ON THE TOPOLOGICAL DEGREE OF REAL POLYNOMIAL VECTOR FIELDS

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1. Introduction. Let $G:\mathbf{R}^n\to\mathbf{R}^n$ be a continuous mapping such that the origin $\mathbf{0}\in\mathbf{R}^n$ is isolated in $G^{-1}(\mathbf{0})$. Then $\deg_0 G$ will denote the local topological degree of G at the origin, i.e. the topological degree of the mapping

$$S_r \ni x \mapsto \frac{G(x)}{\|G(x)\|} \in S^{n-1},$$

where S_r denotes a sphere in $\mathbf{R}^n$ centered at the origin with small radius $r>0$.

We shall prove the following theorem.

THEOREM 1.1 (the Main Theorem). *Let $F:\mathbf{R}^m\times\mathbf{R}\times\mathbf{R}^n\to\mathbf{R}^n$ be a polynomial mapping. For any $(w,t)\in\mathbf{R}^m\times\mathbf{R}$, let $F_{w,t}:\mathbf{R}^n\to\mathbf{R}^n$ denote the vector field $F_{w,t}(x)=F(w,t,x)$. Suppose that*

(a) *$F_{w,t}(\mathbf{0})=\mathbf{0}$ for all $(w,t)\in\mathbf{R}^m\times\mathbf{R}$,*

(b) *there is a proper algebraic set $X\subset\mathbf{R}^m\times\mathbf{R}$ such that $\mathbf{0}\in\mathbf{R}^n$ is isolated in $F_{w,t}^{-1}(\mathbf{0})$ for every $(w,t)\in\mathbf{R}^m\times\mathbf{R}-X$ and so $\deg_0 F_{w,t}$ is defined for all $(w,t)\in\mathbf{R}^m\times\mathbf{R}-X$.*

Then there is a proper algebraic subset $\Sigma\subset\mathbf{R}^m$ and integers μ and ν such that for every $w\in\mathbf{R}^m-\Sigma$ there is $\epsilon>0$ with $\{w\}\times(-\epsilon,\epsilon)\cap X\subset\{(w,0)\}$ and

$$\deg_0 F_{w,-t}+\deg_0 F_{w,t}\equiv\mu \pmod 4$$

and

$$\deg_0 F_{w,-t}-\deg_0 F_{w,t}\equiv\nu \pmod 4$$

provided $0<t<\epsilon$.

In fact the factor $\mathbf{R}^m$ may be replaced by any irreducible real variety.

The theorem gives a new and easy proof of the result by Coste and Kurdyka [**5**] that the Euler characteristic of the link of an irreducible algebraic subset of a real algebraic set is generally constant modulo 4. Recently McCrory and Parusiński [**13**] gave another proof of this fact. Their proof is based on investigation of a relation between complex monodromy and complex conjugation on Milnor fibres. Other theorems concerning the Euler characteristic of the link modulo 4 are presented in [**6**], [**7**], [**17**], [**18**], [**19**], [**21**].

In Section 5 we present another application of the Main Theorem to the bifurcation theory.

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2. Preliminaries. We shall need two theorems concerning topological equisingularity. Both have been proved by Varchenko [**20**], similar results have been presented by Wallace [**22**], Hardt [**9**], Coste [**3**] and Bochnak *et al.* [**2**].

THEOREM 2.1. *Suppose that W is a real algebraic set and $A_1, \ldots, A_s \subset W \times \mathbf{R}^n$ are semialgebraic. Then there is a proper algebraic subset $\Sigma \subset W$ such that, for each connected component $U \subset W - \Sigma$ and a point $w_0 \in U$, there exists a homeomorphism $H : U \times \mathbf{R}^n \to U \times \mathbf{R}^n$ such that $H(w, x) = (w, h(w, x))$ and*

$$H(U \times \mathbf{R}^n \cap A_i) = U \times \{x \in \mathbf{R}^n \mid (w_0, x) \in A_i\} \quad \textit{for} \quad i = 1, \ldots, s.$$

If that is the case we shall say that the projection $\pi : W \times \mathbf{R}^n \to W$ is *topologically equisingular over $W - \Sigma$ with respect to $A_1, \ldots, A_s$.*

We shall say that a complex algebraic set $X \subset \mathbf{C}^n$ is *defined by real polynomials* if there are polynomials $f_1, \ldots, f_p$ having real coefficients such that

$$X = \{z \in \mathbf{C}^n \mid f_1(z) = \ldots = f_p(z) = 0\}.$$

THEOREM 2.2. *Suppose $A_1, \ldots, A_s \subset \mathbf{C}^m \times \mathbf{C}^n$ are complex algebraic sets defined by real polynomials. Then there is a proper complex algebraic set $\Sigma_C \subset \mathbf{C}^m$ defined by real polynomials such that, for every $w_0 \in \mathbf{C}^m - \Sigma_C$, there is an open neighbourhood $U \ni w_0$ and a homeomorphism $H : U \times \mathbf{C}^n \to U \times \mathbf{C}^n$ such that $H(w, z) = (w, h(w, z))$ and*

$$H(U \times \mathbf{C}^n \cap A_i) = U \times \{z \in \mathbf{C}^n \mid (w_0, z) \in A_i\} \quad \textit{for} \quad i = 1, \ldots, s.$$

Clearly $\Sigma = \Sigma_C \cap \mathbf{R}^m$ is a proper real algebraic subset of $\mathbf{R}^m$. Take $w_1, w_2 \in \mathbf{R}^m - \Sigma$. Denote

$$A_i^j = \{z \in \mathbf{C}^n \mid (w_j, z) \in A_i\} \quad \textit{for} \quad i = 1, \ldots, s \quad \textit{and} \quad j = 1, 2.$$

Since $\mathbf{C}^m - \Sigma_C$ is connected then there is a homeomorphism $g : \mathbf{C}^n \to \mathbf{C}^n$ such that $g(A_i^1) = A_i^2$ for $i = 1, \ldots, s$.

