 10), it suffices to show that

$$\liminf_{\tau\to\infty} \int_\tau^{2\tau} \frac{\eta(\zeta)}{(\zeta-\tau)^{1/p}}\,\mathrm{d}\zeta = 0. \tag{A 11}$$

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Suppose on the contrary that there is a positive constant C such that

$$\int_{\tau}^{2\tau} \frac{\eta(\zeta)}{(\zeta-\tau)^{1/p}}\,\mathrm{d}\zeta \geqslant C \quad \text{for all } \tau \text{ sufficiently large.}$$

Let us make the change of variable $\xi = (\zeta-\tau)^{1/p'}$. Then

$$\int_{\tau}^{2\tau} \frac{\eta(\zeta)}{(\zeta-\tau)^{1/p}}\,\mathrm{d}\zeta = p'\int_0^{\tau^{1/p'}} \eta(\xi^{p'}+\tau)\,\mathrm{d}\xi = p'\int_0^{\tau} \eta(\xi^{p'}+\tau^{p'})\,\mathrm{d}\xi. \tag{A 12}$$

For any sufficiently large $R>0$, and fixed $0<\vartheta<1$, we integrate both sides of (A 12) on the interval $\vartheta R^{p-1} \leqslant \tau \leqslant R^{p-1}$ to obtain

$$CR^{p-1} \leqslant \int_{\vartheta R^{p-1}}^{R^{p-1}} \int_0^{\tau} \eta(\xi^{p'}+\tau^{p'})\,\mathrm{d}\xi\,\mathrm{d}\tau \leqslant \int_{\vartheta R^{p-1}}^{R^{p-1}} \int_0^{R^{p-1}} \eta(\xi^{p'}+\tau^{p'})\,\mathrm{d}\xi\,\mathrm{d}\tau. \tag{A 13}$$

We use the generalized sine function S_p and the generalized cosine function C_p to make an appropriate change of variable. For a given $0<p<\infty$, the functions S_p and C_p are defined, respectively, as solutions of

$$(\Psi(u'))' + (p-1)\Psi(u) = 0 \tag{A 14}$$

such that $S_p(0)=0$, $S_p'(0)=1$ and $C_p(0)=1$, $C_p'(0)=0$. Here $\Psi(t) := |t|^{p-2}t$ for $t\in\mathbb{R}$. Let us denote by π_p the number $2\pi/(p\sin(\pi/p))$. We recall some of the basic properties about these functions (see [**13**]):

$$\left.\begin{aligned} S_p'(t) &= C_p(t),\\ C_p'(t) &= -S_p(t),\\ |S_p(t)|^p + |C_p(t)|^p &= 1 \end{aligned}\right\} \quad \text{for } -\tfrac{1}{2}\pi_p < t < \tfrac{1}{2}\pi_p,$$
$$S_p(t), C_p(t) \geqslant 0 \quad \text{for } 0 \leqslant t \leqslant \tfrac{1}{2}\pi_p.$$

We now make the change of variables

$$\xi = r^{p-1}\Psi(C_p(\theta)), \quad \tau = r^{p-1}\Psi(S_p(\theta)), \quad \vartheta^{1/(p-1)}R < r < R, \quad 0<\theta<\tfrac{1}{2}\pi_p.$$

Clearly, we see that

$$\xi^{p'} + \tau^{p'} = r^p.$$

We estimate the Jacobian of the change of coordinates for $r>0$ and $0<\theta<\pi_p/2$ as follows:

$$\begin{aligned} \left|\frac{\partial(\xi,\tau)}{\partial(r,\theta)}\right| &= (p-1)^2 r^{2p-3}((\Psi(S_p(\theta)))^2 + (\Psi(C_p(\theta)))^2)\\ &\leqslant 2(p-1)^2 \max\{1, \vartheta^{(p-2)/(p-1)}\} R^{p-2} r^{p-1}. \end{aligned}$$

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After the change of variables, (A 13) becomes (with constant C independent of R but possibly differing from line to line)

$$\begin{aligned} R^{p-1} &\leqslant CR^{p-2}\int_0^{\pi_p/2}\int_{\vartheta^{1/(p-1)}R}^{R}\eta(r^p)r^{p-1}\,\mathrm{d}r\,\mathrm{d}\theta \\ &= CR^{p-2}\int_0^{\pi_p/2}\int_{\vartheta^{p/(p-1)}R^p}^{R^p}\eta(\rho)\,\mathrm{d}\rho\,\mathrm{d}\theta \\ &\leqslant CR^{p-2}G^{-1}(R^p). \end{aligned}$$

