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ON THE RECONSTRUCTION OF A STAR-SHAPED BODY FROM ITS "HALF-VOLUMES"

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Abstract

The problem is the reconstruction of the shape of an object, whose shell is a surface star-shaped with respect to a point 0, from the knowledge of the volume of every "half-object" obtained by taking any plane through 0. Conditions for the existence and uniqueness of the solution are given. The main result consists in showing that any uniform a-priori bound on the mean curvature of the shell reestablishes continuous dependence on the data for bodies satisfying a certain symmetry condition.

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0. Introduction

In the paper [5] P. Funk considered the following problem. Let Q be a solid in $\mathbf{R}^3$ and let Ω be its boundary. We suppose that Ω is a closed surface *star-shaped* with respect to a point 0 (see [8]): thus any ray issued from 0 meets Ω only once. Any plane through 0 divides Q in two parts; we suppose that we know the volumes of both parts for every plane through 0. The problem consists in reconstructing the shape of Ω from the knowledge of such "half-volumes". This problem will be denoted in the following as Problem (F).

Assuming the point 0 to be the origin of our coordinate system, let $\Phi(z)$ denote the distance from 0 to the point of Ω in the direction z. Thus the points x on Ω are described by the following parametric representation

$$x = \Phi(z)z,$$

where z is the variable point running over the unit sphere $S^2 = \{z \in \mathbf{R}^3: |z| = 1\}$. By introducing the usual spherical coordinate system on S^2, Φ turns out to be a function of (θ, ϕ), where θ is the colatitude and ϕ the east longitude of any point $z \in S^2$. By a solution of Problem (F) we shall mean equivalently the surface Ω or the corresponding function Φ.

Setting $u(z) = \frac{1}{3}\Phi^3(z)$, Problem (F) can be expressed by the following integral equation

$$Lu(z) \equiv \int_{(z, z') \geqslant 0} u(z') d\sigma(z') = f(z), \tag{0.1}$$

where $(\cdot, \cdot)$ denotes the scalar product in $\mathbf{R}^3$, $d\sigma(z')$ denotes the element of area on S^2 and f is a known function on S^2. Actually $f(z)$ is the volume of the part of Q cut out on the side of z by the plane through 0 orthogonal to the direction z. We notice that Lu may be regarded as a generalized Radon transform of u; Problem (F), like several problems of practical interest, falls in the framework of integral geometry problems. A method of solution for equation (0.1) is described in [5]: this method is based upon the fact that if u is a solution of (0.1), then certain averages of u satisfy an Abel equation. The present paper deals with the ill-posedness of Problem (F), in particular with the lack of continuous dependence of the solutions upon the data.

Before summarizing our results, we need to recall two simple preliminary remarks contained in [5].

i) If there exists a solution of equation (0.1), then for every $z \in S^2$ we have

$$f(z) + f(-z) = V, \tag{0.2}$$

where $-z$ is the antipodal point of z and V the volume of Q.

ii) We can decompose u as

$$u(z) = \tilde{u}(z) + w(z) + \frac{V}{4\pi},$$

where

$$\tilde{u}(z) = \frac{1}{2}[u(z) - u(-z)]$$

and

$$w(z) = \frac{1}{2}\left[u(z) + u(-z) - \frac{V}{2\pi}\right].$$

Thus $w(z)$ is an even function on S^2 (that is, $w(-z) = w(z)$) with zero mean value. One can easily verify that $Lw(z) \equiv 0$. This means that the component $w(z)$ has not influence on the data $f(z)$. Therefore, in order to get uniqueness, we shall deal only with solutions of the form

$$u(z) = \tilde{u}(z) + k, \tag{0.3}$$

where $\tilde{u}(z)$ is an odd function on S^2. The constant k will be equal to $V/4\pi$.

We shall see in Section 1 that if f belongs to $H^{3/2}(S^2)$ (the space of functions with derivative of order $3/2$ in $L^2(S^2)$) and satisfies (0.2) then one can construct for equation (0.1) a unique solution $u \in L^2(S^2)$ having the form (0.3).