If that is the case we shall say that the projection $\mathbf{C}^m \times \mathbf{C}^n \to \mathbf{C}^m$ is *topologically equisingular over $\mathbf{C}^m - \Sigma_C$ with respect to $A_1, \ldots, A_s$.*

THEOREM 2.3. *Let $f_1, \ldots, f_s$ be real polynomials, let $V_R = \{x \in \mathbf{R}^n \mid f_1(x) = \ldots = f_s(x) = 0\}$ and let $V_C = \{z \in \mathbf{C}^n \mid f_1(z) = \ldots = f_s(z) = 0\}$. Suppose that the complex germ $(V_C, \mathbf{0})$ is 1-dimensional. Let S_r^{2n-1} denote a sphere in $\mathbf{C}^n$ with radius r centered at $\mathbf{0}$ and let $S_r = S_r^{2n-1} \cap \mathbf{R}^n$. If $r > 0$ is small enough then numbers*

$$\gamma = \# V_R \cap S_r^{2n-1} = \# V_R \cap S_r,$$

$$\Gamma, \textit{ the number of connected components of } V_C \cap S_r^{2n-1},$$

do not depend on r and

$$\gamma \equiv 2\Gamma \pmod 4.$$

Proof. If r is small enough then $V_C \cap S_r^{2n-1}$ is diffeomorphic to a finite disjoint union of circles and complex conjugation acts as involution on $V_C \cap S_r^{2n-1}$. The set of fixed points is $V_R \cap S_r$. If the involution interchanges two different components then they do not contain fixed points. Suppose that S is a component of $V_C \cap S_r^{2n-1}$ preserved by the involution. It is easy to see that the complex conjugation changes the orientation of S, so there are exactly two fixed points in S. Hence $\gamma \equiv 2\Gamma$ (mod 4).

Let $F: \mathbf{R} \times \mathbf{R}^n \to \mathbf{R}^n$ be a polynomial mapping and let $F_C: \mathbf{C} \times \mathbf{C}^n \to \mathbf{C}^n$ be its complexification. For any $t \in \mathbf{C}$, let $F_{C,t}: \mathbf{C}^n \to \mathbf{C}^n$ be a mapping given by $F_{C,t}(z) = F_C(t, z)$.

Suppose that $F_{C,t}(\mathbf{0}) = \mathbf{0}$ and that $F_{C,t}^{-1}(\mathbf{0})$ is finite for all $t \in \mathbf{R}$. From Theorem 2.2, $F_{C,t}^{-1}(\mathbf{0}) \supset \{\mathbf{0}\}$ is finite for all $t \in \mathbf{C}$ with $|t|$ small enough. There is $\epsilon > 0$ such that $F_{C,0}^{-1}(\mathbf{0}) \cap D_\epsilon = \{\mathbf{0}\}$, where $D_\epsilon = \{z \in \mathbf{C}^n \mid \|z\| \le \epsilon\}$. So there is $\delta > 0$ such that $\partial D_\epsilon \cap F_{C,t}^{-1}(\mathbf{0}) = \varnothing$ for all $t \in \mathbf{C}$ with $|t| < \delta$.

Denote $B_\epsilon = D_\epsilon \cap \mathbf{R}^n$ and $S_r^{2n+1} = \{(t, z) \in \mathbf{C} \times \mathbf{C}^n \mid |t|^2 + \|z\|^2 = r^2\}$. Clearly the complex germ $F_C^{-1}(\mathbf{0})$ at $\mathbf{0}$ is 1-dimensional and $\mathbf{C} \times \{\mathbf{0}\} \subset F_C^{-1}(\mathbf{0})$. From Theorem 2.3 we get the following corollary.

COROLLARY 2.4. *If $r > 0$ is small enough then numbers*

$$\gamma' = \#\{-r, r\} \times (B_\epsilon - \{\mathbf{0}\}) \cap F^{-1}(\mathbf{0}),$$

$$\Gamma, \textit{ the number of connected components in } F_C^{-1}(\mathbf{0}) \cap S_r^{2n+1},$$

do not depend on r and

$$\gamma' \equiv 2(\Gamma - 1) \qquad (\text{mod } 4).$$

Let $\mathbf{K}$ denote either $\mathbf{R}$ or $\mathbf{C}$ and let Λ_K denote the space of all n-tuples $(q_1, \ldots, q_n)$, where every $q_i: \mathbf{K}^n \to \mathbf{K}$ is a homogeneous polynomial of degree s. Then Λ_C is the complexification of Λ_R. For $q = (q_1, \ldots, q_n) \in \Lambda_K$ and $f_1, \ldots, f_n \in \mathbf{K}[x_1, \ldots, x_n]$ write

$$G_C^q = (f_1 + q_1, \ldots, f_n + q_n): \mathbf{C}^n \to \mathbf{C}^n$$

and denote $q_i = \sum q_\alpha^i x^\alpha$, where $q_\alpha^i \in \mathbf{K}$ and $x^\alpha = x_1^{\alpha_1} \ldots x_n^{\alpha_n}$. From now on "Jac" will denote the determinant of the Jacobian matrix for mappings $\mathbf{R}^n \to \mathbf{R}^n$ or $\mathbf{C}^n \to \mathbf{C}^n$.

