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Observational interpretation of CASL specifications

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We explore the way in which the refinement of individual 'local' components of a specification relates to the development of a 'global' system from a specification of requirements. The observational interpretation of specifications and refinements adds expressive power and flexibility, but introduces some subtle problems. Our study of these issues is carried out in the context of CASL architectural specifications. We introduce a definition of observational equivalence for CASL models, leading to an observational semantics for architectural specifications for which we prove important properties. Overall, this fulfills the long-standing goal of complementing the standard semantics of CASL specifications with an observational view that supports observational refinement of specifications in combination with CASL-style architectural design.

1. Introduction

There has been a great deal of work in the algebraic specification tradition on formalising the rather intuitive and appealing idea of program development by stepwise refinement, including Ehrig *et al.* (1982), Ganzinger (1983), Schoett (1987) and Sannella and Tarlecki (1988b); for a survey, see Ehrig and Kreowski (1999). There are many issues that make this a difficult problem, and some of them are rather subtle, one example being the relationship between specification structure and program structure, and another being the trade-off between the expressive power of a specification formalism and the ease of reasoning about specifications. Significant complications result when 'observational' or 'behavioural' aspects of specifications are considered, whereby the definition of correctness takes into account only the results of those computations that can be directly observed. An overview covering most of our own contributions is Sannella and Tarlecki (1997) – for more recent work addressing the problem of how to prove correctness of refinement steps, see Bidoit and Hennicker (1998) and Bidoit and Hennicker (2006), for the design of a convenient formalism for writing specifications, see Bidoit *et al.* (2002a), Astesiano *et al.* (2002) and CoFI (2004), and for applications to data refinement in typed λ-calculus, see Honsell *et al.* (2000).

A new angle we explore here is the 'global' effect of refining individual 'local' components of a specification. This involves a well-known technique from algebraic specification, namely the use of pushouts of signatures and amalgamation of models to build large systems by composition of separate interrelated components. The situation becomes considerably more subtle when the observational interpretation of specifications and refinements is brought into the picture.

Part of the answer has already been provided, the main references being Schoett's thesis Schoett (1987) and Schoett (1990), and our own work on formal development in the Extended ML framework (Sannella and Tarlecki 1989); the general ideas go back at least to Hoare (1972). We take another look at these issues here, in the context of the Casl specification formalism (Astesiano *et al.* 2002; CoFI 2004) and, in particular, its *architectural specifications* (Bidoit *et al.* 2002a). Architectural specifications, for describing the modular structure of software systems, are probably the most novel feature of Casl. We view them here as a means of making complex refinement steps by defining a construction for building the overall system from implementations of individually specified units; these may include parametrised units that contribute to this construction.

This paper combines and expands on previous work that was reported on in Bidoit *et al.* (2002a; 2002b; 2004), Baumeister *et al.* (2004) and Schröder *et al.* (2005). It interweaves three strands. The first strand (Sections 2 and 5) recalls the basic semantic concepts of Casl and introduces observational equivalence for Casl models and the induced observational interpretation of Casl basic and structured specifications. In contrast to Bidoit *et al.* (2002b), true Casl models are considered rather than standard many-sorted total algebras.

A second strand (Sections 3 and 6) explores the use of local constructions in an arbitrary global context, and its interaction with an observational view of requirements specifications. In particular, the stability and observational correctness of constructions on Casl models are treated, and practical local criteria to establish both properties are formulated.

The final strand (Sections 4 and 7) provides a careful analysis of the semantics of Casl architectural specifications, taking account of the fact that amalgamability is not ensured for Casl models and linking with the other strands to provide such specifications with an observational semantics. Key invariant properties of the semantics are precisely formulated and proved.

Due to space considerations, we do not deal with full-blown Casl as defined in Mosses (2004), but the addition of unit definitions to the treatment in Bidoit *et al.* (2002b) together with a proper account of dependencies between units means that the extension to full Casl would be routine. The analysis of invariants linking the static semantics and model semantics of architectural specifications in Section 4 provides an essential insight into the semantics of full Casl that was implicit in Baumeister *et al.* (2004); this reiterates Schröder *et al.* (2005, Theorem 2) and provides a basis for an analogous treatment of the observational case in Section 7.

An orthogonal view of the structure of this paper is that Sections 2–4 present a standard treatment of Casl basic and structured specifications, local constructions and their use in a global context, and Casl architectural specifications; a comprehensive observational treatment is then given in Sections 5–7. An example in Section 8, based on one in Bidoit *et al.* (2004), provides a concrete illustration of some of the points that arise.

Overall, this fulfills the long-standing goal of complementing the standard semantics of CASL specifications (Baumeister *et al.* 2004) with an observational view that supports observational refinement of specifications in combination with CASL-style architectural design.

2. CASL **Institution and Specifications**

A basic assumption underpinning algebraic specification and derived approaches to software specification and development is that programs are modelled as algebras (of some kind) with their 'types' captured by algebraic signatures (again, adapted as appropriate). Then specifications include axioms describing the required properties. This leads to quite a flexible framework, which can be tuned as desired to cope with various programming features of interest by selecting the appropriate variation of algebra, signature and axiom. This flexibility has been formalised using the notion of an *institution* (Goguen and Burstall 1992) and related work on the theory of specifications and formal program development (Sannella and Tarlecki 1988a; Sannella and Tarlecki 1997; Bidoit and Hennicker 1993).

Recall that an institution defines a notion of signature together with, for any signature Σ, a set of Σ-sentences, a class of Σ-models equipped with homomorphisms, and a satisfaction relation between Σ-models and Σ-sentences. Moreover, signatures come equipped with signature morphisms, forming a category. Any signature morphism induces a translation of sentences and a translation of models (the latter going in the opposite direction to the morphism). All this can be expressed very concisely using the language of category theory: we require a category **Sig**, a functor $\mathbf{Sen} : \mathbf{Sig} \to \mathbf{Set}$, a (contravariant) functor $\mathbf{Mod} : \mathbf{Sig}^{op} \to \mathbf{Cat}$, and a family of binary relations $\langle \models_\Sigma \subseteq |\mathbf{Mod}(\Sigma)| \times \mathbf{Sen}(\Sigma)\rangle_{\Sigma\in|\mathbf{Sign}|}$. The only semantic requirement is that when we change signatures using a signature morphism, the induced translations of sentences and of models preserve the satisfaction relation.

By now it is standard to base work on specification languages and formal program development on the notion of an institution, so that a clear separation between logic-dependent details and general logic-independent aspects of the work can be achieved. We follow this below, recalling the logical system of CASL (Bidoit and Mosses 2004).

CASL is an algebraic specification language for describing CASL *models*: many-sorted algebras with subsorts, partial and total operations, and predicates. CASL models are classified by CASL *signatures*, which give *sort* names (with their subsorting relation), partial and total *operation* names, and *predicate* names, together with *profiles* of operations and predicates. In CASL models, subsorts and supersorts are linked by implicit *subsort embeddings* that are required to compose with each other and to be compatible with operations and predicates with the same names.

Recalling (and slightly simplifying) some technical detail from Baumeister *et al.* (2004), a CASL signature is a tuple $\Sigma = (S, TF, PF, P, \leqslant)$, where S is a set of sort names, $TF = \langle TF_{ws}\rangle_{ws\in S^+}$ and $PF = \langle PF_{ws}\rangle_{ws\in S^+}$ are, respectively, families of total and partial

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operation names indexed by their profiles (which consist of their arity $w \in S^*$ and result sort $s \in S$), $P = \langle P_w \rangle_{w \in S^*}$ is a family of predicate names, indexed by their arities, and $\leqslant$ is a subsorting preorder on S (a relation that is reflexive and transitive). For simplicity, we bluntly assume that no overloading is allowed, that is, that all the sets in TF, PF and P are mutually disjoint†. We write $f : s_1 \times \cdots \times s_n \to s$ when $s_1, \ldots, s_n, s \in S$ and $f \in TF_{s_1 \ldots s_n s}$; similar notation is used for partial operation names and predicate symbols. If $n = 0$, then f is a constant and we write $f : s$. For Casl signatures $\Sigma = (S, TF, PF, P, \leqslant)$ and $\Sigma' = (S', TF', PF', P', \leqslant')$, a morphism between them, written $\sigma : \Sigma \to \Sigma'$, maps: sort names in S to sort names in S' so that the subsorting preorder is preserved; operation names in $TF \cup PF$ to operation names in $TF' \cup PF'$ so that their totality and profiles are preserved; and predicate names in P to predicate names in P' so that their arities are preserved. This yields a category **Sig** of Casl signatures and their morphisms with the obvious identities and component-wise composition.

Given a Casl signature $\Sigma = (S, TF, PF, P, \leqslant)$, we define its expansion to a many-sorted signature $\Sigma^\#$ that retains the set of sorts S and includes the operation and predicate names from TF, PF and P, adding for all $s \leqslant s'$ in Σ, a new total operation name $em^{s \leqslant s'} : s \to s'$ for subsort embedding, a new partial operation name $pr^{s \leqslant s'} : s' \to s$ for subsort projection, and a new predicate name $in^{s \leqslant s'} : s'$ for subsort membership. Note that $(\_)^\#$ extends to signature morphisms in an obvious way.

Now, a Casl model over the Casl signature $\Sigma = (S, TF, PF, P, \leqslant)$ is a structure M over the signature $\Sigma^\#$, which consists of

— a carrier set $|M|_s$ for each sort $s \in S$,
— a (partial) function $f_M : |M|_{s_1} \times \cdots \times |M|_{s_n} \to |M|_s$ for each of the operation names $f : s_1 \times \cdots \times s_n \to s$ in $\Sigma^\#$ (with f_M being total for total operation names f), and
— a relation $p_M \subseteq |M|_{s_1} \times \cdots \times |M|_{s_n}$ for each predicate name $p : s_1 \times \cdots \times s_n$,

such that for all $s \leqslant s'$ in Σ

— the subsort embedding $em_M^{s \leqslant s'} : |M|_s \to |M|_{s'}$ is injective,
— the subsort projection $pr_M^{s \leqslant s'} : |M|_{s'} \to |M|_s$ is defined exactly on the image of $em_M^{s \leqslant s'}$ as its inverse, and
— the subsort membership predicate $in_M^{s \leqslant s'} \subseteq |M|_{s'}$ holds exactly on the image of $em_M^{s \leqslant s'}$.

Moreover, we require that $em_M^{s \leqslant s}$ is the identity for $s \in S$, and that the embeddings compose, that is, if $s \leqslant s' \leqslant s''$, then $em_M^{s \leqslant s''}$ is the composition of $em_M^{s \leqslant s'}$ and $em_M^{s' \leqslant s''}$.

This yields the class of Casl Σ-models, which form a category **Mod**(Σ) with homomorphisms between $\Sigma^\#$-structures defined as usual, as maps that preserve predicates as well as the definedness and values of operations. A homomorphism is *strong* if it also reflects the predicates and the definedness of operations. Given a Casl Σ-model M, a *submodel* is any Casl Σ-model N with carriers of N included in those of M such that

† This assumption is unrealistic in practical examples, especially when subsorting is involved; Casl deals with this properly, and only imposes a considerably weaker version of this restriction. However, the issues arising are irrelevant for the topics discussed in this paper.

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the inclusion function $|N| \hookrightarrow |M|$ is a strong homomorphism, *cf.* closed subalgebras in Burmeister (1986).

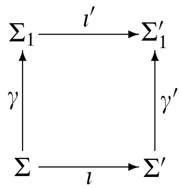
As expected, kernels of homomorphisms between CASL models are *congruences*: equivalence relations on model carriers closed under operations when defined in the model (this also applies to the subsort embeddings and projections). Kernels of strong homomorphisms are *strong congruences*: these are congruences that, in addition, preserve predicates and definedness of operations. Given any CASL Σ-model M and congruence $\simeq$ on it, the quotient of M by $\simeq$ is defined as the quotient of M as a $\Sigma^{\#}$-structure by $\simeq$; it is easy to check that the usual definition yields a $\Sigma^{\#}$-structure that is a CASL Σ-model, and that the natural quotient homomorphism is strong whenever the congruence $\simeq$ is strong.

Any CASL signature morphism $\sigma\colon\Sigma \to \Sigma'$ determines a *reduct* functor from $\mathbf{Mod}(\Sigma')$ to $\mathbf{Mod}(\Sigma)$, where for any Σ'-model $M' \in |\mathbf{Mod}(\Sigma')|$, its reduct $M'|_{\sigma} \in |\mathbf{Mod}(\Sigma)|$ is defined as the $\sigma^{\#}$-reduct of the $(\Sigma')^{\#}$-structure M': any sort, operation or predicate name v in $\Sigma^{\#}$ gets the same interpretation in $M'|_{\sigma}$ as $\sigma^{\#}(v)$ has in M', and similarly for homomorphisms, and for arbitrary relations between carriers of CASL models. This completes the definition of a functor $\mathbf{Mod} : \mathbf{Sig}^{op} \to \mathbf{Cat}$.

It is easy to check that the category $\mathbf{Sig}$ of CASL signatures is (finitely) cocomplete, with colimits of diagrams given in the expected, component-wise way. Note in particular that the subsort preorder in the colimit signature is the transitive closure of the union of the images of the subsort preorders of the signatures in the diagram under the colimit injections. We will assume that some standard construction of pushouts in $\mathbf{Sig}$ is given.

Colimits in $\mathbf{Sig}$ offer a rudimentary way of putting together CASL signatures and basic specifications over them (see below), very much as in the standard algebraic framework (Ehrig and Mahr 1985). When it comes to model theory, however, things are more difficult, since CASL does not ensure that the *amalgamation property* holds.

Definition 2.1 (Amalgamation). A pushout in the category of CASL signatures



ensures amalgamability if for all models $M_1 \in |\mathbf{Mod}(\Sigma_1)|$ and $M' \in |\mathbf{Mod}(\Sigma')|$ such that $M_1|_{\gamma} = M'|_{\iota}$, there exists a unique model $M'_1 \in |\mathbf{Mod}(\Sigma'_1)|$ such that $M'_1|_{\iota'} = M_1$ and $M'_1|_{\gamma'} = M'$. We sometimes write $M_1 \oplus M'$ for such a unique M'_1 and call it the *amalgamation of* M_1 *and* M' when the pushout is clear from the context.

When the signature morphism ι is given and the pushout as above ensures amalgamability, we will refer to the morphism γ as *admissible* (*cf.* Definition 3.3 below).

It is worth stressing that pushouts of CASL signature morphisms between signatures with no proper subsorts (that is, the subsorting preorders are identities) always ensure amalgamability. The potential problems are caused by the built-in requirements of uniqueness and composability of subsort embeddings in CASL models. The simplest example of a pushout that does not ensure amalgamability is when Σ contains just two

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sorts, and both Σ_1 and Σ' expand Σ by adding a new subsort relationship between the two sorts. The pushout signature then coincides with $\Sigma_1 = \Sigma'$ (and so allows for one subsort embedding between the two sorts), and two models over Σ_1 and Σ' with common Σ-reduct amalgamate only if they happen to share the same subsort embedding. Perhaps surprisingly, the problem of whether a pushout (or more generally, a colimit) ensures amalgamability is in general undecidable, but a number of effective algorithms to determine this in various practically relevant cases can be given. However, we do not know any easy syntactic condition that would ensure amalgamability without excluding some cases that naturally arise in practical specifications. For instance, requiring that ι and γ in the diagram above do not introduce new subsorting relationships between sorts from Σ is not sufficient. To see this, consider Σ with just two independent sorts, Σ_1 and Σ' that add, respectively, a new common subsort and a new common supersort for them. Then the resulting pushout does not ensure amalgamability. See Schröder *et al.* (2001), Klin *et al.* (2001) and Schröder *et al.* (2005) for further examples and a more complete study of amalgamability in CASL. Here, we just guard any use of amalgamation with a requirement that the relevant pushout ensures amalgamability.

In the framework of CASL, if a pushout ensures amalgamability (of CASL models, as above), it also ensures amalgamability of homomorphisms.

Lemma 2.2. Suppose that the following pushout

$$\begin{array}{ccc} \Sigma_1 & \xrightarrow{\iota'} & \Sigma'_1 \\ \uparrow{\scriptstyle\gamma} & & \uparrow{\scriptstyle\gamma'} \\ \Sigma & \xrightarrow[\iota]{} & \Sigma' \end{array}$$

ensures amalgamability. Then for all homomorphisms $h_1 : M_1 \to N_1$ in $\mathbf{Mod}(\Sigma_1)$ and $h' : M' \to N'$ in $\mathbf{Mod}(\Sigma')$ such that $h_1|_\gamma = h'|_\iota$ there exists a unique homomorphism $h'_1 : M'_1 \to N'_1$ in $\mathbf{Mod}(\Sigma'_1)$ such that $h'_1|_{\iota'} = h_1$ and $h'_1|_{\gamma'} = h'$. Moreover, h'_1 is strong if both h_1 and h' are strong.

Proof. Let $M'_1 = M_1 \oplus M'$ and $N'_1 = N_1 \oplus N'$ (they are well defined, since the pushout ensures amalgamability). For each sort s_1 in Σ_1, put $(h'_1)_{\iota'(s_1)} = (h_1)_{s_1}$; for each sort s' in Σ', put $(h'_1)_{\gamma'(s')} = (h')_{s'}$. By the construction of pushouts in **Sig**, this yields a well-defined family of functions $(h'_1)_s : |M'_1|_s \to |N'_1|_s$ for sorts s in Σ'_1. The required compatibility with the predicates and operations of the form $(\iota')^\#(f_1)$, for f_1 in $\Sigma_1^\#$, follows from the compatibility of h_1 with the predicates and operations in $\Sigma_1^\#$; and similarly for the predicates and operations of the form $(\gamma')^\#(f')$ for f' in $(\Sigma')^\#$. Consider then a subsort embedding in $(\Sigma'_1)^\#$. Since the subsort relation in Σ'_1 is the transitive closure of the union of the images of the subsort relations in Σ_1 and Σ' under ι' and γ', respectively, the embedding is a composition of embedding operations of the forms considered above – so compatibility follows by an easy induction. The same argument applies for subsort projections in Σ'_1, and then for the subsort membership predicates (which are defined as the domains of the corresponding subsort projections). □

Given a CASL signature Σ, we assume the usual definition of a first-order formula (with quantification and the usual logical connectives) built over *atomic formulae*, which include strong and existential equalities, definedness formulae and predicate applications, over the many-sorted signature $\Sigma^{\#}$, and its satisfaction in a $\Sigma^{\#}$-structure. Adding so-called generation constraints as special, non-first-order sentences yields the set of CASL Σ-sentences, written $\mathbf{Sen}(\Sigma)$. Given a CASL signature morphism $\sigma : \Sigma \to \Sigma'$, the translation of any Σ-sentence $\varphi \in \mathbf{Sen}(\Sigma)$ is defined as usual, and we write it as $\sigma(\varphi)$, see Baumeister *et al.* (2004). This defines a functor $\mathbf{Sen} : \mathbf{Sig} \to \mathbf{Set}$.

As usual for first-order logic, satisfaction is defined for the more general case of formulae with free variables; we write $M[v] \models_\Sigma \varphi$ to state that the Σ-formula φ with free variables in a set X holds in the Σ-model M under the valuation $v : X \to |M|$. The signature subscript in $\models_\Sigma$ is usually left implicit. The notation $(t)_{M[v]}$ is used to denote the value of a term t with variables in X in the model M under the valuation $v : X \to |M|$; this may be undefined when the term involves partial operations. Satisfaction of formulae and evaluation of terms only depend on the valuation of their free variables. We drop the valuation v in this notation for closed terms (terms with no variables) and sentences (formulae with no free variables). The satisfaction of sentences is preserved under signature morphisms: for any $\sigma : \Sigma \to \Sigma'$, $M' \in |\mathbf{Mod}(\Sigma')|$ and $\varphi \in \mathbf{Sen}(\Sigma)$, we have

$$M'|_\sigma \models \varphi \iff M' \models \sigma(\varphi)\,.$$

We consider CASL formulae built over the usual algebraic terms only, so, in particular, CASL conditional terms are excluded (they can be easily eliminated in formulae anyway, see CoFI (2004)).