Thus, we conclude that for some positive constant C, independent of R,

$$R \leqslant CG^{-1}(R^p), \quad R > 0, \text{ sufficiently large.}$$

Thus, for $s = G^{-1}(R^p)$ we see that

$$G(s)^{1/p} \leqslant Cs \quad \text{for } s > 0, \text{ sufficiently large ,}$$

and this contradicts (A 9). Therefore, (A 11) holds, and hence (A 10) must be true. □

A.3. Equivalence of (3.2)–(3.4) for non-decreasing nonlinearity g

Using the fact that $g(s)$ is non-decreasing, one can easily show that for any $s \geqslant t > 0$ the following hold:

$$G(s) - G(t) \geqslant G(s-t) \quad \text{and} \quad G(s) - G(t) \geqslant g(t)(s-t).$$

We now use these inequalities as follows. For $t > 0$,

$$\begin{aligned} \int_t^\infty \frac{1}{(G(s)-G(t))^{1/p}}\,\mathrm{d}s &= \int_t^{2t} \frac{1}{(G(s)-G(t))^{1/p}}\,\mathrm{d}s + \int_{2t}^\infty \frac{1}{(G(s)-G(t))^{1/p}}\,\mathrm{d}s \\ &\leqslant \int_t^{2t} \frac{1}{g(t)^{1/p}(s-t)^{1/p}}\,\mathrm{d}s + \int_{2t}^\infty \frac{1}{(G(s-t))^{1/p}}\,\mathrm{d}s \\ &= \frac{1}{g(t)^{1/p}}\int_0^t \frac{1}{s^{1/p}}\,\mathrm{d}s + \int_t^\infty \frac{1}{G(s)^{1/p}}\,\mathrm{d}s \\ &= p'\left(\frac{t^{p/p'}}{g(t)}\right)^{1/p} + \int_t^\infty \frac{1}{G(s)^{1/p}}\,\mathrm{d}s. \end{aligned}$$

Therefore, we have

$$\int_t^\infty \frac{1}{G(s)^{1/p}}\,\mathrm{d}s \leqslant \int_t^\infty \frac{1}{(G(s)-G(t))^{1/p}}\,\mathrm{d}s \leqslant p'\left(\frac{t^{p/p'}}{g(t)}\right)^{1/p} + \int_t^\infty \frac{1}{G(s)^{1/p}}\,\mathrm{d}s.$$

A.4. Monotonicity of $\Phi_*(t;x,r)$ and $\Phi^*(t;x,r)$ in r

We show that $\Phi_*(t;x,r)$ is non-decreasing in r. Suppose that $r_1 < r_2$. Then, from the definition of h_*, we see that

$$h_*(\zeta;x,r_1) \geqslant h_*(\zeta;x,r_2) \quad \text{for } \zeta > 0.$$

Therefore, for $s \geqslant t > 0$ we have

$$\begin{aligned} H_*(s;x,r_1) - H_*(t;x,r_1) &= \int_t^s h_*(\zeta;x,r_1)\,\mathrm{d}\zeta \\ &\geqslant \int_t^s h_*(\zeta;x,r_2)\,\mathrm{d}\zeta \\ &\geqslant H_*(s;x,r_2) - H_*(t;x,r_2). \end{aligned}$$

Consequently, we have $\Phi_*(t;x,r_1) \leqslant \Phi_*(t;x,r_2)$.