The next theorem has a technical character, in cases $s = 0, 1$ it can be derived from the Sard theorem. The proof has been given in [**1**].

THEOREM 2.5. *Assume that $f_1, \ldots, f_n \in \mathbf{R}[x_1, \ldots, x_n]$. Then for each positive integer s there is a dense semialgebraic set $U \subset \Lambda_R$ such that*

$$\operatorname{Jac} G_C^q(z) \ne 0 \quad \textit{at each} \quad z \in (G_C^q)^{-1}(\mathbf{0}) - \{\mathbf{0}\}$$

for every $q \in U$. Then the semialgebraic set

$$\{q \in \Lambda_R \mid \textit{there is } z \in (G_C^q)^{-1}(\mathbf{0}) - \{\mathbf{0}\} \textit{ with } \operatorname{Jac} G_C^q(z) = 0\}$$

has codimension at least 1 *in Λ_R.*

The next theorem is a parametrized version of the Łojasiewicz inequality. It has been proved by Fekak [**8**].

THEOREM 2.6. *Let A and B be semialgebraic closed subsets in $\mathbf{R}^m \times \mathbf{R}^s$. For $w \in \mathbf{R}^m$ denote $A_w = \{x \in \mathbf{R}^s \mid (w, x) \in A\}$ and $B_w = \{x \in \mathbf{R}^s \mid (w, x) \in B\}$. Then there is a finite family S_i of disjoint semialgebraic sets covering $\mathbf{R}^m$, a family h_i of semialgebraic continuous functions defined on $S_i \times \mathbf{R}^s \cap A$ and positive rational numbers p_i such that: if $w \in S_i$ then $d(x, A_w \cap B_w)^{p_i} \le h_i(w, x) d(x, B_w)$ for every $x \in A_w$, d denoting the Euclidean distance in $\mathbf{R}^s$.*

If $B = \mathbf{R}^m \times \mathbf{R} \times \{0\}$ then applying Theorems 2.1 and 2.6 one may prove the following result.

PROPOSITION 2.7. *Suppose $A \subset \mathbf{R}^m \times \mathbf{R} \times \mathbf{R}$ is a closed semialgebraic set. For each $w \in \mathbf{R}^m$, denote*

$$A'_w = \text{Closure}(A_w - \{w\} \times \mathbf{R} \times \{0\}).$$

Suppose that there is a proper algebraic subset $\Sigma \subset \mathbf{R}^m$ such that for every $w \in \mathbf{R}^m - \Sigma$ there is $\epsilon > 0$ such that

$$\{w\} \times (-\epsilon, \epsilon) \times \{0\} \cap A'_w \subset \{(w, 0, 0)\}.$$

Then there is a positive constant σ such that, for every $w \in \mathbf{R}^m - \Sigma$, there are $C > 0$ and $\delta > 0$ such that

$$|y| \geq C\,|r|^\sigma$$

for every $(w, r, y) \in A$ with $|r| < \delta$ and $y \neq 0$.

3. Proof of the Main Theorem.

LEMMA 3.1. *Set $s = m + 1$. Suppose that $F : \mathbf{R}^s \times \mathbf{R}^n \to \mathbf{R}^n$ is a polynomial mapping. For any $w \in \mathbf{R}^s$, let $F_w : \mathbf{R}^n \to \mathbf{R}^n$ be a mapping given by $F_w(x) = F(w, x)$. Suppose that*

(a) *$F_w(\mathbf{0}) = \mathbf{0}$ for every $w \in \mathbf{R}^s$,*

(b) *there is a proper algebraic subset $\Sigma \subset \mathbf{R}^s$ such that $\mathbf{0} \in \mathbf{R}^n$ is isolated in $F_w^{-1}(\mathbf{0})$ for every $w \in \mathbf{R}^s - \Sigma$ and so, for all $w \in \mathbf{R}^s - \Sigma$, the local topological degree* $\deg_0 F_w$ *is defined.*

Then there is a polynomial mapping $G : \mathbf{R}^s \times \mathbf{R}^n \to \mathbf{R}^n$, its complexification $G_C : \mathbf{C}^s \times \mathbf{C}^n \to \mathbf{C}^n$ and a proper algebraic $\Sigma' \subset \mathbf{R}^s$ with $\Sigma \subset \Sigma'$ such that

(i) *$G_C(w, \mathbf{0}) = \mathbf{0}$ for all $w \in \mathbf{C}^m$,*

(ii) *for the mapping $G_{C,w} : \mathbf{C}^n \to \mathbf{C}^n$ given by $G_{C,w}(z) = G_C(w, z)$, $G_{C,w}^{-1}(\mathbf{0})$ is finite for all $w \in \mathbf{C}^m$ (in particular $\mathbf{0}$ is isolated in $G_{C,w}^{-1}(\mathbf{0})$) and, moreover, if $F_w \equiv \mathbf{0}$ then $G_{C,w}^{-1}(\mathbf{0}) = \{\mathbf{0}\}$,*

(iii) *for every $w \in \mathbf{R}^s - \Sigma'$, if $z \in G_{C,w}^{-1}(\mathbf{0}) - \{\mathbf{0}\}$ then* $\operatorname{Jac} G_{C,w}(z) \neq 0$,