We introduce a more general form of conditional terms, as follows, but without allowing them in formulae. Given a CASL signature Σ, a *conditional term* of sort s with variables in X is of the form $c = \langle(\phi_i, t_i)\rangle_{i \geqslant 0}$, where for $i \geqslant 0$, ϕ_i are formulae with variables in X, and t_i are terms of sort s with variables in X. Given a Σ-model M and a valuation $v : X \to |M|$, the value $c_{M[v]}$ of such a conditional term c is $(t_k)_{M[v]}$ for the least $k \geqslant 0$ such that $M[v] \models \phi_k$, or is undefined if no such $k \geqslant 0$ exists. Note that the infinitary unfolding of any recursive definition can be captured by such a conditional term. Therefore we use these conditional terms to model arbitrary computations, even though they go well beyond what programming languages offer: arbitrary formulae are used as conditions without regard to decidability, the sequence of conditions and terms need not even be recursively enumerable, and so on. Some of this generality will be excluded by requirements arising from the discussion in Sections 5 and 6.1.

We use these conditional terms to generalise derived signature morphisms (Goguen *et al.* 1978). A *derived signature morphism* $\delta : \Sigma \to \Sigma'$ maps the partial operation symbols $f : s_1 \times \ldots \times s_n \to s$ in Σ to conditional Σ'-terms of sort $\delta(s)$ with the variables $\{x_1 : \delta(s_1), \ldots, x_n : \delta(s_n)\}$. Evidently, such a derived signature morphism $\delta : \Sigma \to \Sigma'$ still determines a reduct function $\_|_\delta$ mapping Σ'-models to Σ-models. In general, this does *not* extend to a reduct functor between model categories, since values of conditional terms with arbitrary conditions need not be preserved by homomorphisms (but see the comment following Lemma 5.5).

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The basic level of CASL includes *declarations* to introduce components of signatures and *axioms* to give properties that characterise *models* of a specification. Consequently, a basic CASL specification SP amounts to a definition of a signature Σ and a set of axioms $\Phi \subseteq \mathbf{Sen}(\Sigma)$. It denotes the class $[\![SP]\!] \subseteq |\mathbf{Mod}(\Sigma)|$ of SP-models, which are those Σ-models that *satisfy* all the axioms in Φ:

$$[\![SP]\!] = \{M \in |\mathbf{Mod}(\Sigma)| \mid M \models \Phi\}.$$

Apart from basic specifications as above, CASL provides ways of building complex specifications out of simpler ones by means of various *structuring constructs*. These include translation, hiding, union, and both free and loose forms of extension. *Generic specifications* and their *instantiations* with pushout-style semantics (Burstall and Goguen 1980; Ehrig and Mahr 1985) are also provided. Structured specifications built using these constructs are given a compositional semantics where each specification SP determines a signature $Sig[SP]$ and a class $[\![SP]\!] \subseteq |\mathbf{Mod}(Sig[SP])|$ of models. Most of the details, given in Baumeister *et al.* (2004), are irrelevant for the purposes of this paper. It is enough to know that for any specification SP and signature morphism $\sigma : Sig(SP) \to \Sigma'$, we write SP **with** σ for the translation of SP along σ, with semantics given by $Sig[SP \textbf{ with } \sigma] = \Sigma'$ and $[\![SP \textbf{ with } \sigma]\!] = \{M' \in |\mathbf{Mod}(\Sigma')| \mid M'|_\sigma \in [\![SP]\!]\}$, and for any two specifications SP_1 and SP_2 with common signature, we write SP_1 **and** SP_2 for their union, with semantics given by $Sig[SP_1 \textbf{ and } SP_2] = Sig[SP_1] = Sig[SP_2]$ and $[\![SP_1 \textbf{ and } SP_2]\!] = [\![SP_1]\!] \cap [\![SP_2]\!]$. Note that union in CASL generalises this by allowing $Sig[SP_1] \neq Sig[SP_2]$.

3. Software components and their correctness

The intended use of CASL, as of any such specification formalism, is to specify programs. Each CASL specification should determine a class of programs that correctly realise the specified requirements. To fit this into the formal view of CASL specifications, programs must be written in a programming language having a semantics that assigns to each program its *denotation* as a CASL model[†]. Then each program P determines a CASL signature $Sig[P]$ and a model $[\![P]\!] \in |\mathbf{Mod}(Sig[P])|$. Any specification SP is then a description of its admissible realisations: a program P is a *(correct) realisation* of SP if $Sig[P] = Sig[SP]$ and $[\![P]\!] \in [\![SP]\!]$.

We will now consider component-based systems, that is, systems obtained by assembling components, rather than 'monolithic' programs. We take a rather restrictive view of components, namely *software components* (understood as pieces of code) in contrast with *system components* (understood as self-contained processors with their own hardware and software interacting with each other and the environment by exchanging messages across linking interfaces). However, our view is consistent with the best accepted definition in

[†] This may be rather indirect, and in general involves a non-trivial abstraction step. It has not yet been attempted for any real programming language, but see (Schröder and Mossakowski 2002) for an outline of how this could be done for Haskell. See also the pre-CASL work on Extended ML (Kahrs *et al.* 1997), and see Larch (Guttag and Horning 1993) for another attempt to link a specification language with various programming languages.

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the software industry, see Szyperski (1998): a (software) component is an independently-deployable unit of composition with contractually specified interfaces and fully explicit context dependencies.

To capture this, we will assume that a software component ΔP determines a 'parameter' signature, say Σ, corresponding to the symbols required by the component, and a 'result' signature, say Σ', corresponding to the symbols provided by the component, together with a signature morphism $\iota:\Sigma \to \Sigma'$ relating the 'parameter' signature to the 'result' signature. In this way $\iota:\Sigma \to \Sigma'$ corresponds to the (syntactic part of the) interface of the software component.

Then the software component ΔP determines a function $F = [\![\Delta P]\!]$ from CASL Σ-models to CASL Σ'-models. This function may be partial, see below. When assembled with (applied to) a sub-system P (determining a CASL signature $Sig[P] = \Sigma$ and a model $[\![P]\!] \in |\mathbf{Mod}(\Sigma)|$), the software component ΔP 'extends' P to a larger system, say $\Delta P(P)$, with signature $Sig[\Delta P(P)] = \Sigma'$, determining a CASL model $[\![\Delta P(P)]\!] \in |\mathbf{Mod}(\Sigma')|$. It is intuitively clear that the software component 'preserves' the sub-system it is applied to, so $[\![\Delta P(P)]\!]|_\iota = [\![P]\!]$.

Thus a software component ΔP determines a semantic object F called a *local construction* according to the definition below. Since software components preserve their arguments, we assume that such constructions are *persistent*: the argument of a construction is always fully included in its result, without modification[†] – note that this assumption holds for all constructions that can be declared and specified in CASL, see Section 4. In fact, we generalise CASL somewhat by considering arbitrary signature morphisms rather than just inclusions.

Definition 3.1 (Local construction). Given a signature morphism $\iota:\Sigma \to \Sigma'$, a *local construction along* ι is a persistent partial function $F:|\mathbf{Mod}(\Sigma)| \rightharpoonup |\mathbf{Mod}(\Sigma')|$ (for each $M \in dom(F)$, $F(M)|_\iota = M$). We write $\mathbf{Mod}(\Sigma \xrightarrow{\iota} \Sigma')$ for the class of all local constructions along ι.

We will not dwell here on how particular local constructions are defined. Free functor semantics for parametrised specifications is one way to proceed, with the persistency requirement giving rise to additional proof obligations (Ehrig and Mahr 1985). Perhaps closer to ordinary programming, any 'definitional' derived signature morphism $\delta:\Sigma' \to \Sigma$ that defines Σ'-components in terms of Σ-components naturally gives rise to a local construction, since the induced reduct function $\_|_\delta:|\mathbf{Mod}(\Sigma)| \to |\mathbf{Mod}(\Sigma')|$ is a local construction along a signature morphism $\iota:\Sigma \to \Sigma'$ whenever $\iota;\delta = id_\Sigma$[‡].

Of course, we are interested in specifications of software components, that is, in 'semantic' specifications of the parameter required by the component and of its result

[†] Otherwise we would have to indicate explicitly any 'sharing' between the argument and result of each construction, and explain how such sharing is preserved by the various ways of putting together constructions, as was painfully spelled out in Sannella and Tarlecki (1989). If necessary, superfluous components of models constructed using persistent constructions can be discarded at the end using the reduct along a signature inclusion.

[‡] The composition of derived signature morphisms can be defined in the obvious way, and equality of two derived signature morphisms is understood here semantically.

(and not just in the 'syntactic' specification of its interface given by $\iota : \Sigma \to \Sigma'$). Thus, in our algebraic setting, we will specify a software component by a pair of specifications SP and SP', written $SP \xrightarrow{\iota} SP'$, where SP specifies the symbols and the properties required of them by the component, SP' specifies the symbols and the properties provided by the component, together with a signature morphism $\iota : Sig[SP] \to Sig[SP']$ relating the parameter signature to the result signature. Indeed, we require ι to be a *specification morphism* $\iota : SP \to SP'$, that is, for all $M' \in [\![SP']\!]$, $M'|_\iota \in [\![SP]\!]$. This amounts to demanding that the result specification SP' includes the properties of the parameter required by the parameter specification SP. The fact that the result actually has those properties is guaranteed by the persistency of the local construction.

The following definition states when a local construction F, determined by a software component ΔP, is a correct realisation of a given component specification. (We refer to this as *literal* correctness by contrast with the *observational* correctness of Definition 6.9 given later.)

Definition 3.2 (Literal correctness). A local construction F along $\iota : Sig[SP] \to Sig[SP']$ is *literally correct with respect to SP and SP'* if for all models $M \in [\![SP]\!]$, we have $M \in dom(F)$ and $F(M) \in [\![SP']\!]$. We write $[\![SP \xrightarrow{\iota} SP']\!]$ for the class of all local constructions along ι that are literally correct with respect to SP and SP'.

Hence, to realise the component specification $SP \xrightarrow{\iota} SP'$, we should provide a software component ΔP that extends any realisation P of SP to a realisation $P' = \Delta P(P)$ of SP'. The basic semantic property required is that for all programs P such that $[\![P]\!] \in [\![SP]\!]$, $\Delta P(P)$ is a program that extends P and realises SP' (semantically, $[\![\Delta P(P)]\!]|_\iota = [\![P]\!]$ and $[\![\Delta P(P)]\!] \in [\![SP']\!]$). This amounts to requiring that the partial function $F \in \mathbf{Mod}(\Sigma \xrightarrow{\iota} \Sigma')$ determined by ΔP preserves its argument whenever it is defined, that it is defined on (at least) all models in $[\![SP]\!]^\dagger$, and that it yields a result in $[\![SP']\!]$ when applied to a model in $[\![SP]\!]$.

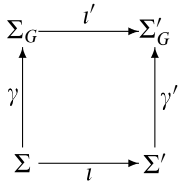
There is a crucial difference here between monolithic self-contained programs and software components: while monolithic programs are modelled as Casl models, software components are modelled as (possibly partial) functions mapping Casl models of the parameter specification SP to Casl models of the result specification SP'.

The next important idea is that when assembling components, in general, a given component will not be applied to a sub-system providing *exactly* what is required by the component, but will be applied to a sub-system providing *at least, and, in general, more than* is required.

Technically, this means that we need to look at constructions that map Σ-models to Σ'-models, but applied to parts cut out of 'larger' Σ_G-models, where this 'cutting out' is given as the reduct with respect to a signature morphism $\gamma : \Sigma \to \Sigma_G$ that fits the local argument signature into its global context.

† Intuitively, $\Delta P(P)$ is 'statically' well formed if P has the correct signature, but needs to be defined only for arguments that realise SP.

Throughout the rest of the paper we will repeatedly refer to the signatures and morphisms in the following pushout diagram:



where the local construction is along the bottom of the diagram, 'cutting out' its argument from a larger model uses the signature morphism on the left, and the resulting global construction is along the top.

Definition 3.3 (Admissibility and global construction). Given a local construction F along a signature morphism $\iota\colon\Sigma \to \Sigma'$, a morphism $\gamma\colon\Sigma \to \Sigma_G$ fitting Σ into a 'global' signature Σ_G is *admissible* if the pushout of ι and γ above ensures amalgamability. Then, for any Σ_G-model $\mathscr{G} \in |\mathbf{Mod}(\Sigma_G)|$, we define the *global result* $F_G(\mathscr{G})$ of applying F to $\mathscr{G}$ by reference to the pushout diagram above, using the amalgamation property: if $\mathscr{G}|_\gamma \in dom(F)$, then $F_G(\mathscr{G}) = \mathscr{G} \oplus F(\mathscr{G}|_\gamma)$, otherwise $F_G(\mathscr{G})$ is undefined.

This determines a *global construction* $F_G\colon|\mathbf{Mod}(\Sigma_G)| \rightharpoonup |\mathbf{Mod}(\Sigma'_G)|$, which is persistent along $\iota'\colon\Sigma_G \to \Sigma'_G$.

This way of 'lifting' a persistent function to a larger context through a '*fitting morphism*' using signature pushout and amalgamation is well established in the algebraic specification tradition, and goes back at least to 'parametrised specifications' with free functor semantics, see Ehrig and Mahr (1985). The extra requirement here is that only admissible fitting morphisms are permitted, turning amalgamability into a (static) requirement for correct application of a local construction in a given context, which is to be discharged using the machinery of Schröder *et al.* (2001), Klin *et al.* (2001) and Schröder *et al.* (2005).

Then an obvious issue is whether a software component that realises a component specification $SP \xrightarrow{\iota} SP'$, when combined with a sub-system that realises a specification SP_G, actually provides a system that realises a given specification SP'_G. The corresponding correctness condition is provided by the following theorem.

Theorem 3.4. If we are given a local construction $F \in [\![SP \xrightarrow{\iota} SP']\!]$, a specification SP_G with admissible fitting morphism $\gamma\colon Sig[SP] \to Sig[SP_G]$, and a specification SP'_G with $Sig[SP'_G] = \Sigma'_G$, then the induced global construction F_G along $\iota'\colon\Sigma_G \to \Sigma'_G$ is literally correct with respect to SP_G and SP'_G, that is, $F_G \in [\![SP_G \xrightarrow{\iota'} SP'_G]\!]$, provided

— $[\![SP_G]\!] \subseteq [\![SP \textbf{ with } \gamma]\!]$, and
— $[\![(SP' \textbf{ with } \gamma') \textbf{ and } (SP_G \textbf{ with } \iota')]\!] \subseteq [\![SP'_G]\!]$.

Proof. Let $\mathscr{G} \in [\![SP_G]\!]$. Then $\mathscr{G}|_\gamma \in [\![SP]\!]$, so $\mathscr{G}|_\gamma \in dom(F)$ and $F(\mathscr{G}|_\gamma) \in [\![SP']\!]$. Consequently, $F_G(\mathscr{G}) \in [\![SP' \textbf{ with } \gamma']\!] \cap [\![SP_G \textbf{ with } \iota']\!]$. □

Informally, this directly captures a 'bottom-up' process of building component-based systems, whereby we start with SP_G, a specification of a 'global' assembly of components built so far, find a local construction (a component) $F \in [\![SP \xrightarrow{\iota} SP']\!]$ with a fitting

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morphism γ that satisfies the first condition, and define SP'_G such that the second condition is satisfied (for example, by simply taking $SP'_G = (SP'$ **with** $\gamma')$ **and** $(SP_G$ **with** $\iota'))$, thus obtaining a specification of the global assembly of components with the new component built using F added. When proceeding 'top-down', we start with the global requirements specification SP'_G. To use a local construction (a component) $F \in [\![SP \xrightarrow{\iota} SP']\!]$, we have to decide which part of the requirements it is going to implement by providing a signature morphism $\gamma' : Sig[SP'] \to Sig[SP'_G]$, then we construct the 'pushout complement' $\gamma : Sig[SP] \to \Sigma_G$, $\iota' : \Sigma_G \to Sig[SP'_G]$ for ι and γ', and finally devise a specification SP_G with $Sig[SP_G] = \Sigma_G$ such that both conditions are satisfied. Then SP_G is the requirements specification for the components that remain to be implemented.

4. Architectural specifications

Using local constructions for global implementations of specifications, we have moved only one step away from a monolithic global view of specifications and constructions used to implement them. The notion of an *architectural specification* (Bidoit *et al.* 2002a) as introduced for Casl takes us much further. An architectural specification *prescribes* a decomposition of the task of implementing a requirements specification into a number of subtasks to implement specifications of 'modular components' (called *units*) of the system under development. The units may be parametrised, and then we can identify them with local constructions; non-parametrised units are just models. Another essential part of an architectural specification is a prescription of how the units, once developed, are to be put together using a few simple operators. One of these is the application of a parametrised unit, which corresponds exactly to the lifting of a local construction to a larger context studied above. Thus, an architectural specification may be thought of as a definition of a complex construction to be used in a top-down development process to implement a requirements specification by a number of specifications (of non-parametrised units), where the construction uses a number of specified local constructions that are to be developed as well.

For the sake of readability, we will discuss here a simplified version of Casl architectural specifications, with a limited (but representative) number of constructs, based on a version used in Schröder *et al.* (2001; 2005); a generalisation to full architectural specifications (including unit renaming, units with multiple parameters, local unit definitions, etc.) would be tedious but rather straightforward, except perhaps for the 'unguarded import' mechanism, see Hoffman (2001). Our version of architectural specifications is defined as follows.

Architectural specifications: ASP ::= **arch spec** UDD^+ **result** T;

UDD ::= Dcl | Dfn

An architectural specification consists of a (non-empty) list of unit declarations or definitions followed by a unit result term.

Unit declarations: Dcl ::= $U : SP$ | $U : SP_1 \xrightarrow{\iota} SP_2$

A unit declaration introduces a unit name with its type, which is either a specification or a specification of a parametrised unit, determined by a specification of its parameter and its result that extends the parameter *via* a signature morphism ι.

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Unit definitions: $Dfn ::= U = T$

A unit definition introduces a (non-parametrised) unit and gives its value by a unit term.

Unit terms: $T ::= U \mid$ **reduce** T **by** $\sigma \mid U[T$ **fit** $\gamma] \mid T_1$ **and** T_2

A unit term is either a (non-parametrised) unit name, or a unit restricted with respect to a signature morphism, or a unit application with an argument that fits *via* a signature morphism γ, or an amalgamation of units.

Following the semantics of full CASL (see Baumeister *et al.* (2004); see also Schröder *et al.* (2001) and Schröder *et al.* (2005)), we give the semantics of this CASL fragment in two stages: first we give its *extended static semantics*[†] and then its *literal model semantics*. (We refer to this as the *literal* model semantics by contrast with the *observational* model semantics of Section 7.)

For the extended static semantics we need a concept of *static context*, which carries signatures for the units declared or defined within an architectural specification, together with information on their mutual dependencies. Analogously, for the model semantics we need a concept of *environment*, which carries the semantics of the units named in the corresponding static context.

When discussing the application of local constructions to global models in Section 3, we viewed the global context as a single monolithic model over a single 'global' signature. Unfortunately, this view cannot be maintained in the context of architectural specifications in CASL. The technical reason is that CASL does not ensure amalgamation over arbitrary colimits of signature diagrams, as pointed out in Section 2. Indeed, if amalgamability were ensured for arbitrary colimits of signature diagrams, we could always represent the global context of all the (non-parametrised) units declared or defined so far by a monolithic global model over a single global signature, and many of the technicalities below become rather simpler, see Bidoit *et al.* (2002b) and Tarlecki (2003). As things are, for architectural specifications in CASL, static information about (non-parametrised) units declared or defined in an architectural specification will be stored in signature diagrams, with nodes labelled by unit signatures and edges labelled by signature morphisms that capture dependencies between units.

More formally, we view a signature diagram as a graph morphism from its *shape* $\mathbf{I}$ to the category of CASL signatures, $D : \mathbf{I} \to \mathbf{Sig}$. We write $|D|$ for the set of nodes of $\mathbf{I}$, and $m : i \to j$ in D for an edge m with source i and target j in $\mathbf{I}$. The extension of diagrams is understood as usual. Two diagrams D_1, D_2 *disjointly extend* D if both D_1 and D_2 extend D and the intersection of their shapes is the shape of D. If this is the case, the union $D_1 \cup D_2$ is well defined. As usual, disjointness of diagram extensions may be ensured by choosing the new nodes and edges appropriately.

