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PERIODIC PEAKONS AND CALOGERO–FRANÇOISE FLOWS*

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Abstract It has long been known that a number of periodic completely integrable systems are associated to hyperelliptic curves, for which the Abel map linearizes the flow (at least in part). We show that this is true for a relatively recent such system: the periodic discrete reduction of the shallow water equation derived by Camassa and Holm. The associated spectral problem has the same form and evolves in the same way as the spectral problem for a family of finite-dimensional non-periodic Hamiltonian flows introduced by Calogero and Françoise. We adapt the Weyl function method used earlier by us to solve the peakon problem to give an explicit solution to both the periodic discrete Camassa–Holm system and the (non-periodic) Calogero–Françoise system in terms of theta functions.

Keywords: spectral problems; Weyl function; Abel map

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1. Introduction

One of the striking discoveries made in the study of the Korteweg–de Vries equation (KdV) and other infinite-dimensional integrable systems is that methods of algebraic geometry can be used to linearize and integrate the periodic versions of many such systems, as well as such finite-dimensional counterparts as the Toda lattice (see [**15, 17, 18, 22**–**26, 28, 30**]). In outline, these problems are integrable because they can be expressed as a Lax equation for the evolution of a linear operator. In the periodic case there is an algebraic curve, or family of such curves, attached to a spectral problem for the operator, and the Abel map linearizes the flow of certain spectral data.

In this paper we derive analogous results for two more recently discovered finite-dimensional integrable systems. One system is the periodic discrete reduction of the Camassa–Holm equation. The second is the (non-periodic) Calogero–Françoise system, which generalizes the finite-dimensional reduction of the Camassa–Holm equation. In general, the scattering problem for the Calogero–Françoise system is essentially identical to that of periodic discrete Camassa–Holm. Again, there is an associated algebraic curve, and data that linearizes under the Abel map.

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The Camassa–Holm equation is one of a class of (formally) integrable equations discovered by Fokas and Fuchssteiner [**19**]. It was derived as a shallow water wave equation and studied in detail by Camassa, Holm and collaborators [**1**, **2**, **10**, **11**]. For this equation the analogues of KdV solitons and multisolitons are weak solutions, known as peakons, antipeakons and multipeakons, that have corners. Explicit formulae for general multipeakons were found in [**3**, **4**]. The smooth periodic case has been studied, for example, in [**1**, **2**, **12**–**14**, **27**, **29**]. The present authors found formulae for the periodic two peakon and peakon/antipeakon pair problems in terms of Weierstrass elliptic functions [**6**].

In this paper we present explicit solutions for an arbitrary number of peakons and/or antipeakons in terms of Riemann theta functions. The spectral problem is treated in § 2, and a corresponding Floquet matrix is analysed in § 3. Theta function representations of the spectral data are obtained in §§ 4–6.

The (non-periodic) Calogero–Françoise systems have, in general, a scattering matrix that evolves in the same way as the Floquet matrix for the finite-dimensional periodic Camassa–Holm system. In §§ 7 and 8 we carry over the methods of the earlier sections to these cases.

The flows considered here may have singularities in finite time, depending on the initial conditions and other parameters. The theta function formulae are valid until singularities occur, and they sometimes allow reasonable continuation past singularities. In § 9 we make some observations about the dynamics in various cases.

2. Periodic Camassa–Holm equation: spectral problem(s)

The linear spectral problem associated to the Camassa–Holm evolution has the form

$$L(\lambda)\varphi \equiv D^2\varphi - \nu^2\varphi - 2\nu\lambda m(x)\varphi = 0, \qquad D = \frac{\mathrm{d}}{\mathrm{d}x}, \tag{2.1}$$

with $\nu > 0$. An operator of this form is compatible with a generalized Lax evolution

$$-2\nu\lambda m_t = \frac{\mathrm{d}}{\mathrm{d}t}L(\lambda) = [L(\lambda), B(\lambda)] + 2u_x L(\lambda), \tag{2.2}$$

where

$$B(\lambda) = \left\{\frac{1}{2\nu\lambda} - u(x)\right\} D + \tfrac{1}{2}u_x(x). \tag{2.3}$$

In fact, Equation (2.2) is equivalent to the Camassa–Holm evolution

$$m_t = u_x m + (um)_x, \qquad 2m_x = 4\nu^2 u_x - u_{xxx} \tag{2.4}$$

(cf. the argument in [**5**]). Replacing the second equation with the relation $2m = 4\nu^2 u - u_{xx}$ and substituting in the first equation gives the Camassa–Holm evolution in the form

$$4\nu^2 u_t - u_{xxt} = 2u_x(4\nu^2 u - u_{xx}) + u(4\nu^2 u_{xx} - u_{xxxx}). \tag{2.5}$$

Multipeakon/antipeakon solutions of (2.4) on the line correspond to discrete measures

$$m(x,t) = \sum_j m_j(t)\delta(x - x_j(t)); \tag{2.6}$$

here (2.4) and the spectral problem (2.1) must be interpreted in the sense of distributions (cf. [**4**]). The associated function u is

$$u(x,t) = \frac{1}{2\nu}\sum_j m_j(t)\mathrm{e}^{-2\nu|x-x_j(t)|}. \tag{2.7}$$

In this discrete case the Camassa–Holm evolution is equivalent to the finite-dimensional Hamiltonian system with variables x_j, $1 \leqslant j \leqslant d$, dual variables m_j, $1 \leqslant j \leqslant d$, and Hamiltonian

$$H(x_1,\ldots,x_d,m_1,\ldots,m_d) = \frac{1}{2}\sum_{j,k=1}^{d} m_j m_k G_\nu(x_j - x_k), \tag{2.8}$$

where $G_\nu = \mathrm{e}^{-2\nu|x|}/2\nu$ is the integrable fundamental solution for $2\nu^2 - D^2/2$.

We assume here that m is periodic with respect to x, with period $X > 0$ and is supported on d points in each period interval. Thus the index j runs through the integers, and

$$x_{j+d} = x_j + X, \quad m_{j+d} = m_j, \quad j \in \mathbb{Z}, \tag{2.9}$$

and (at least for most values of t)

$$m_j(t) \neq 0, \quad x_j(t) < x_{j+1}(t), \quad j \in \mathbb{Z}.$$

This is equivalent to the problem with d particles on a circle of length X, and again the Camassa–Holm evolution is equivalent to a finite-dimensional Hamiltonian system. Here the Hamiltonian is

$$H(x_1,\ldots,x_d,m_1,\ldots,m_d) = \frac{1}{2}\sum_{j,k=1}^{d} m_j m_k G_{\nu,X}(x_j - x_k), \tag{2.10}$$

where $G_{\nu,X}$ is the periodization of G_ν,

$$G_{\nu,X}(x) = \frac{1}{2\nu}\sum_{j=-\infty}^{\infty} \mathrm{e}^{-2\nu|x-jX|}. \tag{2.11}$$

The series is easily summed to give

$$G_{\nu,X}(x) = \frac{\cosh(2\nu|x| - \nu X)}{2\nu\sinh\nu X}, \quad |x| \leqslant X. \tag{2.12}$$

We consider the spectral problem (2.1) at fixed time $t = 0$, and write $x_j = x_j(0)$, $m_j = m_j(0)$. We are assuming

$$x_1 < x_2 < \cdots < x_d < x_{d+1} = x_1 + X, \qquad \prod_{j=1}^{d} m_j \neq 0. \tag{2.13}$$

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Suppose that φ is a solution of (2.1). On each interval $I_j = (x_{j-1}, x_j)$, φ is a linear combination $a_j e^{\nu x} + b_j e^{-\nu x}$. Equation (2.1) itself translates into a continuity equation and a jump condition:

$$\begin{aligned} a_{j+1}e^{\nu x_j} + b_{j+1}e^{-\nu x_j} &= a_j e^{\nu x_j} + b_j e^{-\nu x_j}, \\ a_{j+1}e^{\nu x_j} - b_{j+1}e^{-\nu x_j} &= a_j e^{\nu x_j} - b_j e^{-\nu x_j} + \lambda m_j(a_j e^{\nu x_j} + b_j e^{-\nu x_j}). \end{aligned} \tag{2.14}$$

Thus the transition is given by

$$\begin{bmatrix} a_{j+1} \\ b_{j+1} \end{bmatrix} = \begin{bmatrix} 1+\lambda m_j & \lambda m_j e^{-2\nu x_j} \\ -\lambda m_j e^{2\nu x_j} & 1-\lambda m_j \end{bmatrix} \begin{bmatrix} a_j \\ b_j \end{bmatrix}. \tag{2.15}$$

Denoting the transition matrix in (2.15) by $T_j(\lambda)$, the Floquet matrix for this problem, giving the transition from the representation on the interval $(x_d - X, x_1) = (x_0, x_1)$ to the representation on the interval $(x_d, x_1 + X) = (x_d, x_d + 1)$, is the product

$$\Phi(\lambda) = T_d(\lambda)T_{d-1}(\lambda)\cdots T_1(\lambda). \tag{2.16}$$

It follows by induction that

$$\Phi(\lambda) = I + \lambda \begin{bmatrix} M & M_- \\ -M_+ & -M \end{bmatrix} + O(\lambda^2), \quad \lambda \to 0, \tag{2.17}$$

and

$$\Phi(\lambda) = \lambda^d \prod_{j=1}^{d} m_j \prod_{j=2}^{d}(1 - e^{-2\nu(x_j - x_{j-1})}) \begin{bmatrix} 1 & e^{-2\nu x_1} \\ -e^{2\nu x_d} & -e^{2\nu(x_d - x_1)} \end{bmatrix} + O(\lambda^{d-1}), \quad \lambda \to \infty, \tag{2.18}$$

where

$$M = \sum_{j=1}^{d} m_j, \qquad M_\pm = \sum_{j=1}^{d} m_j e^{\pm 2\nu x_j}. \tag{2.19}$$

Thus the entries $\Phi_{ij}(\lambda)$ are polynomials of degree d in λ. Since $\det T_j = 1$, it follows that $\det \Phi = 1$.

Of course, one could analyse the spectral problem using any choice of d consecutive indices. For example, starting at x_k, the corresponding quantities $M_\pm$ would be

$$M_\pm(k) = \sum_{j=k}^{k+d-1} m_j e^{\pm 2\nu x_j}.$$

We note that the data $\{x_j\}$, $\{m_j\}$ can be reconstructed from $d+1$ consecutive values of $M_\pm(k)$.

Proposition 2.1. *We have*

$$e^{4\nu x_k} = -e^{-4\nu X}\frac{M_+(k+1) - M_+(k)}{M_-(k+1) - M_-(k)}$$

and

$$m_k^2 = -\frac{[M_+(k+1) - M_+(k)][M_-(k+1) - M_-(k)]}{4\sinh^2(\nu X)}.$$

Proof. These identities follow easily from the identities

$$M_\pm(k+1) - M_\pm(k) = e^{\pm 2\nu x_{k+d}} m_{k+d} - e^{\pm 2\nu x_k} m_k$$

and the periodicity conditions $x_{k+d} = x_k + X$, $m_{k+d} = m_k$. □

In the remainder of this section we discuss the periodic and anti-periodic spectra of the operator (2.2) and their relationship to certain other spectral data.