According to the uniqueness condition described above, in what follows we shall restrict ourselves to considering *only* surfaces Ω whose corresponding representation $\Phi(z)$ satisfies $\Phi^3(z) + \Phi^3(-z) =$ constant. Thus, solving equation (0.1) yields the function $\Phi(z)$ and Ω can be uniquely reconstructed. Unfortunately, such a solution Ω of Problem (F) cannot depend continuously on the data f. One can verify the instability by the following example. Let Ω_n be the surfaces of revolution represented respectively by the functions

$$\Phi_n(z) = \Psi_n(\theta) = \left(3[2 + \sin(n\cos\theta)]\right)^{1/3};$$

let Q_n be the solid enclosed by Ω_n and $u_n = \frac{1}{3}\Phi_n^3$. The corresponding "half-volumes" are expressed by

$$f_n(z) = v_n(\theta) = 4\pi + 2\int_{\pi/2}^{\theta} J_1(n\sin t)\,dt,$$

where $J_1(t)$ is the Bessel function of the first order. An easy computation shows that

$$\|f_0 - f_n\|_{L^p(S^2)} \to 0 \quad \text{as } n \to +\infty \qquad (\text{for any } 1 \leqslant p \leqslant +\infty),$$

while the volume of the symmetric difference between Q_0 and Q_n,

$$\operatorname{vol}\left(Q_0 \,\Delta\, Q_n\right) = \|u_0 - u_n\|_{L^1(S^2)},$$

is bounded from below by a positive constant independent of n.

The main purpose of the present paper is to find an *a-priori geometrical bound* for the solutions of Problem (F) in order to restore the continuous dependence on the data. The final result we shall prove (Section 4) is the following: *solutions within any class of C^2-surfaces whose mean curvatures are uniformly bounded depend continuously on the data* (in the sense that if Ω_n, Ω are the solutions corresponding to the data f_n, f and f_n converges to f in $L^2(S^2)$, as $n \to +\infty$, then the volume of the symmetric difference $Q_n \,\Delta\, Q$ converges to 0, where Q_n, Q are the solids enclosed by Ω_n, Ω). This conclusion is obtained in several steps. First of all (Section 1) we shall show that the stability of the solutions of (0.1) can be restored if we assume an a-priori bound for $\|(-\Delta_S)^\alpha u\|_{L^2(S^2)}$, where α is any positive real number and $-\Delta_S$ denotes the Laplace-Beltrami operator on S^2. Then we prove that starting from an a-priori uniform bound for the mean curvature of the

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solutions of Problem (F) we can suitably bound the norms $\|(-\Delta_S)^\alpha u\|_{L^2(S^2)}$. We obtain such a result by proving two inequalities:

i) an interpolatory estimate, for $0 < \alpha < 1/4$, of $\|(-\Delta_S)^\alpha u\|_{L^2(S^2)}$ in terms of $\sup_{z\in S^2}|u(z)|$ and $\|Du\|_{L^1(S^2)}$ (Section 2);

ii) an isoperimetric inequality stating that the volume enclosed by a star-shaped C^2-surface can be bounded in terms of the area of the surface and the maximum of the mean curvature (Section 3).

1. The equation $Lu = f$

Let u, $f \in L^2(S^2)$. Then $u(z)$ and $f(z)$ can be expanded in a series of normalized spherical harmonics

$$u(z) = \sum_{l=0}^{+\infty} \sum_{n=-l}^{l} u_l^n Y_l^n(z), \qquad f(z) = \sum_{l=0}^{+\infty} \sum_{n=-l}^{l} f_l^n Y_l^n(z), \tag{1.1}$$

convergent in $L^2(S^2)$, where

$$Y_l^n(\theta, \phi) = (-1)^n \left(\frac{2l+1}{4\pi}\right)^{1/2} \left[\frac{(l-n)!}{(l+n)!}\right]^{1/2} P_l^n(\cos\theta) e^{in\phi},$$

with $P_l^n(t)$ denoting the associated Legendre functions. Let us verify that *if u is a solution of equation* (0.1) *then*

$$f_l^n = B_l u_l^n, \tag{1.2}$$

where

$$B_l = \begin{cases} \dfrac{\pi^{3/2}}{\Gamma\left(\frac{l}{2} + \frac{3}{2}\right)\Gamma\left(1 - \frac{l}{2}\right)} & \text{if } l = 0, 1, 3, 5, \ldots, \\ 0 & \text{otherwise}. \end{cases} \tag{1.3}$$

In order to deduce (1.2), (1.3) we can apply an argument by G. Backus in [1] (see also [4]). Let z_N be the north-pole of S^2. One may verify that

$$LY_l^n(z_N) = \begin{cases} \dfrac{\pi\sqrt{2l+1}}{2\Gamma\left(\frac{l}{2} + \frac{3}{2}\right)\Gamma\left(1 - \frac{l}{2}\right)} & \text{if } n = 0 \text{ and } l = 0, 1, 3, 5, \ldots, \\ 0 & \text{otherwise} \end{cases} \tag{1.4}$$