For any diagram $D : \mathbf{I} \to \mathbf{Sig}$, a family $\mathscr{M} = \langle M_i \rangle_{i \in |D|}$ of models is called *D-coherent* if for each $i \in |D|$, $M_i \in |\mathbf{Mod}(D(i))|$, and for each $m : i \to j$ in $\mathbf{I}$, $M_i = M_j|_{D(m)}$; this is extended to $|D|$-indexed families of model morphisms in the obvious way. Given a

[†] Baumeister *et al.* (2004) makes a distinction between the *static semantics* of architectural specifications, which ignores dependencies between terms and hence does not contribute to the analysis of amalgamability, and the *extended static semantics*.

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D-coherent family $\mathscr{M} = \langle M_i \rangle_{i \in |D|}$, we write $\mathscr{M}_i$ for M_i, $i \in |D|$. We let $\mathbf{Mod}(D)$ be the category with D-coherent model families as objects and D-coherent families of model morphisms as morphisms (with the obvious component-wise composition).

D ensures amalgamability for D', where D' extends D, if any D-coherent model family can be uniquely extended to a D'-coherent model family. It is easy to see that Definition 2.1 is in fact a special case of this notion[†].

An *extended static context* $\mathscr{C}_{st} = (P_{st}, \mathscr{B}_{st}, D)$, in which CASL phrases are elaborated, consists of a static context for parametrised units P_{st} mapping parametrised unit names to signature morphisms (from the parameter to the result signature), a global context diagram D, and a static context for non-parametrised units $\mathscr{B}_{st}$ mapping non-parametrised unit names to nodes in D. From any such extended static context we can extract a *static context* $ctx(\mathscr{C}_{st}) = (P_{st}, B_{st})$ by preserving the static context P_{st} for parametrised units and building a direct static context B_{st} for non-parametrised units that extracts their signatures from $\mathscr{B}_{st}$ and D (that is, $B_{st}(U) = D_{\mathscr{B}_{st}(U)}$). $\mathscr{C}_{st}^{\varnothing}$ stands for the 'empty' extended static context that consists of the empty parametrised and non-parametrised unit contexts, and of the empty context diagram. Extension (or inclusion) of extended static contexts, written $\mathscr{C}_{st} \subseteq \mathscr{C}'_{st}$, is defined component-wise, as expected. We refer to unit names in $dom(P_{st})$ as parametrised unit names in $\mathscr{C}_{st}$, and to those in $dom(\mathscr{B}_{st})$ as non-parameterised unit names in $\mathscr{C}_{st}$.

Figure 1 gives rules to derive semantic judgments of the following forms:

— $\vdash ASP \rhd\!\!\rhd ((P_{st}, B_{st}), \Sigma)$
The architectural specification ASP yields a static context describing the units declared or defined in ASP, and the signature of the result unit.

— $\vdash UDD^{+} \rhd\!\!\rhd \mathscr{C}_{st}$
The sequence UDD^{+} of unit declarations and definitions yields an extended static context $\mathscr{C}_{st}$.

— $\mathscr{C}_{st} \vdash UDD \rhd\!\!\rhd \mathscr{C}'_{st}$
The unit declaration or definition UDD in the extended static context $\mathscr{C}_{st}$ yields a new extended static context $\mathscr{C}'_{st}$ extending $\mathscr{C}_{st}$.

— $(P_{st}, \mathscr{B}_{st}, D) \vdash T \rhd\!\!\rhd (i, D')$
The unit term T in the extended static context $(P_{st}, \mathscr{B}_{st}, D)$ yields a new context diagram D' extending D and a node i in D' that carries the signature of the unit term T.

To follow the rules for unit application and amalgamation, it may be helpful to look at Figure 2, where the corresponding global context diagrams are sketched.

It is worth noting that in the rule for parametrised unit application, the requirement that D' ensures amalgamability for D'' is weaker than requiring that the pushout used

[†] In spite of Lemma 2.2 and its obvious generalisation to colimits of arbitrary signature diagrams, we do not know whether in the framework of CASL it is always the case that if D ensures amalgamability for D', then the similar property also holds for D-coherent families of model morphisms; we conjecture that this is the case. However, in this paper we need only a few special cases of this, where D' arises from D essentially by adding a surjective cone, and so a proof similar to that for Lemma 2.2 goes through; the same is true for a similar generalisation of Lemma 5.6 below.

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$$\frac{\vdash UDD^{+} \rhd\!\!\rhd \mathcal{C}_{st} \qquad \mathcal{C}_{st} \vdash T \rhd\!\!\rhd (i, D)}{\vdash \textbf{arch spec } UDD^{+} \textbf{ result } T \rhd\!\!\rhd (ctx(\mathcal{C}_{st}), D(i))}$$

$$\frac{\begin{array}{c} \mathcal{C}^{\emptyset}_{st} \vdash UDD_1 \rhd\!\!\rhd (\mathcal{C}_{st})_1 \\ \cdots \\ (\mathcal{C}_{st})_{n-1} \vdash UDD_n \rhd\!\!\rhd (\mathcal{C}_{st})_n \end{array}}{\vdash UDD_1 \ldots UDD_n \rhd\!\!\rhd (\mathcal{C}_{st})_n}$$

$$\frac{\begin{array}{c} U \notin (dom(P_{st}) \cup dom(\mathcal{B}_{st})) \\ D' \text{ extends } D \text{ by a new node } i \text{ with } D'(i) = Sig(SP) \end{array}}{(P_{st}, \mathcal{B}_{st}, D) \vdash U{:}\, SP \rhd\!\!\rhd (P_{st}, \mathcal{B}_{st} + \{U \mapsto i\}, D')}$$

$$\frac{\begin{array}{c} \iota : Sig(SP_1) \rightarrow Sig(SP_2) \\ U \notin (dom(P_{st}) \cup dom(\mathcal{B}_{st})) \end{array}}{(P_{st}, \mathcal{B}_{st}, D) \vdash U{:}\, SP_1 \xrightarrow{\iota} SP_2 \rhd\!\!\rhd (P_{st} + \{U \mapsto \iota\}, \mathcal{B}_{st}, D)}$$

$$\frac{(P_{st}, \mathcal{B}_{st}, D) \vdash T \rhd\!\!\rhd (i, D') \qquad U \notin (dom(P_{st}) \cup dom(\mathcal{B}_{st}))}{(P_{st}, \mathcal{B}_{st}, D) \vdash U = T \rhd\!\!\rhd (P_{st}, \mathcal{B}_{st} + \{U \mapsto i\}, D')}$$

$$\frac{U \in dom(\mathcal{B}_{st})}{(P_{st}, \mathcal{B}_{st}, D) \vdash U \rhd\!\!\rhd (\mathcal{B}_{st}(U), D)}$$

$$\frac{\begin{array}{c} \mathcal{C}_{st} \vdash T \rhd\!\!\rhd (i, D) \qquad \sigma{:}\, \Sigma \rightarrow D(i) \\ D' \text{ extends } D \text{ by a new node } j \text{ and a new edge } m{:}\, j \rightarrow i \text{ with } D'(m) = \sigma \end{array}}{\mathcal{C}_{st} \vdash \textbf{reduce } T \textbf{ by } \sigma \rhd\!\!\rhd (j, D')}$$

$$\frac{\begin{array}{c} P_{st}(U) = \iota{:}\, \Sigma \rightarrow \Sigma' \qquad \mathcal{C}_{st} \vdash T \rhd\!\!\rhd (i, D) \qquad \gamma{:}\, \Sigma \rightarrow D(i) \\ (\iota', \gamma') \text{ is a pushout of } (\gamma, \iota) \\ D' \text{ extends } D \text{ by new nodes } j, k \text{ and edges } m{:}\, j \rightarrow i, n{:}\, j \rightarrow k \\ \text{with } D'(m) = \gamma, D'(n) = \iota \\ D'' \text{ extends } D' \text{ by a new node } l \text{ and edges } r{:}\, i \rightarrow l, s{:}\, k \rightarrow l \\ \text{with } D''(r) = \iota', D''(s) = \gamma' \\ D' \text{ ensures amalgamability for } D'' \end{array}}{\mathcal{C}_{st} \vdash U[T \textbf{ fit } \gamma] \rhd\!\!\rhd (l, D'')}$$

$$\frac{\begin{array}{c} (P_{st}, \mathcal{B}_{st}, D) \vdash T_1 \rhd\!\!\rhd (i_1, D_1) \qquad (P_{st}, \mathcal{B}_{st}, D) \vdash T_2 \rhd\!\!\rhd (i_2, D_2) \\ D_1 \text{ and } D_2 \text{ are disjoint extensions of } D \\ D' \text{ extends } D_1 \cup D_2 \text{ by a new node } j \text{ and edges } m_1{:}\, i_1 \rightarrow j, m_2{:}\, i_2 \rightarrow j \text{ with} \\ D'(j) = D_1(i_1) \cup D_2(i_2), D'(m_1){:}\, D_1(i_1) \hookrightarrow D'(j), D'(m_2){:}\, D_2(i_2) \hookrightarrow D'(j) \\ D_1 \cup D_2 \text{ ensures amalgamability for } D' \end{array}}{(P_{st}, \mathcal{B}_{st}, D) \vdash T_1 \textbf{ and } T_2 \rhd\!\!\rhd (j, D')}$$

Fig. 1. Extended static semantics.

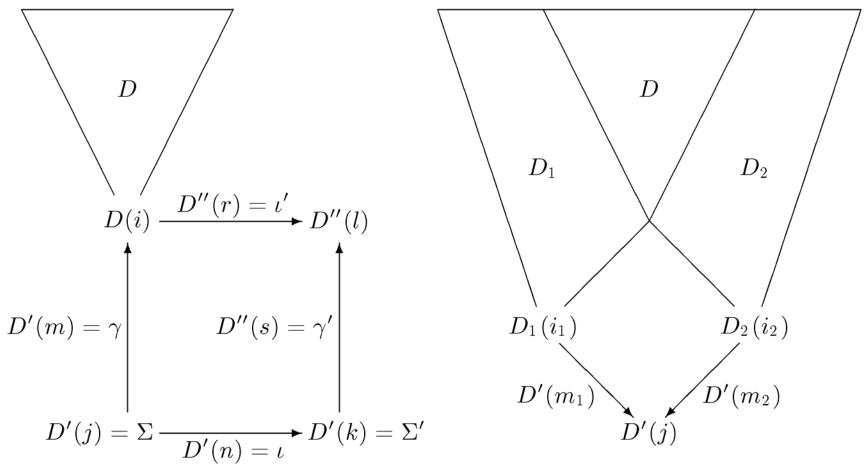

Fig. 2. Unit application and amalgamation diagrams.

in this rule ensures amalgamability: even if it does not, the global context in which the application is carried out may impose additional constraints on the models involved that ensure amalgamability.

Note also that the rule for unit amalgamation does not require that the amalgamated units have common signatures: the resulting unit will be built over the union of the two signatures, provided this union is defined[†] and that the two units built can be uniquely amalgamated to yield a unit over this union signature. This is ensured by the final condition in the rule, which requires that the dependencies between units captured in the diagram $D_1 \cup D_2$ ensure amalgamability of the two models involved. This requires, in particular, that these models share the interpretation of the symbols in the intersection of their signatures.

In the model semantics, we work with *contexts* $\mathscr{C}$ that are classes of *unit environments* E. Unit environments map unit names to either local constructions (for parametrised units) or to individual models (for non-parametrised units). *Unit evaluators UEv* map unit environments to models.

Given an extended static context $\mathscr{C}_{st} = (P_{st}, \mathscr{B}_{st}, D)$, a unit environment E *fits* $\mathscr{C}_{st}$ if:

— for each $U \in dom(P_{st})$, $E(U)$ is a local construction along $P_{st}(U)$; and
— there is a D-coherent family of models $\mathscr{M} \in |\mathbf{Mod}(D)|$ such that for each $U \in dom(\mathscr{B}_{st})$, $E(U) = \mathscr{M}_{\mathscr{B}_{st}(U)}$ – we say then that $\mathscr{M}$ *witnesses* E *in* $\mathscr{C}_{st}$.

We write $ucx(\mathscr{C}_{st})$ for the class of all unit environments that fit $\mathscr{C}_{st}$. Note that if $\mathscr{C}_{st} \subseteq \mathscr{C}'_{st}$, then $ucx(\mathscr{C}'_{st}) \subseteq ucx(\mathscr{C}_{st})$.

Two unit environments $E_1, E_2 \in ucx(\mathscr{C}_{st})$ *coincide in* $\mathscr{C}_{st}$, written $E_1 =_{\mathscr{C}_{st}} E_2$, if for all (parametrised and non-parametrised) unit names U in $\mathscr{C}_{st}$, $E_1(U) = E_2(U)$.

[†] The union is defined in the obvious, component-wise manner, with the subsort preorder given as the transitive closure of the two preorders in the component signatures – however, this may fail to yield a CASL signature due to overloading of operation and predicate names that may arise, which we have disallowed here. The union may also fail to be defined with CASL's treatment of overloading, albeit for more subtle reasons.

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$$\frac{\vdash UDD^{+} \Rightarrow \mathcal{C} \qquad \mathcal{C} \vdash T \Rightarrow UEv}{\vdash \textbf{arch spec } UDD^{+} \textbf{ result } T \Rightarrow (\mathcal{C}, UEv)}$$

$$\frac{\begin{array}{c}\mathcal{C}^{\emptyset} \vdash UDD_1 \Rightarrow \mathcal{C}_1 \\ \cdots \\ \mathcal{C}_{n-1} \vdash UDD_n \Rightarrow \mathcal{C}_n\end{array}}{\vdash UDD_1 \ldots UDD_n \Rightarrow \mathcal{C}_n}$$

$$\frac{}{\mathcal{C} \vdash U\colon SP \Rightarrow \mathcal{C} \times \{U \mapsto [\![SP]\!]\}}$$

$$\frac{}{\mathcal{C} \vdash U\colon SP_1 \xrightarrow{\iota} SP_2 \Rightarrow \mathcal{C} \times \{U \mapsto [\![SP_1 \xrightarrow{\iota} SP_2]\!]\}}$$

$$\frac{\mathcal{C} \vdash T \Rightarrow UEv}{\mathcal{C} \vdash U = T \Rightarrow \{E + \{U \mapsto UEv(E)\} \mid E \in \mathcal{C}\}}$$

$$\frac{}{\mathcal{C} \vdash U \Rightarrow \lambda E \in \mathcal{C} \cdot E(U)}$$

$$\frac{\mathcal{C} \vdash T \Rightarrow UEv}{\mathcal{C} \vdash \textbf{reduce } T \textbf{ by } \sigma \Rightarrow \lambda E \in \mathcal{C} \cdot UEv(E)|_{\sigma}}$$

$$\frac{\begin{array}{c}\mathcal{C} \vdash T \Rightarrow UEv \qquad \text{for each } E \in \mathcal{C},\ UEv(E)|_{\gamma} \in dom(E(U)) \\ \text{for each } E \in \mathcal{C},\ UEv(E) \oplus E(U)(UEv(E)|_{\gamma}) \text{ is well-defined}\end{array}}{\mathcal{C} \vdash U[T \textbf{ fit } \gamma] \Rightarrow \lambda E \in \mathcal{C} \cdot UEv(E) \oplus E(U)(UEv(E)|_{\gamma})}$$

$$\frac{\begin{array}{c}\mathcal{C} \vdash T_1 \Rightarrow UEv_1 \qquad \mathcal{C} \vdash T_2 \Rightarrow UEv_2 \\ \text{for each } E \in \mathcal{C}, \text{ there is a unique } M \in |\mathbf{Mod}(\Sigma)| \text{ such that} \\ M|_{\iota_1} = UEv_1(E), M|_{\iota_2} = UEv_2(E) \\ UEv = \{E \mapsto M \mid E \in \mathcal{C}, M|_{\iota_1} = UEv_1(E), M|_{\iota_2} = UEv_2(E)\}\end{array}}{\mathcal{C} \vdash T_1 \textbf{ and } T_2 \Rightarrow UEv}$$

Fig. 3. Literal model semantics.

Proposition 4.1. If $E_1 =_{\mathscr{C}_{st}} E_2$, any family that witnesses E_1 in $\mathscr{C}_{st}$, also witnesses E_2 in $\mathscr{C}_{st}$.

A context $\mathscr{C} \subseteq ucx(\mathscr{C}_{st})$ is *closed in* $\mathscr{C}_{st}$ if for all unit environments $E_1 \in \mathscr{C}$ and $E_2 \in ucx(\mathscr{C}_{st})$, we have $E_1 =_{\mathscr{C}_{st}} E_2$ implies $E_2 \in \mathscr{C}$.

$\mathscr{C}^{\varnothing} = ucx(\mathscr{C}^{\varnothing}_{st})$ is the context that constrains no unit name. Given a context $\mathscr{C}$, a unit name U and a class of units $\mathscr{V}$, we write $\mathscr{C} \times \{U \mapsto \mathscr{V}\}$ for $\{E + \{U \mapsto V\} \mid E \in \mathscr{C}, V \in \mathscr{V}\}$, where $E + \{U \mapsto V\}$ maps U to V and otherwise behaves like E.

Figure 3 gives rules to derive semantic judgments of the following forms:

— $\vdash ASP \Rightarrow (\mathscr{C}, UEv)$

The architectural specification ASP yields a context $\mathscr{C}$ with environments providing interpretations for the units declared and defined in ASP, and a unit evaluator that for each such environment determines the result unit.

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— $\vdash UDD^+ \Rightarrow \mathscr{C}$

The sequence UDD^+ of unit declarations and definitions yields a context $\mathscr{C}$.

— $\mathscr{C} \vdash UDD \Rightarrow \mathscr{C}'$

The unit declaration or definition UDD in the context $\mathscr{C}$ yields a new context $\mathscr{C}'$.

— $\mathscr{C} \vdash T \Rightarrow UEv$

The unit term T in the context $\mathscr{C}$ yields a unit evaluator UEv that when given an environment (in $\mathscr{C}$) yields the unit resulting from the evaluation of T in this environment.

The rules rely on a successful run of the extended static semantics; this allows us to use the static concepts and notation introduced there. The crossed-out premises in the rules are crucial properties that are guaranteed to hold for phrases for which the extended static semantics yields a result: this is a consequence of the following theorem.

Theorem 4.2. The following invariants link the extended static semantics and model semantics:

(1) If $\vdash ASP \rhd ((P_{st}, B_{st}), \Sigma)$ and $\vdash ASP \Rightarrow (\mathscr{C}, UEv)$, then there is an extended static context $\mathscr{C}_{st}$ such that $ctx(\mathscr{C}_{st}) = (P_{st}, B_{st})$ and $\mathscr{C} \subseteq ucx(\mathscr{C}_{st})$, $\mathscr{C}$ is closed in $\mathscr{C}_{st}$, and for each $E \in \mathscr{C}$, $E \in dom(UEv)$ and $UEv(E) \in |\mathbf{Mod}(\Sigma)|$. Moreover, for $E_1, E_2 \in \mathscr{C}$, if $E_1 =_{\mathscr{C}_{st}} E_2$, then $UEv(E_1) = UEv(E_2)$.

(2) If $\vdash UDD^+ \rhd \mathscr{C}_{st}$ and $\vdash UDD^+ \Rightarrow \mathscr{C}$, then $\mathscr{C} \subseteq ucx(\mathscr{C}_{st})$ and $\mathscr{C}$ is closed in $\mathscr{C}_{st}$.

(3) If $\mathscr{C}_{st} \vdash UDD \rhd \mathscr{C}'_{st}$ and $\mathscr{C} \vdash UDD \Rightarrow \mathscr{C}'$, where $\mathscr{C} \subseteq ucx(\mathscr{C}_{st})$ and $\mathscr{C}$ is closed in $\mathscr{C}_{st}$, then $\mathscr{C}' \subseteq ucx(\mathscr{C}'_{st})$, $\mathscr{C}' \subseteq \mathscr{C}$, $\mathscr{C}'$ is closed in $\mathscr{C}'_{st}$ and for each unit environment $E \in \mathscr{C}$ and model family $\mathscr{M}$ that witnesses E in $\mathscr{C}_{st}$, there is $E' \in \mathscr{C}'$ such that $E =_{\mathscr{C}_{st}} E'$ and an extension of $\mathscr{M}$ witnesses E' in $\mathscr{C}'_{st}$.