Note that $\varphi(x+X) \equiv \varphi(x)$ if and only if the coefficients of φ with respect to the basis $e^{\nu x}$, $e^{-\nu x}$ on the interval $(x_d, x_1 + X)$ are the same as the coefficients with respect to the basis $e^{\nu(x+X)}$, $e^{-\nu(x+X)}$ on the interval $(x_d - X, x_1)$. Therefore, in order to compare wave functions φ near $x = x_1$ and near $x = x_1 + X$ with periodicity in mind, we renormalize on the subinterval $(x_d - X, x_1)$ by writing φ as a linear combination of $e^{\nu(x+X)}$ and $e^{-\nu(x+X)}$ rather than $e^{\nu x}$ and $e^{-\nu x}$. The associated transition matrix that relates the coefficients of φ on $(x_d, x_1 + X)$ to the coefficients with respect to the revised basis on $(x_d - X, x_1)$ is the renormalized Floquet matrix

$$\Psi(\lambda) = \Phi(\lambda)E_{\nu X} = \Phi(\lambda)\begin{bmatrix} e^{\nu X} & 0 \\ 0 & e^{-\nu X}\end{bmatrix}. \tag{2.20}$$

We show later that the trace

$$P(\lambda) = \operatorname{tr}\Psi(\lambda) = \Psi_{11}(\lambda) + \Psi_{22}(\lambda) \tag{2.21}$$

is invariant under the flow (2.4).

Choose a constant a so that $x_d - X < a < x_1$ and consider the operator $L(\lambda)$ on the interval $[a, a+X]$. The *periodic spectrum* consists of those λ for which there is a solution $\varphi \neq 0$ of (2.1) with periodic boundary conditions: $\varphi(a+X) = \varphi(a)$, $D\varphi(a+X) = D\varphi(a)$. The *anti-periodic spectrum* consists of those λ for which there is a solution $\varphi \neq 0$ such that $\varphi(a+X) = -\varphi(a)$, $D\varphi(a+X) = -D\varphi(a)$. Note that 0 cannot lie in the periodic or anti-periodic spectrum. Let φ be a wave function and let $v = (c, d)^{\mathrm{t}}$, where

$$\varphi(x) = ce^{\nu x} + de^{-\nu x}, \quad x \in (x_d - X, x_1).$$

Then φ is periodic if and only if v is an eigenvector of $\Psi(\lambda)$ with eigenvalue 1, and anti-periodic if and only if v is an eigenvector of $\Psi(\lambda)$ with eigenvalue -1. Thus the periodic and anti-periodic spectra consist of those λ such that $\Psi(\lambda) - \lambda I$ or $\Psi(\lambda) + \lambda I$ are singular, respectively.

The *auxiliary spectrum* consists of those λ for which there is a solution $\varphi \neq 0$ that satisfies one of the following pairs of boundary conditions:

$$D\varphi(a) - \nu\varphi(a) = 0 = D\varphi(a+X) - \nu\varphi(a+X), \tag{2.22}$$

$$D\varphi(a) + \nu\varphi(a) = 0 = D\varphi(a+X) + \nu\varphi(a+X). \tag{2.23}$$

Note that 0 is in the auxiliary spectrum, since the wave functions $\mathrm{e}^{\nu x}$ and $\mathrm{e}^{-\nu x}$ satisfy (2.22) and (2.23), respectively. We remark here that an auxiliary spectrum typically plays a role in the analysis of such periodic problems, but usually it is taken to be the Dirichlet spectrum, e.g. in the Camassa–Holm case (see [**14**]).

Lemma 2.2. *The periodic and anti-periodic spectra of $L(\lambda)$ are real and non-zero, and coincide with the zeros of $P(\lambda) - 2$ and with the zeros of $P(\lambda) + 2$, respectively.*

The spectra with respect to the boundary conditions (2.22) and (2.23) are real and coincide with the zeros of Ψ_{21} and Ψ_{12}, respectively.

Proof. To prove reality, we note that if $L(\lambda)\varphi = 0$, then

$$\begin{aligned} 2\nu\lambda \sum_{j=1}^{d} |\varphi(x_j)|^2 m_j &= \int_a^{a+X} [D^2\varphi(x) - \nu^2\varphi(x)]\bar{\varphi}(x)\,\mathrm{d}x \\ &= D\varphi \cdot \bar{\varphi}|_a^{a+X} - \int_a^{a+X} \{|D\varphi(x)|^2 + \nu^2|\varphi(x)|^2\}\,\mathrm{d}x. \end{aligned} \tag{2.24}$$

If φ satisfies periodic or anti-periodic boundary conditions, then the boundary terms vanish and it follows that $\varphi \equiv 0$ or λ is real and non-zero. If φ satisfies (2.22), then

$$D\varphi \cdot \bar{\varphi}|_a^{a+X} = \nu\varphi\bar{\varphi}|_a^{a+X} = \nu \int_a^{a+X} \{D\varphi(x)\overline{\varphi(x)} + \varphi(x)\overline{D\varphi(x)}\}\,\mathrm{d}x. \tag{2.25}$$

The same calculation holds for boundary conditions (2.23), with a change of sign. Therefore, with $D\varphi \pm \nu\varphi = 0$ at a and $a + X$, we may sum (2.24) and (2.25) to obtain

$$2\nu\lambda \sum_{j=1}^{d} |\varphi(x_j)|^2 m_j = -\int_a^{a+X} |D\varphi(x) \pm \nu\varphi(x)|^2\,\mathrm{d}x.$$

Again, it follows that λ is real.

We observed above that the periodic and anti-periodic spectra consist of those λ such that $\Psi(\lambda) - I$ and $\Psi(\lambda) + I$ are singular, respectively. Since $\det \Psi = 1$, these conditions are equivalent to $P(\lambda) = 2$ and $P(\lambda) = -2$, respectively. A solution $\varphi \neq 0$ satisfies (2.22) if and only if the initial coefficients have the form $(c, 0) = v^{\mathrm{t}}$, and the final coefficients $(\Psi(\lambda)v)^{\mathrm{t}}$ have the form $(d, 0)$, so the necessary and sufficient condition is $\Psi_{21} = 0$; the proof for boundary conditions (2.23) is similar. □

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Theorem 2.3. *Counting multiplicity, $P^2 - 4$ has $2d$ real roots. Let these be numbered $\lambda_1, \dots, \lambda_{2d}$ in increasing order.*

(a) *Each of the pairs $\lambda_{2j-1}, \lambda_{2j}$ contains one periodic and one anti-periodic eigenvalue. Each of the pairs $\lambda_{2j}, \lambda_{2j+1}$ consists of two periodic or two anti-periodic eigenvalues, or of a root of multiplicity two.*

(b) *The roots of $\lambda^{-1}\Psi_{12}$ and $\lambda^{-1}\Psi_{21}$ are real and simple. Each (possibly degenerate) interval $I_j = [\lambda_{2j}, \lambda_{2j+1}]$, $j = 1, \dots, d-1$, contains exactly one root of $\lambda^{-1}\Psi_{12}$ and one root of $\lambda^{-1}\Psi_{21}$*

(c) *If all the m_j are positive (respectively, negative), then all the λ_j are negative (respectively, positive). If exactly k of the m_j are positive, $1 \leqslant k < d$, then 0 lies in the interval $I_k = [\lambda_{2k}, \lambda_{2k-1}]$.*

Proof. The proof is by induction on d. For $d = 1$, it follows from (2.17) and (2.20) that

$$P(\lambda) = \operatorname{tr}\Psi(\lambda) = 2\cosh\nu X + 2\lambda m_1 \sinh\nu X.$$

Therefore, the roots of $P \pm 2$ are

$$-\frac{1}{2m_1\cosh(\nu X/2)}, \qquad -\frac{1}{2m_1\sinh(\nu X/2)}.$$

This verifies (a) and (c) when $d = 1$; (b) is vacuous.

Suppose now that (a), (b), and (c) hold for a given $d \geqslant 1$. We claim that these conditions are stable under continuous, sign-preserving changes of the m_j. Under such a perturbation, the roots λ_j of $P^2 - 4$ move continuously (in the complex plane), but they remain non-zero. The roots of $\lambda^{-1}\Psi_{12}$ and of $\lambda^{-1}\Psi_{21}$ remain real, so long as they remain simple. The identity

$$\begin{aligned} P^2 - 4 &= (\Psi_{11} + \Psi_{22})^2 - 4(\Psi_{11}\Psi_{22} - \Psi_{21}\Psi_{12}) \\ &= (\Psi_{11} - \Psi_{22})^2 + 4\Psi_{21}\Psi_{12} \end{aligned} \tag{2.26}$$

implies that each of these latter roots lies in a (maximal) interval where $P+2$ and $P-2$ have the same sign. Therefore, the (possibly degenerate) interval I_j that contains such a root cannot vanish under perturbation. It follows that the roots λ_j remain real under perturbation, $2 \leqslant j \leqslant 2d-1$, and that the roots of $\lambda^{-1}\Psi_{12}$ and of $\lambda^{-1}\Psi_{21}$ remain (real and) simple. Moreover, the configuration of these intervals with respect to the origin remains the same, since $\lambda_j \neq 0$ throughout. As for λ_1 and λ_{2d}, note that $P^2 - 4$ is negative between λ_1 and λ_2 and has no roots to the left, so either $P = -2$ at λ_2 and $P \to +\infty$ as $\lambda \to -\infty$, or $P = 2$ at λ_2 and $P \to +\infty$ as $\lambda \to -\infty$. This behaviour is stable also, so the real root λ_1 cannot vanish. The same argument applies to λ_{2d}.

In view of these remarks, we may carry out the inductive step by assuming the truth of (a), (b), (c) for a trace P_0 associated to a configuration with x_j, m_j, $1 \leqslant j < d$, and verifying them for P_ε associated to the configuration with additional $x_d \in (x_{d-1}, x_1 + X)$ and $m_d = \varepsilon \approx 0$. By stability, the result then carries over to arbitrary values of m_d.

If $m_d = \varepsilon$ is small enough, then the previous stability argument implies that there are roots of $P_\varepsilon \pm 2$ near the roots of $P_0 \pm 2$, and the same is true for the auxiliary spectra. Note that (2.18) and (2.20) imply that the coefficient of the leading coefficient in P_ε is a positive multiple of

$$\prod_{j=1}^{d} m_j (\mathrm{e}^{\nu X} - \mathrm{e}^{2\nu(x_d - x_1) - \nu X}).$$

The last factor on the right is positive, so the leading coefficient has the same sign as the product of the m_j. It follows easily that if ε is sufficiently small and positive, $P_\varepsilon \pm 2$ will each have one 'new' root lying far to the left of the roots of $P_0 \pm 2$. Similarly, if ε is small but negative, each will have one 'new' root lying far to the right of the roots of $P_0 \pm 2$. Moreover, the configuration of roots of $P_\varepsilon \pm 2$ will continue to satisfy (a) and (c).

The new $\lambda^{-1}\Psi_{12}$ and $\lambda^{-1}\Psi_{21}$ will also have roots near the old roots and 'new' roots lying far to the left or right, according as ε is positive or negative. To complete the proof, we need to show that these roots lie closer to the origin than do the new roots of $P_\varepsilon \pm 2$.