That $\Phi^*(t;x,r)$ is non-increasing in r can be shown in a similar manner. □

References

1. C. Bandle and M. Marcus, Large solutions of semilinear elliptic equations: existence uniqueness and asymptotic behavior, *J. Analysis Math.* **58** (1992), 9–24.
2. C. Bandle and M. Marcus, Asymptotic behavior of solutions and their derivatives for semilinear elliptic problems with blowup on the boundary, *Annales Inst. H. Poincaré* **12** (1995), 155–171.
3. L. Bieberbach, $\Delta u = \mathrm{e}^u$ und die automorphen Funktionen, *Math. Annalen* **77**(2) (1916), 173–212.
4. S. Cano-Casanova and J. López-Gómez, Blow-up rates of radially symmetric large solutions, *J. Math. Analysis Applic.* **352**(1) (2009), 166–174.
5. F. Cîrstea and V. Rădulescu, Existence and uniqueness of blow-up solutions for a class of logistic equations, *Commun. Contemp. Math.* **4**(3) (2002), 559–586.
6. F. Cîrstea and V. Rădulescu, Boundary blow-up in nonlinear elliptic equations of Bieberbach–Rademacher type, *Trans. Am. Math. Soc.* **359**(7) (2007), 3275–3286.
7. G. Diaz and R. Letelier, Explosive solutions of quasilinear elliptic equations: existence and uniqueness, *Nonlin. Analysis* **20** (1993), 97–125.
8. E. DiBenedetto, $C^{1+\alpha}$ local regularity of weak solutions of degenerate elliptic equations, *Nonlin. Analysis* **7**(8) (1983), 827–850.
9. G. Dinca, P. Jebelean and J. Mawhin, Variational and topological methods for Dirichlet problems with p-Laplacian, *Portugaliae Math.* **58** (2001), 339–378.
10. M. Dindoš, Large solutions for Yamabe and similar problems on domains in Riemannian manifolds, *Trans. Am. Math. Soc.* **363**(10) (2011), 5131–5178.
11. Y. Du and Z. Guo, Boundary blow-up solutions and their applications in quasilinear elliptic equations, *J. Analysis Math.* **89** (2003), 277–302.
12. S. Dumont, L. Dupaigne, O. Goubet and V. Rădulescu, Back to the Keller–Osserman condition for boundary blow-up solutions, *Adv. Nonlin. Studies* **7**(2) (2007), 271–298.
13. Á. Elbert, A half-linear second order differential equation, in *Qualitative theory of differential equations (Szeged, 1979)*, Colloquia Mathematica Societatis János Bolyai, Volume 30, pp. 153–180 (North-Holland, Amsterdam, 1981).
14. J. García-Melián, R. Letelier and J. C. Sabina de Lis, Uniqueness and asymptotic behaviour for functions of semilinear problems with boundary blow-up, *Proc. Am. Math. Soc.* **129** (2001), 3593–3602.
15. J. García-Melián, J. D. Rossi and J. C. Sabina de Lis, Large solutions for the Laplacian with a power nonlinearity given by a variable exponent, *Annales Inst. H. Poincaré Analyse Non Linéaire* **26**(3) (2009), 889–902.
16. J. García-Melián, J. C. Sabina de Lis, and J. D. Rossi, Existence, asymptotic behavior and uniqueness for large solutions to $\Delta u = \mathrm{e}^{q(x)u}$, *Adv. Nonlin. Studies* **9**(2) (2009), 395–424.