(4) If $\mathscr{C}_{st} \vdash T \rhd (i, D')$ and $\mathscr{C} \vdash T \Rightarrow UEv$ with $\mathscr{C} \subseteq ucx(\mathscr{C}_{st})$, then for each unit environment $E \in \mathscr{C}$ and model family $\mathscr{M}$ that witnesses E in $\mathscr{C}_{st}$, there is an extension of $\mathscr{M}$ to a D'-coherent model family $\mathscr{M}' \in |\mathbf{Mod}(D')|$ such that $\mathscr{M}'_i = UEv(E)$. Moreover, for $E_1, E_2 \in \mathscr{C}$, if $E_1 =_{\mathscr{C}_{st}} E_2$, then $UEv(E_1) = UEv(E_2)$.

Proof.

(4) We use induction on the structure of the unit term. The fact that the value of the unit evaluator on an environment does not change when it does not depend on the values in the environment not mentioned in the static context (for $E_1, E_2 \in \mathscr{C}$, if $E_1 =_{\mathscr{C}_{st}} E_2$, then $UEv(E_1) = UEv(E_2)$) follows easily in each case by using the induction hypothesis.

The case of unit name is trivial, and the case of unit reduct is very easy.

Consider the case of unit application, when the unit term is of the form $U[T \textbf{ fit } \gamma]$. Adjusting the notation slightly to fit the corresponding rules (for unit application) in Figures 1 and 3 (we will rely implicitly below on the notation used in these rules), assume that $\mathscr{C} \subseteq ucx(\mathscr{C}_{st})$, $\mathscr{C}_{st} \vdash U[T \textbf{ fit } \gamma] \rhd (l, D'')$ and $\mathscr{C} \vdash U[T \textbf{ fit } \gamma] \Rightarrow UEv'$, where $UEv'(E) = UEv(E) \oplus E(U)(UEv(E)|_{\gamma})$ for $E \in \mathscr{C}$. Consequently, all the premises of the corresponding rules (for unit application) in Figures 1 and 3 must hold. Let $E \in \mathscr{C}$ and $\mathscr{M}$ be a model family that witnesses E in $\mathscr{C}_{st}$. By the induction hypothesis, there is an extension $\mathscr{M}^T \in |\mathbf{Mod}(D)|$ of $\mathscr{M}$ such that $\mathscr{M}^T_i = UEv(E)$.

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Let $\mathscr{M}'$ extend $\mathscr{M}^T$ by putting $\mathscr{M}'_j = UEv(E)|_\gamma$ and $\mathscr{M}'_k = E(U)(UEv(E)|_\gamma)$ (since $UEv(E)|_\gamma \in dom(E(U))$, the latter is well defined). Then $\mathscr{M}' \in |\mathbf{Mod}(D')|$. Since D' ensures amalgamability for D'', we have $\mathscr{M}'$ uniquely extends to $\mathscr{M}'' \in |\mathbf{Mod}(D'')|$, yielding $\mathscr{M}''_l|_{\iota'} = \mathscr{M}'_i$ and $\mathscr{M}''_l|_{\gamma'} = \mathscr{M}'_k$, that is, $\mathscr{M}''_l = UEv(E) \oplus E(U)(UEv(E)|_\gamma)$, which completes the proof for this case.

For the case of unit amalgamation, when the unit term is of the form T_1 **and** T_2, assume $\mathscr{C} \subseteq ucx(\mathscr{C}_{st})$, $\mathscr{C}_{st} \vdash T_1$ **and** $T_2 \rhd\!\!\rhd (j, D')$ and $\mathscr{C} \vdash T_1$ **and** $T_2 \Rightarrow UEv$, where $\mathscr{C}_{st} = (P_{st}, \mathscr{B}_{st}, D)$. Consequently, all the premises of the corresponding rules (for unit amalgamation) in Figures 1 and 3 must hold; we refer below to the notation used in the rules. Let $E \in \mathscr{C}$ and $\mathscr{M}$ be a model family that witnesses E in $\mathscr{C}_{st}$. By the induction hypothesis, there are extensions $\mathscr{M}^1 \in |\mathbf{Mod}(D_1)|$ and $\mathscr{M}^2 \in |\mathbf{Mod}(D_2)|$ of $\mathscr{M}$ such that $\mathscr{M}^1_{i_1} = UEv_1(E)$ and $\mathscr{M}^2_{i_2} = UEv_2(E)$. Since D_1 and D_2 are disjoint extensions of D, we have $\mathscr{M}^1 \cup \mathscr{M}^2$ is a $(D_1 \cup D_2)$-coherent family of models. Now, since $D_1 \cup D_2$ ensures amalgamability for D', we have $\mathscr{M}^1 \cup \mathscr{M}^2$ extends uniquely to a D'-coherent family $\mathscr{M}' \in |\mathbf{Mod}(D')|$, necessarily with $\mathscr{M}'_j|_{D'(m_1)} = \mathscr{M}^1_{i_1}$ and $\mathscr{M}'_j|_{D'(m_2)} = \mathscr{M}^2_{i_2}$, that is, $\mathscr{M}'_j = UEv(E)$, which completes the proof of item (4).

(3) This follows by inspection of the rules; the cases of unit declarations are easy. The case of unit definitions relies on item (4) as follows. Assume that $\mathscr{C} \subseteq ucx(\mathscr{C}_{st})$ and $\mathscr{C}$ is closed in $\mathscr{C}_{st}$, $\mathscr{C}_{st} = (P_{st}, \mathscr{B}_{st}, D)$. To derive $\mathscr{C}_{st} \vdash UDD \rhd\!\!\rhd \mathscr{C}'_{st}$ and $\mathscr{C} \vdash UDD \Rightarrow \mathscr{C}'$, where UDD is of the form $U = T$, we must have $(P_{st}, \mathscr{B}_{st}, D) \vdash T \rhd\!\!\rhd (i, D')$, $U \notin (dom(P_{st}) \cup dom(\mathscr{B}_{st}))$, and $\mathscr{C} \vdash T \Rightarrow UEv$, with $\mathscr{C}'_{st} = (P_{st}, \mathscr{B}_{st} + \{U \mapsto i\}, D')$ and $\mathscr{C}' = \{E + \{U \mapsto UEv(E)\} \mid E \in \mathscr{C}\}$. Now, for each $E \in \mathscr{C}$ and model family $\mathscr{M} \in |\mathbf{Mod}(D)|$ that witnesses E in $\mathscr{C}_{st}$, by item (4) there exists an extension $\mathscr{M}' \in |\mathbf{Mod}(D')|$ of $\mathscr{M}$ with $\mathscr{M}'_i = UEv(E)$. $\mathscr{M}'$ witnesses $E + \{U \mapsto UEv(E)\}$ in $\mathscr{C}'_{st}$. Consequently, we have $\mathscr{C}' \subseteq ucx(\mathscr{C}'_{st})$. Moreover, since $\mathscr{C}$ is closed in $\mathscr{C}_{st}$ and $U \notin (dom(P_{st}) \cup dom(\mathscr{B}_{st}))$, we have $(E + \{U \mapsto UEv(E)) \in \mathscr{C}$, which shows $\mathscr{C}' \subseteq \mathscr{C}$. Finally, $\mathscr{C}'$ is closed in $\mathscr{C}'_{st}$ since $\mathscr{C}$ is closed in $\mathscr{C}_{st}$.

(2) This follows from item (3) by an obvious induction on the length of the sequence of unit declarations and definitions.

(1) This follows from items (2) and (4) by inspection of the rules. Namely, to derive the assumptions for ASP of the form **arch spec** UDD^+ **result** T, we must have $\vdash UDD^+ \rhd\!\!\rhd \mathscr{C}_{st}$ and $\vdash UDD^+ \Rightarrow \mathscr{C}$, as well as $\mathscr{C}_{st} \vdash T \rhd\!\!\rhd (i, D)$ and $\mathscr{C} \vdash T \Rightarrow UEv$, with $(P_{st}, B_{st}) = ctx(\mathscr{C}_{st})$ and $\Sigma = D(i)$. The thesis now follows directly from items (2) and (4). □

The invariants in Theorem 4.2 ensure that the crossed out premises of the unit amalgamation rule and of the parametrised unit application rule in the literal model semantics follow from the other premises of the rule and the premises of the corresponding rules of the extended static semantics.

5. Observational equivalence for CASL models

Up to this point we have followed the usual interpretation for basic specifications given as sets of axioms over some signature, which is to require models of such a basic specification

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to satisfy all of its axioms. This is what is captured by the notion of literal correctness (Definition 3.2) and the literal model semantics of Figure 3. However, in many practical examples this turns out to be overly restrictive. The point is that only a subset of the sorts in the signature of a specification are typically intended to be directly observable, while the others are treated as internal with properties of their elements made visible only through *observations*: terms producing a result of an observable sort, and predicates. Often there are models that do not satisfy the axioms 'literally', but in which all observations nevertheless deliver the required results. This calls for a relaxation of the interpretation of specifications, as advocated in numerous 'observational' or 'behavioural' approaches, going back at least to Giarratana *et al.* (1976) and Reichel (1981). Two general approaches are possible:

— introduce an 'internal' *observational indistinguishability* relation between elements in the carrier of each model, and re-interpret equality in the axioms as indistinguishability; or

— introduce an 'external' *observational equivalence* relation on models over each signature, and re-interpret specifications by closing their class of models under such equivalence.

It turns out that under some acceptable technical conditions, these two approaches are closely related and coincide for most basic specifications (Bidoit *et al.* 1995; Bidoit and Tarlecki 1996). We follow the second approach here.

From now on we will assume that the set of observable sorts is empty and so predicates are the only observations. Because of this decision, there is no need to parametrise the definitions below by a chosen set of observable sorts. This departs from standard approaches to observational equivalence in the usual algebraic frameworks, where choosing a non-empty set of observable sorts is crucial if we are to have any observations at all. Moreover, it is appropriate for this set to vary in the process of modular development, where some sorts must be locally considered as observable (for example, the parameter sorts in specifications of local constructions). The former is taken care of by assuming that appropriate predicates are introduced into the specifications considered. For instance, to make a generated sort observable, it is enough to introduce the 'equality predicate' on this sort into the specification†. The latter will be achieved in a technically different way here, see Definition 6.9 below and the subsequent comment.

We should also note here that for each CASL signature Σ and sort s in Σ we have $s \leqslant s$, so we also have a predicate $in^{s \leqslant s} : s$, which holds for all its arguments in any CASL model. This means that given a Σ-term t of sort s, we have $in^{s \leqslant s}(t)$ holds if and only if t has a defined value. Consequently, observing predicates in CASL models covers observing definedness of terms.

Given a CASL signature Σ, an *observation* is an atomic predicate formula ϕ of the form $p(t_1, \ldots, t_n)$, where $p : s_1 \times \cdots \times s_n$ is a predicate symbol in $\Sigma^\#$ and for $i = 1, \ldots, n$, we have

† Some free datatype definitions in CASL ensure that the new sort is observable even though no equality predicate is explicitly introduced. This is the case when there is a subsort for each alternative and selectors for each non-constant constructor. This means that enough observations are available to distinguish between any two data values, provided the other argument sorts for the constructors are observable (come with enough observations to distinguish between any data of these sorts).

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t_i is a $\Sigma^\#$-term of sort s_i. The observation $p(t_1,\dots,t_n)$ is *closed* if all the terms t_i, $i=1,\dots,n$, are closed (contain no variables). Given a sort s in Σ, the observation $p(t_1,\dots,t_n)$ is *for sort* s if it contains a unique variable $z{:}s$ of sort s (and no other variables at all). We will then often write $\phi(z)$ to indicate the variable explicitly, and for a $\Sigma^\#$-term t of sort s, we write $\phi(t)$ for the result of substituting t for z in ϕ.

Definition 5.1 (Observational equivalence). Given a CASL signature Σ, two Σ-models $M,N \in |\mathbf{Mod}(\Sigma)|$ are *observationally equivalent*, written $M \equiv N$, if for all closed observations ϕ,

$$M \models \phi \iff N \models \phi\,.$$

It is trivial to see that observational equivalence is indeed an equivalence on CASL models over any signature Σ.

In the following we will work with a technically different but equivalent definition of observational equivalence, where the equivalence of two models is 'witnessed' by a relation between them; this has been worked out in detail (for partial algebras without predicates) in Schoett (1987), *cf.* 'simulations' in Milner (1971) and 'weak homomorphisms' in Ginzburg (1968).

Definition 5.2 (Correspondence). Consider a signature Σ. A *correspondence* between two Σ-models $M,N \in |\mathbf{Mod}(\Sigma)|$, written $\rho{:}M \bowtie N$, is a relation $\rho \subseteq |M| \times |N|$ that

— *is closed under the operations*: for $f{:}s_1 \times \ldots \times s_n \to s$ in $\Sigma^\#$, $a_1 \in |M|_{s_1}$, $\dots$, $a_n \in |M|_{s_n}$ and $b_1 \in |N|_{s_1}$, $\dots$, $b_n \in |N|_{s_n}$, if $(a_1,b_1) \in \rho_{s_1}$, $\dots$, $(a_n,b_n) \in \rho_{s_n}$, then $f_M(a_1,\dots,a_n)$ is defined if and only if $f_M(b_1,\dots,b_n)$ is defined, and if this is the case, then $(f_M(a_1,\dots,a_n), f_N(b_1,\dots,b_n)) \in \rho_s$; and

— *preserves and reflects the predicates*: for $p{:}s_1 \times \ldots \times s_n$ in $\Sigma^\#$, $a_1 \in |M|_{s_1}$, $\dots$, $a_n \in |M|_{s_n}$ and $b_1 \in |N|_{s_1}$, $\dots$, $b_n \in |N|_{s_n}$, if $(a_1,b_1) \in \rho_{s_1}$, $\dots$, $(a_n,b_n) \in \rho_{s_n}$, then $p_M(a_1,\dots,a_n) \iff p_N(b_1,\dots,b_n)$.

In the rest of the paper we will rely on the following equivalence without further mention.

Theorem 5.3. Given a CASL signature Σ, Σ-models $M,N \in |\mathbf{Mod}(\Sigma)|$ are observationally equivalent if and only if there is a correspondence between them.

Proof. Let $M \equiv N$. Define a relation $\rho \subseteq |M| \times |N|$ to contain, for each sort s in Σ, all and only pairs of the form (t_M, t_N), for all closed $\Sigma^\#$-terms t of sort s such that the value of t is defined in both M and N. To check that ρ is a correspondence between M and N, consider for $i=1,\dots,n$, $a_i \in |M|_{s_i}$ and $b_i \in |N|_{s_i}$ such that $(a_i,b_i) \in \rho_{s_i}$, so that $a_i = (t_i)_M$ and $b_i = (t_i)_N$ for some $\Sigma^\#$-term t_i of sort s_i. Now consider $f{:}s_1 \times \ldots \times s_n \to s$ in $\Sigma^\#$. Since $M \equiv N$, $M \models in^{s \leqslant s}(f(t_1,\dots,t_n))$ if and only if $N \models in^{s \leqslant s}(f(t_1,\dots,t_n))$; so, $f_M(a_1,\dots,a_n)$ is defined if and only if $f_M(b_1,\dots,b_n)$ is defined, and if this is the case, then, by definition, $(f_M(a_1,\dots,a_n), f_N(b_1,\dots,b_n)) \in \rho_s$ (since $(f(t_1,\dots,t_n))_M = f_M(a_1,\dots,a_n)$ and $(f(t_1,\dots,t_n))_N = f_N(b_1,\dots,b_n)$). Similarly, for $p{:}s_1 \times \ldots \times s_n$ in $\Sigma^\#$, we have $M \models p(t_1,\dots,t_n)$ if and only if $N \models p(t_1,\dots,t_n)$, which shows the equivalence of $p_M(a_1,\dots,a_n)$ and $p_N(b_1,\dots,b_n)$, and completes the proof of $\rho{:}M \bowtie N$.

Now consider a correspondence $\rho : M \bowtie N$. Using the correspondence properties, by simple induction on the term structure, for any closed $\Sigma^{\#}$-term t, one can prove that t_M is defined if and only if t_N is defined, and if this is the case, $(t_M, t_N) \in \rho$. Now, given any closed observation $p(t_1, \ldots, t_n)$, by symmetry, it is enough to prove that if $M \models p(t_1, \ldots, t_n)$, then $N \models p(t_1, \ldots, t_n)$. Suppose $M \models p(t_1, \ldots, t_n)$. Then for $i = 1, \ldots, n$, we have $(t_i)_M$ is defined, so $(t_i)_N$ is defined and $((t_i)_M, (t_i)_N) \in \rho$. Moreover, $p_M((t_1)_M, \ldots, (t_n)_M)$ holds, so, by the correspondence property, $p_N((t_1)_N, \ldots, (t_n)_N)$ holds as well. Thus $N \models p(t_1, \ldots, t_n)$. □

It is easy to check that isomorphisms (and, in particular, identities) are correspondences and that the class of correspondences is closed under composition.

Correspondences between Casl models may be replaced by spans of strong homomorphisms. Namely, given a span of strong homomorphisms $(h_M : K \to M, h_N : K \to N)$, putting $\rho = h_M^{-1} ; h_N$, that is, $\rho_s = \{(h_M(c), h_N(c)) \mid c \in |K|_s\}$ for each sort s in Σ, yields a correspondence $\rho : M \bowtie N$. In the opposite direction, we have the following proposition.

Proposition 5.4. For any Casl signature Σ, any Σ-models M, N and any correspondence $\rho : M \bowtie N$, there is a Σ-model K and strong Σ-homomorphisms $h_M : K \to M$ and $h_N : K \to N$ such that $\rho = h_M^{-1} ; h_N$.

Proof. To define K, first put $|K|_s = \rho_s \subseteq |M|_s \times |N|_s$ for each sort s in Σ. The operations in K are then defined component-wise using the operations in M and N, respectively. The predicates in K are defined using either the first components and the predicates in M, or (equivalently) the second components and the predicates in N. The correspondence properties of ρ ensure that no problems arise, and that the projection functions $h_M : K \to M$ and $h_N : K \to N$ are strong Σ-homomorphisms. □

This proposition implies directly that the reduct of a correspondence along a signature morphism (defined in the obvious way) is a correspondence. More interestingly, this extends to derived signature morphisms with observable conditions.

Consider a signature Σ. A conditional Σ-term $\langle(\phi_i, t_i)\rangle_{i \geqslant 0}$ is *observationally sensible* if for all $i \geqslant 0$, we have ϕ_i are *observers*, that is, Boolean combinations of observations. A derived signature morphism $\delta : \Sigma' \to \Sigma$ is *observationally sensible* if it maps Σ'-operations to observationally sensible terms.

Lemma 5.5. Let $\delta : \Sigma' \to \Sigma$ be an observationally sensible derived signature morphism, and let $\rho : M \bowtie N$ be a correspondence between Σ-models $M, N \in |\mathbf{Mod}(\Sigma)|$. Then $\rho|_\delta : M|_\delta \bowtie N|_\delta$ is a correspondence also.

It follows that reducts with respect to observationally sensible derived signature morphisms extend to strong homomorphisms.

The view of correspondences as spans of homomorphisms also leads to an easy extension to correspondences of the amalgamation property given in Lemma 2.2 for homomorphisms.

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Lemma 5.6. Suppose that the pushout

$$\begin{array}{ccc} \Sigma_1 & \xrightarrow{\iota'} & \Sigma'_1 \\ \uparrow{\scriptstyle\gamma} & & \uparrow{\scriptstyle\gamma'} \\ \Sigma & \xrightarrow[\iota]{} & \Sigma' \end{array}$$

ensures amalgamability. Then for all correspondences $\rho_1 : M_1 \bowtie N_1$ in $\mathbf{Mod}(\Sigma_1)$ and $\rho' : M' \bowtie N'$ in $\mathbf{Mod}(\Sigma')$ such that $\rho_1|_\gamma = \rho'|_\iota$ there exists a unique correspondence $\rho'_1 : M'_1 \bowtie N'_1$ in $\mathbf{Mod}(\Sigma'_1)$ such that $\rho'_1|_{\iota'} = \rho_1$ and $\rho'_1|_{\gamma'} = \rho'$, where $M'_1 = M_1 \oplus M'$ and $N'_1 = N_1 \oplus N'$.