To lighten the notational burden, let $y_j = \mathrm{e}^{2\nu x_j}$. By (2.16) we have, for $m_d = \varepsilon \neq 0$, that up to a scalar factor, $\Phi(\lambda)$ is

$$\begin{aligned} &\varepsilon\lambda^d \begin{bmatrix} 1 - y_{d-1}y_d^{-1} & y_1^{-1} - y_{d-1}(y_1 y_d)^{-1} \\ y_{d-1} - y_d & y_{d-1}y_1^{-1} - y_d y_1^{-1} \end{bmatrix} \\ &\quad + \lambda^{d-1} \begin{bmatrix} 1 & y_1^{-1} \\ -y_{d-1} & -y_{d-1}y_1^{-1} \end{bmatrix} + O(\varepsilon\lambda^{d-1} + \lambda^{d-2}) \\ &\qquad = \varepsilon(y_d - y_{d-1})\lambda^d \begin{bmatrix} y_d^{-1} & (y_1 y_d)^{-1} \\ -1 & -y_1^{-1} \end{bmatrix} + \lambda^{d-1} \begin{bmatrix} 1 & y_1^{-1} \\ -y_{d-1} & -y_{d-1}y_1^{-1} \end{bmatrix} \\ &\qquad\qquad + O(\varepsilon\lambda^{d-1} + \lambda^{d-2}). \end{aligned} \tag{2.27}$$

Therefore, the new zeros of $\Psi_{21} = \Phi_{21}\mathrm{e}^{\nu X}$ and $\Psi_{12} = \Phi_{12}\mathrm{e}^{-\nu X}$ occur at

$$\lambda = -\frac{1}{\varepsilon(y_d - y_{d-1})} y_{d-1} + O(1), \qquad \lambda = -\frac{1}{\varepsilon(y_d - y_{d-1})} y_d + O(1), \tag{2.28}$$

respectively. By (2.20), up to a scalar factor $P(\lambda)$ is

$$\varepsilon(y_d - y_{d-1})\mathrm{e}^{-X}\lambda^d \left(\frac{\mathrm{e}^{2\nu X}}{y_d} - \frac{1}{y_1}\right) + \lambda^{d-1}\left(\mathrm{e}^{2\nu X} - \frac{y_{d-1}}{y_1}\right) + O(\varepsilon\lambda^{d-1} + \lambda^{d-1}).$$

Therefore, the new zeros of $P \pm 2$ occur at

$$\lambda = -\frac{1}{\varepsilon(y_d - y_{d-1})} \frac{\mathrm{e}^{2\nu X} y_1 - y_{d-1}}{\mathrm{e}^{2\nu X} y_1 - y_d} y_d + O(1). \tag{2.29}$$

It follows from (2.28) and (2.29) that, for small $m_d = \varepsilon$, the new zeros of Ψ_{21} and Ψ_{12} are closer to the origin than the new zeros of $P \pm 2$. This completes the proof. □

3. Spectral analysis of Ψ; algebraic curve and Weyl function

Throughout this and the following three sections, we assume that $d > 1$ and that the x_j, m_j take 'generic' values. For example, we assume that the roots λ_j of $P^2 - 4$ are simple, and that zero is a simple root of Ψ_{12} and of Ψ_{21}. The last pair of assumptions is equivalent to assuming that $M_\pm \neq 0$ (see (2.17)). In view of Theorem 2.3, these assumptions imply that all roots of Ψ_{12} and of Ψ_{21} are simple. More such assumptions will be made, some tacitly, as we proceed.

For given λ, the eigenvalues $\mu_\pm(\lambda)$ of the matrix $\Psi(\lambda)$ are the solutions of the equation $\mu^2 - P(\lambda)\mu + 1 = 0$; therefore, they define a single-valued function

$$\mu(\lambda, z) = \tfrac{1}{2}(P(\lambda) + z) \tag{3.1}$$

on the curve

$$\Gamma = \Gamma_P = \{(\lambda, z) \in \mathbb{C}^2 : z^2 = P(\lambda)^2 - 4\}. \tag{3.2}$$

The curve Γ is elliptic (genus 1) if $d = 2$, hyperelliptic of genus $g = d-1$ if $d > 2$. We refer to [**16**] for various results from the theory of hyperelliptic curves and theta functions.

We represent Γ as a double cover of the Riemann sphere by cutting the sphere along the real intervals $[\lambda_{2j-1}, \lambda_{2j}]$, $j = 1, \dots, d$. As $\lambda \to \infty$,

$$z = \pm P(\lambda) + O(\lambda^{-d}). \tag{3.3}$$

The functions z and μ can be considered as single-valued functions on this double cover.

We take the 'upper' and 'lower' sheets of the cover to be those on which $z \sim P$ and $z \sim -P$ at infinity, respectively. For convenience, we adopt the following notational convention: λ denotes a point of the Riemann sphere, $\lambda^\pm$ the corresponding point on the double cover Γ, on the upper $(+)$ or lower $(-)$ sheet. Thus (3.1) implies that

$$\left.\begin{aligned} \mu(\lambda^+) &= P(\lambda) + O(\lambda^{-d}), \qquad &\lambda \to \infty, \\ \mu(\lambda^-) &= O(\lambda^{-d}), \qquad &\lambda \to \infty. \end{aligned}\right\} \tag{3.4}$$

Lemma 3.1. *If λ_0 is a zero of $\Psi_{12}\Psi_{21}$, then $z(\lambda_0^+) = \Psi_{11}(\lambda_0) - \Psi_{22}(\lambda_0)$ and thus $\mu(\lambda_0^+) = \Psi_{11}(\lambda_0)$. Similarly, $z(\lambda_0^-) = \Psi_{22}(\lambda_0) - \Psi_{11}(\lambda_0)$ and thus $\mu(\lambda_0^-) = \Psi_{22}(\lambda_0)$.*

In particular,

$$z(0^\pm) = \pm 2 \sinh \nu X, \qquad \mu(0^\pm) = \mathrm{e}^{\pm \nu X}. \tag{3.5}$$

Proof. According to (2.26), $z^2 = (\Psi_{11} - \Psi_{22})^2$ at roots of $\Psi_{12}\Psi_{21} = \Phi_{12}\Phi_{21}$. To determine the sign, we use a continuity argument, treating the period X as a parameter. Note that Φ is independent of X, and as $X \to \infty$, $\mathrm{e}^{-\nu X} P \to \Phi_{11}$ at each point. Taking $\lambda \sim \infty$, we find

$$\mathrm{e}^{-\nu X} z(\lambda^+) \sim \Phi_{11} \sim \mathrm{e}^{-\nu X}(\Psi_{11} - \Psi_{22}).$$

The relation must hold with this choice of sign on the entire upper sheet, and, in particular, at λ_0^+. The same argument applies to λ_0^-. This verifies the asserted choice of signs.

It follows from (2.26) and (2.17) that, as $\lambda \to 0$,

$$\left.\begin{aligned} \Psi_{11} &= \mathrm{e}^{\nu X} + O(\lambda), & \Psi_{22} &= \mathrm{e}^{-\nu X} + O(\lambda), \\ \Psi_{12} &= \lambda \mathrm{e}^{-\nu X} M_- + O(\lambda^2), & \Psi_{21} &= -\lambda \mathrm{e}^{\nu X} M_+ + O(\lambda^2). \end{aligned}\right\} \tag{3.6}$$

The identities (3.5) are a consequence. □

Except possibly at the branch points λ_j, $\Psi(\lambda^\pm) - \mu(\lambda^\pm)$ has a one-dimensional null space, so there is an induced map from Γ to complex projective space which we normalize by taking the eigenvector in the form $(1, w(\lambda^\pm))^{\mathrm{t}}$. The *Weyl function* $w(\lambda^\pm)$ is characterized by the matrix equation

$$\Psi(\lambda^\pm)\begin{bmatrix} 1 \\ w(\lambda^\pm) \end{bmatrix} = \mu(\lambda^\pm)\begin{bmatrix} 1 \\ w(\lambda^\pm) \end{bmatrix},$$

which leads to two equivalent equations for w,

$$w = \frac{\mu - \Psi_{11}}{\Psi_{12}} = \frac{\Psi_{21}}{\mu - \Psi_{22}}. \tag{3.7}$$

Theorem 3.2. *The Weyl function w is meromorphic on the curve Γ. Its zeros are simple and occur at the points on the upper sheet that correspond to zeros of Ψ_{21}; its poles are simple and occur at the points on the lower sheet that correspond to zeros of Ψ_{12}.*

Proof. It follows from Equations (3.7) that w is meromorphic and that its zeros on either sheet correspond to a subset of the zeros of Ψ_{21}, while its poles on either sheet correspond to a subset of the zeros of Ψ_{12}.

Lemma 3.1 and Equations (3.7) imply that the potential poles on the upper sheet and potential zeros on the lower sheet do not occur, while the potential zeros on the upper sheet and poles on the lower sheet do occur. □

We turn next to the behaviour of w at the points $0^\pm$ and $\infty^\pm$ on the upper and lower sheets.

Theorem 3.3. *The Weyl function has the properties*

$$w(\lambda^+) = -\frac{\lambda M_+ \mathrm{e}^{\nu X}}{2\sinh \nu X} + O(\lambda^2), \quad \lambda \to 0, \tag{3.8}$$

$$w(\lambda^-) = -\frac{2\sinh \nu X}{\lambda M_- \mathrm{e}^{-\nu X}} + O(1), \quad \lambda \to 0, \tag{3.9}$$

$$w(\lambda^+) = -\mathrm{e}^{2\nu x_d}\left(1 - \frac{1}{\lambda m_d} + \frac{\mathrm{e}^{2\nu x_d}}{(\lambda m_d)^2(\mathrm{e}^{2\nu x_d} - \mathrm{e}^{2\nu x_{d-1}})}\right) + O(\lambda^{-3}), \quad \lambda \to \infty, \tag{3.10}$$

$$w(\lambda^-) = -\mathrm{e}^{2\nu x_{d+1}}\left(1 - \frac{1}{\lambda m_1} - \frac{\mathrm{e}^{2\nu x_2}}{(\lambda m_1)^2(\mathrm{e}^{2\nu x_2} - \mathrm{e}^{2\nu x_1})}\right) + O(\lambda^{-3}), \quad \lambda \to \infty. \tag{3.11}$$

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Proof. According to (3.5) and (3.6), near 0^+, $\mu - \Psi_{22} \sim e^{\nu X} - e^{-\nu X}$, while near 0^-, $\mu - \Psi_{11} \sim e^{-\nu X} - e^{\nu X}$. In view of (3.6) and (3.7), the results (3.8) and (3.9) follow.