(iv) *for the mapping $G_w : \mathbf{R}^n \to \mathbf{R}^n$ given by $G_w(x) = G(w, x)$,* $\deg_0 F_w = \deg_0 G_w$ *for every $w \in \mathbf{R}^s - \Sigma'$.*

Proof. Denote

$$X = \{(w, x, r, y)\} \in \mathbf{R}^s \times \mathbf{R}^n \times \mathbf{R} \times \mathbf{R} \mid \|x\| = r, y = \|F(w, x)\|\}.$$

Let A denote the projection of X on $\mathbf{R}^s \times \mathbf{R} \times \mathbf{R}$. Clearly X is closed, semialgebraic and the projection of X to $\mathbf{R}^s \times \mathbf{R} \times \mathbf{R}$ is proper. Then A is closed and semialgebraic.

For every $w \in \mathbf{R}^s - \Sigma$, $\mathbf{0} \in \mathbf{R}^n$ is isolated in $F_w^{-1}(\mathbf{0})$, so there is $\epsilon > 0$ such that $\{\mathbf{0}\} = \{x \in \mathbf{R}^n \mid \|x\| < \varepsilon, F(w, x) = \mathbf{0}\}$ and then

$$\{w\} \times (-\epsilon, \epsilon) \times \{0\} \cap A = \{(w, 0, 0)\}.$$

From Proposition 2.7, there is a positive constant σ such that if $w \in \mathbf{R}^s - \Sigma$ then

$$|y| \geq C\,|r|^\sigma$$

for some $C > 0$ and every $(w, r, y) \in A$ with $|r|$ small enough. That means that

$$\|F(w, x)\| \geq C\,\|x\|^\sigma$$

for every $w \in \mathbf{R}^s - \Sigma$ and $\|x\| < \delta$, where $C = C(w) > 0$ and $\delta = \delta(w) > 0$.

Suppose $d > \sigma$ is an integer. So if $q_1(x), \ldots, q_n(x)$ are homogeneous polynomials of degree d and

$$G(w, x) = F(w, x) + (q_1(x), \ldots, q_n(x))$$

then, for every $w \in \mathbf{R}^s - \Sigma$, $\mathbf{0}$ is isolated in $G_w^{-1}(\mathbf{0})$ and $\deg_0 F_w = \deg_0 G_w$.

Let Λ denote the space of all n-tuples $(q_1, \ldots, q_n)$, where each q_i is a real homogeneous polynomial of degree d. For $q = (q_1, \ldots, q_n) \in \Lambda$ and $w \in \mathbf{R}^s$, let $G^q_{C,w} : \mathbf{C}^n \to \mathbf{C}^n$ denote a map given by

$$G^q_{C,w}(z) = F_C(w, z) + (q_1(z), \ldots, q_n(z)).$$

Let

$$P = \{(q, w) \in \Lambda \times \mathbf{R}^s \mid \text{there is } z \in (G^q_{C,w})^{-1}(\mathbf{0}) - \{\mathbf{0}\} \text{ with } \operatorname{Jac} G^q_{C,w}(z) = 0\}.$$

Clearly P is semialgebraic.

From Theorem 2.5, the codimension of $P \cap \Lambda \times \{w\}$ in $\Lambda \times \{w\}$ is at least 1 for every $w \in \mathbf{R}^s$. Then the codimension of P in $\Lambda \times \mathbf{R}^s$ is at least 1 too. This implies that there is an open dense semialgebraic $\Lambda_0 \subset \Lambda$ such that, for any $q \in \Lambda_0$, the codimension of $P \cap \{q\} \times \mathbf{R}^s$ in $\{q\} \times \mathbf{R}^s$ is at least 1. Choose $\bar{q} \in \Lambda_0$. Let P' denote the projection of $P \cap \{\bar{q}\} \times \mathbf{R}^s$ on $\mathbf{R}^s$, let Σ' denote the smallest algebraic set which contains P' and Σ. Define

$$G(w, x) = F(w, x) + (\bar{q}_1(x), \ldots, \bar{q}_n(x)),$$

where $\bar{q} = (\bar{q}_1, \ldots, \bar{q}_n)$.

Since the codimension of P' in $\mathbf{R}^s$ is at least 1 then Σ' is a proper algebraic subset of $\mathbf{R}^s$. Clearly, if $w \in \mathbf{R}^s - \Sigma'$ then $(\bar{q}, w) \notin P$. So, if $z \in G_{C,w}^{-1}(\mathbf{0}) - \{\mathbf{0}\}$ then $\operatorname{Jac} G_{C,w}(z) \neq 0$. As we have shown before, $\deg_0 F_w = \deg_0 G_w$. Hence conditions (iii) and (iv) hold.