Proof. A direct proof mimics the proof of Lemma 2.2. □

Note though that this does not ensure that amalgamation preserves observational equivalence.

Counterexample 5.7. Let Σ be a signature with a single sort s, and let Σ_1 extend Σ by a constant $a{:}s$. Since there are no predicates in Σ_1, all Σ_1-models in which the constant a is defined are observationally equivalent. Let Σ' extend Σ by a unary predicate $p{:}s$; since there are no closed observations over Σ', all Σ'-models are observationally equivalent. The pushout signature of the two extensions of Σ is the signature Σ'_1 with sort s, constant $a{:}s$ and predicate $p{:}s$. Clearly, not all Σ'_1-models with defined values of a are observationally equivalent – there is a new closed observation here, namely $p(a)$.

To make the counterexample explicit, let M_1 be a Σ_1-model with a single element, $|M_1|_s = \{x\}$, and $a_{M_1} = x$. Let M' and M'' be Σ'-models such that $M'|_\Sigma = M''|_\Sigma = M_1|_\Sigma$ and $p_{M'}(x)$ holds while $p_{M''}(x)$ does not hold. We still have $M' \equiv M''$ (and trivially $M_1 \equiv M_1$). However, $(M_1 \oplus M') \not\equiv (M_1 \oplus M'')$.

Observational equivalence can also be characterised in terms of internal indistinguishability. Namely, consider a CASL signature Σ and Σ-model $M \in |\mathbf{Mod}(\Sigma)|$. Let $\langle M \rangle$ be the generated submodel of M having all and only the defined values in M of closed $\Sigma^\#$-terms as elements of the carrier. For any sort s in Σ, given $a, a' \in |\langle M \rangle|_s$, we say that *$a$ and a' are observationally indistinguishable in M*, written $a \approx_M a'$, if for all observations ϕ for sort s,

$$M[z \mapsto a] \models \phi \iff N[z \mapsto a'] \models \phi\,.$$

Thus defined, *observational indistinguishability* on M, $\approx_M \subseteq |\langle M \rangle| \times |\langle M \rangle|$, is the largest strong congruence on $\langle M \rangle$. The *observational quotient* of M, written $M/{\approx}$, is the quotient of $\langle M \rangle$ by $\approx_M$.

Theorem 5.8. Consider a CASL signature Σ. Two Σ-models are observationally equivalent if and only if their observational quotients are isomorphic.

Proof. For all CASL models M, since there is a natural strong homomorphism from $\langle M \rangle$ to $M/{\approx}$, which is a correspondence between M and $M/{\approx}$, we have that $M \equiv M/{\approx}$.

Therefore, given two CASL models $M, N \in |\mathbf{Mod}(\Sigma)|$ with isomorphic observational quotients $M/{\approx}$ and $N/{\approx}$, we get $M \equiv N$.

Now suppose that $M \equiv N$. Then for any closed $\Sigma^{\#}$-term t of a sort s, the value t_M of t in M is defined if and only if the value t_N of t in N is defined. Moreover, if this is the case, then for any observation $\phi(z)$ for sort s

$$M[z \mapsto t_M] \models \phi(z) \iff M \models \phi(t) \iff N \models \phi(t) \iff N[z \mapsto t_N] \models \phi(z).$$

It follows that for any closed $\Sigma^{\#}$-terms t and t' of a common sort s, if their values are defined in M (and hence in N as well)

$$t_M \approx_M t'_M \iff t_N \approx_N t'_N.$$

Consequently, a function that for each closed $\Sigma^{\#}$-term t with defined value in M maps the equivalence class of t_M with respect to $\approx_M$ to the equivalence class of t_N with respect to $\approx_N$ is a well-defined, bijective, strong homomorphism, and hence an isomorphism, between $M/{\approx}$ and $N/{\approx}$. □

Corollary 5.9. Consider a CASL signature Σ. Σ-models M and N are observationally equivalent if and only if they have submodels with common strong quotients, that is, there exist submodels M' of M and N' of N and strong congruences $\simeq$ on M' and $\simeq'$ on N' such that the quotients of M' by $\simeq$ and of N' by $\simeq'$ are isomorphic.

6. Observational correctness and stability

The observational concepts introduced in Section 5 above motivate a new interpretation of specifications. For any specification SP with $Sig[SP] = \Sigma$, we define its *observational interpretation* by abstracting from the standard interpretation as follows:

$$[\![SP]\!]_{\equiv} = \{M \in |\mathbf{Mod}(\Sigma)| \mid M \equiv N \text{ for some } N \in [\![SP]\!]\}.$$

Given this, the most obvious way to re-interpret the correctness of local constructions (Definition 3.2) in order to take advantage of the observational interpretation of specifications is to modify the earlier definition by requiring $[\![SP]\!]_{\equiv} \subseteq dom(F)$ and $F([\![SP]\!]_{\equiv}) \subseteq [\![SP']\!]_{\equiv}$. This works, but misses a crucial point: when using a realisation of a specification, we would like to pretend that it satisfies the specification literally, even if when we actually implement it, we are permitted to supply a model that is correct only up to observational equivalence. This leads to a different notion of observational correctness of a local construction, for which we would just require $[\![SP]\!] \subseteq dom(F)$ and $F([\![SP]\!]) \subseteq [\![SP']\!]_{\equiv}$. This relaxation has a price: observationally correct local constructions do not automatically compose! The crucial insight required for resolving this problem came from Schoett (1987), who noticed that well-behaved constructions satisfy the *stability* property described in the following section.

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6.1. *Stability*

Definition 6.1 (Stability). A construction $F\colon |\mathbf{Mod}(\Sigma)| \rightharpoonup |\mathbf{Mod}(\Sigma')|$ is *stable* if it preserves observational equivalence, that is, for any models $M, N \in |\mathbf{Mod}(\Sigma)|$ such that $M \equiv N$, if $M \in dom(F)$, then $N \in dom(F)$ and $F(M) \equiv F(N)$.

The rest of this subsection is devoted to an analysis of conditions that ensure the stability of constructions when they arise through the use of local constructions, as in Section 3. The problem is that we want to restrict attention to conditions that are essentially local to the local constructions involved, rather than conditions that refer to all the possible global contexts in which such a construction can be used.

We will start with the local version of the stability property for local constructions, aiming for the stability of any use of local constructions in an admissible global context.

Definition 6.2 (Local stability). A local construction F along $\iota\colon \Sigma \to \Sigma'$ is *locally stable* if for any Σ-models $M, N \in |\mathbf{Mod}(\Sigma)|$ and correspondence $\rho\colon M \bowtie N$, $M \in dom(F)$ if and only if $N \in dom(F)$ and, moreover, if this is the case, there exists a correspondence $\rho'\colon F(M) \bowtie F(N)$ that extends ρ (that is, $\rho'|_\iota = \rho$).

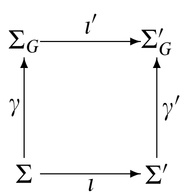
Clearly, local stability implies stability. Trivial identity constructions are locally stable, and composition of locally stable constructions is locally stable as well. Local stability is also preserved under observational equivalence of constructions, which is defined as follows.

Local constructions F_1, F_2 along $\iota\colon \Sigma \to \Sigma'$ are *observationally equivalent*, written $F_1 \equiv F_2$, if $dom(F_1) = dom(F_2)$ and for each $M \in dom(F_1)$ there exists a correspondence $\rho\colon F_1(M) \bowtie F_2(M)$ with reduct $\rho|_\iota$ being the identity on M.

Proposition 6.3. Let F_1 and F_2 be observationally equivalent local constructions along $\iota\colon \Sigma \to \Sigma'$. Then, if F_1 is locally stable, so is F_2.

Proof. Consider models $M, N \in |\mathbf{Mod}(\Sigma)|$ with correspondence $\rho\colon M \bowtie N$. Suppose $M \in dom(F_2)$. Then $M \in dom(F_1)$, and so $N \in dom(F_1) = dom(F_2)$. Since F_1 is locally stable, there is a correspondence $\rho'\colon F_1(M) \bowtie F_1(N)$ with $\rho'|_\iota = \rho$. From $F_1 \equiv F_2$, we get correspondences $\rho_M\colon F_2(M) \bowtie F_1(M)$ and $\rho_N\colon F_1(N) \bowtie F_2(N)$ with the identity reducts $\rho_M|_\iota$ and $\rho_N|_\iota$. This yields a correspondence $(\rho_M;\rho';\rho_N)\colon F_2(M) \bowtie F_2(N)$ with reduct $(\rho_M;\rho';\rho_N)|_\iota = \rho$. □

Most crucially though, local stability (*unlike* stability in general) is preserved under lifting local constructions to a global application context, which is, as usual, given by the following pushout diagram:



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Lemma 6.4. If F is a locally stable construction along $\iota : \Sigma \to \Sigma'$, then for any signature Σ_G and admissible fitting morphism $\gamma : \Sigma \to \Sigma_G$, the induced global construction $F_G : |\mathbf{Mod}(\Sigma_G)| \rightharpoonup |\mathbf{Mod}(\Sigma'_G)|$ along $\iota' : \Sigma_G \to \Sigma'_G$ is locally stable as well.

Proof. Consider a correspondence $\rho_G : \mathscr{G} \bowtie \mathscr{H}$ between models $\mathscr{G}, \mathscr{H} \in |\mathbf{Mod}(\Sigma_G)|$. Its reduct is a correspondence $\rho_G|_\gamma : \mathscr{G}|_\gamma \bowtie \mathscr{H}|_\gamma$, so $\mathscr{G}|_\gamma \in dom(F)$ if and only if $\mathscr{H}|_\gamma \in dom(F)$, and consequently $\mathscr{G} \in dom(F_G)$ if and only if $\mathscr{H} \in dom(F_G)$. Suppose $\mathscr{G}|_\gamma \in dom(F)$. Then there exists a correspondence $\rho' : F(\mathscr{G}|_\gamma) \bowtie F(\mathscr{H}|_\gamma)$ with $\rho'|_\iota = \rho_G|_\gamma$. Amalgamation of ρ_G and ρ' yields a correspondence $\rho'_G : F_G(\mathscr{G}) \bowtie F_G(\mathscr{H})$ such that $\rho'_G|_{\iota'} = \rho_G$, see Lemma 5.6. $\square$

Corollary 6.5. If F is a locally stable construction along $\iota : \Sigma \to \Sigma'$, then for any signature Σ_G and admissible fitting morphism $\gamma : \Sigma \to \Sigma_G$, the induced global construction $F_G : |\mathbf{Mod}(\Sigma_G)| \rightharpoonup |\mathbf{Mod}(\Sigma'_G)|$ along $\iota' : \Sigma_G \to \Sigma'_G$ is stable.

This establishes a sufficient local condition (local stability) that ensures that a local construction induces a stable global construction in every possible context of use. Imposing an additional requirement on the correspondences involved yields an auxiliary notion, which we will use to prove that this is both sufficient and necessary.

Given a CASL signature Σ, a correspondence $\rho : M \bowtie N$ is *closed* if whenever $(a, b) \in \rho$, $(a', b) \in \rho$ and $(a, b') \in \rho$, then $(a', b') \in \rho$. The following proposition is easy.

Proposition 6.6. For any correspondence $\rho : M \bowtie N$ there is a least closed correspondence $\widehat{\rho} : M \bowtie N$ that contains ρ.

Consequently, two Σ-models are behaviourally equivalent if and only if there is a closed correspondence between them.

Theorem 6.7. For any local construction F along $\iota : \Sigma \to \Sigma'$, the following conditions are equivalent:

(1) F is locally stable.
(2) F induces a stable global construction in every possible (also infinitary) context of use, that is, for every admissible fitting morphism $\gamma : \Sigma \to \Sigma_G$, the induced global construction $F_G : |\mathbf{Mod}(\Sigma_G)| \rightharpoonup |\mathbf{Mod}(\Sigma'_G)|$ along $\iota' : \Sigma_G \to \Sigma'_G$ is stable.
(3) F extends closed correspondences, that is, for every closed correspondence $\widehat{\rho} : M \bowtie N$ in $\mathbf{Mod}(\Sigma)$, $M \in dom(F)$ if and only if $N \in dom(F)$, and if this is the case, there exists a closed correspondence $\widehat{\rho'} : F(M) \bowtie F(N)$ in $\mathbf{Mod}(\Sigma')$ that extends $\widehat{\rho}$ (that is, $\widehat{\rho'}|_\iota = \widehat{\rho}$).

Proof.

(1) $\Longrightarrow$ (2) See Corollary 6.5.

(2) $\Longrightarrow$ (3) Consider a closed correspondence $\widehat{\rho} : M \bowtie N$ in $\mathbf{Mod}(\Sigma)$. Construct the extension Σ_G of Σ by adding

— for each sort s in Σ and $(a, b) \in \widehat{\rho}_s$, a (total) constant $!^{a,b,s} : s$,

— for each sort s in Σ and $b \in |N|_s$, a predicate $?^{b,s} : s$, and

— for each sort s in Σ, a predicate $?^s : s$,

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and let $\gamma : \Sigma \to \Sigma_G$ be the signature inclusion. The admissibility of γ is easy to check. Now construct the following expansions M_G and N_G of M and N, respectively:

— for each sort s in Σ and $(a, b) \in \widehat{\rho}_s$, $!^{a,b,s}_{M_G} = a$ and $!^{a,b,s}_{N_G} = b$;

— for each sort s in Σ and $b \in |N|_s$, $?^{b,s}_{M_G}(a)$ holds if and only if $(a, b) \in \widehat{\rho}_s$; $?^{b,s}_{N_G}(b')$ holds if and only if there exists $a \in |M|_s$ such that $(a, b) \in \widehat{\rho}_s$ and $(a, b') \in \widehat{\rho}_s$;

— for each sort s in Σ and $a \in |M|_s$, $?^{s}_{M_G}(a)$ holds, and for each $b \in |N|_s$ $?^{s}_{N_G}(b)$ holds if and only if there exists $a \in |M|_s$ such that $(a, b) \in \widehat{\rho}_s$.

It is easy to check that $\widehat{\rho} : M_G \bowtie N_G$ is a correspondence: closedness of $\widehat{\rho} : M \bowtie N$ is needed to establish that $\widehat{\rho}$ preserves and reflects the $?^{b,s}$ predicates. Moreover, $\widehat{\rho}$ is the only correspondence between M_G and N_G: any such correspondence includes $\widehat{\rho}$ because it must preserve the $!^{a,b,s}$ constants, and it is included in $\widehat{\rho}$ because it must preserve and reflect the $?^{b,s}$ and $?^{s}$ predicates.

Hence, $M_G \in dom(F_G)$ if and only if $N_G \in dom(F_G)$. So we also have $M \in dom(F)$ if and only if $N \in dom(F)$. Moreover, if this is the case, there is a correspondence $\rho_G : F_G(M_G) \bowtie F_G(N_G)$ in $\mathbf{Mod}(\Sigma'_G)$, and the uniqueness of the correspondence $\widehat{\rho} : M_G \bowtie N_G$ in $\mathbf{Mod}(\Sigma_G)$ implies that $\rho_G|_{\iota'} = \widehat{\rho}$. Consider the least closed correspondence $\widehat{\rho_G} : F_G(M_G) \bowtie F_G(N_G)$ that includes ρ_G. Then we also have $\widehat{\rho_G}|_{\iota'} = \widehat{\rho}$, so we obtain $\widehat{\rho_G}|_{\gamma'} : F(M) \bowtie F(N)$ with $(\widehat{\rho_G}|_{\gamma'})|_{\iota} = \widehat{\rho}$.

(3) $\Longrightarrow$ (1) Consider a correspondence $\rho : M \bowtie N$ in $\mathbf{Mod}(\Sigma)$. By Proposition 5.4, we have a Σ-model K and strong Σ-homomorphisms $h_M : K \to M$ and $h_N : K \to N$ such that $\rho = h_M^{-1};h_N$. Since h_M^{-1} and h_N are closed correspondences, by (3), $M \in dom(F)$ if and only if $K \in dom(F)$ if and only if $N \in dom(F)$, and if this is the case, we have correspondences $\rho_M : F(M) \bowtie F(K)$ and $\rho_N : F(K) \bowtie F(N)$ that extend h_M^{-1} and h_N, respectively. Then the correspondence $\rho_M;\rho_N : F(M) \bowtie F(N)$ extends ρ. □

The following is a corollary of Lemma 5.5.

Corollary 6.8. Let $\delta : \Sigma' \to \Sigma$ be an observationally sensible derived signature morphism and $\iota : \Sigma \to \Sigma'$ be a signature morphism such that $\iota;\delta = id_\Sigma$. Then the reduct $\_|_\delta : |\mathbf{Mod}(\Sigma)| \to |\mathbf{Mod}(\Sigma')|$ is a local construction that is locally stable.

The above corollary supports the point put forward in Schoett (1987) that stable constructions are those that respect modularity in the software construction process. That is, such constructions can use the components provided by their imported parameters, but they cannot take advantage of their particular internal properties. This is the point of the requirement that δ should be observationally sensible: any branching in the code must be governed by directly observable properties. This turns (local) stability into a directive for language design, rather than a condition to be checked on a case-by-case basis: in a language with good modularisation facilities, all constructions that one can code should be locally stable.

6.2. *Observational correctness*

We now turn again to the issue of correctness of local constructions with respect to given specifications.

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Definition 6.9 (Observational correctness). We say a local construction F along $\iota\colon Sig[SP] \to Sig[SP']$ is *observationally correct with respect to SP and SP'* if for every model $M \in [\![SP]\!]$, we have $M \in dom(F)$ and there exists a model $M' \in [\![SP']\!]$ and correspondence $\rho'\colon M' \bowtie F(M)$ such that $\rho'|_\iota$ is the identity.

We write $[\![SP \xrightarrow{\iota} SP']\!]_\equiv$ for the class of all locally stable constructions along ι that are observationally correct with respect to SP and SP'.

By imposing the restriction in this definition that ρ' is the identity on the carriers of the parameter sorts, we have in fact 'locally' introduced a set of sorts that act as directly observable for the purposes of verification of the local construction considered.

It follows that if $F \in [\![SP \xrightarrow{\iota} SP']\!]_\equiv$, there is some $F' \in [\![SP \xrightarrow{\iota} SP']\!]$ such that $dom(F') = dom(F)$, and for each $M \in [\![SP]\!]$, there is a correspondence $\rho\colon F'(M) \bowtie F(M)$ that is the identity on sorts of the form $\iota(s)$ for s in Σ. However, in general, $[\![SP \xrightarrow{\iota} SP']\!] \not\subseteq [\![SP \xrightarrow{\iota} SP']\!]_\equiv$, as literally correct local constructions need not be stable. Moreover, it may happen that there are no stable observationally correct constructions, even if there are literally correct ones: that is, we may have $[\![SP \xrightarrow{\iota} SP']\!]_\equiv = \emptyset$ even if $[\![SP \xrightarrow{\iota} SP']\!] \neq \emptyset$. This was, perhaps, first pointed out in Bernot (1987), though in a different framework.

Counterexample 6.10. Let SP_1 have a sort s with two constants $a, b\colon s$, and let SP_2 enrich SP_1 by a new sort o with predicate $p : o \times o$, two (total) constants $c, d\colon o$ and axiom $p(c, d) \iff a = b$. Then $[\![SP_1 \to SP_2]\!]$ is non-empty, with any construction in it mapping models satisfying $a = b$ to those that satisfy $p(c, d)$, and models satisfying $a \neq b$ to those that do not satisfy $p(c, d)$. But none of these constructions is stable!

To see this, consider any construction $F \in [\![SP_1 \xrightarrow{\iota} SP_2]\!]$, 'singleton' model $M \in [\![SP_1]\!]$ (where $a_M = b_M$) and two-element model $N \in [\![SP_2]\!]$ with $a_N \neq b_N$. Clearly, $M \equiv N$. However, there is no correspondence between $F(M)$ and $F(N)$: it would have to link $c_{F(M)}$ with $c_{F(N)}$ and $d_{F(M)}$ with $d_{F(N)}$, which is impossible since $F(M) \models p(c, d)$ while $F(N) \not\models p(c, d)$.