As $\lambda^+ \to \infty^+$, $\mu - \Psi_{22} = \Psi_{11} + O(\lambda^{-d})$, so

$$w(\lambda^+) = \frac{\Psi_{21}}{\Psi_{11}} + O(\lambda^{-2d}) = \frac{\Phi_{21}}{\Phi_{11}} + O(\lambda^{-2d}). \tag{3.12}$$

By (2.18), the last quotient in (3.12) approaches $-e^{2\nu x_d}$, so we have obtained the principal term in (3.10). However, all terms in (3.10) can be obtained in a different way. Starting with the vector $v_0 = (1,0)^t$, use the transition matrices T_j of (2.15) to define vectors $v_j(\lambda) = T_j(\lambda)v_{j-1}$. Set $v_j = (a_j, b_j)^t$ and $r_j = b_j/a_j$. Then $r_d = \Phi_{21}/\Phi_{11}$. At each stage, r_j is regular at ∞. By the construction, and the form of T_d, we have

$$\begin{aligned}\frac{\Phi_{21}}{\Phi_{11}} &= r_d \\ &= \frac{-\lambda m_d e^{2\nu x_d} a_{d-1} + (1-\lambda m_d) b_{d-1}}{(1+\lambda m_d) a_{d-1} + \lambda m_d e^{-2\nu x_d} b_{d-1}} \\ &= \frac{-\lambda m_d e^{2\nu x_d} + (1-\lambda m_d) r_{d-1}}{1 + \lambda m_d + \lambda m_d e^{-2\nu x_d} r_{d-1}} \\ &= -e^{2\nu x_d}\left(1 - \frac{1}{\lambda m_d} + \frac{e^{2\nu x_d}}{(\lambda m_d)^2 (e^{2\nu x_d} + r_{d-1}) + \lambda m_d e^{2\nu x_d}}\right).\end{aligned} \tag{3.13}$$

This gives the first two terms of the expansion (3.10). The same calculation applies to r_{d-1}, so its leading term is $-e^{2\nu x_{d-1}}$, and we obtain the third term in the expansion.

Similarly, as $\lambda^- \to \infty^-$, $\mu - \Psi_{11} = -\Psi_{11} + O(\lambda^{-d})$, so

$$w(\lambda^-) \sim -\frac{\Psi_{11}}{\Psi_{12}} = -e^{2\nu X}\frac{\Phi_{11}}{\Phi_{12}}.$$

It follows from this and (2.18) that the leading term in (3.11) is $e^{2\nu x_{d+1}}$. Further terms may be computed as above, by taking row vectors $v_0 = (1,0)$, $v_j = v_{j-1}T_{d-j+1} = (a_j, b_j)$, so that $a_d/b_d = \Phi_{11}/\Phi_{12}$. We omit the details. □

4. Theta function representations, I

In the two-sheet representation of the genus $g = d-1$ curve Γ, we use the left-most g cuts to determine cycles $a_1, \dots, a_{d-1}$. We choose dual cycles $b_1, \dots, b_g$, to obtain a standard basis for the homology of Γ. Then there are unique holomorphic one-forms ω_j with

$$\int_{a_k} \omega_j = 2\pi i \delta_{jk}, \quad \int_{b_k} \omega_j = B_{jk}, \quad j,k = 1,\dots,g.$$

The Jacobi variety $J(\Gamma)$ is $\mathbb{C}^g/\Lambda$, where Λ is the lattice

$$\Lambda = \left\{ w \in \mathbb{C}^g : w_j = 2\pi i m_j + \sum_k n_{kj} B_{jk},\ m_k, n_{kj} \in \mathbb{Z} \right\}.$$

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The Abel map $A : \Gamma \to J(\Gamma)$ is determined by choosing a point $p_0 \in \Gamma$ and defining

$$A_k(p) = \int_{p_0}^{p} \omega_k, \quad p \in \Gamma, \quad k = 1, \dots, g.$$

The same notation is commonly used for the induced map from the symmetric product $S^{d-1}\Gamma$ to $J(\Gamma)$, but to avoid confusion we denote the latter by A^{s},

$$A^{\mathrm{s}}(p_1, \dots, p_g) = \sum_{j=1}^{g} A(p_j).$$

We will take $p_0 = \infty^+$, so $A(\infty^+) = 0$.

Let λ_{1j}, $j = 1, \dots, g = d-1$, denote the non-zero roots of Φ_{12}, and let λ_{2j}, $j = 1, \dots, g$, denote the non-zero roots of Φ_{21}. According to 3.2, the zeros and poles of the meromorphic function w are simple and occur at

$$0^+, \lambda_{21}^+, \dots, \lambda_{2g}^+, \qquad 0^-, \lambda_{11}^-, \dots, \lambda_{1g}^-. \tag{4.1}$$

By a theorem of Abel, the sum of the values of the Abel map at the zeros equals the sum of the values at the poles,

$$A(0^+) + \sum_{j=1}^{g} A(\lambda_{2j}^+) = A(0^-) + \sum_{j=1}^{g} A(\lambda_{1j}^-). \tag{4.2}$$

Let $\theta : J(\Gamma) \to \mathbb{C}$ be the Riemann theta function and K the Riemann vector in $J(\Gamma)$; these have the property that if the function

$$f(p) = \theta(A(p) - A^{\mathrm{s}}(p_1, \dots, p_g) - K)$$

is not identically zero, then its zeros are $p_1, \dots, p_g$ (see [**16**, Theorem 2.4.2]).

There is a unique Abelian differential (meromorphic one-form) of the third kind $\omega_{0^+0^-}$ with residue ± 1 at $0^\pm$ and integral 0 on each cycle a_k.

Theorem 4.1. *The Weyl function w is*

$$w(\lambda^\pm) = C \frac{\theta(A(\lambda^\pm) - \xi^+)}{\theta(A(\lambda^\pm) - \xi^-)} \exp\biggl(\int_{\infty^+}^{\lambda^\pm} \omega_{0^+0^-}\biggr), \tag{4.3}$$

where

$$\left.\begin{aligned} \xi^+ &= A^{\mathrm{s}}(\lambda_{21}^+, \dots, \lambda_{2g}^+) + K, \\ \xi^- &= A^{\mathrm{s}}(\lambda_{11}^-, \dots, \lambda_{1g}^-) + K = \xi^+ + A(0^+) - A(0^-), \end{aligned}\right\} \tag{4.4}$$

and the constant C is given by

$$C = -\frac{M_+ \mathrm{e}^{\nu X}}{2\sinh \nu X} \cdot \frac{\theta(A(0^+) - \xi^-)}{\theta(A(0^+) - \xi^+)} = -\frac{2\sinh \nu X}{M_- \mathrm{e}^{-\nu X}} \cdot \frac{\theta(A(0^-) - \xi^-)}{\theta(A(0^-) - \xi^+)}. \tag{4.5}$$

Proof. Properties of the theta function imply that the function on the right in (4.3) is single valued and meromorphic on Γ. According to the previous remarks, this function has the same zeros and poles as w; therefore, by the Riemann–Roch theorem, w has the form (4.3), for some choice of C. To determine C, we use (3.8), (3.9), and the fact that the right-most factor in the product on the right in (4.3) is $\lambda + O(\lambda^2)$ as $\lambda^+ \to 0^+$ and $1/\lambda + O(1)$ as $\lambda^- \to 0^-$. The results are the two expressions (4.5). □

For later use we extract from (4.4) the identity

$$A(0^+) - \xi^- = A(0^-) - \xi^+. \tag{4.6}$$

As a corollary, we obtain an explicit theta function representation of x_d and a (less explicit) representation of m_d. We give here the formula for the former, the latter will be discussed in § 6. Recall that $A(\infty^+) = 0$.

Corollary 4.2. *The position x_d can be obtained from*

$$\mathrm{e}^{2\nu x_d} = \frac{M_+ \mathrm{e}^{\nu X}}{2\sinh \nu X} \cdot \frac{\theta(A(0^+) - \xi^-)\theta(\xi^+)}{\theta(A(0^+) - \xi^+)\theta(\xi^-)}. \tag{4.7}$$

Remark 4.3. Theorem 3.3 leads to corresponding representations of x_1 and m_1 from the first two terms in the expansion of w at ∞^-. Moreover, the proof shows that x_2 and m_2 can be obtained from the next two terms, and so on, but the relationships become more complicated. Thus it is natural to try to obtain the other m_j and x_j by examining the Weyl functions associated with different starting positions. This is the discrete analogue of a method used for continuous periodic problems of this kind, e.g. by Constantin and McKean [**14**] for the Camassa–Holm equation.

Lemma 4.4. *Let $\tilde{\Psi}$, $\tilde{P}$, and $\tilde{w}$ denote the Floquet matrix, trace and Weyl function associated to the starting position x_2 in place of x_1. Then $\tilde{P} = P$, so that the associated curve $\tilde{\Gamma} = \Gamma$.*

Proof. That $\tilde{P} = P$ follows from the fact that these polynomials have the same roots (the periodic and anti-periodic spectra), and the same value at $\lambda = 0$. For a purely algebraic proof, note that $E_{\nu X}^{-1} T_1 E_{\nu X} = T_{d+1}$. Therefore,

$$\begin{aligned} \tilde{\Psi} &= \tilde{\Phi} E_{\nu X} \\ &= T_{d+1} \cdots T_2 E_{\nu X} \\ &= T_{d+1} \Phi T_1^{-1} E_{\nu X} \\ &= T_{d+1} \Psi E_{\nu X}^{-1} T_1^{-1} E_{\nu X} \\ &= T_{d+1} \Psi T_{d+1}^{-1}. \end{aligned} \tag{4.8}$$

Therefore, the traces $\tilde{P}$ and P are the same. □

Lemma 4.5. *There is a meromorphic function on Γ whose zeros are simple and equal to the non-zero poles of w, and whose poles are simple and equal to the non-zero poles of $\tilde{w}$, in addition to a simple zero at ∞^+ and a simple pole at ∞^-.*

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Proof. The identity (4.8) shows that the vector $(a,b)^{\mathrm{t}} = T_{d+1}^{-1}(1,\tilde{w})^{\mathrm{t}}$ is an eigenvector for Ψ, so the quotient $b/a = w$. Now

$$a = (1-\lambda m_{d+1}) - \lambda m_{d+1}\mathrm{e}^{-2\nu x_{d+1}}\tilde{w} = 1 - \lambda m_{d+1}(1+\mathrm{e}^{-2\nu x_{d+1}}\tilde{w}). \tag{4.9}$$

This shows that a is meromorphic. It is regular at $0^{\pm}$, so its finite poles are the non-zero poles of $\tilde{w}$, namely the points on the lower sheet that correspond to non-zero roots of $\tilde{\Psi}_{12}$. Moreover, it follows from (4.9) and (3.10) (for $\tilde{w}$) that a has a simple zero at ∞^{+}. Similarly, it follows from (4.9) and (3.11) that a has a simple pole at ∞^{-}. Since $b/a = w$, it follows that the zeros of a are included among the non-zero poles of w. Since the number of zeros must equal the number of poles, we conclude that all the non-zero poles of w are zeros of a. □

Theorem 4.6. *The Weyl function w_k that corresponds to the configuration*

$$x_{k+1}, x_{k+2}, \dots, x_{k+d}$$

is

$$w_k(\lambda^{\pm}) = C_k \frac{\theta(A(\lambda^{\pm}) - \xi^{+} + kA(\infty^{-}))}{\theta(A(\lambda^{\pm}) - \xi^{-} + kA(\infty^{-}))} \exp\biggl(\int_{\infty^{+}}^{\lambda^{\pm}} \omega_{0^{+}0^{-}}\biggr), \tag{4.10}$$

for some choice of the constant C_k.