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(use, for instance, 7.126 of [7]). Let $\bar{z}$ be any point on S^2 distinct from z_N and let g be a rotation of $\mathbf{R}^3$ which carries $\bar{z}$ into z_N. Setting $z' = g(z)$ gives

$$Y_l^n(z) = \sum_{m=-l}^{l} a_l^{nm} Y_l^m(z'), \tag{1.5}$$

where

$$a_l^{nm} = \int_{S^2} Y_l^n(z) \bar{Y}_l^m(z')\, d\sigma(z).$$

From (1.4), (1.5) it follows that

$$LY_l^n(\bar{z}) = \begin{cases} \dfrac{\pi\sqrt{2l+1}}{2\Gamma\left(\frac{l}{2}+\frac{3}{2}\right)\Gamma\left(1-\frac{l}{2}\right)} a_l^{no} & \text{if } l = 0, 1, 3, 5, \ldots, \\ 0 & \text{otherwise.} \end{cases} \tag{1.6}$$

The addition theorem for spherical harmonics implies that

$$a_l^{no} = 2\left(\frac{\pi}{2l+1}\right)^{1/2} Y_l^n(\bar{z}). \tag{1.7}$$

Therefore, by substituting (1.7) in (1.6), we deduce (1.2), (1.3).

Remark. For $\lambda = 0, 1, 2, \ldots$, one has

$$B_{2\lambda+1} = (-1)^{\lambda} \frac{\sqrt{\pi}\,\Gamma(\lambda + 1/2)}{\Gamma(\lambda + 2)};$$

hence, as $\lambda \to +\infty$, $B_{2\lambda+1}$ is of the same order as $\lambda^{-3/2}$. This fact can be expressed by saying that L^{-1} behaves like $(-\Delta_S)^{3/4}$, a derivative of order 3/2. We define

$$(-\Delta_S)^{\alpha} f(z) = \sum_{l=1}^{+\infty} [l(l+1)]^{\alpha} \sum_{n=-l}^{l} f_l^n Y_l^n(z),$$

α being any positive number. This explains the instability phenomenon noticed before.

In order to restore the stability we need an a-priori estimate for the solutions in the form (0.3) of equation (0.1). Let α be a positive number and suppose that the solution u of our equation belongs to $H^{\alpha}(S^2)$, that is, $\|(-\Delta_S)^{\alpha/2} u\|_{L^2(S^2)} < +\infty$. By repeating the procedure of [2], Section 2, one has

$$\left(\frac{\|\tilde{u}\|_{L^2(S^2)}}{\|(-\Delta_S)^{\alpha/2} u\|_{L^2(S^2)}}\right)^{2+3/\alpha} \leqslant \frac{\sum_{l=1}^{+\infty} [l(l+1)]^{-3/2} \sum_{n=-l}^{l} |u_l^n|^2}{\|(-\Delta_S)^{\alpha/2} u\|_{L^2(S^2)}^2}, \tag{1.8}$$

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where $\tilde{u}(z) = u(z) - u_0^0/2\sqrt{\pi}$. Since

$$[l(l+1)]^{-3/2} < \frac{B_l^2}{8\pi}, \quad \text{for every } l \text{ odd,*}$$

from (1.8) we deduce the estimate

$$\|\tilde{u}\|^2_{L^2(S^2)} \leqslant (8\pi)^{-2\alpha/(2\alpha+3)} \|(-\Delta_S)^{\alpha/2} u\|^{6/(2\alpha+3)}_{L^2(S^2)} \|\tilde{f}\|^{4\alpha/(2\alpha+3)}_{L^2(S^2)}, \tag{1.9}$$

where $\tilde{f}(z) = f(z) - f_0^0/2\sqrt{\pi}$.

The above results can be summarized by the following theorem.

THEOREM 1. *If $f \in H^{3/2}(S^2)$ and $f(z) + f(-z)$ is constant for all $z \in S^2$, then there exists a unique solution $u \in L^2(S^2)$ of equation* (0.1) *such that $u(z) + u(-z)$ is constant for all $z \in S^2$. Such a solution is given by*

$$u(z) = \frac{f_0^0}{2\pi^{3/2}} + \sum_{l=0}^{+\infty} (B_{2l+1})^{-1} \sum_{n=-2l-1}^{2l+1} f_{2l+1}^n Y_{2l+1}^n(z), \tag{1.10}$$

where the f_j^n and B_j are defined respectively by (1.1) *and* (1.3), *and the convergence of the series is in the L^2-sense. Moreover*

(1.11)

$$\|u\|^2_{L^2(S^2)} \leqslant \left(\frac{f_0^0}{2\pi}\right)^2 + (8\pi)^{-2\alpha/(2\alpha+3)} \|(-\Delta_S)^{\alpha/2} u\|^{6/(2\alpha+3)}_{L^2(S^2)} \left\| f - \frac{f_0^0}{2\sqrt{\pi}} \right\|^{4\alpha/(2\alpha+3)}_{L^2(S^2)},$$

provided $u \in H^\alpha(S^2)$.