The crucial issue here is how specifications of local constructions can be used when the local constructions are lifted to an admissible global context, which is captured by the following pushout diagram:

$$\begin{array}{ccc} \Sigma_G & \xrightarrow{\iota'} & \Sigma'_G \\ \uparrow{\scriptstyle \gamma} & & \uparrow{\scriptstyle \gamma'} \\ Sig[SP] & \xrightarrow[\iota]{} & Sig[SP'] \end{array}$$

Lemma 6.11. Consider a local construction F along $\iota\colon Sig[SP] \to Sig[SP']$ that is observationally correct with respect to SP and SP', $F \in [\![SP \xrightarrow{\iota} SP']\!]_\equiv$. Then, for every global signature Σ_G and admissible fitting morphism $\gamma\colon Sig[SP] \to \Sigma_G$, and every $\mathscr{G} \in [\![SP \textbf{ with } \gamma]\!]$, we have $\mathscr{G} \in dom(F_G)$, and there is some $\mathscr{G}' \in [\![SP' \textbf{ with } \gamma']\!]$ such that $\mathscr{G}'|_{\iota'} = \mathscr{G}$ and $\mathscr{G}' \equiv F_G(\mathscr{G})$.

Proof. We have $\mathscr{G}|_\gamma \in [\![SP]\!]$, so $\mathscr{G}|_\gamma \in dom(F)$ and there is $M' \in [\![SP']\!]$ and a correspondence $\rho'\colon M' \bowtie F(\mathscr{G}|_\gamma)$ with identity reduct $\rho'|_\iota$. Consider the Σ'_G-model $\mathscr{G}' = \mathscr{G} \oplus M'$.

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Then the identity $id_{\mathscr{G}}: \mathscr{G} \bowtie \mathscr{G}$ and $\rho': M' \bowtie F(\mathscr{G}|_{\gamma})$ amalgamate to a correspondence $\rho'_G: \mathscr{G}' \bowtie F_G(\mathscr{G})$, which proves $F_G(\mathscr{G}) \equiv \mathscr{G}' \in [\![SP' \textbf{ with } \gamma']\!]$. □

If $F \in [\![SP \xrightarrow{\iota} SP']\!]_{\equiv}$ and $\gamma: Sig[SP] \to \Sigma_G$ is admissible, then, by Lemma 6.11, we obtain $[\![SP \textbf{ with } \gamma]\!] \subseteq dom(F_G)$ and $F_G([\![SP \textbf{ with } \gamma]\!]) \subseteq [\![SP' \textbf{ with } \gamma']\!]_{\equiv}$, and by Corollary 6.5, F_G is stable. Given two 'global' specifications SP_G with $Sig[SP_G] = \Sigma_G$ and SP'_G with $Sig[SP'_G] = \Sigma'_G$, we have $F_G \in [\![SP_G \xrightarrow{\iota'} SP'_G]\!]_{\equiv}$ whenever $[\![SP_G]\!] \subseteq [\![SP \textbf{ with } \gamma]\!]_{\equiv}$ and $[\![SP' \textbf{ with } \gamma']\!] \subseteq [\![SP'_G]\!]_{\equiv}$. But while the former requirement is quite acceptable, the latter is in fact impossible to achieve in practice since it implicitly requires that all the global requirements must follow (up to observational equivalence) from the result specification for the local construction, independent of the argument. More practical requirements are obtained by generalising Theorem 3.4 to the observational setting as follows.

Theorem 6.12. Assuming a local construction $F \in [\![SP \xrightarrow{\iota} SP']\!]_{\equiv}$, a specification SP_G with admissible fitting morphism $\gamma: Sig[SP] \to Sig[SP_G]$, and a specification SP'_G with $Sig[SP'_G] = \Sigma'_G$, if

(i) $[\![SP_G]\!] \subseteq [\![SP_G \textbf{ and } (SP \textbf{ with } \gamma)]\!]_{\equiv}$ and
(ii) $[\![(SP' \textbf{ with } \gamma') \textbf{ and } (SP_G \textbf{ with } \iota')]\!] \subseteq [\![SP'_G]\!]_{\equiv}$,

then for every $\mathscr{G} \in [\![SP_G]\!]$, we have $\mathscr{G} \in dom(F_G)$ and $F_G(\mathscr{G}) \in [\![SP'_G]\!]_{\equiv}$, hence $F_G \in [\![SP_G \xrightarrow{\iota'} SP'_G]\!]_{\equiv}$.

Proof. Let $\mathscr{G} \in [\![SP_G]\!]$. Then $\mathscr{G} \equiv \mathscr{H}$ for some $\mathscr{H} \in [\![SP_G]\!] \cap [\![SP \textbf{ with } \gamma]\!]$ by (i). By Lemma 6.11, $F_G(\mathscr{H}) \equiv \mathscr{H}'$ for some $\mathscr{H}' \in [\![SP' \textbf{ with } \gamma']\!]$ with $\mathscr{H}'|_{\iota'} = \mathscr{H} \in [\![SP_G]\!]$. Hence $\mathscr{H}' \in [\![SP'_G]\!]_{\equiv}$ by (ii). By stability of F_G (Corollary 6.5), $\mathscr{G} \in dom(F_G)$ and $F_G(\mathscr{G}) \equiv F_G(\mathscr{H}) \equiv \mathscr{H}'$, so $F_G(\mathscr{G}) \in [\![SP'_G]\!]_{\equiv}$. This completes the proof, since F_G is locally stable by Lemma 6.4. □

Requirement (i) is perhaps the only surprising assumption in this theorem. Note though that it follows straightforwardly from the inclusion of literal model classes $[\![SP_G]\!] \subseteq [\![SP \textbf{ with } \gamma]\!]$ (or, equivalently, $[\![SP_G]\!]|_{\gamma} \subseteq [\![SP]\!]$), which is often easiest to verify. However, condition (i) is strictly stronger in general than the perhaps more expected $[\![SP_G]\!] \subseteq [\![SP \textbf{ with } \gamma]\!]_{\equiv}$. This weaker condition turns out to be sufficient (and is in fact equivalent to (i)) if we also assume that the two specifications involved are *behaviourally consistent* (Bidoit *et al.* 1995), that is, closed under observational quotients. When this is not the case, the use of this weaker condition would have to be paid for by a stronger version of (ii):

$$[\![SP' \textbf{ with } \gamma']\!]_{\equiv} \cap [\![SP_G \textbf{ with } \iota']\!] \subseteq [\![SP'_G]\!]_{\equiv},$$

which seems even less convenient to use than (i). Overall, we need a way to pass information on the global context from SP_G to SP'_G independently from the observational interpretation of the local construction and its correctness, and this must result in some inconvenience of verification on either the parameter or the result side.

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$$\overline{\mathcal{C} \vdash U{:}\, SP \stackrel{\equiv}{\Longrightarrow} \mathcal{C} \times \{U \mapsto [\![SP]\!]_{\equiv}\}}$$

$$\overline{\mathcal{C} \vdash U{:}\, SP_1 \stackrel{\iota}{\longrightarrow} SP_2 \stackrel{\equiv}{\Longrightarrow} \mathcal{C} \times \{U \mapsto [\![SP_1 \stackrel{\iota}{\longrightarrow} SP_2]\!]_{\equiv}\}}$$

Fig. 4. Observational model semantics – the modified rules.

7. Observational interpretation of architectural specifications

In this section we discuss an observational interpretation of the architectural specifications introduced in Section 4. The extended static semantics remains unchanged – observational interpretation of specifications does not affect their static properties. We provide, however, a new *observational model semantics*, with judgments written as $\_ \vdash \_ \stackrel{\equiv}{\Longrightarrow} \_$.

To begin with, the effect of unit declarations has to be modified, taking into account observational interpretation of the specifications involved, as discussed in Sections 5 and 6. The new rules follow in Figure 4. No other modifications are necessary: all the remaining rules are the same for the observational and literal model semantics. This should not be surprising: the interpretation of the constructs on unit terms remains the same, all we change is the interpretation of unit specifications. Moreover, the observational model semantics can be linked to the extended static semantics in exactly the same way as in the case of the literal model semantics: the invariants stated in Theorem 4.2 carry over without change. We will not repeat here either the unmodified rules, or Theorem 4.2 for the observational model semantics.

The fact that nearly all the rules remain the same does not mean that the two semantics quite coincide: at the point in the model semantics where verification is performed, the resulting verification conditions for literal and observational model semantics differ. Namely, in the rule for parametrised unit application, the premise

$$\text{for each } E \in \mathscr{C}, UEv(E)|_{\gamma} \in dom(E(U))$$

checks whether what we can conclude about the argument ensures that it is indeed in the domain of the parametrised unit. Suppose the corresponding unit declaration was $U : SP_1 \stackrel{\iota}{\longrightarrow} SP_2$. Then in the literal model semantics this requirement reduces to

$$\text{for each } E \in \mathscr{C}, UEv(E)|_{\gamma} \in [\![SP_1]\!]\,.$$

Now, in the observational model semantics, this is replaced by a more permissive condition (since the parametrised units considered are locally stable, their domains are closed under observational equivalence):

$$\text{for each } E \in \mathscr{C}, UEv(E)|_{\gamma} \in [\![SP_1]\!]_{\equiv}\,.$$

Of course, the situation is complicated by the fact that the contexts $\mathscr{C}$ from which environments are taken are different in the two semantics. In the simplest case, where the argument T is given as a unit name previously declared with a specification SP, for the literal model semantics the above verification condition amounts to $[\![SP]\!] \subseteq [\![SP_1]\!]$, while for the observational model semantics we get, as expected, $[\![SP]\!] \subseteq [\![SP_1]\!]_{\equiv}$ (which is equivalent to $[\![SP]\!]_{\equiv} \subseteq [\![SP_1]\!]_{\equiv}$).

This relaxation of verification conditions is not of merely theoretical interest: it is not difficult to find statically correct architectural specifications ASP (that is, $\vdash ASP \rhd\!\!\!\rhd (\mathscr{C}_{st}, \Sigma)$ for some extended static context $\mathscr{C}_{st}$ and signature Σ) that are observationally correct (that is, $\vdash ASP \stackrel{\equiv}{\Longrightarrow} (\mathscr{C}_\flat, UEv_\flat)$ for some unit context $\mathscr{C}_\flat$ and evaluator $UEv_\flat$) but *are not* literally correct (that is, for *no* unit context $\mathscr{C}$ and evaluator UEv can we derive $\vdash ASP \Rightarrow (\mathscr{C}, UEv)$). For instance, along the lines of the discussion above, one may take

arch spec ASP =
  units $U : SP_1 \stackrel{\iota}{\longrightarrow} SP_2$;
    $T : SP$
  result $U[T]$

where $Sig[SP] = Sig[SP_1]$, $[\![SP]\!] \subseteq [\![SP_1]\!]_\equiv$ but $[\![SP]\!] \not\subseteq [\![SP_1]\!]$.

A complete study of verification conditions for architectural specifications is beyond the scope of this paper; see Hoffman (2001) and Mossakowski *et al.* (2004) for work in this direction, which still has to be combined with the observational interpretation as given by the semantics here and presented in the simpler setting of Section 6. In the rest of this paper we will concentrate on some aspects of the relationship between the literal and observational model semantics and on the stability of the unit constructions introduced in Section 4.

Our first aim is to show that constructions that can be defined by architectural specifications are (locally) stable. In order to state this precisely, we need some more notation and terminology, as constructions are captured here by unit evaluators operating on environments rather than on individual units.

For any extended static context $\mathscr{C}_{st} = (P_{st}, \mathscr{B}_{st}, D)$, environments $E_1, E_2 \in ucx(\mathscr{C}_{st})$ are *observationally equivalent in* $\mathscr{C}_{st}$, written $E_1 \equiv_{\mathscr{C}_{st}} E_2$, if for each unit name U in $\mathscr{C}_{st}$, $E_1(U) \equiv E_2(U)$. A unit environment $E \in ucx(\mathscr{C}_{st})$ is *stable in* $\mathscr{C}_{st}$ if for each parametrised unit name U in $\mathscr{C}_{st}$, we have that $E(U)$ is locally stable. By Proposition 6.3, the class of environments that are stable in $\mathscr{C}_{st}$ is closed under observational equivalence in $\mathscr{C}_{st}$. We write $ucx_\flat(\mathscr{C}_{st})$ for the class of all unit environments that fit $\mathscr{C}_{st}$ and are stable in $\mathscr{C}_{st}$.

A *D-coherent correspondence* between the two D-coherent model families $\mathscr{M}^1, \mathscr{M}^2 \in |\mathbf{Mod}(D)|$, written $\rho : \mathscr{M}^1 \bowtie \mathscr{M}^2$, is a family of correspondences $\rho_i : \mathscr{M}^1_i \bowtie \mathscr{M}^2_i$ for $i \in |D|$ such that $\rho_i = \rho_j|_{D(m)}$ for each $m : i \to j$ in D.

Two unit environments $E_1, E_2 \in ucx_\flat(\mathscr{C}_{st})$ are *coherently equivalent in* $\mathscr{C}_{st}$, written $E_1 \bowtie_{\mathscr{C}_{st}} E_2$, if for all parametrised unit names U in $\mathscr{C}_{st}$, we have $E_1(U) \equiv E_2(U)$, and there are D-coherent families of models $\mathscr{M}_1$ and $\mathscr{M}_2$ with a D-coherent correspondence $\rho : \mathscr{M}_1 \bowtie \mathscr{M}_2$ such that $\mathscr{M}_1$ and $\mathscr{M}_2$ witness E_1 and E_2, respectively, in $\mathscr{C}_{st}$.

Then, given a unit context $\mathscr{C} \subseteq ucx(\mathscr{C}_{st})$, we write $Cl^{\mathscr{C}_{st}}_{\equiv}(\mathscr{C})$ for the class of all unit environments that in $\mathscr{C}_{st}$ are stable and coherently equivalent to a unit environment in $\mathscr{C}$. It is then clear that $Cl^{\mathscr{C}_{st}}_{\equiv}(\mathscr{C}) \subseteq ucx_\flat(\mathscr{C}_{st})$.

Returning to the stability of the constructions defined by architectural specifications, we want to show that if $\vdash ASP \rhd\!\!\!\rhd (\mathscr{C}_{st}, \Sigma)$ and $\vdash ASP \stackrel{\equiv}{\Longrightarrow} (\mathscr{C}_\flat, UEv_\flat)$, then the unit evaluator $UEv_\flat$ is stable, that is, it maps observationally equivalent environments to observationally equivalent models. Unfortunately, this cannot be proved by a simple induction on the structure of the unit terms involved, relying on the fact that (locally) stable

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constructions are closed under composition. The trouble is with amalgamation, since, in general, amalgamation is not stable: informally, joining the signatures of two models may introduce new observations for either or both of them – see Counterexample 5.7.

However, the key point here is that amalgamation in unit terms in architectural specifications is not used as a construction on its own, but it just identifies a new part of the global context that has been constructed earlier. Since the constructions used to build genuinely new components of the global context are locally stable, such use of amalgamation can do no harm.

The following lemma captures the essential stability property of the unit evaluators built for unit terms by the observational model semantics.

Lemma 7.1. Assume $\mathscr{C}_{st} \vdash T \rhd\!\!\rhd (i, D')$ and $\mathscr{C}_\flat \vdash T \stackrel{\equiv}{\Longrightarrow} UEv_\flat$ with $\mathscr{C}_\flat \subseteq ucx_\flat(\mathscr{C}_{st})$, where $\mathscr{C}_{st} = (P_{st}, \mathscr{B}_{st}, D)$. The unit evaluator $UEv_\flat$ is locally stable in the following sense.

Consider any $E_1, E_2 \in \mathscr{C}_\flat$ such that $E_1 \bowtie_{\mathscr{C}_{st}} E_2$, and $\mathscr{M}_1, \mathscr{M}_2 \in |\mathbf{Mod}(D)|$ that witness E_1 and E_2, respectively, in $\mathscr{C}_{st}$. Any D-coherent correspondence $\rho : \mathscr{M}^1 \bowtie \mathscr{M}^2$ can be extended to a D'-coherent correspondence $\rho' : \mathscr{M}'_1 \bowtie \mathscr{M}'_2$ between model families $\mathscr{M}'_1, \mathscr{M}'_2 \in |\mathbf{Mod}(D')|$ that extend $\mathscr{M}_1$ and $\mathscr{M}_2$, respectively, and satisfy $(\mathscr{M}'_1)_i = UEv_\flat(E_1)$ and $(\mathscr{M}'_2)_i = UEv_\flat(E_2)$.

Proof. We use induction on the structure of the unit term. The cases when the term is a unit name or a unit reduction are trivial.

Consider the case of parametrised unit application. Using the notation as in the corresponding rules of the extended static semantics and of the (observational) model semantics in Figures 1 and 3, respectively, consider $E_1, E_2 \in \mathscr{C}_\flat$ such that $E_1 \bowtie_{\mathscr{C}_{st}} E_2$ and a coherent correspondence $\rho : \mathscr{M}_1 \bowtie \mathscr{M}_2$ between model families $\mathscr{M}^1$, $\mathscr{M}^2$ that witness E_1 and E_2, respectively, in $\mathscr{C}_{st}$. By the induction hypothesis, ρ can be extended to a D-coherent correspondence $\rho^T : \mathscr{M}_1^T \bowtie \mathscr{M}_2^T$, where $\mathscr{M}_1^T$ extends $\mathscr{M}_1$, $\mathscr{M}_2^T$ extends $\mathscr{M}_2$, $(\mathscr{M}_1^T)_i = UEv(E_1)$ and $(\mathscr{M}_2^T)_i = UEv(E_2)$. Then, ρ^T extends to a D'-coherent correspondence $\rho' : \mathscr{M}'_1 \bowtie \mathscr{M}'_2$, where $(\mathscr{M}'_1)_j = UEv(E_1)|_\gamma$, $(\mathscr{M}'_1)_k = E_1(U)(UEv(E_1)|_\gamma)$, and similarly for $\mathscr{M}'_2$ (by local stability of either $E_1(U)$ or $E_2(U)$, and the fact that $E_1(U) \equiv E_2(U)$). Now, we can extend $\mathscr{M}'_1$ and $\mathscr{M}'_2$ to D''-coherent model families $\mathscr{M}''_1$ and $\mathscr{M}''_2$, respectively, by putting $(\mathscr{M}''_1)_l = UEv(E_1)|_\gamma \oplus E_1(U)(UEv(E_1)|_\gamma)$, and similarly for $\mathscr{M}''_2$. Moreover, as in Lemma 5.6, following the proof of Lemma 2.2, we can extend ρ' to a coherent correspondence $\rho'' : \mathscr{M}''_1 \bowtie \mathscr{M}''_2$.

Finally, consider the case of unit amalgamation. Again using the notation as in the corresponding rules of the extended static semantics and of the (observational) model semantics in Figures 1 and 3, respectively, consider $E_1, E_2 \in \mathscr{C}_\flat$ such that $E_1 \bowtie_{\mathscr{C}_{st}} E_2$ and a coherent correspondence $\rho : \mathscr{M}_1 \bowtie \mathscr{M}_2$ between model families $\mathscr{M}_1$, $\mathscr{M}_2$ that witness E_1 and E_2, respectively, in $\mathscr{C}_{st}$. By the induction hypothesis, ρ can be extended to a D_1-coherent correspondence $\rho^{T_1} : \mathscr{M}_1^{T_1} \bowtie \mathscr{M}_2^{T_1}$, where $(\mathscr{M}_1^{T_1})$ extends $\mathscr{M}_1$, $(\mathscr{M}_2^{T_1})$ extends $\mathscr{M}_2$, $(\mathscr{M}_1^{T_1})_i = UEv_1(E_1)$ and $(\mathscr{M}_2^{T_1})_i = UEv_1(E_2)$. Similarly, ρ can be extended to a D_2-coherent correspondence $\rho^{T_2} : \mathscr{M}_1^{T_2} \bowtie \mathscr{M}_2^{T_2}$, where $(\mathscr{M}_1^{T_2})$ extends $\mathscr{M}_1$, $(\mathscr{M}_2^{T_2})$ extends $\mathscr{M}_2$, $(\mathscr{M}_1^{T_2})_i = UEv_2(E_1)$ and $(\mathscr{M}_2^{T_2})_i = UEv_2(E_2)$. Now, since D_1 and D_2 are disjoint extensions of D, ρ^{T_1} and ρ^{T_2} can be put together to form a $(D_1 \cup D_2)$-coherent correspondence between $\mathscr{M}_1^{T_1} \cup \mathscr{M}_1^{T_2}$ and $\mathscr{M}_2^{T_1} \cup \mathscr{M}_2^{T_2}$, respectively. To complete the proof, we proceed

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as in the previous case, following Lemma 5.6 generalised as indicated in the footnote on page 338; this is possible since the union of CASL signatures is built by taking the union of their respective sets of sort, operation and predicate names and forming the transitive closure of the union of the subsort preorders. Consequently, no new sorts, operations or predicates are added in the resulting model, since everything there was constructed 'earlier' while evaluating T_1 and T_2. □

We can strengthen the invariant concerning the semantics of unit declarations and definitions by adding the following property.