We may assume that d is so large that for every $w \in \mathbf{R}^s$ we have

$$\lim \|F_C(w, z)\| / \|z\|^d = 0 \quad \text{as} \quad \|z\| \to \infty, \quad \text{where} \quad z \in \mathbf{C}^m.$$

Since Λ_0 is open and dense in Λ we may assume that

$$\{z \in \mathbf{C}^n \mid \bar{q}_1(z) = \ldots = \bar{q}_n(z) = 0\} = \{\mathbf{0}\}.$$

Then there is $C_1 > 0$ such that $\|(\bar{q}_1(z), \ldots, \bar{q}_n(z))\| \geq C_1 \|z\|^d$ for every $z \in \mathbf{C}^n$ with $\|z\|$ large enough. Thus, for every $w \in \mathbf{R}^s$, a complex algebraic set

$$G_{C,w}^{-1}(\mathbf{0}) = \{z \in \mathbf{C}^n \mid F_C(w, z) = -(\bar{q}_1(z), \ldots, \bar{q}_n(z))\}$$

is bounded, and so finite. Moreover, if $F_w \equiv \mathbf{0}$ then $G_{C,w}(z) = (\bar{q}_1(z), \ldots, \bar{q}_n(z))$ and then $G_{C,w}^{-1}(\mathbf{0}) = \{\mathbf{0}\}$. Hence condition (ii) holds.

Proof of Theorem 1.1. Of course we may change F and enlarge X as long as we do not change local topological degrees at $\mathbf{0}$ over an open dense semialgebraic subset in $\mathbf{R}^m \times \mathbf{R}$. So, according to Lemma 3.1, we may assume that

(a) $\mathbf{R}^m \times \{0\} \subset X$,

(b) $\{z \in \mathbf{C}^n \mid F_C(w, t, z) = \mathbf{0}\}$ is finite for all $(w, t) \in \mathbf{C}^m \times \mathbf{C}$, in particular, $\mathbf{0}$ is isolated in $F_{w,t}^{-1}(\mathbf{0})$ and $\dim_C F_C^{-1}(\mathbf{0}) = m + 1$,

(c) if $(w, 0) \in \mathbf{R}^m \times \{0\}$ then $F_{w,0}^{-1}(\mathbf{0}) = \{\mathbf{0}\}$,

(d) for every $(w, t) \in \mathbf{R}^m \times \mathbf{R} - X$, if $z \in F_{C,w,t}^{-1}(\mathbf{0}) - \{\mathbf{0}\}$ then $\operatorname{Jac} F_{C,w,t}(z) \neq 0$.

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From (b), each irreducible component of $F_C^{-1}(\mathbf{0})$ has dimension at most $m+1$. So we may also suppose that $X \times \{\mathbf{0}\} \subset \mathbf{R}^m \times \mathbf{R} \times \mathbf{R}^n$ contains

$$\text{Zariski closure } (F_C^{-1}(\mathbf{0}) - \mathbf{C}^m \times \mathbf{C} \times \{\mathbf{0}\}) \cap \mathbf{R}^m \times \mathbf{R} \times \{\mathbf{0}\}.$$

Thus

(e) if (w_1, t_1) and (w_2, t_2) belong to the same connected component of $\mathbf{R}^m \times \mathbf{R} - X$ then

$$\deg_0 F_{w_1,t_1} = \deg_0 F_{w_2,t_2}.$$

Let $X_C \subset \mathbf{C}^m \times \mathbf{C}$ denote the complexification of X. From Theorem 2.2 there exists a proper algebraic subset $\Sigma_C \subset \mathbf{C}^m$, defined by real polynomials, such that the projection $\mathbf{C}^m \times \mathbf{C} \times \mathbf{C}^n \to \mathbf{C}^m$ is topologically equisingular over $\mathbf{C}^m - \Sigma_C$ with respect to $X_C \times \{\mathbf{0}\}$, $F_C^{-1}(\mathbf{0})$, $\mathbf{C}^m \times \mathbf{C} \times \{\mathbf{0}\}$ and $\mathbf{C}^m \times \{0\} \times \{\mathbf{0}\}$.

Denote $\Sigma = \Sigma_C \cap \mathbf{R}^m$. Thus Σ is a proper algebraic subset of $\mathbf{R}^m$, and then $\mathbf{R}^m - \Sigma$ is open and dense in $\mathbf{R}^m$.

From (c) we conclude that there is an integer μ_1 such that

$$\deg_0 F_{w,0} = \mu_1 \quad \text{for all} \quad w \in \mathbf{R}^m.$$

Choose $w_0 \in \mathbf{R}^m - \Sigma$. Let $\{w_0\} \times S_r^{2n+1} = \{w_0\} \times \{(t, z) \in \mathbf{C} \times \mathbf{C}^n \mid |t|^2 + \|z\|^2 = r^2\}$, let $\Gamma(w_0)$ denote the number of connected components in $F_C^{-1}(\mathbf{0}) \cap \{w_0\} \times S_r^{2n+1}$, where $r > 0$ is small, and let $\mu_2 = 2(\Gamma(w_0) - 1)$. Because of topological equisingularity with respect to $F_C^{-1}(\mathbf{0})$ and $\mathbf{C}^m \times \{0\} \times \{\mathbf{0}\}$, for every $w \in \mathbf{R}^m - \Sigma$, we have

$$2(\Gamma(w) - 1) = 2(\Gamma(w_0) - 1) = \mu_2. \tag{1}$$

We have $\dim_C X_C \leq m$. Because of topological equisingularity with respect to $X_C \times \{\mathbf{0}\}$ and $\mathbf{C}^m \times \mathbf{C} \times \{\mathbf{0}\}$, if $w \in \mathbf{R}^m - \Sigma$ then $\{w\} \times \mathbf{C} \cap X_C$ is finite. In particular there is $\epsilon > 0$ with $\{w\} \times (-\epsilon, \epsilon) \cap X = \{(w, 0)\}$. Since $\mathbf{0}$ is isolated in $F_{w,0}^{-1}(\mathbf{0})$ then there is $\delta > 0$ such that $F_{w,0}^{-1}(\mathbf{0}) \cap B_\delta = \{\mathbf{0}\}$, where $B_\delta = \{x \in \mathbf{R}^n \mid \|x\| \leq \delta\}$. We may suppose that

$$\partial B_\delta \cap F_{w,t}^{-1}(\mathbf{0}) = \varnothing \quad \text{for all} \quad -\epsilon < t < \epsilon.$$