Theorem 1 tells us how we may choose the functional classes for the solutions u and the data f in order to get existence and uniqueness for equation (0.1). Moreover, estimate (1.11) implies that in every subset of solutions satisfying in addition the condition $\|(-\Delta_S)^\alpha u\|_{L^2(S^2)} \leqslant \beta$, where β is any fixed constant, the continuous dependence on the data holds. The remaining part of the paper is devoted to showing that if we replace the above "analytical" a-priori bound by one of geometrical meaning, namely a uniform a-priori bound on the mean curvature of the solutions of Problem (F), then the stability for this problem is guaranteed. This conclusion is achieved by using the results of the next two sections.

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* The constant $1/8\pi$ is the best possible.

2. An interpolatory estimate

This section is devoted to proving the following theorem.

THEOREM 2. *Let u be a function on S^2 such that u is bounded and $|Du|$ is summable on S^2. Then for any $\alpha \in (0, \frac{1}{2})$ we have*

$$\|(-\Delta_S)^{\alpha/2}u\|_{L^2(S^2)} \leqslant \gamma\Big(\sup_{z\in S^2}|u(z)|\Big)^{1-\alpha}\|Du\|^{\alpha}_{L^1(S^2)}, \tag{2.1}$$

where γ is a constant independent of u.

REMARK. An estimate like (2.1) does not hold when $\alpha > \frac{1}{2}$. This fact can be checked, for instance, by the help of the function $u(z) = v(\theta) = |\cos\theta|^{\alpha-1/2}$. For such a function $\sup|u|$ and $\|Du\|_{L^1(S^2)}$ are both finite, while $\|(-\Delta_S)^{\alpha/2}u\|_{L^2(S^2)} = +\infty$.

For the proof of Theorem 2 we need a preliminary result. Let $G(x)$, $x \equiv (x_1, x_2, x_3) \in \mathbf{R}^3$, denote the matrix

$$G(x) = e^{x_1h_1+x_2h_2+x_3h_3},$$

where

$$h_1 = \begin{pmatrix} 0 & 0 & 0\\ 0 & 0 & -1\\ 0 & 1 & 0\end{pmatrix},\qquad h_2 = \begin{pmatrix} 0 & 0 & 1\\ 0 & 0 & 0\\ -1 & 0 & 0\end{pmatrix},\qquad h_3 = \begin{pmatrix} 0 & -1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}.$$

One can verify that $G(x)$ represents a rotation of $\mathbf{R}^3$ through an angle $|x|$ around the direction $x/|x|$ (see [3]). Let us prove the following

LEMMA. *Let α be a fixed real number, $0 < \alpha < 1$. For every nonconstant function $u \in H^\alpha(S^2)$ we have*

$$m \leqslant \frac{\int_{\mathbf{R}^3}|x|^{-(2\alpha+3)}\,dx\int_{S^2}[u(G(x)z) - u(z)]^2\,d\sigma(z)}{16\pi q\|(-\Delta_S)^{\alpha/2}u\|^2_{L^2(S^2)}} \leqslant M, \tag{2.2}$$

where m and M denote respectively the inf *and the* sup *of the sequence $\{(2l+1)^{-1}[l(l+1)]^{-\alpha}\sum_{k=1}^{l}k^{2\alpha}\}$, with $l \geqslant 1$, and*

$$q = \begin{cases} \pi/2 & \text{if } \alpha = \frac{1}{2},\\ -\cos(\pi\alpha)\Gamma(-2\alpha) & \text{otherwise}.\end{cases} \tag{2.3}$$

PROOF. Let us expand u in spherical harmonics

$$u(z) = \sum_{l=0}^{+\infty} \sum_{n=-l}^{l} u_l^n Y_l^n(z), \tag{2.4}$$

and recall that for any l and n we have (see (1.5))

$$Y_l^n(G(x)z) = \sum_{j=-l}^{l} c_l^{nj}(x) Y_l^j(z), \tag{2.5}$$

where

$$c_l^{nj}(x) = \int_{S^2} Y_l^n(G(x)z) \overline{Y}_l^j(z)\, d\sigma(z).$$

Notice that

$$\sum_{k=-l}^{l} c_l^{nk}(x) \bar{c}_l^{jk}(x) = \delta_{nj}. \tag{2.6}$$

For fixed l let us consider the $(2l+1) \times (2l+1)$ matrix

$$C_l(x) = \left(c_l^{nk}(x)\right),$$

with $-l \leq n, k \leq l$. The map T: $G(x) \to C_l(x)$ is an irreducible unitary representation of the group of proper rotations of $\mathbf{R}^3$ (see [3]). Thus the matrix $C_l(x)$ can be expressed as