Proof. It is enough to prove this for $k=1$. It follows from Theorem 4.3 and Lemma 4.4 that the Weyl function $\tilde{w}$ in this case has such a theta function expression related to the same curve Γ. To complete the proof, we only need to show that the respective non-zero poles λ_{1j}^{-} and $\tilde{\lambda}_{1j}^{-}$ satisfy

$$\sum_{j=1}^{g} A(\tilde{\lambda}_{1j}^{-}) + A(\infty^{-}) = \sum_{j=1}^{g} A(\lambda_{1j}^{-}).$$

But this is a consequence of Lemma 4.5, together with the normalization $A(\infty^{+}) = 0$. □

5. Time dependence of the spectra under the Camassa–Holm flow

We now consider evolution under the flow. Equation (2.2) implies

$$L(\lambda)(\dot{\varphi} + B(\lambda)\varphi) = 0, \tag{5.1}$$

where the dot denotes differentiation with respect to time. By (2.3),

$$\begin{aligned} B(\lambda)(a\mathrm{e}^{\nu x} + b\mathrm{e}^{-\nu x}) = \frac{1}{2\lambda}(a\mathrm{e}^{\nu x} - b\mathrm{e}^{-\nu x}) - a\mathrm{e}^{\nu x}\sum_{j=1}^{d} m_j[\nu G(x-x_j) - \tfrac{1}{2}G'(x-x_j)] \\ + b\mathrm{e}^{-\nu x}\sum_{j=1}^{d} m_j[\nu G(x-x_j) + \tfrac{1}{2}G'(x-x_j)], \end{aligned} \tag{5.2}$$

where $G = G_{\nu,X}$.

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Equation (2.11) implies

$$\left.\begin{aligned}\nu G(x-x_j)+\tfrac{1}{2}G'(x-x_j)&=\begin{cases}\mathrm{e}^{2\nu(x-x_j)}\cdot\dfrac{\mathrm{e}^{\nu X}}{2\sinh\nu X}, & x_d-X<x<x_1,\\[2mm] \mathrm{e}^{2\nu(x-x_j)}\cdot\dfrac{\mathrm{e}^{-\nu X}}{2\sinh\nu X}, & x_d<x<x_1+X,\end{cases}\\ \nu G(x-x_j)-\tfrac{1}{2}G'(x-x_j)&=\begin{cases}\mathrm{e}^{2\nu(x_j-x)}\cdot\dfrac{\mathrm{e}^{-\nu X}}{2\sinh\nu X}, & x_d-X<x<x_1,\\[2mm] \mathrm{e}^{2\nu(x_j-x)}\cdot\dfrac{\mathrm{e}^{\nu X}}{2\sinh\nu X}, & x_d<x<x_1+X.\end{cases}\end{aligned}\right\}\tag{5.3}$$

Combining (5.2) and (5.3), we find that the vector representation of $B(\lambda)\varphi$ in terms of that of φ is given by

$$B(\lambda):\begin{bmatrix}a\\ b\end{bmatrix}\to B_-(\lambda)\begin{bmatrix}a\\ b\end{bmatrix}=\begin{bmatrix}\dfrac{1}{2\lambda} & \dfrac{M_-\mathrm{e}^{\nu X}}{2\sinh\nu X}\\ -\dfrac{M_+\mathrm{e}^{-\nu X}}{2\sinh\nu X} & -\dfrac{1}{2\lambda}\end{bmatrix}\begin{bmatrix}a\\ b\end{bmatrix},\tag{5.4}$$

for $x_d-X<x<x_1$, and

$$B(\lambda):\begin{bmatrix}a\\ b\end{bmatrix}\to B_+(\lambda)\begin{bmatrix}a\\ b\end{bmatrix}=\begin{bmatrix}\dfrac{1}{2\lambda} & \dfrac{M_-\mathrm{e}^{-\nu X}}{2\sinh\nu X}\\ -\dfrac{M_+\mathrm{e}^{\nu X}}{2\sinh\nu X} & -\dfrac{1}{2\lambda}\end{bmatrix}\begin{bmatrix}a\\ b\end{bmatrix},\tag{5.5}$$

for $x_d<x<x_1+X$.

Let us take solutions $\varphi_1=\mathrm{e}^{\nu x}$ and $\varphi_2=\mathrm{e}^{-\nu x}$ in the interval $x_d-X<x<x_1$, so that the two vector representations give the identity matrix on (x_d-X,x_1) and the matrix $\Phi(\lambda)$ on (x_d,x_1+X). In view of the preceding, Equation (5.1) on (x_d,x_1+X) gives

$$\dot{\Phi}(\lambda)+B_+(\lambda)\Phi(\lambda)=\Phi(\lambda)B(\lambda),$$

while (5.1) on (x_d-X,x_1) gives

$$B_-(\lambda)=B(\lambda).$$

We note that $B_+(\lambda)=E_{\nu X}^{-1}B_-(\lambda)E_{\nu X}$, so the preceding two equations may be combined with (2.20) to give

$$\dot{\Psi}(\lambda)=[\Psi(\lambda),B_+(\lambda)]=\Psi(\lambda)B_+(\lambda)-B_+(\lambda)\Psi(\lambda).\tag{5.6}$$

One consequence is that the polynomial $P=\operatorname{tr}\Psi$ is an invariant of the motion.

The evolution of the off-diagonal entries of Ψ is

$$\left.\begin{aligned}\dot{\Psi}_{12}&=-\frac{1}{\lambda}\Psi_{12}+\frac{M_-\mathrm{e}^{-\nu X}}{2\sinh\nu X}(\Psi_{11}-\Psi_{22}),\\ \dot{\Psi}_{21}&=\frac{1}{\lambda}\Psi_{21}+\frac{M_+\mathrm{e}^{\nu X}}{2\sinh\nu X}(\Psi_{11}-\Psi_{22}).\end{aligned}\right\}\tag{5.7}$$

The momentum $M=\sum m_j$ is constant; this follows by direct computation using Hamilton's equations for the Hamiltonian (2.10), or because the linear term of the invariant polynomial $P(\lambda)$ has coefficient $2\sinh(\nu X)M$. Equations (5.7) allow us to relate the flow of $M_\pm$ to spectral data.

Theorem 5.1. *We have*

$$\left.\begin{aligned}\frac{\dot M_+}{M_+}&=-M\coth(\nu X)-\sum_{j=1}^{g}\frac{1}{\lambda_{2j}},\\ \frac{\dot M_-}{M_-}&=M\coth(\nu X)+\sum_{j=1}^{g}\frac{1}{\lambda_{1j}}.\end{aligned}\right\}\tag{5.8}$$

Proof. As before, let λ_{1j} and λ_{2j}, $j=1,\dots,g$, denote the roots of Ψ_{12}/λ and of Ψ_{21}/λ, respectively. We continue to assume that these roots are simple and non-zero. In view of (2.17) and (2.20),

$$\left.\begin{aligned}\Psi_{12}&=\lambda M_-\mathrm{e}^{-\nu X}\prod_{j=1}^{g}\biggl(1-\frac{\lambda}{\lambda_{1j}}\biggr),\\ \Psi_{21}&=-\lambda M_+\mathrm{e}^{\nu X}\prod_{j=1}^{g}\biggl(1-\frac{\lambda}{\lambda_{2j}}\biggr).\end{aligned}\right\}\tag{5.9}$$

It follows that

$$\frac{\dot M_-}{M_-}=\lim_{\lambda\to0}\frac{\dot\Psi_{12}}{\Psi_{12}},\qquad \frac{\dot M_+}{M_+}=\lim_{\lambda\to0}\frac{\dot\Psi_{21}}{\Psi_{21}}.\tag{5.10}$$

To evaluate the limits, we turn to (5.7). Note that (2.17) and (2.20) imply that

$$\Psi_{11}-\Psi_{22}=2\sinh\nu X+2\cosh\nu XM\lambda+O(\lambda^2).$$

This and (5.9) imply that

$$\begin{aligned}\frac{M_-\mathrm{e}^{-\nu X}}{2\sinh\nu X}\cdot\frac{\Psi_{11}-\Psi_{22}}{\Psi_{12}}&=\frac{1}{\lambda}\biggl\{\frac{1+M\lambda\coth(\nu X)+\cdots}{1-\lambda\sum{\lambda_{1j}}^{-1}+\cdots}\biggr\},\\ \frac{M_+\mathrm{e}^{\nu X}}{2\sinh\nu X}\cdot\frac{\Psi_{11}-\Psi_{22}}{\Psi_{21}}&=-\frac{1}{\lambda}\biggl\{\frac{1+M\lambda\coth(\nu X)+\cdots}{1-\lambda\sum{\lambda_{2j}}^{-1}+\cdots}\biggr\}.\end{aligned}$$

Combining these equations with (5.7), we find that (5.10) becomes (5.8). □

Theorem 5.2. *The roots $\{\lambda_{1j}\}$ of Ψ_{12} and the roots $\{\lambda_{2j}\}$ of Ψ_{21} evolve according to*

$$\left.\begin{aligned}\dot\lambda_{1j}&=\frac{1}{\prod_{k\neq j}(1-\lambda_{1j}/\lambda_{1k})}\cdot\frac{\Psi_{11}(\lambda_{1j})-\Psi_{22}(\lambda_{1j})}{2\sinh\nu X},\\ \dot\lambda_{2j}&=-\frac{1}{\prod_{k\neq j}(1-\lambda_{2j}/\lambda_{2k})}\cdot\frac{\Psi_{11}(\lambda_{2j})-\Psi_{22}(\lambda_{2j})}{2\sinh\nu X}.\end{aligned}\right\}\tag{5.11}$$

Proof. Evaluating (5.7) at the roots gives

$$\left.\begin{aligned}\dot{\Psi}_{12}(\lambda_{1j}) &= \frac{M_- \mathrm{e}^{-\nu X}}{2\sinh \nu X}(\Psi_{11}-\Psi_{22}),\\ \dot{\Psi}_{21}(\lambda_{2j}) &= \frac{M_+ \mathrm{e}^{\nu X}}{2\sinh \nu X}(\Psi_{11}-\Psi_{22}).\end{aligned}\right\} \tag{5.12}$$

On the other hand, differentiating (5.9) and then evaluating at the roots gives

$$\left.\begin{aligned}\dot{\Psi}_{12}(\lambda_{1j}) &= M_- \mathrm{e}^{-\nu X}\prod_{k\neq j}\left(1-\frac{\lambda_{1j}}{\lambda_{1k}}\right)\dot{\lambda}_{1j},\\ \dot{\Psi}_{21}(\lambda_{2j}) &= -M_+ \mathrm{e}^{\nu X}\prod_{k\neq j}\left(1-\frac{\lambda_{2j}}{\lambda_{2k}}\right)\dot{\lambda}_{2j}.\end{aligned}\right\} \tag{5.13}$$

Combining (5.12) and (5.13) gives (5.11). □

We may rewrite (5.11) as a system of equations on the curve Γ. Indeed, at a root of $\Psi_{12}\Psi_{21}$, as noted earlier, $\Psi_{11}-\Psi_{22}=\pm z$ (see (2.26)). Therefore, Equations (5.11) can be written as

$$\left.\begin{aligned}\dot{\lambda}^{\pm}_{1j} &= \pm\frac{1}{\prod_{k\neq j}(1-\lambda_{1j}/\lambda_{1k})}\cdot\frac{z(\lambda^{\pm}_{1j})}{2\sinh\nu X},\\ \dot{\lambda}^{\pm}_{2j} &= \mp\frac{1}{\prod_{k\neq j}(1-\lambda_{2j}/\lambda_{2k})}\cdot\frac{z(\lambda^{\pm}_{2j})}{2\sinh\nu X}.\end{aligned}\right\} \tag{5.14}$$

According to Theorem 3.2, the zeros and poles of w correspond to these roots. We have the following result.