Corollary 7.2. Let $\mathscr{C}_{st} \vdash UDD \rhd \mathscr{C}'_{st}$ and $\mathscr{C}_\flat \vdash UDD \stackrel{\equiv}{\Longrightarrow} \mathscr{C}'_\flat$ with $\mathscr{C}_\flat \subseteq ucx_\flat(\mathscr{C}_{st})$. Then $\mathscr{C}'_\flat \subseteq ucx_\flat(\mathscr{C}'_{st})$, $\mathscr{C}'_\flat \subseteq \mathscr{C}_\flat$, and for any unit environments $E'_1, E'_2 \in \mathscr{C}'_\flat$ such that $E'_1 \equiv_{\mathscr{C}'_{st}} E'_2$, whenever $E'_1 \bowtie_{\mathscr{C}_{st}} E'_2$, we also have $E'_1 \bowtie_{\mathscr{C}'_{st}} E'_2$.

Proof. The statement follows by easy inspection of the rules, using Lemma 7.1 for the case of unit definitions. □

Corollary 7.3. Let $\vdash UDD^+ \rhd \mathscr{C}_{st}$ and $\vdash UDD^+ \stackrel{\equiv}{\Longrightarrow} \mathscr{C}_\flat$. Then $\mathscr{C}_\flat \subseteq ucx_\flat(\mathscr{C}_{st})$ and for any unit environments $E_1, E_2 \in \mathscr{C}_\flat$, if $E_1 \equiv_{\mathscr{C}_{st}} E_2$, then $E_1 \bowtie_{\mathscr{C}_{st}} E_2$.

Proof. For the empty extended static context $\mathscr{C}^{\varnothing}_{st}$, any environment in $\mathscr{C}^{\varnothing}$ is witnessed by the empty family of models, so any two such environments are coherently equivalent in $\mathscr{C}^{\varnothing}_{st}$. Therefore, by Corollary 7.2 and an easy induction on the length of the sequence of unit declarations and definitions, for any $E_1, E_2 \in \mathscr{C}_\flat$ such that $E_1 \equiv_{\mathscr{C}_{st}} E_2$ as in the premise of the corollary, we have $E_1 \bowtie_{\mathscr{C}_{st}} E_2$. □

Corollary 7.4. If $\vdash ASP \rhd (\mathscr{C}_{st}, \Sigma)$ and $\vdash ASP \stackrel{\equiv}{\Longrightarrow} (\mathscr{C}_\flat, UEv_\flat)$, then $\mathscr{C}_\flat \subseteq ucx_\flat(\mathscr{C}_{st})$, and for any unit environments $E_1, E_2 \in \mathscr{C}_\flat$ such that $E_1 \equiv_{\mathscr{C}_{st}} E_2$, we have $UEv_\flat(E_1) \equiv UEv_\flat(E_2)$.

Proof. By Corollary 7.3, we have that, for any $E_1, E_2 \in \mathscr{C}_\flat$ such that $E_1 \equiv_{\mathscr{C}_{st}} E_2$ as in the premise here, $E_1 \bowtie_{\mathscr{C}_{st}} E_2$. The conclusion then follows by the stability property in Lemma 7.1. □

As already mentioned, the observational semantics is more permissive than the literal model semantics: the existence of a successful derivation of an observational meaning for an architectural specification does not in general imply that its literal model semantics is defined. Moreover, the observational semantics may 'lose' some results permitted by the literal model semantics – see Counterexample 6.10. However, if an architectural specification has a literal model semantics, its observational semantics is defined as well, and up to observational equivalence, nothing new is added. The following theorem captures the essential links between literal model semantics and observational model semantics.

Theorem 7.5. The following relationships between the literal and observational model semantics hold:

(1) If $\vdash ASP \rhd ((P_{st}, B_{st}), \Sigma)$ and $\vdash ASP \Rightarrow (\mathscr{C}, UEv)$, then $\vdash ASP \stackrel{\equiv}{\Longrightarrow} (\mathscr{C}_\flat, UEv_\flat)$ with $\mathscr{C}_\flat \subseteq Cl^{\mathscr{C}_{st}}_{\equiv}(\mathscr{C})$, and for each unit environment $E \in \mathscr{C}$ that is stable in $\mathscr{C}_{st}$, $E \in \mathscr{C}_\flat$ and $UEv_\flat(E) = UEv(E)$.

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(2) If $\vdash UDD^+ \rhd\!\!\!\rhd \mathscr{C}_{st}$ and $\vdash UDD^+ \Rightarrow \mathscr{C}$, then $\vdash UDD^+ \stackrel{\equiv}{\Longrightarrow} \mathscr{C}_\flat$, where $\mathscr{C}_\flat \subseteq Cl_{\equiv}^{\mathscr{C}_{st}}(\mathscr{C})$ and $\mathscr{C}_\flat$ contains all unit environments $E \in \mathscr{C}$ that are stable in $\mathscr{C}_{st}$.

(3) If $\mathscr{C}_{st} \vdash UDD \rhd\!\!\!\rhd \mathscr{C}'_{st}$ and $\mathscr{C} \vdash UDD \Rightarrow \mathscr{C}'$, where $\mathscr{C} \subseteq ucx(\mathscr{C}_{st})$, then for any $\mathscr{C}_\flat \subseteq Cl_{\equiv}^{\mathscr{C}_{st}}(\mathscr{C})$ that contains all unit environments $E \in \mathscr{C}$ that are stable in $\mathscr{C}_{st}$, $\mathscr{C}_\flat \vdash UDD \stackrel{\equiv}{\Longrightarrow} \mathscr{C}'_\flat$, where $\mathscr{C}'_\flat \subseteq Cl_{\equiv}^{\mathscr{C}'_{st}}(\mathscr{C}')$ and $\mathscr{C}'_\flat$ contains all unit environments $E' \in \mathscr{C}'$ that are stable in $\mathscr{C}'_{st}$.

(4) If $\mathscr{C}_{st} \vdash T \rhd\!\!\!\rhd (i, D')$ and $\mathscr{C} \vdash T \Rightarrow UEv$ with $\mathscr{C} \subseteq ucx(\mathscr{C}_{st})$, then for any $\mathscr{C}_\flat \subseteq Cl_{\equiv}^{\mathscr{C}_{st}}(\mathscr{C})$ that contains all unit environments $E \in \mathscr{C}$ that are stable in $\mathscr{C}_{st}$, $\mathscr{C}_\flat \vdash T \stackrel{\equiv}{\Longrightarrow} UEv_\flat$ and for $E \in \mathscr{C} \cap \mathscr{C}_\flat$, we have $UEv_\flat(E) = UEv(E)$.

Proof.

(4) We use induction on the structure of the unit term. As usual, the cases when the term is a unit name or a unit reduction are easy.
Consider the case of unit application, when the unit term is of the form $U[T \textbf{ fit } \gamma]$. Assume then that $\mathscr{C} \subseteq ucx(\mathscr{C}_{st})$, $\mathscr{C}_{st} \vdash U[T \textbf{ fit } \gamma] \rhd\!\!\!\rhd (l, D'')$ and $\mathscr{C} \vdash U[T \textbf{ fit } \gamma] \Rightarrow UEv'$, with $UEv'(E) = UEv(E) \oplus E(U)(UEv(E)|_\gamma)$ for $E \in \mathscr{C}$. Consequently, all the premises of the corresponding rules (for unit application) in Figures 1 and 3 must hold; we refer below to the notation used in the rules. Now take any $\mathscr{C}_\flat \subseteq Cl_{\equiv}^{\mathscr{C}_{st}}(\mathscr{C})$ that contains all unit environments $E \in \mathscr{C}$ that are stable in $\mathscr{C}_{st}$. By the induction hypothesis, $\mathscr{C}_\flat \vdash T \stackrel{\equiv}{\Longrightarrow} UEv_\flat$, and for $E \in \mathscr{C} \cap \mathscr{C}_\flat$, $UEv_\flat(E) = UEv(E)$. Now consider any $E_\flat \in \mathscr{C}_\flat \subseteq Cl_{\equiv}^{\mathscr{C}_{st}}(\mathscr{C})$, with some $E \in \mathscr{C}$ such that $E_\flat \bowtie_{\mathscr{C}_{st}} E$. Then $E \in \mathscr{C} \cap \mathscr{C}_\flat$. We have $E_\flat(U) \equiv E(U)$, $UEv_\flat(E) = UEv(E)$, and since by Lemma 7.1 $UEv(E) \equiv UEv(E_\flat)$ and observational equivalence is preserved by reducts, from $UEv(E)|_\gamma \in dom(E(U))$, we obtain $UEv_\flat(E_\flat)|_\gamma \in dom(E_\flat(U))$. Thus, we can derive $\mathscr{C}_\flat \vdash U[T \textbf{ fit } \gamma] \stackrel{\equiv}{\Longrightarrow} UEv'_\flat$, where for $E_\flat \in \mathscr{C}_\flat$, $UEv'_\flat(E_\flat) = UEv_\flat(E_\flat) \oplus E_\flat(U)(UEv_\flat(E_\flat)|_\gamma)$. Now, for $E \in \mathscr{C} \cap \mathscr{C}_\flat$, since $UEv_\flat(E) = UEv(E)$, it follows that $UEv'_\flat(E) = UEv'(E)$, which completes the proof for this case.
The proof for the case of unit amalgamation, when the unit term is of the form T_1 **and** T_2, proceeds along similar lines: assume $\mathscr{C} \subseteq ucx(\mathscr{C}_{st})$, $\mathscr{C}_{st} \vdash T_1 \textbf{ and } T_2 \rhd\!\!\!\rhd (j, D')$ and $\mathscr{C} \vdash T_1 \textbf{ and } T_2 \Rightarrow UEv$. Consequently, all the premises of the corresponding rules (for unit amalgamation) in Figures 1 and 3 must hold; we refer below to the notation used in the rules. Now take any $\mathscr{C}_\flat \subseteq Cl_{\equiv}^{\mathscr{C}_{st}}(\mathscr{C})$ that contains all unit environments $E \in \mathscr{C}$ that are stable in $\mathscr{C}_{st}$. By the induction hypothesis, $\mathscr{C}_\flat \vdash T_1 \stackrel{\equiv}{\Longrightarrow} UEv^1_\flat$, $\mathscr{C}_\flat \vdash T_2 \stackrel{\equiv}{\Longrightarrow} UEv^2_\flat$, and for $E \in \mathscr{C} \cap \mathscr{C}_\flat$, $UEv^1_\flat(E) = UEv_1(E)$ and $UEv^2_\flat(E) = UEv_2(E)$. Then $\mathscr{C}_\flat \vdash T_1 \textbf{ and } T_2 \stackrel{\equiv}{\Longrightarrow} UEv_\flat$, where for $E_\flat \in \mathscr{C}_\flat$, we have $UEv_\flat(E_\flat)$ amalgamates $UEv^1_\flat(E_\flat)$ and $UEv^2_\flat(E_\flat)$. Clearly now, by the definition of UEv in the model semantics, for $E \in \mathscr{C} \cap \mathscr{C}_\flat$, since $UEv^1_\flat(E) = UEv_1(E)$ and $UEv^2_\flat(E) = UEv_2(E)$, we conclude that $UEv_\flat(E) = UEv(E)$, which completes the proof of item (4).

(3) This item follows by inspection of the rules; the cases of unit declarations are easy. The case of unit definition relies on item (4) as follows. Assume that $\mathscr{C} \subseteq ucx(\mathscr{C}_{st})$ and $\mathscr{C}$ is closed in $\mathscr{C}_{st} = (P_{st}, \mathscr{B}_{st}, D)$. To derive $\mathscr{C}_{st} \vdash UDD \rhd\!\!\!\rhd \mathscr{C}'_{st}$ and $\mathscr{C} \vdash UDD \Rightarrow \mathscr{C}'$, where UDD is of the form $U = T$, we must have $(P_{st}, \mathscr{B}_{st}, D) \vdash T \rhd\!\!\!\rhd (i, D')$,

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$U \notin (dom(P_{st}) \cup dom(\mathscr{B}_{st}))$, and $\mathscr{C} \vdash T \Rightarrow UEv$, with $\mathscr{C}'_{st} = (P_{st}, \mathscr{B}_{st} + \{U \mapsto i\}, D')$ and $\mathscr{C}' = \{E + \{U \mapsto UEv(E)\} \mid E \in \mathscr{C}\}$. Now take any $\mathscr{C}_\flat \subseteq Cl_{\equiv}^{\mathscr{C}_{st}}(\mathscr{C})$ that contains all unit environments $E \in \mathscr{C}$ that are stable in $\mathscr{C}_{st}$. By item (4), $\mathscr{C}_\flat \vdash T \stackrel{\equiv}{\Longrightarrow} UEv_\flat$, and for $E \in \mathscr{C} \cap \mathscr{C}_\flat$, $UEv_\flat(E) = UEv(E)$. Hence, $\mathscr{C}_\flat \vdash U = T \Rightarrow \mathscr{C}'_\flat$ with $\mathscr{C}'_\flat = \{E_\flat + \{U \mapsto UEv_\flat(E_\flat)\} \mid E_\flat \in \mathscr{C}_\flat\}$. To see that $\mathscr{C}'_\flat \subseteq Cl_{\equiv}^{\mathscr{C}'_{st}}(\mathscr{C}')$, consider any $E_\flat \in \mathscr{C}_\flat \subseteq Cl_{\equiv}^{\mathscr{C}_{st}}(\mathscr{C})$, with some $E \in \mathscr{C}$ such that $E_\flat \bowtie_{\mathscr{C}_{st}} E$. By Lemma 7.1, $E_\flat + \{U \mapsto UEv_\flat(E_\flat)\}$ is coherently equivalent in $\mathscr{C}'_{st}$ to $E + \{U \mapsto UEv_\flat(E)\}$, which is the same as $E + \{U \mapsto UEv(E)\}$. This shows that $E_\flat + \{U \mapsto UEv_\flat(E_\flat)\}$ is indeed in $Cl_{\equiv}^{\mathscr{C}'_{st}}(\mathscr{C}')$. Finally, if for some $E \in \mathscr{C}$, we have $E + \{U \mapsto UEv(E)\}$ is stable in $\mathscr{C}'_{st}$, then E is stable in $\mathscr{C}_{st}$ and hence is in $\mathscr{C}_\flat$. Then, since $UEv_\flat(E) = UEv(E)$ by item (4), we also have that $E + \{U \mapsto UEv(E)\}$ is in $\mathscr{C}'_\flat$.

(2) This follows from item (3) by an easy induction on the length of the sequence of unit declarations and definitions. To begin, note that every environment in $\mathscr{C}^\varnothing$ is stable in the empty static context $\mathscr{C}_{st}^\varnothing$ and is witnessed in $\mathscr{C}_{st}^\varnothing$ by the empty family of models, so $\mathscr{C}^\varnothing = Cl_{\equiv}^{\mathscr{C}_{st}^\varnothing}(\mathscr{C}^\varnothing)$.

(1) This now follows easily. In order to derive the assumptions for ASP of the form **arch spec** UDD^+ **result** T, we must have $\vdash UDD^+ \rhd \mathscr{C}_{st}$ and $\vdash UDD^+ \Rightarrow \mathscr{C}$, as well as $\mathscr{C}_{st} \vdash T \rhd (i, D)$ and $\mathscr{C} \vdash T \Rightarrow UEv$, with $(P_{st}, B_{st}) = ctx(\mathscr{C}_{st})$ and $\Sigma = D(i)$. So, by item (2), we have $\vdash UDD^+ \stackrel{\equiv}{\Longrightarrow} \mathscr{C}_\flat$, where $\mathscr{C}_\flat \subseteq Cl_{\equiv}^{\mathscr{C}_{st}}(\mathscr{C})$ and $\mathscr{C}_\flat$ contains all unit environments $E \in \mathscr{C}$ that are stable in $\mathscr{C}_{st}$. By item (4) in turn, $\mathscr{C}_\flat \vdash T \stackrel{\equiv}{\Longrightarrow} UEv_\flat$, and for each unit environment $E \in \mathscr{C}$ stable in $\mathscr{C}_{st}$, $UEv_\flat(E) = UEv(E)$ (since $E \in \mathscr{C} \cap \mathscr{C}_\flat$ then). □

Corollary 7.6. If $\vdash ASP \rhd (\mathscr{C}_{st}, \Sigma)$ and $\vdash ASP \Rightarrow (\mathscr{C}, UEv)$, then $\vdash ASP \stackrel{\equiv}{\Longrightarrow} (\mathscr{C}_\flat, UEv_\flat)$, where for every $E_\flat \in \mathscr{C}_\flat$ there exists $E \in \mathscr{C}$ such that $E_\flat \equiv_{\mathscr{C}_{st}} E$ and $UEv_\flat(E_\flat) \equiv UEv(E)$.

Proof. Given the assumptions, by Theorem 7.5, $\vdash ASP \stackrel{\equiv}{\Longrightarrow} (\mathscr{C}_\flat, UEv_\flat)$ with $\mathscr{C}_\flat \subseteq Cl_{\equiv}^{\mathscr{C}_{st}}(\mathscr{C})$ and for each $E \in \mathscr{C}$ that is stable in $\mathscr{C}_{st}$, we have $E \in \mathscr{C}_\flat$ and $UEv_\flat(E) = UEv(E)$. Hence, for each $E_\flat \in \mathscr{C}_\flat$ there is a stable environment $E \in \mathscr{C}$ such that $E_\flat \bowtie_{\mathscr{C}_{st}} E$ and $UEv(E) = UEv_\flat(E)$. It follows that $E_\flat \equiv_{\mathscr{C}_{st}} E$ and, by Corollary 7.4, $UEv_\flat(E) \equiv UEv_\flat(E_\flat)$, which yields $UEv(E) \equiv UEv_\flat(E_\flat)$. □

8. Example

The following example illustrates some of the points in the paper. We hope that the notation of CASL is understandable without further explanation; otherwise, see CoFI (2004).