When $0 < t < \epsilon$, denote

$$p(w) = \#\{(w, \pm t, x) \mid x \in F_{w,\pm t}^{-1}(\mathbf{0}) - \{\mathbf{0}\} \text{ and } \operatorname{Jac} F_{w,\pm t}(x) > 0\},$$
$$n(w) = \#\{(w, \pm t, x) \mid x \in F_{w,\pm t}^{-1}(\mathbf{0}) - \{\mathbf{0}\} \text{ and } \operatorname{Jac} F_{w,\pm t}(x) < 0\}.$$

If t is small enough then $p(w)$ and $n(w)$ do not depend on t. Because of (d), it is easy to see that

$$p(w) - n(w) + \deg_0 F_{w,-t} + \deg_0 F_{w,t} = 2 \deg_0 F_{w,0} = 2\mu_1. \tag{2}$$

From Corollary 2.4 and (1) we get

$$p(w) + n(w) \equiv \mu_2 \qquad (\text{mod } 4).$$

From now on we assume that $\operatorname{Jac} F_{w,0} = 0$ for all $w \in \mathbf{R}^m$. In the other case the proof is very easy. Then a polynomial mapping $H : \mathbf{R}^m \times \mathbf{R} \times (\mathbf{R}^n \times \mathbf{R}) \to \mathbf{R}^n \times \mathbf{R}$ given by

$$H(w, t, (x, y)) = (F(w, t, x), y^2 + \operatorname{Jac} F_{w,t}(x))$$

also satisfies conditions (b), (c) and (d). Now we may apply again all arguments presented above so as to prove, after eventually enlarging Σ, that there is an integer μ_3 such that

$$2n(w) \equiv \mu_3 \quad (\mathrm{mod}\, 4) \quad \text{for all} \quad w \in \mathbf{R}^m - \Sigma.$$

Hence

$$\deg_0 F_{w,-t} + \deg_0 F_{w,t} \equiv 2\mu_1 - \mu_2 + \mu_3 \quad (\mathrm{mod}\, 4)$$

for all $w \in \mathbf{R}^m - \Sigma$. The proof of the second congruence is similar.

4. The Euler characteristic of a link. The next theorem has been proved in [**16**].

THEOREM 4.1. *Let $f : \mathbf{R}^n \to \mathbf{R}$ be a polynomial with $f(\mathbf{0}) = 0$, having a critical point at $\mathbf{0}$. Let $A_\pm = \{x \in S_r \mid \pm f(x) \geq 0\}$, where $r > 0$ is small. There are positive constants C, σ and r_0 such that if $x \in S_r$, with $0 < r < r_0$, is a critical point of a restricted function $f \mid S_r$ and $f(x) \neq 0$ then $|f(x)| \geq Cr^\sigma$. Let k be a positive integer such that $2k > \sigma$, let $f_\pm(x) = \pm f(x) - \|x\|^{2k}$ and let $F_\pm = \mathrm{grad}\, f_\pm : (\mathbf{R}^n, \mathbf{0}) \to (\mathbf{R}^n, \mathbf{0})$. Then $\mathbf{0}$ is isolated in $F_\pm^{-1}(\mathbf{0})$ and*

$$\chi(A_-) = 1 - \deg_0 F_+ \quad \textit{and} \quad \chi(A_+) = 1 - \deg_0 F_-.$$

COROLLARY 4.2. *For $t \in \mathbf{R}$, denote $f_t(x) = tf(x) - \|x\|^{2k}$ and $F_t = \mathrm{grad}\, f_t : (\mathbf{R}^n, \mathbf{0}) \to (\mathbf{R}^n, \mathbf{0})$. Then $\mathbf{0}$ is isolated in $F_t^{-1}(\mathbf{0})$ for any $t \in \mathbf{R}^n$ and, for every $t > 0$,*

$$\chi(A_-) = 1 - \deg_0 F_t, \qquad \chi(A_+) = 1 - \deg_0 F_{-t}.$$

PROPOSITION 4.3. *Let $f : \mathbf{R}^m \times \mathbf{R}^n \to \mathbf{R}$ be a polynomial. For any $w \in \mathbf{R}^m$, let $f_w : \mathbf{R}^n \to \mathbf{R}$ be given by $f_w(x) = f(w, x)$ and suppose that $f_w(\mathbf{0}) = 0$. Then there is a positive constant σ such that, for every $w \in \mathbf{R}^m$, there are $C = C(w) > 0$ and $r(w) > 0$ such that if $x \in S_r$, with $0 < r < r(w)$, is a critical point of a restricted function $f_w \mid S_r$ and $f_w(x) \neq 0$ then*

$$|f_w(x)| \geq Cr^\sigma.$$

Proof. Set $\omega_{ij} = x_i \partial f/\partial x_j - x_j \partial f/\partial x_i$ when $1 \leq i, j \leq n$. Then $x \in S_r$ is a critical point of the restricted function $f_w \mid S_r$ if and only if $\omega_{ij}(w, x) = 0$ for all i, j.