Theorem 5.3. *The Abel map for the curve (3.2) linearizes the flow of zeros and poles of w,*

$$\left.\begin{aligned}\xi^+(t) &\equiv A^{\mathrm{s}}(\lambda^+_{2j}(t),\dots,\lambda^+_{2g}(t))+K=\xi^+ + t\kappa,\\ \xi^-(t) &\equiv A^{\mathrm{s}}(\lambda^-_{1j}(t),\dots,\lambda^-_{1g}(t))+K=\xi^- + t\kappa,\end{aligned}\right\} \tag{5.15}$$

where K is the vector of Riemann constants and κ is a fixed element of $J(\Gamma)$.

Proof. It is enough to prove

$$\frac{\mathrm{d}}{\mathrm{d}t}A^{\mathrm{s}}(\lambda^+_{21}(t),\dots,\lambda^+_{2g}(t)) = \text{const.}, \tag{5.16}$$

since (4.2) shows that the difference $\xi^+(t)-\xi^-(t)$ is constant. The kth equation of the system (5.16) is

$$\sum_{j=1}^{g}\frac{\mathrm{d}}{\mathrm{d}t}\int_{p_0}^{\lambda_{2j}}\omega_k = \text{const.} \tag{5.17}$$

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The ω_j are linear combinations of the one-forms $\alpha_m = \lambda^{m-1} \mathrm{d}\lambda^{\pm}/z$, $m = 1, \dots, g$. Therefore, it suffices to prove the analogue of (5.17) for each of the α_m. We shall show that if $p_j(t) = \lambda_{2j}^{+}(t)$, then

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\{\sum_{j=1}^{g}\int_{\infty^+}^{p_j(t)} \alpha_m\right\} = -\frac{\delta_{m1}}{2\sinh \nu X}. \tag{5.18}$$

In fact, according to (5.14), the derivative here is

$$-\frac{1}{2\sinh \nu X}\sum_{j=1}^{g}\frac{(\lambda_{2j})^{m-1}}{\prod_{k\neq j}(1-\lambda_{2j}/\lambda_{2k})}. \tag{5.19}$$

Let

$$R_m(\lambda) = \lambda^{m-2}\prod_{k=1}^{g}\left(1-\frac{\lambda}{\lambda_{2k}}\right)^{-1}, \quad m = 1, \dots, g.$$

Since R_m is $O(\lambda^{m-g})$ as $\lambda \to \infty$, the sum of its residues is zero. For $m = 2, \dots, g$, the sum in (5.19) is this sum of the residues of R_m, and hence is zero. For $m = 1$, R_1 has an additional residue 1 at $\lambda = 0$, so the sum in (5.19) is $-\frac{1}{2}\sinh \nu X$. This proves (5.16). □

Writing

$$\omega_k = \sum_{m=1}^{g} c_{km}\alpha_m, \tag{5.20}$$

we note that the above argument evaluates the constant $\kappa \in J(\Gamma)$ as

$$\kappa = -\frac{1}{2\sinh \nu X}(c_{11}, c_{21}, \dots, c_{g1}). \tag{5.21}$$

6. Theta function representations, II

Theorems 5.1 and 5.3 make it possible to find theta function representations of $M_{\pm}$.

Theorem 6.1. *The functions $M_{\pm}(t)$ can be represented as*

$$M_+(t) = C_+ \mathrm{e}^{-ct}\frac{\theta(A(0^+) - \xi^+ - t\kappa)}{\theta(A(0^-) - \xi^+ - t\kappa)}, \tag{6.1}$$

$$M_-(t) = C_- \mathrm{e}^{ct}\frac{\theta(A(0^-) - \xi^- - t\kappa)}{\theta(A(0^+) - \xi^- - t\kappa)}. \tag{6.2}$$

Here we have used the notation of Theorem 5.3. The constant c is

$$c = \sum_{j=1}^{g}\frac{1}{2\pi \mathrm{i}}\int_{a_j}\lambda^{-1}\omega_j + M\coth(\nu X), \tag{6.3}$$

while the constants $C_{\pm}$ are

$$C_+ = M_+(0)\frac{\theta(A(0^-) - \xi^+)}{\theta(A(0^+) - \xi^+)}, \qquad C_- = M_-(0)\frac{\theta(A(0^+) - \xi^-)}{\theta(A(0^-) - \xi^-)}.$$

Proof. For $p \in \Gamma$, set

$$F_{\pm}(p) = \theta(A(p) - \xi^{\pm} - t\kappa), \qquad \gamma = \sum_{j=1}^{g} \frac{1}{2\pi \mathrm{i}} \int_{a_j} \lambda^{-1}\omega_j.$$

The constants $C_{\pm}$ are chosen so that the formulae (6.1), (6.2) are correct at $t = 0$. In view of Theorem 5.1, it is enough to show that

$$\frac{\mathrm{d}\log F_+}{\mathrm{d}t}(0^+) - \frac{\mathrm{d}\log F_+}{\mathrm{d}t}(0^-) = \gamma - \sum_{j=1}^{g} \frac{1}{\lambda_{2j}}, \tag{6.4}$$

$$\frac{\mathrm{d}\log F_-}{\mathrm{d}t}(0^-) - \frac{\mathrm{d}\log F_-}{\mathrm{d}t}(0^+) = -\gamma + \sum_{j=1}^{g} \frac{1}{\lambda_{1j}}. \tag{6.5}$$

The function F_+ has simple zeros precisely at the points $\lambda_{2j}^{+}(t)$ (see [**16**, Lemma 2.4.2]). The cycles $\{a_j\}$, $\{b_j\}$ may be chosen so that their complement $\tilde{\Gamma}$ is simply connected. Computing residues, it follows that the integral

$$\frac{1}{2\pi \mathrm{i}} \int_{\partial\tilde{\Gamma}} \frac{\mathrm{d}F}{\lambda F} = \sum_{j=1}^{g} \frac{1}{\lambda_{2j(t)}} + \frac{\mathrm{d}}{\mathrm{d}\lambda} \log F_+(0^+) + \frac{\mathrm{d}}{\mathrm{d}\lambda} \log F_-(0^-). \tag{6.6}$$

On the other hand, because of the jump properties of the theta function on the cycles, the integral (6.6) is constant and equal to γ. We turn to the last two summands on the right in (6.6). First,

$$\frac{\mathrm{d}\log F_+}{\mathrm{d}\lambda} = \frac{\mathrm{d}}{\mathrm{d}\lambda} \log \theta(A(\lambda) - \xi^+ - t\kappa) = \frac{1}{F_+} \sum_{k=1}^{g} \frac{\partial\theta}{\partial z_k} \frac{\mathrm{d}A_k}{\mathrm{d}\lambda},$$

where $(z_1, z_2, \dots, z_g)$ are the standard coordinates in $\mathbb{C}^g/\Lambda$. It follows from (5.20) and the definition of the Abel map that

$$\frac{\mathrm{d}A_k}{\mathrm{d}\lambda} = \sum_{m=1}^{g} c_{km} \frac{\lambda^{m-1}}{z}.$$

Evaluating at $0^{\pm}$, we obtain

$$\frac{\mathrm{d}A_k}{\mathrm{d}\lambda}(0^{\pm}) = \pm \frac{c_{k1}}{2\sinh \nu X} = \mp \kappa_k.$$

Therefore,

$$\gamma = \sum_{j=1}^{g} \frac{1}{\lambda_{2j(t)}} - \frac{1}{F_+(0^+)} \sum_{k=1}^{g} \kappa_k \frac{\partial\theta}{\partial z_k}(0^+) + \frac{1}{F_+(0^-)} \sum_{k=1}^{g} \kappa_k \frac{\partial\theta}{\partial z_k}(0^-). \tag{6.7}$$

The time derivative in (6.4) is

$$\frac{\mathrm{d}\log F_+}{\mathrm{d}t} = -\frac{1}{F_+} \sum_{k=1}^{g} \kappa_k \frac{\partial\theta}{\partial z_k}.$$

Combining this with (6.7) gives (6.4). The same argument proves (6.5). □

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We can use this representation of M_+ or of M_- to obtain a theta function representation of the flow of the Weyl function.

Theorem 6.2. *The Weyl function $w(\lambda^\pm, t)$ has the form*

$$w(\lambda^\pm, t) = C\mathrm{e}^{-ct}\frac{\theta(A(\lambda^\pm) - \xi^+ - t\kappa)}{\theta(A(\lambda^\pm) - \xi^- - t\kappa)}\exp\biggl(\int_{\infty^+}^{\lambda^\pm}\omega_{0^+0^-}\biggr), \tag{6.8}$$

where c is given by (6.3).

Proof. According to Theorems 4.1 and 5.3,

$$w(\lambda^\pm, t) = C(t)\frac{\theta(A(\lambda^\pm) - \xi^+ - t\kappa)}{\theta(A(\lambda^\pm) - \xi^- - t\kappa)}\exp\biggl(\int_{\infty^+}^{\lambda^\pm}\omega_{0^+0^-}\biggr). \tag{6.9}$$

To identify the function $C(t)$, we note first that

$$\frac{w(0^+, t)}{w(0^+, 0)} = \frac{C(t)}{C(0)}\frac{\theta(A(0^+) - \xi^+ - t\kappa)}{\theta(A(0^+) - \xi^- - t\kappa)}\frac{\theta(A(0^+) - \xi^-)}{\theta(A(0^+) - \xi^+)}. \tag{6.10}$$

On the other hand, Equations (3.8) and (6.1) imply that the left-hand side of (6.10) is equal to

$$\frac{M_+(t)}{M_+(0)} = \mathrm{e}^{-ct}\frac{\theta(A(0^+) - \xi^+ - t\kappa)}{\theta(A(0^-) - \xi^+ - t\kappa)}\frac{\theta(A(0^-) - \xi^+)}{\theta(A(0^+) - \xi^+)}. \tag{6.11}$$

It follows from the equality of (6.10) and (6.11) that $C(t)$ has the form

$$C(t) = C\mathrm{e}^{-ct}\frac{\theta(A(0^+) - \xi^- - t\kappa)}{\theta(A(0^-) - \xi^+ - t\kappa)}.$$

By (4.6), the arguments of the two theta functions are the same, so their quotient is 1. Combining the resulting equation with (6.9), we obtain (6.8). □

As in § 4, this can be carried over to other starting positions.

Theorem 6.3. *The Weyl function w_k that corresponds to the configuration of positions $x_{k+1}, \dots, x_{k+d}$ has the form*

$$w_k(\lambda^\pm, t) = C_k\mathrm{e}^{-ct}\frac{\theta(A(\lambda^\pm) - \xi^+ - t\kappa + kA(\infty^-))}{\theta(A(\lambda^\pm) - \xi^- - t\kappa + kA(\infty^-))}\exp\biggl(\int_{\infty^+}^{\lambda^\pm}\omega_{0^+0^-}\biggr), \tag{6.12}$$

where c is given by (6.3) and C_k is independent of time.

Proof. The argument is essentially the same as in Theorem 6.2, combining Theorem 4.6 and an appropriate version of Theorem 3.3, with M_+ replaced by $M_+(k)$. □

In view of Theorem 3.3, generalized to $k = -j$, $j = 0, 1, \dots, d-1$, we can now write formulae for the positions x_{d-j}, as well as the momenta m_{d-j}.