$$C_l(x) = e^{i(H_1x_1 + H_2x_2 + H_3x_3)}, \tag{2.7}$$

where

$$H_1 = \frac{1}{2}\begin{bmatrix} 0 & \eta_{-l+1} & 0 & \cdot & 0 \\ \eta_{-l+1} & 0 & \eta_{-l+2} & \cdot & 0 \\ 0 & \eta_{-l+2} & 0 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \eta_l \\ 0 & 0 & \cdot & \eta_l & 0 \end{bmatrix}, \qquad \eta_k = [(l+k)(l-k+1)]^{1/2},$$

$$H_2 = \frac{1}{2}i\begin{bmatrix} 0 & \eta_{-l+1} & 0 & \cdot & 0 \\ -\eta_{-l+1} & 0 & \eta_{-l+2} & \cdot & 0 \\ 0 & -\eta_{-l+2} & 0 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \eta_l \\ 0 & 0 & \cdot & -\eta_l & 0 \end{bmatrix},$$

$$H_3 = -\begin{bmatrix} l & & & & \\ & l-1 & & 0 & \\ & & \ddots & & \\ & 0 & & -(l-1) & \\ & & & & -l \end{bmatrix}$$

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(see [6]). An easy computation, involving (2.4), (2.5), (2.6), (2.7) and the Parseval identity, yields

$$(2.8)\quad \int_{\mathbf{R}^3} |x|^{-(2\alpha+3)}\,dx \int_{S^2} [u(G(x)z) - u(z)]^2\,d\sigma(z) = 2\,\mathrm{Re} \sum_{l=1}^{+\infty} \mathrm{tr}(U_l \times A_l),$$

where tr stands for "trace" and U_l, A_l are matrices defined by

$$U_l = (U_l^{nk}), \qquad U_l^{nk} = \bar{u}_l^n u_l^k, \quad -l \leqslant n, k \leqslant l,$$

$$(2.9)\qquad A_l = \int_{\mathbf{R}^3} |x|^{-(2\alpha+3)}\,[I - e^{i(H_1x_1 + H_2x_2 + H_3x_3)}]\,dx$$

(I is the $(2l+1)$-dimensional identity matrix).

We shall show that

$$(2.10)\qquad A_l = a_l I,$$

where

$$(2.11)\qquad a_l = \frac{8\pi}{2l+1} q \sum_{k=1}^{l} k^{2\alpha},$$

q being defined by (2.3). Therefore, by putting (2.10) in (2.8) and taking into account that

$$\|(-\Delta_S)^{\alpha/2} u\|^2_{L^2(S^2)} = \sum_{l=1}^{+\infty} [l(l+1)]^\alpha \sum_{n=-l}^{l} |u_l^n|^2,$$

one obtains the inequalities (2.2) from (2.11). To deduce (2.10), (2.11) we rewrite (2.9) as

$$(2.12)\qquad A_l = \int_0^{+\infty} \rho^{-(2\alpha+1)}\,d\rho \int_0^{2\pi} d\phi \int_0^{\pi} [I - e^{i\rho\Lambda(\theta,\phi)}] \sin\theta\,d\theta,$$

where $\Lambda(\theta, \phi) = H_1 \sin\theta \cos\phi + H_2 \sin\theta \sin\phi + H_3 \cos\theta$. If ζ is a rotation which carries $(\sin\theta\ \cos\phi, \sin\theta\ \sin\phi, \cos\theta)$ into $(0, 0, 1)$, and $T_\zeta = (t^{nj})$, $-l \leqslant n, j \leqslant l$, denotes the corresponding matrix in our representation, then

$$(2.13)\qquad T_\zeta \Lambda T_\zeta^{-1} = H_3$$

(see [6]). Thus, because of the spectral theorem, we have

$$(2.14)\qquad e^{i\rho\Lambda(\theta,\phi)} = \sum_{k=-l}^{l} e^{i\rho k} E_k(\theta, \phi),$$

where the $E_k = (\varepsilon_k^{nj})$, $-l \leqslant n, j \leqslant l$, are projection matrices. By (2.13) we have $\varepsilon_k^{nj} = \bar{t}^{kn} t^{kj}$, which in turn implies

$$(2.15)\qquad \int_0^{2\pi} d\phi \int_0^{\pi} E_k(\theta, \phi) \sin\theta\,d\theta = \frac{4\pi}{2l+1} I,$$