Let

$$X = \{(w, x, r, y) \in \mathbf{R}^m \times \mathbf{R}^n \times \mathbf{R} \times \mathbf{R} \mid \|x\| = r, \text{ all } \omega_{ij}(w, x) = 0 \text{ and } y = f_w(x)\}.$$

Then X is closed, semialgebraic. Let A denote the projection of X on $\mathbf{R}^m \times \mathbf{R} \times \mathbf{R}$. The projection restricted to X is proper, so A is closed, semialgebraic.

If $w \in \mathbf{R}^m$ then $A \cap \{w\} \times \{0\} \times \mathbf{R} = \{(w, 0, 0)\}$, if $r > 0$ then

$$A_{w,r} = A \cap \{w\} \times \{r\} \times \mathbf{R} = \{w\} \times \{r\} \times \{\text{critical values of } f_w \mid S_r\}.$$

According to Milnor [**14**], the set of critical values of any polynomial restricted to an algebraic manifold is finite. Hence $A_{w,r}$ is finite for all $(w, r) \in \mathbf{R}^m \times \mathbf{R}$. Then $A_w = A \cap \{w\} \times \mathbf{R} \times \mathbf{R}$ is a 1-dimensional, closed and semialgebraic set for all $w \in \mathbf{R}^m$.

Denote $A'_w = \text{Closure}(A_w - \{w\} \times \mathbf{R} \times \{0\})$. Clearly, for every $w \in \mathbf{R}^m$ there is $\epsilon > 0$ such that

$$\{w\} \times (-\epsilon, \epsilon) \times \{0\} \cap A'_w \subset \{(w, 0, 0)\}.$$

From Proposition 2.7, there is a positive constant σ such that for every $w \in \mathbf{R}^m$ there are $C = C(w) > 0$ and $r(w) > 0$ such that

$$|y| \geq Cr^\sigma$$

for every $(w, r, y) \in A$ with $|r| < r(w)$ and $y \neq 0$.

Suppose that $w \in \mathbf{R}^m$, $0 < r < r(w)$, $x \in S_r$ is a critical point of $f_w \mid S_r$ and $f_w(x) \neq 0$. Then $(w, x, r, f_w(x)) \in X$, so $(w, r, f_w(x)) \in A$, and then

$$|f_w(x)| \geq Cr^\sigma.$$

Let f and f_w be as above and let $A_\pm(w) = \{x \in S_r \mid \pm f_w(x) \geq 0\}$ for $r = r(w) > 0$ small enough.

The next theorem has been proved, in a more general version, by Coste and Kurdyka [**5**].

Theorem 4.4. *There is a proper algebraic subset $\Sigma \subset \mathbf{R}^m$ and an integer μ such that*

$$\chi(A_+(w)) + \chi(A_-(w)) \equiv \mu \pmod 4$$

for all $w \in \mathbf{R}^m - \Sigma$.

Proof. From Proposition 4.3 there is a positive constant σ such that for every $w \in \mathbf{R}^m$ there are $C = C(w) > 0$ and $r(w) > 0$ such that if $x \in S_r$, with $0 < r < r(w)$, is a critical point of the restricted function $f_w \mid S_r$ and $f_w(x) \neq 0$ then $|f_w(x)| \geq Cr^\sigma$.

Suppose that k is an integer such that $2k > \sigma$. For $(w, t) \in \mathbf{R}^m \times \mathbf{R}$, let $f_{w,t} : \mathbf{R}^n \to \mathbf{R}$ be given by

$$f_{w,t}(x) = tf_w(x) - \|x\|^{2k}.$$

Let $F : \mathbf{R}^m \times \mathbf{R} \times \mathbf{R}^n \to \mathbf{R}^n$ be a polynomial mapping given by $F(w, t, x) = \operatorname{grad} f_{w,t}(x)$. According to Corollary 4.2, $\mathbf{0} \in \mathbf{R}^n$ is isolated in $F^{-1}_{w,t}(\mathbf{0})$ for all $(w, t) \in \mathbf{R}^m \times \mathbf{R}$. Moreover, if $t > 0$ then

$$\chi(A_+(w)) + \chi(A_-(w)) = 2 - \deg_0 F_{w,t} - \deg_0 F_{w,-t}.$$

To finish the proof it is enough to apply Theorem 1.1.

The next theorem has been proved by Coste [**4**] in the case when X has codimension 2 and by Coste and Kurdyka [**5**] for arbitrary X. In fact they have proved a more general version where the factor $\mathbf{R}^m$ may be replaced by any irreducible algebraic set. However the proof given here provides the essential part of their proof. After changing necessary technical details the reader may follow the argument presented in this paper so as to prove the general version of the Coste-Kurdyka theorem.

Theorem 4.5. *Suppose that $X \subset \mathbf{R}^m \times \mathbf{R}^n$ is a proper algebraic set with $\mathbf{R}^m \times \{0\} \subset X$. For $w \in \mathbf{R}^m$, let $L(w) = X \cap \{w\} \times S_r$, for $r > 0$ small enough. Then there is a proper* *algebraic subset $\Sigma \subset \mathbf{R}^m$ and an integer μ such that*

$$\chi(L(w)) \equiv \mu \pmod 4$$

for all $w \in \mathbf{R}^m - \Sigma$.