We start with a simple specification of sets of strings; we will not go into any details of a specification of strings, just remarking that any standard specification would typically be monomorphic (with a unique model, up to isomorphism) and would certainly provide the equality predicate for strings.

spec STRING = **sort** *String*
...
pred eq_S : *String* × *String* ;
axiom $\forall s, s' : String \bullet eq_S(s, s') \iff s = s'$
...

spec STRINGSET = STRING
then sort *Set*
ops *empty* : *Set* ;
add : *String* × *Set* → *Set*
pred *present* : *String* × *Set*
$\forall\ s, s' : String, t : Set$
- $add(s, add(s, t)) = add(s, t)$
- $add(s, add(s', t)) = add(s', add(s, t))$
- $\neg present(s, empty)$
- $present(s, add(s, t))$
- $s \neq s' \implies (\ present(s, add(s', t)) \iff present(s, t)\ )$

We now provide a more elaborate version of the requirements this specification captures, introducing the idea of using a hash table implementation of sets.

spec INT = **sort** *Int*
...
pred eq_N : *Int* × *Int* ;
axiom $\forall n, n' : Int \bullet eq_N(n, n') \iff n = n'$
...

spec ELEM = **sort** *Elem*

spec ARRAY[ELEM] = ELEM **and** INT
then sort *Array* [*Elem*]
ops *empty* : *Array* [*Elem*];
put : *Int* × *Elem* × *Array* [*Elem*] → *Array* [*Elem*];
take : *Int* × *Array* [*Elem*] →? *Elem*
pred *used* : *Int* × *Array* [*Elem*]
$\forall\ i, j : Int ; e, e' : Elem ; a : Array[Elem]$
- $i \neq j \implies put(i, e', put(j, e, a)) = put(j, e, put(i, e', a))$
- $put(i, e', put(i, e, a)) = put(i, e', a)$
- $\neg used(i, empty)$
- $used(i, put(i, e, a))$
- $i \neq j \implies (\ used(i, put(j, e, a)) \iff used(i, a)\ )$
- $take(i, put(i, e, a)) = e$

spec ELEMKEY = ELEM **and** INT
then op *hash* : *Elem* → *Int*

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spec HASHTABLE[ELEMKEY] = ELEMKEY **and** ARRAY[ELEM]
then ops $add : Elem \times Array[Elem] \rightarrow Array[Elem]$;
$putnear : Int \times Elem \times Array[Elem] \rightarrow Array[Elem]$
preds $present : Elem \times Array[Elem]$
$isnear : Int \times Elem \times Array[Elem]$
$\forall\ i : Int; e : Elem; a : Array[Elem]$
- $add(e, a) = putnear(hash(e), e, a)$
- $\neg used(i, a) \implies putnear(i, e, a) = put(i, e, a)$
- $used(i, a) \wedge take(i, a) = e \implies putnear(i, e, a) = a$
- $used(i, a) \wedge take(i, a) \neq e \implies putnear(i, e, a) = putnear(succ(i), e, a)$
- $present(e, a) \iff isnear(hash(e), e, a)$
- $\neg used(i, a) \implies \neg isnear(i, e, a)$
- $used(i, a) \wedge take(i, a) = e \implies isnear(i, e, a)$
- $used(i, a) \wedge take(i, a) \neq e \implies (\ isnear(i, e, a) \iff isnear(succ(i), e, a)\ )$

spec STRINGKEY = STRING **and** INT
then op $hash : String \rightarrow Int$

spec STRINGHASHTABLE =
HASHTABLE[STRINGKEY] **with** $Array[String] \mapsto Set$
reveal $String, Set, empty, add, present$

STRINGHASHTABLE does not literally ensure all the requirements imposed by the original specification STRINGSET: the second axiom (commutativity of adding elements to a set) fails in some models of STRINGHASHTABLE. Still, it is easy to check that $[\![$STRINGHASHTABLE$]\!] \subseteq [\![$STRINGSET$]\!]_{\equiv}$, so every future (observationally-correct) realisation of STRINGHASHTABLE is an observationally-correct realisation of STRINGSET[†].

STRINGHASHTABLE is structured in a fairly natural way, building on a generic specification of arrays that is presumably already available, and including a generic specification of hash tables that may be reused in the future.

However, the structure of STRINGHASHTABLE must not be viewed as an obligatory prescription of the structure of the final implementation. For example, we may decide to adopt the architectural specification STRINGHASHTABLEDESIGN given below as an alternative structure.

The architectural specification uses the CASL construct **given** to mark units that are imported by other units. Formally, a sequence of declarations like

N : INT; S : STRING;
SK : STRINGKEY **given** S, N;

abbreviates

N : INT; S : STRING;
SK' : INT × STRING → STRINGKEY;
$SK = SK'[N][S]$;

where a new generic construction SK' is introduced and immediately applied to the imported units.

[†] Note that dropping the first two axioms in STRINGSET yields a specification with a class of models that coincides with $[\![$STRINGSET$]\!]_{\equiv}$ – in fact, we could have started with such an observationally-closed version of the specification, without making any use of observational correctness at this stage.

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arch spec STRINGHASHTABLEDESIGN =
units N : INT;
S : STRING;
SK : STRINGKEY **given** S, N;
A : ELEM $\rightarrow$ ARRAY[ELEM] **given** N;
ASK $= A[SK$ **fit** $Elem \mapsto String]$;
HT : STRINGHASHTABLE **given** $\{ASK$ **with** $Array[String] \mapsto Set\}$
result HT **reveal** $String, Set, empty, add, present$

The above architectural specification captures a modular design of the system to be built as follows. Components N and S are to be defined, implementing specifications INT and STRING, respectively. Presumably, these would be predefined in any practical programming language. Then, N and S are put together and extended by a definition of a hash function *hash*, yielding a new component SK. However, as explained above, the **given** notation used here really means that we are to provide a construction (a generic unit SK') that yields such a component for any realisations of INT and STRING. Another component to be provided is a generic unit A to implement arrays indexed by integers and storing data of any sort (*Elem*, to be instantiated when A is applied to an argument component). Again, this is to be given by a construction A' that works for any implementation of INT, but is then instantiated with the specific implementation given by N. This is then used to build a component ASK, which implements arrays of strings (with a hash function) by instantiating A with SK. In turn, ASK (with the main sort renamed to *Set* to fit the top level names given in the original requirement specification) will be extended to a component implementing STRINGHASHTABLE – again, this is to be built using a construction HT', independently of the details of ASK, for an arbitrary implementation of ARRAY[STRINGKEY]. Finally, the overall result will be given by exporting from this component only the required sorts, operations and predicate.

Note that the structure here differs from the structure of STRINGHASHTABLE in an essential way, since we have chosen to forego genericity of hash tables (for arbitrary elements), implementing them for the special case of strings.

Further development might lead to a final implementation in Standard ML, including the following modules. The task of extracting Standard ML signatures (`ARRAY_SIG`, and so on, using boolean functions for predicates) from the corresponding CASL signatures of the specifications given above is left for the reader. We assume though that the implementations `N` of INT and `S` of STRING, which we do not spell out here, use the Standard ML built-in types `int` and `string`, respectively. These are so-called *equality types* in Standard ML, and come with the built-in (infix) equality function `_=_`, which should replace eq_N and eq_S in the corresponding Standard ML signatures. We also omit a component `SK` that implements a hash function `hash`; any total function from strings to integers will do, although, of course, a good hash function will produce an even distribution of hash values. We compress consecutive instantiations of `A'` (first to `N` and then to `SK`) into a single functor application. Finally, we will incorporate the

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final adjustment to the overall result signature (the **reveal** construct in the result unit in StringHashTableDesign) and the renaming of arrays to sets (in the **given** part of HT) directly into the definition of the functor HT' used to build the resulting hash table of strings.

```
functor A'(structure N: INT_SIG and E : ELEM_SIG) : ARRAY_SIG =
  struct
    open N E
    type array = int -> elem
    exception unused
    fun empty(i) = raise unused
    fun put(i,e,a)(j) = if i=j then e else a(j)
    fun take(i,a) = a(i)
    fun used(i,a) = (a(i); true) handle unused => false
  end

structure ASK =
  struct
    structure Astring =
       A'(structure N=N and E=struct type elem=SK.string end)
    open Astring
    open SK
  end

functor HT'(structure ASK: ASK_SIG) : STRING_HASH_TABLE_SIG =
  struct
    open ASK
    type set = array
    fun putnear(i,s,t) =
       if used(i,t)
       then if take(i,t)=s then t else putnear(i+1,s,t)
       else put(i,s,t)
    fun add(s,t) = putnear(hash(s),s,t)
    fun isnear(i,s,t) =
       used(i,t) andalso (take(i,t)=s orelse isnear(i+1,s,t))
    fun present(s,t) = isnear(hash(s),s,t)
  end

structure HT = HT'(structure ASK=ASK)
```

The functor A' is literally correct with respect to Int **and** Elem and Array[Elem]. To be more precise, the semantic function on the models determined by A' extends any model

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in [[INT **and** ELEM]] to a model in [[ARRAY[ELEM]]] such that

$$[\![\mathtt{A'}]\!] \in [\![\text{INT } \mathbf{and} \text{ ELEM} \xrightarrow{\iota} \text{ARRAY[ELEM]}]\!],$$

where ι is the obvious signature inclusion. Similarly, the structure HT satisfies the axioms of STRINGHASHTABLE literally (at least on the reachable part, and assuming the use of extensional equality on functions).

The reader might want to check that STRINGHASHTABLEDESIGN is a statically correct architectural specification†: we can derive

$$\vdash \text{STRINGHASHTABLEDESIGN} \rhd ((P_{st}, B_{st}), \Sigma)$$

where P_{st} binds the generic units declared in STRINGHASHTABLE (including those implicitly introduced by expanding the **given** construct for imports), B_{st} maps the non-generic unit names in STRINGHASHTABLE to their signatures, and Σ is the signature of the result unit (the signature of STRINGHASHTABLE). Moreover, the (literal) model semantics also works, so we have

$$\vdash \text{STRINGHASHTABLEDESIGN} \Rightarrow (\mathscr{C}, UEv).$$

Here, the context $\mathscr{C}$ contains all environments that map unit names declared and defined in STRINGHASHTABLEDESIGN to their realisations so that declared units satisfy their specifications and the defined units are built from the units given in the environment as prescribed by their respective definitions. Then, the unit evaluator *UEv* maps any such environment in $\mathscr{C}$ to a model as determined by the result unit definition. In particular, the environment determined by the Standard ML functor and structure definitions given above is in $\mathscr{C}$, and *UEv* maps it to the expected system realisation.

However, even though the above functor A' implementing arrays is correct, we might want to use a completely different array implementation, for instance, because it is given as a highly optimised module in a library. Various useful 'tricks' in the code might then be expected. Here is an example where each entry in the array includes its history of updates:

```
functor Atrick(structure N: INT_SIG and E : ELEM_SIG) : ARRAY_SIG =
  struct
    open E
    type array = int -> elem list
    fun empty(i) = nil
    fun put(i,e,a)(j) = if i=j then e::a(j) else a(j)
    fun take(i,a) = let val e::_=a(i) in e end
    fun used(i,a) = not(null a(i))
  end
```

Then, Atrick given here is not literally correct with respect to INT **and** ELEM and ARRAY[ELEM], since it violates the axiom $put(i, e', put(i, e, a)) = put(i, e', a)$, but it is

† For example, the HETS tool, see www.informatik.uni-bremen.de/cofi/hets/, could be used.

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observationally correct: $[\![\texttt{Atrick}]\!] \in [\![\text{INT } \textbf{and} \text{ ELEM} \xrightarrow{\iota} \text{ARRAY[ELEM]}]\!]_{\equiv}$. Similarly, the extra flexibility that observational correctness offers would allow us, for instance, to change the code for `HT'` to count the number of insertions of each string, yielding a new functor `HTtrick'`. The structure

```
structure HTtrick = HTtrick'(structure ASK=ASK)
```

violates the axiom $used(i,a) \wedge take(i,a) = s \implies putnear(i,s,a) = a$, but, again, it is observationally correct: $[\![\texttt{HTtrick}]\!] \in [\![\text{STRINGHASHTABLE}]\!]_{\equiv}$.

The unit environment determined by `Atrick'` and `HTtrick'` is not in the context $\mathscr{C}$ given by the literal model semantics of STRINGHASHTABLE. However, under the observational semantics, we have

$$\vdash \text{STRINGHASHTABLEDESIGN} \stackrel{\equiv}{\Longrightarrow} (\mathscr{C}_\flat, UEv_\flat),$$

where $\mathscr{C}_\flat$ contains the environment that is determined by `Atrick'` and `HTtrick'`. Moreover, $UEv_\flat$ (which essentially coincides with UEv given by the literal model semantics above, but works on a different domain) maps such an environment to a model of the whole system that is an observationally correct realisation of the original specification STRINGHASHTABLE, as expected.

The Standard ML functors above define locally stable constructions: they respect encapsulation since they do not use any properties of their arguments other than what is spelled out in their parameter signatures. Indeed, all closed functors (which do not refer to external structure definitions) in Standard ML define locally stable constructions.

Returning to the idea inherent in the structure of the STRINGHASHTABLE specification, we will try to build our implementation using a generic construction for hash tables. That structure may be captured by the following architectural specification:

arch spec STRINGHASHTABLEDESIGN′ =
  units N : INT;
    A : ELEM → ARRAY[ELEM] **given** N;
    HT_{gen} : ELEMKEY × ARRAY[ELEM] → HASHTABLE[ELEMKEY];
    S : STRING;
    SK : STRINGKEY **given** S, N;
  result $HT_{gen}[SK$ **fit** $Elem \mapsto String][A[S]]$ **with** $Array[String] \mapsto Set$
      reveal $String, Set, empty, add, present$

This is again a correct architectural specification, and, indeed, we get

$$\vdash \text{STRINGHASHTABLEDESIGN}' \rhd ((P'_{st}, B'_{st}), \Sigma)$$
$$\vdash \text{STRINGHASHTABLEDESIGN}' \Rightarrow (\mathscr{C}', UEv')$$
$$\vdash \text{STRINGHASHTABLEDESIGN}' \stackrel{\equiv}{\Longrightarrow} (\mathscr{C}'_\flat, UEv'_\flat).$$

The extended static semantics and the literal model semantics work as expected (we encourage the reader to try to describe the resulting contexts). However, perhaps unexpectedly, we get $\mathscr{C}'_\flat = \emptyset$, so the above architectural specification is observationally inconsistent!

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The trouble is, of course, with the specification of generic hash tables. One might try to implement it as follows:

```
functor HTgen
    (structure EK : ELEM_KEY_SIG and A : ARRAY_ELEM_KEY_SIG
         sharing type EK.elem=A.elem) : HASH_TABLE_ELEM_KEY_SIG =
  struct
    open EK A
    fun putnear(i,e,a) =
       if used(i,a)
       then if take(i,a)=e then a else putnear(i+1,e,a)
       else put(i,e,a)
    fun add(e,a) = putnear(hash(e),e,a)
    fun isnear(i,e,a) =
       used(i,a) andalso (take(i,a)=e orelse isnear(i+1,e,a))
    fun present(e,a) = isnear(hash(e),e,a)
  end
```

Unfortunately, the construction defined by `HTgen` is not locally stable, and, in fact, `HTgen` is not correct code in Standard ML, since it requires equality on `elem` (in `take(i,a)=e`), which is not provided by `ELEM_KEY_SIG`. This problem is not accidental: there is no locally stable construction, and hence no Standard ML functor, satisfying the required specification. Consequently, there are no stable environments in context $\mathscr{C}'$ resulting from the literal model semantics, leading to the observational inconsistency of StringHashTableDesign$'$ ($\mathscr{C}'_\flat = \emptyset$). Even though what is a reasonable structure for the requirements specification, as expressed in StringHashTable, led to an inappropriate modular design StringHashTableDesign$'$, this is in fact good news. While allowing for a more relaxed interpretation of the axioms in (result) specifications as long as their observable consequences are ensured, the observational semantics marked as inconsistent a specification that cannot be implemented in a reasonable programming language in which no tricky means are available for violating the modular structure.

Of course, this does not mean that there is no good design that would require a generic implementation of hash tables. A simple way to achieve this would be to modify the above architectural specification to add 'equality' on *Elem* by introducing an equality predicate (for instance, in ElemKey). Notice that this is different from requiring *Elem* to be an equality type as in Standard ML, since this predicate is not necessarily constrained to be interpreted as the identity. Consequently, we should then use this predicate, rather than identity, to compare elements stored in HashTable. One point of architectural specifications is that such a change of structure is an important design decision that deserves to be recorded explicitly. The new specifications would be as follows:

spec ElemKeyEq = Elem **and** Int
  then op $hash : Elem \to Int$;
    pred $eq_E : Elem \times Elem$

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spec HASHTABLEEQ[ELEMKEYEQ] = ELEMKEYEQ **and** ARRAY[ELEM]
  then ops $add : Elem \times Array[Elem] \to Array[Elem]$;
    $putnear : Int \times Elem \times Array[Elem] \to Array[Elem]$
  preds $present : Elem \times Array[Elem]$
    $isnear : Int \times Elem \times Array[Elem]$
  $\forall\, i : Int; e : Elem; a : Array[Elem]$
  • $add(e, a) = putnear(hash(e), e, a)$
  • $\neg used(i, a) \Longrightarrow putnear(i, e, a) = put(i, e, a)$
  • $used(i, a) \wedge eq_E(take(i, a), e) \Longrightarrow putnear(i, e, a) = a$
  • $used(i, a) \wedge \neg eq_E(take(i, a), e) \Longrightarrow putnear(i, e, a) = putnear(succ(i), e, a)$
  • $present(e, a) \iff isnear(hash(e), e, a)$
  • $\neg used(i, a) \Longrightarrow \neg isnear(i, e, a)$
  • $used(i, a) \wedge eq_E(take(i, a), e) \Longrightarrow isnear(i, e, a)$
  • $used(i, a) \wedge \neg eq_E(take(i, a), e) \Longrightarrow (\; isnear(i, e, a) \iff isnear(succ(i), e, a)\;)$

The architectural design might then look as follows:

arch spec STRINGHASHTABLEDESIGNEQ =
  units N : INT;
    A : ELEM → ARRAY[ELEM] **given** N;
    HT_{gen} : ELEMKEYEQ × ARRAY[ELEM] → HASHTABLEEQ[ELEMKEYEQ];
    S : STRING;
    SK : STRINGKEY **given** S, N;
  result $HT_{gen}[SK$ **fit** $Elem \mapsto String][A[S]]$ **with** $Array[String] \mapsto Set$
    reveal $String, Set, empty, add, present$

The following Standard ML functor then provides a generic implementation of hash tables for any type of elements with an equality function, yielding a locally stable construction that is (observationally) correct with respect to ELEMKEYEQ **and** ARRAY[ELEM] and HASHTABLEEQ[ELEMKEYEQ]:

```
functor HTEQgen
    (structure EK : ELEM_KEY_EQ_SIG and A : ARRAY_ELEM_KEY_SIG
         sharing type EK.elem=A.elem) : HASH_TABLE_ELEM_KEY_EQ_SIG =
  struct
    open EK A
    fun putnear(i,e,a) =
       if used(i,a)
       then if eq_E(take(i,a),e) then a else putnear(i+1,e,a)
       else put(i,e,a)
    fun add(e,a) = putnear(hash(e),e,a)
    fun isnear(i,e,a) =
       used(i,a) andalso (eq_E(take(i,a),e) orelse isnear(i+1,e,a))
    fun present(e,a) = isnear(hash(e),e,a)
  end
```

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9. Conclusions and further work

The overall goal of this paper was to provide an observational view of Casl specifications that supports observational refinement of specifications in combination with Casl-style architectural design. This has been achieved, and spelled out in detail for a simplified version of Casl architectural specifications. Extending this to full Casl architectural specifications (by allowing multiple parameters for parametrised units, adding unit translations, and so on) is straightforward. Imports of units defined by arbitrary unit expressions are the only potential source of difficulty. But the methodologically well-justified case of this, where the import can be given an explicit specification, is easily dealt with as in Section 8.

Although we have worked in the specific setting of Casl signatures and models, formulated as an institution in Section 2, it should be clear that much of the above applies to a wide range of institutions. Rather than attempting to spell out the appropriate notion of 'institution with extra structure', we just note that surprisingly little appears to be required. A notion of *observational model morphisms* that is closed under composition and reduct, plus some extra categorical structure to identify 'correspondences' as certain spans of such morphisms, seems necessary and sufficient to formulate most of the material presented. The need for additional structure is obviated by the fact that the technical development makes no reference to a set of observable sorts, in contrast to standard approaches to the observational interpretation of specifications. In the context of Casl (where one can treat a sort as observable by introducing an 'equality predicate' on it) this is adequate. It may well not be adequate in institutions of much more limited expressive power, but it is not clear that such institutions are of genuine practical importance. Links with indistinguishability relations using factorisation properties, like Theorem 5.8, may require the richer context of *concrete institutions*, where model categories are equipped with a concretisation structure subject to a number of technical requirements as in Bidoit and Tarlecki (1996), or alternatively may follow the ideas of Popescu and Roşu (2005).

A challenging issue is now to understand how far the concepts developed for our somewhat simplified view of software components as local constructions on Casl models can be inspiring for a more general view of components involving some form of external communication. While this is clearly beyond the scope of this paper, we can, nevertheless, imagine that a promising direction of future research would be to look for an adequate counterpart of (local) stability in this more general setting.

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