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for every k (see, for instance, [9]). Therefore from (2.12), (2.14) and (2.15) we deduce

$$A_l = \left[\frac{8\pi \sum_{k=1}^{l} k^{2\alpha}}{2l+1} \int_0^{+\infty} \rho^{-(2\alpha+1)} (1 - \cos\rho)\, d\rho \right] I. \tag{2.16}$$

The latter integral equals $\pi/2$ if $\alpha = 1/2$, and equals $-\cos(\pi\alpha)\Gamma(-2\alpha)$ otherwise. Thus (2.10) and (2.11) are proven and the lemma is proved. The estimates (2.2) of the above lemma correct the equality (A.1) of the Appendix in the paper [2] (the A_l's were wrongly evaluated there). That equality does not affect however any result presented in [2].

PROOF OF THEOREM 2. The first inequality of (2.2) can be rewritten as

$$\|(-\Delta_S)^{\alpha/2} u\|^2_{L^2(S^2)} \leqslant \frac{1}{16\pi q m} \int_0^{+\infty} \rho^{-2\alpha-1} F(\rho)\, d\rho, \tag{2.17}$$

where

$$F(\rho) = \int_{S^2} d\sigma(z') \int_{S^2} [u(G(\rho z')z) - u(z)]^2\, d\sigma(z).$$

Firstly let us show that for every $\rho \geqslant 0$

$$|F'(\rho)| \leqslant 16\pi \Big(\sup_{z \in S^2} |u(z)| \Big) \|Du\|_{L^1(S^2)}. \tag{2.18}$$

It is easy to see that

$$|F'(\rho)| \leqslant 2 \Big(\sup_{z \in S^2} |u(z)| \Big) \int_{S^2} d\sigma(z') \int_{S^2} \left| \frac{d}{d\rho} u(G(\rho z')z) \right| d\sigma(z). \tag{2.19}$$

But

$$\begin{aligned} \left| \frac{d}{d\rho} u(G(\rho z')z) \right| &\leqslant |Du(G(\rho z')z)| \cdot \left| \frac{d}{d\rho} (G(\rho z')z) \right| \\ &= |Du(G(\rho z')z)| \cdot |[G(\rho z') W(z')] z|, \end{aligned}$$

where $W(z') = z'_1 h_1 + z'_2 h_2 + z'_3 h_3$. One can verify directly that $|W| = 1$, where $|\cdot|$ denotes the Euclidean matrix norm. Then $|GW| = 1$, since G is unitary. Thus we obtain

$$\left| \frac{d}{d\rho} u(G(\rho z')z) \right| \leqslant |Du(G(\rho z')z)|. \tag{2.20}$$

Since

$$\int_{S^2} |Du(G(\rho z')z)|\, d\sigma(z) = \int_{S^2} |Du(z)|\, d\sigma(z),$$

from (2.19), (2.20) we can deduce (2.18).

Let us fix now $R > 0$. Integrating by parts yields

$$\int_0^R \rho^{-2\alpha-1} F(\rho)\, d\rho \leqslant \frac{1}{2\alpha} \int_0^R \rho^{-2\alpha} F'(\rho)\, d\rho,$$

where we used the fact that F, by (2.18), is a Lipschitz function. Therefore

$$\int_0^R \rho^{-2\alpha-1} F(\rho)\, d\rho \leqslant \frac{R^{1-2\alpha}}{2\alpha(1-2\alpha)} \Big(\sup_{0 \leqslant \rho \leqslant R} |F'(\rho)| \Big).$$

Since

$$\int_R^{+\infty} \rho^{-2\alpha-1} F(\rho)\, d\rho \leqslant \frac{R^{-2\alpha}}{2\alpha} \Big(\sup_{\rho \geqslant R} |F(\rho)| \Big),$$

one has

$$\int_0^{+\infty} \rho^{-2\alpha-1} F(\rho)\, d\rho \leqslant \frac{R^{-2\alpha}}{2\alpha} \left[\frac{R}{1-2\alpha} \sup_{\rho \geqslant 0} |F'(\rho)| + \sup_{\rho \geqslant 0} |F(\rho)| \right]. \tag{2.21}$$

Minimizing the right-hand side of (2.21) with respect to R then yields

$$\int_0^{+\infty} \rho^{-2\alpha-1} F(\rho)\, d\rho \leqslant \frac{(2\alpha)^{-1-2\alpha}}{1-2\alpha} \Big(\sup_{\rho \geqslant 0} |F'(\rho)| \Big)^{2\alpha} \Big(\sup_{\rho \geqslant 0} |F(\rho)| \Big)^{1-2\alpha}. \tag{2.22}$$

Notice that

$$|F(\rho)| \leqslant (8\pi)^2 \Big(\sup_{z \in S^2} |u(z)| \Big)^2. \tag{2.23}$$

Therefore, by using (2.17), (2.22), (2.18) and (2.23), one obtains inequality (2.1).