Theorem 6.4. *Let $0 \leqslant j \leqslant d-1$. Then the positions x_{d-j} can be obtained from*

$$x_{d-j} = c_{d-j} - \frac{ct}{2\nu} + \frac{1}{2\nu}\log\frac{\theta(\xi^+ + t\kappa + jA(\infty^-))}{\theta(\xi^- + t\kappa + jA(\infty^-))}, \tag{6.13}$$

where c is given by (6.3) and the c_{d-j} are independent of time. If the local parameter around ∞^+ is $\sigma = 1/\lambda$, then $1/m_{d-j}$ can be computed from

$$\frac{1}{m_{d-j}} = D - \frac{\mathrm{d}}{\mathrm{d}\sigma}\log\frac{\theta(A(\lambda^+) - \xi^+ - t\kappa - jA(\infty^-))}{\theta(A(\lambda^+) - \xi^- - t\kappa - jA(\infty^-))}\bigg|_{\sigma=0}, \tag{6.14}$$

where D is a time-independent constant.

Proof. By Theorem 3.3 generalized to $k = -j$,

$$w_{-j}(\lambda^+) = -\mathrm{e}^{2\nu x_{d-j}}\bigg(1 - \frac{1}{\lambda m_{d-j}} + O(\lambda^{-2})\bigg), \quad \lambda \to \infty. \tag{6.15}$$

Hence $-w_j(\infty^+) = \mathrm{e}^{2\nu x_{d-j}}$, and the formula for the positions x_{d-j} follows from Theorem 6.3. Similarly, to get the momenta m_{d-j}, one writes (6.15) as

$$\frac{w_{-j}(\lambda^+)}{w_{-j}(\infty^+)} = 1 - \frac{1}{\lambda m_{d-j}} + O(\lambda^{-2}). \tag{6.16}$$

The formula for the momenta follows then from the Taylor expansion in $\sigma = 1/\lambda$ about $\sigma = 0$ and Theorem 6.3. □

7. Calogero–Françoise flows

Calogero [**7**] and Calogero and Françoise [**9**] introduced a family of completely integrable finite-dimensional Hamiltonian systems with Hamiltonian in a form that generalizes (2.10),

$$H(x_1, \dots, x_d, m_1, \dots, m_d) = \frac{1}{2}\sum_{j,k=1}^{d} m_j m_k G_{\nu,\beta}(x_j - x_k), \tag{7.1}$$

where

$$G_{\nu,\beta}(x) = \frac{\beta_+}{2\nu}\mathrm{e}^{2\nu|x|} + \frac{\beta_-}{2\nu}\mathrm{e}^{-2\nu|x|}. \tag{7.2}$$

Here, ν and $\beta = (\beta_-, \beta_+)$ are complex parameters. We shall assume that $\beta_+ \neq \beta_-$, so that $G_{\nu,\beta}$ is not a smooth function of x.

Remark 7.1. The Hamiltonians considered by Calogero and Françoise included an additive constant, and also two limiting cases. The additive constant may be removed by changing to a moving frame [**6**]. One limiting case is the case $\beta_- = \beta_+$ of smooth G (see [**7, 8**]). The other limiting case is

$$G(x) = ax^2 + b|x| + c,$$

associated to the Hunter–Saxton equation (see [**5, 20, 21**]).

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We assume here that $\beta_- - \beta_+ = 1$, so that $(D^2 - 4\nu^2)G = -2\delta(x)$; a rescaling in (perhaps complex) time will accomplish this. It will also be convenient to assume that $\beta_-\beta_+ \neq 0$.

As before, the operator

$$L(\lambda) = D^2 - \nu^2 - 2\nu\lambda m(x) \tag{7.3}$$

is compatible with the generalized Lax evolution (2.2), where $B(\lambda)$ is given by (2.3). For sufficiently smooth u, Equation (2.2) is equivalent to the conditions

$$m_t = u_x m + (um)_x, \qquad 2m_x = 4\nu^2 u_x - u_{xxx}. \tag{7.4}$$

Suppose that m is a discrete measure of the form (2.6), $x_1 < \cdots < x_d$. Up to an additive constant, any even fundamental solution for $(-D^3 + 4\nu^2 D)u = 2Dm$ has the form $G_{\nu,\beta}$. Thus, up to a choice of moving frame, system (7.4) for u is equivalent to Hamilton's equation for the Hamiltonian (7.1).

In analogy with the periodic problem considered in the previous sections, we set

$$M_\pm = \sum_{j=1}^{d} \mathrm{e}^{\pm 2\nu x_j} m_j. \tag{7.5}$$

Then $u(x) = \sum m_j G_{\nu,\beta}(x - x_j)$ has precise asymptotics,

$$\left.\begin{aligned} u(x) &= \frac{\beta_-}{2\nu} M_- \mathrm{e}^{2\nu x} + \frac{\beta_+}{2\nu} M_+ \mathrm{e}^{-2\nu x}, \qquad x < x_1, \\ u(x) &= \frac{\beta_+}{2\nu} M_- \mathrm{e}^{2\nu x} + \frac{\beta_-}{2\nu} M_+ \mathrm{e}^{-2\nu x}, \qquad x > x_d. \end{aligned}\right\} \tag{7.6}$$

As before, M is constant under the flow.

We may analyse the spectral problem

$$D^2\varphi - \nu^2\varphi = 2\nu\lambda m\varphi \tag{7.7}$$

in the same way as for the periodic case above. Set $x_0 = -\infty$, $x_{d+1} = +\infty$. On each interval (x_j, x_{j+1}), a solution of (7.7) is a linear combination $a_j \mathrm{e}^{\nu x} + b_j \mathrm{e}^{-\nu x}$. Equation (7.7) is formally identical to (2.1), so again it leads to Equations (2.14) and (2.15). Again, denoting the transition matrix in (2.15) by $T_j(\lambda)$, we see that the scattering matrix for this problem is given by the product

$$\Phi(\lambda) = T_d(\lambda) T_g(\lambda) \cdots T_1(\lambda) = I + \lambda \begin{bmatrix} M & M_- \\ -M_+ & -M \end{bmatrix} + O(\lambda^2). \tag{7.8}$$

As before, the entries $\Phi_{ij}(\lambda)$ are polynomials of degree d in λ, and Φ has determinant 1.

We now consider evolution under the flow. Equation (2.2) implies (5.1). It follows from (7.6) and an argument like that in § 2 that the matrix representation is is

$$B(\lambda): \begin{bmatrix} a \\ b \end{bmatrix} \to B_-(\lambda) \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} \dfrac{1}{2\nu\lambda} & \beta_- M_- \\ -\beta_+ M_+ & -\dfrac{1}{2\nu\lambda} \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}, \quad x < x_1, \tag{7.9}$$

$$B(\lambda): \begin{bmatrix} a \\ b \end{bmatrix} \to B_+(\lambda) \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} \dfrac{1}{2\nu\lambda} & \beta_+ M_- \\ -\beta_- M_+ & -\dfrac{1}{2\nu\lambda} \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}, \quad x > x_d. \tag{7.10}$$

Take solutions $\varphi_1 = \mathrm{e}^{\nu x}$, $\varphi_2 = \mathrm{e}^{-\nu x}$ for $x < x_1$, so that the two vector representations give the identity matrix for $x < x_1$ and the matrix $\Phi(\lambda)$ for $x > x_d$. As before, we deduce that

$$\dot{\Phi}(\lambda) + B_+(\lambda)\Phi(\lambda) = \Phi(\lambda)B_-(\lambda).$$

Note that

$$\operatorname{diag}(\beta_-, \beta_+)B_+(\lambda) = B_-(\lambda)\operatorname{diag}(\beta_-, \beta_+).$$

For convenience later, we choose γ so that

$$\gamma^2 = \frac{\beta_-}{\beta_+}$$

and set

$$\Psi = \Phi \begin{bmatrix} \gamma & 0 \\ 0 & \gamma^{-1} \end{bmatrix}.$$

Then

$$\dot{\Psi}(\lambda) = [\Psi(\lambda), B_+(\lambda)] = \Psi(\lambda)B_+(\lambda) - B_+(\lambda)\Psi(\lambda). \tag{7.11}$$

Thus the polynomial

$$P(\lambda) = \Psi_{11}(\lambda) + \Psi_{22}(\lambda) = \gamma\Phi_{11}(\lambda) + \gamma^{-1}\Phi_{22}(\lambda) \tag{7.12}$$

is an invariant of the motion. The evolution of the off-diagonal entries is

$$\dot{\Psi}_{12} = -\frac{1}{\lambda}\Psi_{12} + \beta_+ M_-(\Psi_{11} - \Psi_{22}), \tag{7.13}$$

$$\dot{\Psi}_{21} = \frac{1}{\lambda}\Psi_{21} + \beta_- M_+(\Psi_{11} - \Psi_{22}). \tag{7.14}$$

It follows that the roots of Ψ_{12} and of Ψ_{21} evolve by (5.11), while the quantities $M_\pm$ evolve according to (5.8). Moreover,

$$\Psi_{12}(\lambda)\Psi_{21}(\lambda) = 0 \quad \text{implies } \{\Psi_{11}(\lambda) - \Psi_{22}(\lambda)\}^2 = P(\lambda)^2 - 4.$$

8. Linearization and solution of the Calogero–Françoise flows

The observation of the previous section show that the Hamiltonian system associated to (7.1) may be analysed in exactly the same way as the periodic Camassa–Holm problem. We consider the associated hyperelliptic curve

$$\Gamma = \{\xi = (\lambda, z) \in \mathbb{C}^2 : z^2 = P(\lambda)^2 - 4\}. \tag{8.1}$$

The analogues of Theorems 5.3 and 6.1 hold for the Calogero–Françoise flows. The missing step in the analysis would appear to be Proposition 2.1. This result was based on the assumption of periodicity, so that one could use any of the positions x_k, $k \in \mathbb{Z}$, as the first of d consecutive locations. To overcome this obstacle, we begin by assuming that ν is positive, and note just how close the connection between the discrete periodic problem and the Calogero–Françoise problem really is.

Proposition 8.1. *Suppose that $\nu > 0$ and $\beta_- > 1$. Let*

$$X = \frac{1}{2\nu} \log\left\{\frac{\beta_-}{\beta_+}\right\}. \tag{8.2}$$

Then Green's functions $G_{\nu,X}$ of (2.12) for the periodic Camassa–Holm problem and Green's function $G_{\nu,\beta}$ of (7.2) for the Calogero–Françoise flow are identical on the interval $-X \leqslant x \leqslant X$.

Proof. For a given period $X > 0$, $G_{\nu,X}$ clearly has the form (7.2) on the interval $-X \leqslant x \leqslant X$, with $\beta_\pm = \mathrm{e}^{\mp\nu X}/2\sinh \nu X$. Solving for X in terms of β_- yields (8.2). □

It follows that, under these assumptions, Φ and Ψ coincide with the same matrices for periodic Camassa–Holm. Moreover, the Calogero–Françoise dynamics correspond to a 'window' of the periodic Camassa–Holm dynamics.

Corollary 8.2. *Suppose that $\nu > 0$, $\beta_- > 1$. Let X be given by (8.2). Suppose that the initial positions $x_1(0) < \cdots < x_d(0)$ for the flow associated to the Hamiltonian (7.1) are such that $x_d(0) - x_1(0) < X$. Define $x_j(0)$ and $m_j(0)$ for all integers j by periodizing,*

$$x_{j+d}(0) = x_j(0) + X, \qquad m_{j+d}(0) = m_j.$$

For small enough $|t|$, the Calogero–Françoise flow coincides with the flow of $x_1, \dots, x_d$, $m_1, \dots, m_d$ for the periodic discrete Camassa–Holm flow of period X.