17. D. Gilbarg and N. Trudinger, *Elliptic partial differential equations of second order* (Springer, 1983).
18. F. Gladiali and G. Porru, Estimates for explosive solutions to p-Laplace equations, in *Progress in Partial Differential Equations, Pont-à-Mousson, 1997*, Volume 1, pp. 117–127, Pitman Research Notes in Mathematics Series, Volume 383 (Longman, New York, 1998).
19. B. Guan and H. Y. Jian, The Monge–Ampère equation with infinite boundary value, *Pac. J. Math.* **216**(1) (2004), 77–94.
20. J. Heinonen, T. Kilpeläinen and O. Martio, *Nonlinear potential theory of degenerate elliptic equations*, Oxford Mathematical Monographs (Clarendon, Oxford, 1993).
21. L. Hörmander, *The analysis of linear partial differential operators I* (Springer, 1983).
22. H. Y. Jian, Hessian equations with infinite Dirichlet boundary value, *Indiana Univ. Math. J.* **55**(3) (2006), 1045–1062.
23. J. B. Keller, On solutions of $\Delta u = f(u)$, *Commun. Pure Appl. Math.* **10** (1957), 503–510.
24. A. V. Lair, A necessary and sufficient condition for existence of large solutions to semilinear elliptic equations, *J. Math. Analysis Applic.* **240**(1) (1999), 205–218.
25. A. V. Lair and A. Wood, Existence of entire large positive solutions of semilinear elliptic systems, *J. Diff. Eqns* **164** (2000), 380–394.
26. A. C. Lazer and P. J. McKenna, On a problem of Bieberbach and Rademacher, *Nonlin. Analysis* **21** (1993), 327–335.
27. A. C. Lazer and P. J. McKenna, Asymptotic behavior of solutions of boundary blowup problems, *Diff. Integ. Eqns* **7**(3–4) (1994), 1001–1019.
28. M. C. Leon, Existence results for quasilinear problems via ordered sub- and supersolutions, *Annales Fac. Sci. Toulouse Math.* **6**(4) (1997), 591–608
29. C. Loewner and L. Nirenberg, Partial differential equations invariant under the conformal or projective transformations, in *Contributions to analysis: a collection of papers dedicated to Lipman Bers*, pp. 245–272 (Academic Press, 1974).
30. J. Lopez-Gomez, Large solutions, metasolutions and asymptotic behavior of the regular positive solutions of sublinear parabolic problems, *Electron. J. Diff. Eqns* **Conference 5** (2000), 135–171.
31. J. Lopez-Gomez, Varying stoichometric exponents I: classical steady states and metasolutions, *Adv. Nonlin. Studies* **3** (2003), 327–354.
32. J. Lopez-Gomez, The boundary blow-up rate of large solutions, *J. Diff. Eqns* **195** (2003), 25–45.
33. J. Lopez-Gomez, Optimal uniqueness theorems and exact blow-up rate of large solutions, *J. Diff. Eqns* **224** (2006), 385–439.
34. J. Lopez-Gomez, Uniqueness of radially symmetric large solutions, *Discrete Contin. Dynam. Syst.* (Suppl.) (2007), 677–686.
35. M. Marcus and L. Véron, Uniqueness of solutions with blowup at the boundary of a class of nonlinear elliptic equations, *C. R. Acad. Sci. Paris Sér. I* **317**(6) (1993), 559–563.
36. M. Marcus and L. Véron, Uniqueness and asymptotic behavior of solutions with boundary blowup for a class of nonlinear elliptic equations, *Annales Inst. H. Poincaré Analyse Non Linéaire* **14**(2) (1997), 237–274.
37. J. Matero, The Bieberbach–Rademacher problem for the Monge–Ampère operator, *Manuscr. Math.* **962**(3) (1996), 379–391.
38. J. Matero, Quasilinear elliptic equations with boundary blow-up, *J. Analysis Math.* **69** (1996), 229–247.
39. A. Olofsson, Apriori estimates of Osserman–Keller type, *Diff. Integ. Eqns* **16**(6) (2003), 737–756.
40. R. Osserman, On the inequality $\Delta u \geqslant f(u)$, *Pac. J. Math.* **7** (1957), 1641–1647.

41. H. Rademacher, Einige besondere probleme der parttiellen Differentialgleichungen, in *Die Differential- und Integralgleichungen der Mechanik und Physik I*, 2nd edn (ed. P. Frank and R. von Mises), pp. 838–845 (Rosenberg, New York, 1943).
42. V. Rădulescu, Singular phenomena in nonlinear elliptic problems: from blow-up boundary solutions to equations with singular nonlinearities, in *Handbook of differential equations: stationary partial differential equations* (ed. M. Chipot), Volume IV, pp. 485–593 (North-Holland, Amsterdam, 2007).
43. P. Salani, Boundary blow-up problems for Hessian equations, *Manuscr. Math.* **96** (1998), 281–294.
44. J. L. Vázquez, A strong maximum principle for some quasilinear elliptic equations, *Appl. Math. Optim.* **12**(3) (1984), 191–202.
45. L. Véron, Semilinear elliptic equations with uniform blow-up on the boundary, *J. Analysis Math.* **59** (1992), 231–250.
46. L. Véron, Large solutions of elliptic equations with strong absorption, in *Elliptic and parabolic problems*, Progress in Nonlinear Differential Equations and Their Applications, Volume 63, pp. 453–464 (Birkhäuser, 2005).
47. Z. Zhang, The asymptotic behaviour of solutions with boundary blow-up for semilinear elliptic equations with nonlinear gradient terms, *Nonlin. Analysis* **62**(6) (2005), 1137–1148.
48. Z. Zhang, Y. Ma, L. Mi and X. Li, Blow-up rates of large solutions for elliptic equations, *J. Diff. Eqns* **249**(1) (2010), 180–199.