3. An isoperimetric inequality

Let Ω be any closed star-shaped (with respect to the point 0, assumed as the origin of our coordinate system) C^2-surface in $\mathbf{R}^3$ and let Q be the solid enclosed by Ω. We orient Ω by always choosing the normal pointing *inward*; in this way if Ω is convex its mean curvature will be positive. Let us denote by $N(\omega)$ the unit normal to Ω at ω and by $H(\omega)$ the mean curvature of Ω at ω. We want to obtain an estimate of the area $\mathbf{A}$ of Ω in terms of the volume V of Q and the maximum of H.

THEOREM 3. *We have*

$$\mathbf{A} \leqslant 3 \Big(\max_{\omega \in \Omega} H(\omega) \Big) V; \tag{3.1}$$

the sign = holds if and only if Ω is a sphere.

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PROOF. Let us introduce the following support function of Ω

$$s(\omega) = (\omega, -N(\omega)), \qquad \omega \in \Omega.$$

The function $s(\omega)$ is, for a general surface, the oriented distance from the origin of the tangent plane at ω. Since Ω is star-shaped with respect to 0 we have $s(\omega) > 0$ for every $\omega \in \Omega$. Our proof is based upon the following Minkowski's formula (see [8]):

$$\mathbf{A} = \int_\Omega s(\omega) H(\omega)\, d\sigma(\omega), \tag{3.2}$$

where $d\sigma(\omega)$ denotes the element of area on Ω.

If we denote by Ω^+ the subset of Ω where H is positive, from (3.2) we deduce that

$$\mathbf{A} \leqslant \Big(\max_{\omega \in \Omega} H(\omega)\Big) \int_{\Omega^+} s(\omega)\, d\sigma(\omega). \tag{3.3}$$

But

$$\int_{\Omega^+} s(\omega)\, d\sigma(\omega) = 3V^+, \tag{3.4}$$

where V^+ denotes the volume of the solid

$$Q^+ = \{x \in \mathbf{R}^3 : x = t\omega, \forall \omega \in \Omega^+, \forall t \in [0,1]\}.$$

Since $Q^+ \subseteq Q$, we have $V^+ \leqslant V$. Therefore from (3.3), (3.4) we deduce (3.1). Clearly, if Ω is a sphere then (3.1) is actually an equality. Conversely, if in (3.1) the equality sign holds then $H(\omega)$ must be constant. Therefore Ω is a sphere, by the Liebmann-Aleksandrov theorem (see, for instance, [8]).

REMARK 1. The proof shows that (3.1) remains valid if we replace V by V^+.

REMARK 2. If we deal with a plane closed star-shaped C^2-curve τ, the analog of (3.1) is

$$l \leqslant 2\Big(\max_{t \in \tau} k(t)\Big) a,$$

where l is the length of τ, a is the area enclosed by τ and $k(t)$ is the curvature of τ at t.

4. Conclusion

Now we are in position to prove a stability result for the solutions of Problem (F) under the assumption of an a-priori uniform bound on the mean curvature. For any fixed positive real μ, let us denote by $\mathbf{M}_\mu$ the set of all closed star-shaped (with respect to the same point 0) C^2-surfaces Ω such that:

i) $\Phi^3(z) + \Phi^3(-z)$ is a constant function on S^2, where $\Phi(z)$ is the representation of Ω;

ii) $H(\omega) \leqslant \mu$, for every $\omega \in \Omega$, where $H(\omega)$ is the mean curvature of Ω at ω.

The following theorem holds.

THEOREM 4. *Let Ω_1, Ω_2 be solutions in $\mathbf{M}_\mu$ of Problem* (F) *corresponding to the data f_1, f_2, and let Q_1, Q_2 be the solids enclosed by Ω_1, Ω_2. For any real α, $0 < \alpha < \frac{1}{2}$, there exists a constant K, depending only on α and μ, such that*

$$(4.1)\quad \left[\operatorname{vol}\left(Q_1 \,{}^{\Delta}\, Q_2\right)\right]^2 \leqslant 4\left(f_1^{\bullet} - f_2^{\bullet}\right)^2 + K\left(f_1^{\bullet} + f_2^{\bullet}\right)^{(2\alpha+6)/(2\alpha+3)} \left\|\tilde{f}_1 - \tilde{f}_2\right\|_{L^2(S^2)}^{4\alpha/(2\alpha+3)},$$

where $\operatorname{vol}(Q_1 \,{}^{\Delta}\, Q_2)$ *denotes the volume of the symmetric difference $Q_1 \,{}^{\Delta}\, Q_2$, $f_1^{\bullet}$, $f_2^{\bullet}$ denote the mean values on S^2 of f_1, f_2 and $\tilde{f}_i(z) = f_i(z) - f_i^{\bullet}$, $i = 1, 2$.*