(Actually, the flows, suitably interpreted, coincide for all time. This is explained in the next section.)

The formulae obtained from Propositions 6.1 and 2.1 are analytic in ν, X and the initial data $x_j(0)$, $m_j(0)$. The analytic continuations will continue to provide the solutions to any of the Calogero–Françoise flows considered here, so long as the flow remains non-singular.

9. Remarks on dynamics and singularities

One can prove by induction that the scattering matrix Φ of (7.8) has leading term of the same form as in the Camassa–Holm case,

$$\Phi(\lambda) = \lambda^d \prod_{j=1}^{d} m_j \prod_{j=2}^{d} (1 - \mathrm{e}^{-2\nu(x_j - x_{j-1})}) \begin{bmatrix} 1 & \mathrm{e}^{-2\nu x_1} \\ -\mathrm{e}^{2\nu x_d} & -\mathrm{e}^{2\nu(x_d - x_1)} \end{bmatrix} + O(\lambda^g), \quad \lambda \to \infty. \tag{9.1}$$

Therefore, the leading coefficient of the invariant polynomial $P = \operatorname{tr} \Psi$ is

$$\prod_{j=1}^{d} m_j \prod_{j=2}^{d} (1 - \mathrm{e}^{-2\nu(x_j - x_{j-1})})(\gamma - \gamma^{-1} \mathrm{e}^{2\nu(x_d - x_1)}). \tag{9.2}$$

Under the assumptions of Proposition 8.1, $\gamma = \mathrm{e}^{\nu X}$, so the leading coefficient of P is a multiple of

$$\prod_{j=1}^{d} m_j \prod_{j=2}^{d} (1 - \mathrm{e}^{-2\nu(x_j - x_{j-1})})(1 - \mathrm{e}^{2\nu(x_d - x_1 - X)}). \tag{9.3}$$

Theorem 9.1. *Consider the periodic discrete Camassa–Holm equation with parameter $\nu > 0$, with d positions x_j in a period X. Suppose that the $m_j(0)$ all have the same sign. Then the solution exists for all time, and the successive distances $x_{j+1}(t) - x_j(t)$ are bounded away from zero.*

Proof. Since the sum $M = \sum_{j=1}^{d} m_j$ is a constant of the motion, it follows that as long as the m_j have the same sign, they are bounded. The fact that the expression (9.3) is constant implies that the m_j cannot vanish, and thus cannot change sign. Therefore, each of the factors in (9.2) is bounded, so each must also be bounded away from zero. This implies that the $x_{j+1} - x_j$ are bounded away from zero, $j = 1, \dots, g$. By periodicity, $-(x_d - x_1 - X) = x_{d+1} - x_d$, so all these differences are bounded away from zero. Existence of the solution for all time follows from Hamilton's equations, given that $G_{\nu,X}$ is bounded and that the m_j remain bounded. □

The same proof, in conjunction with Proposition 8.1 and Corollary 8.2, gives the following result.

Theorem 9.2. *Under the assumptions of Proposition 8.1, if the $m_j(0)$ all have the same sign, then the Calogero–Françoise flow exists for all time, provided $x_d(0) - x_1(0) \neq X$. Moreover, $x_d(t) - x_1(t) - X$ is bounded away from zero.*

If $x_d - x_1 < X$, then the flow coincides for all time with the flow of $x_1, \dots, x_d$, $m_1, \dots, m_d$ for the periodic discrete Camassa–Holm flow having period X.

In the non-periodic Camassa–Holm flow, if the m_j do not all have the same sign, then there are 'collisions'. At a certain time t_0, one or more distinct pairs satisfy

$$x_{j+1}(t) - x_j(t) = a_j(t - t_0)^2 + O((t - t_0)^3),$$
$$m_j(t) = \frac{b_j}{t - t_0} + O(1), \qquad m_{j+1}(t) = -\frac{b_j}{t - t_0} + O(1).$$

It follows that the function $u = (1/2\nu)\sum m_j e^{-2\nu|x-x_j|}$ remains continuous through the transition from $t < t_0$ to $t > t_0$, and one may consider m_j and m_{j+1} to have exchanged signs. We shall show elsewhere that this result is valid in the periodic case as well.

This has a particularly interesting implication for the Calogero–Françoise flows of Corollary 8.2. If the m_j do not have the same sign, then in one or both time directions there will be internal collisions within the window that contains x_1 and x_d. In addition, in one or both time directions it will happen that $x_d - x_1 \to X$ as $t \to t_0$. Then there is a corresponding blow-up of m_1 and m_d. Once again there is a natural continuation past t_0, in which m_d and m_1 have exchanged signs, and $x_d - x_1$ decreases. Similarly, if $x_d - x_1 > X$ and $m_1 m_d < 0$, then in one time direction one will have $x_d - x_1$ decrease to X and then increase, while m_1 and m_2 blow up and exchange signs, and u remains well behaved.

Finally, we note that for other values of ν or β_-, the behaviour of Calogero–Françoise flows will be quite different. With $d = 2$ and

$$\nu = 1, \qquad \beta_- = -\beta_+ = \tfrac{1}{2}, \qquad m_1 m_2 > 0,$$

the separation $x_2 - x_1$ goes to infinity in finite time, in both time directions, while with

$$\nu = i, \qquad \beta_- = -\beta_+ = \tfrac{1}{2}, \qquad m_1 m_2 > 0,$$

solutions exist for all time, and $x_2 - x_1$, m_1, m_2 and $x_1 + x_2 + Kt$ are periodic in time for some constant K (see [**6**]).

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References

1. M. S. Alber, R. Camassa, Y. Fedorov, D. D. Holm and J. E. Marsden, On billiard solutions of nonlinear PDEs, *Phys. Lett.* A **264** (1999), 171–178.
2. M. S. Alber, R. Camassa and M. E. Gekhtman, *On billiard weak solutions of nonlinear PDEs and Toda flows*, CRM Proceedings Lecture Notes, vol. 25, pp. 1–11 (Providence, RI, American Mathematical Society, 1999).
3. R. Beals, D. H. Sattinger and J. Szmigielski, Multi-peakons and a theorem of Stieltjes, *Inv. Probl.* **15** (1999), L1–L4.
4. R. Beals, D. H. Sattinger and J. Szmigielski, Multipeakons and the classical moment problem, *Adv. Math.* **154** (2000), 229–257.
5. R. Beals, D. H. Sattinger and J. Szmigielski, Inverse scattering solutions of the Hunter–Saxton equation, *Applic. Analysis* **78** (2001), 255–269.
6. R. Beals, D. H. Sattinger and J. Szmigielski, Calogero–Françoise flows and periodic peakons, *Theoret. Math. Phys.* **133** (2002), 1631–1646.
7. F. Calogero, An integrable Hamiltonian system, *Phys. Lett.* A **201** (1995), 306–310.
8. F. Calogero, *Classical many-body problems amenable to exact treatments*, Lecture Notes in Physics, vol. 66 (Springer, New York, 2001).

9. F. Calogero and J. P. Françoise, A completely integrable Hamiltonian system, *J. Math. Phys.* **37** (1996), 2863–2871.
10. R. Camassa and D. D. Holm, An integrable shallow water equation with peaked solitons, *Phys. Rev. Lett.* **71** (1993), 1661–1664.
11. R. Camassa, D. D. Holm and J. M. Hyman, A new integrable shallow water equation, *Adv. Appl. Mech.* **31** (1994), 1–33.
12. A. Constantin, On the spectral problem for the periodic the Camassa–Holm equation, *J. Math. Analysis Applic.* **210** (1997), 215–230.
13. A. Constantin, Quasiperiodicity with respect to time of spatially periodic finite-gap solutions of the Camassa–Holm equation, *Bull. Sci. Math.* **122** (1998), 487–494.
14. A. Constantin and H. P. McKean, A shallow water equation on the circle, *Commun. Pure Appl. Math.* **52** (1999), 949–982.
15. B. A. Dubrovin, Periodic problems for the Korteweg–de Vries equation in the class of finite-zone potentials, *Funkt. Analysis Prilozh.* **9** (1975), 41–51 (transl. *Funct. Analysis Applic.* **9** (1975), 215–223).
16. B. A. Dubrovin, Theta-functions and nonlinear equations. *Usp. Mat. Nauk* **36** (1981), 11–80 (transl. *Russ. Math. Surv.* **32** (1981), 11–92).
17. B. A. Dubrovin and S. P. Novikov, Periodic and conditionally periodic analogues of the multi-soliton solutions of the Korteweg–de Vries equation, *Zh. Eksp. Teoret. Fiz.* **67** (1974), 2131–2143 (transl. *Sov. Phys. JETP* **40** (1974), 1058–1063).
18. B. A. Dubrovin, V. B. Matveev and S. P. Novikov, Non-linear equations of Korteweg–de Vries type, finite-zone operators, and Abelian varieties, *Usp. Mat. Nauk* **31** (1977), 55–136 (transl. *Russ. Math. Surv.* **31** (1977), 185–213).
19. A. S. Fokas and B. Fuchssteiner, Symplectic structures and their Bäcklund transformations and hereditary symmetries, *Physica* D **4** (1981), 47–66.
20. J. K. Hunter and R. Saxton, Dynamics of director fields, *SIAM J. Appl. Math.* **51** (1991), 1498–1521.
21. J. K. Hunter and Y. Zheng, On a completely integrable nonlinear hyperbolic variational equation, *Physica* D **79** (1994), 361–386.
22. A. P. Its and V. B. Matveev, Schrödinger operators with a finite-zone spectrum and the N-soliton solutions of the Korteweg–de Vries equation, *Teor. Mat. Fiz.* **23** (1975), 51–67.
23. M. Kac and P. van Moerbeke, A complete solution of the periodic Toda problem, *J. Math. Phys.* **72** (1974), 2879–2880.
24. I. M. Krichever, An algebraic-geometric construction of the Zakharav–Shabat equations and their periodic solutions, *Dokl. Akad. Nauk* **227** (1976), 291–294 (transl. *Sov. Math. Dokl.* **17** (1976) 394–397).
25. I. M. Krichever and S. P. Novikov, Methods of algebraic geometry in the theory of nonlinear equations, *Usp. Mat. Nauk* **32** (1977), 183–208 (transl. *Russ. Math. Surv.* **32**, 185–213).
26. P. Lax, Periodic solutions of Korteweg–de Vries equation, *Commun. Pure Appl. Math.* **28** (1975), 141–188.
27. G. Misiolek, A shallow water equation as a geodesic flow on the Bott–Virasoro group, *J. Geom. Phys.* **24** (1998), 203–208.
28. S. P. Novikov, A periodic problem for the Korteweg–de Vries equation, *Funkt. Analysis Prilozh.* **8** (1974), 54–66 (transl. *Funct. Analysis Applic.* **8** (1974), 236–246).
29. T. Qian and M. Tang, Peakons and periodic cusp waves in a generalized Camassa–Holm equation, *Chaos Solitons Fractals* **12** (2001), 1347–1360.
30. M. Toda, *Theory of nonlinear lattices*, Springer Series in Solid-State Science, vol. 20 (Springer, 1981).