Proof. There are polynomials $f_1(w,x),\dots,f_s(w,x)$ such that

$$\begin{aligned} X &= \{(w,x) \mid f_1(w,x) = \dots = f_s(w,x) = 0\} \\ &= \{(w,x) \mid f_1^2(w,x) + \dots + f_s^2(w,x) = 0\}. \end{aligned}$$

Denote $f = f_1^2 + \dots + f_s^2$. Then $A_+(w) = S_r$ and $A_-(w) = L(w)$. Thus the theorem is a consequence of Theorem 4.4.

5. Applications to the bifurcation theory. In this section we show how to apply the Krasnosielski theorem and the Main Theorem in order to prove existence of bifurcation points for polynomial families of vector fields.

There are other theorems which provide information on flows in terms of topological degrees of vector fields, e.g. the Poincaré-Bendixson theorem (see [**10**]). One may apply the Main Theorem in these cases too and formulate theorems similar to that one presented in this section.

Let $F : \mathbf{R} \times \mathbf{R}^n \to \mathbf{R}^n$ be a continuous family of vector fields with $F(t, \mathbf{0}) = \mathbf{0}$ for all $t \in \mathbf{R}$. Denote $F_t = F(t, \cdot)$. We shall say that *$t = 0$ is a bifurcation point for F_t* if $(0, \mathbf{0}) \in \text{Closure}(F^{-1}(\mathbf{0}) - \mathbf{R} \times \{\mathbf{0}\})$.

The next theorem is a version of the well-known Krasnosielksi theorem (see [**11**], [**15**]).

THEOREM 5.1. *Suppose that there is $\epsilon > 0$ such that $\mathbf{0}$ is isolated in $F_t^{-1}(\mathbf{0})$ and* $\deg_0 F_{-t} \neq \deg_0 F_t$ *for all t with $0 < |t| < \epsilon$. Then $t = 0$ is a bifurcation point for F_t.*

We also have the following theorem.

THEOREM 5.2. *Let $F : \mathbf{R}^m \times \mathbf{R} \times \mathbf{R}^n \to \mathbf{R}^n$ be a polynomial mapping. Suppose that*

(a) *$F_{w,t}(\mathbf{0}) = \mathbf{0}$ for all $(w,t) \in \mathbf{R}^m \times \mathbf{R}$ and there is an algebraic subset $X \subset \mathbf{R}^m \times \mathbf{R}$ such that $\mathbf{0} \in \mathbf{R}^n$ is isolated in $F_{w,t}^{-1}(\mathbf{0})$ for every $(w,t) \in \mathbf{R}^m \times \mathbf{R} - X$,*

(b) *there is an open non-empty set $U \subset \mathbf{R}^m$, and $\epsilon > 0$ such that* $\deg_0 F_{w,t} \not\equiv \deg_0 F_{w,-t} \pmod 4$ *for all $w \in U$ with $(w,t) \notin X$ and $0 < |t| < \epsilon$.*
Then there is a proper algebraic subset $\Sigma \subset \mathbf{R}^m$ such that $t = 0$ is a bifurcation point for $F_{w,t}$ for every $w \in \mathbf{R}^m - \Sigma$.

Proof. According to the Main Theorem there is an integer ν and a proper algebraic $\Sigma \subset \mathbf{R}^m$ such that $\deg_0 F_{w,-t} - \deg_0 F_{w,t} \equiv \nu \pmod 4$ for every $w \in \mathbf{R}^m - \Sigma$. Since $\mathbf{R}^m - \Sigma$ is open and dense then $U - \Sigma$ is non-empty and then $\nu \not\equiv 0 \pmod 4$. Thus $\deg_0 F_{w,t} \not\equiv \deg_0 F_{w,-t} \pmod 4$ for every $w \in \mathbf{R}^m - \Sigma$. From the Krasnosielski Theorem, $t = 0$ is a bifurcation point for $F_{w,t}$.

EXAMPLE. Let $F : \mathbf{R} \times \mathbf{R} \times \mathbf{R}^2 \to \mathbf{R}^2$ be given by

$$F(w,t,x_1,x_2) = (p_1(w)x_1^2 - p_2(w)tx_2^2 + w^3t^3x_1x_2 \quad + t^4x_2^3, q(w)x_1x_2 - wt^2x_2^2 - t^5x_1^4),$$

where $p_1(w)$ and $p_2(w)$ are polynomials of degree 4 such that $\lim_{w\to\infty} p_1(w)/w^4 = \lim_{w\to\infty} p_2(w)/w^4 = 1$ and $q(w)$ is a polynomial of degree at least 2 with $\lim_{w\to\infty} q(w) = +\infty$.

Let $X = \{(w,t) \in \mathbf{R}\times\mathbf{R} \mid t = 0\}$. If $t \neq 0$ then $F^{-1}_{C,w,t}(\mathbf{0})$ is a bounded complex algebraic set, and then it is finite. So $\mathbf{0}$ is isolated in $F^{-1}_{w,t}(\mathbf{0})$ for all $(w,t) \in \mathbf{R}\times\mathbf{R} - X$. The reader may easily check that if $w \gg 0$ and $0 < |t| \ll 1$ then $\deg_0 F_{w,t}$ equals $\deg_0 H_t$, where $H_t(x_1, x_2) = (x_1^2 - tx_2^2, x_1x_2)$.

We have $\deg_0 H_t = 0$ for $t < 0$ and $\deg_0 H_t = 2$ for $t > 0$. According to Theorem 5.2, there is a proper algebraic (and hence finite) $\Sigma \subset \mathbf{R}$ such that $t = 0$ is a bifurcation point for $F_{w,t}$ for every $w \in \mathbf{R} - \Sigma$.

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