COROLLARY. *Let Ω_n, Ω be solutions in $\mathbf{M}_\mu$ of Problem* (F) *corresponding to the data f_n, f and let Q_n, Q be the solids enclosed by Ω_n, Ω. If $\|f_n - f\|_{L^2(S^2)} \to 0$, as $n \to +\infty$, then* $\operatorname{vol}(Q_n \,{}^{\Delta}\, Q) \to 0$.

PROOF OF THEOREM 4. Let $\Phi_1(z)$, $\Phi_2(z)$ be the representations of the surfaces Ω_1, Ω_2; let us set $\frac{1}{3}\Phi_i^3(z) = u_i(z)$, $i = 1, 2$. By applying inequality (2.1) to $u_1 - u_2$, we obtain

$$(4.2)\quad \left\|(-\Delta_S)^{\alpha/2}(u_1 - u_2)\right\|_{L^2(S^2)} \leqslant \gamma\left(\max_{z\in S^2} u_1(z) + \max_{z\in S^2} u_2(z)\right)^{1-\alpha} \left(\|Du_1\|_{L^1(S^2)} + \|Du_2\|_{L^1(S^2)}\right)^{\alpha}.$$

Notice that

$$\int_{S^2} \Phi_i(z)\left[\Phi_i^2(z) + |D\Phi_i(z)|^2\right]^{1/2} d\sigma(z) = \mathbf{A}_i, \qquad i = 1, 2,$$

where $\mathbf{A}_i$ denotes the area of the surface Ω_i. Therefore

$$(4.3)\qquad \|Du_i\|_{L^1(S^2)} \leqslant \left(\max_{z\in S^2} \Phi_i(z)\right)\mathbf{A}_i, \qquad i = 1, 2.$$

For every $z \in S^2$ we have

$$u_i(z) + u_i(-z) = \frac{V_i}{2\pi}, \qquad i = 1,2,$$

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where V_i is the volume of Q_i; thus, since the $u_i(z)$'s are nonnegative functions, we deduce that

(4.4)
$$\max_{z \in S^2} u_i(z) \leqslant \frac{V_i}{2\pi}, \qquad i = 1, 2.$$

By using (4.4) and (3.1), from (4.3) we obtain

(4.5)
$$\|Du_i\|_{L^1(S^2)} \leqslant \frac{3^{4/3}}{(2\pi)^{1/3}} \mu V_i^{4/3}, \qquad i = 1, 2.$$

Therefore (4.2), (4.4) and (4.5) imply

(4.6)
$$\|(-\Delta_S)^{\alpha/2}(u_1 - u_2)\|_{L^2(S^2)} \leqslant \mu'(V_1 + V_2)^{1+\alpha/3},$$

where $\mu' = 3^{4\alpha/3}(2\pi)^{2\alpha/3-1}\gamma\mu^\alpha$. Notice that, for every $z \in S^2, f_i(z) + f_i(-z) = V_i$, $i = 1, 2$; thus

(4.7)
$$V_i = 2f_i^*, \qquad i = 1, 2.$$

Finally, let us apply inequality (1.11) to $u_1 - u_2$ and use (4.6), (4.7). By taking into account that

$$\mathrm{vol}(Q_1 \,\Delta\, Q_2) = \int_{S^2} |u_1(z) - u_2(z)| \, d\sigma(z),$$

one can deduce the estimate (4.1).

REMARK. From the above proof it is clear that any set of solutions such that the corresponding areas are uniformly bounded is a class of stability for Problem (F). Moreover, by applying the Hölder inequality to (3.2) and by using the fact that the support function $s(\omega)$ is bounded above by the maximum of the representation $\Phi(z)$, it is easy to deduce that

$$\mathbf{A} \leqslant \int_\Omega |H(\omega)|^p \, d\sigma(\omega) \Big(\max_{z \in S^2} \Phi(z) \Big)^p,$$

for any $p > 1$. Therefore, if we assume an a-priori uniform bound on $\|H\|_{L^p(\Omega)}$, by the arguments of the proof of Theorem 4 the continuous dependence is also guaranteed.

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