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UNCERTAINTY PRINCIPLES LIKE HARDY'S THEOREM ON SOME LIE GROUPS

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Abstract

We extend an uncertainty principle due to Cowling and Price to Euclidean spaces, Heisenberg groups and the Euclidean motion group of the plane. This uncertainty principle is a generalisation of a classical result due to Hardy. We also show that on the real line this uncertainty principle is almost equivalent to Hardy's theorem.

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0. Introduction

In the vast literature on *uncertainty principles in harmonic analysis* (see [3, 5]), the central theme is the impossibility of *simultaneous smallness* of a nonzero function f and its Fourier transform $\hat{f}$, where $\hat{f}$ is defined by

$$\hat{f}(y) = \int_{\mathbb{R}} f(x)e^{-2\pi i x y}dx.$$

A large number of results, beginning with a classical theorem of Hardy (Theorem 1 below), show such impossibility when *smallness* is interpreted as *sharp decay*.

In this paper we concern ourselves with results of this kind on certain Lie groups. We begin by stating the main results of this genre for the real line.

THEOREM 1 (Hardy). *Let $f : \mathbb{R} \to \mathbb{C}$ be measurable and for all x, y*

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(i) $|f(x)| \leq Ce^{-a\pi x^2}$,
(ii) $|\hat{f}(y)| \leq Ce^{-b\pi y^2}$,

where $C, a, b > 0$. *If* $ab > 1$ *then* $f = 0$ *almost everywhere. If* $ab = 1$ *then* $f(x) = Ce^{-a\pi x^2}$. *If* $ab < 1$ *then there exist infinitely many linearly independent functions satisfying* (i) *and* (ii).

THEOREM 2 (Cowling and Price). *Let* $f : \mathbb{R} \to \mathbb{C}$ *be measurable and*

(i) $\|e_a f\|_{L^p(\mathbb{R})} < \infty$,
(ii) $\|e_b \hat{f}\|_{L^q(\mathbb{R})} < \infty$,

where $a, b > 0$, $e_k(x) = e^{k\pi x^2}$ *and* $\min(p, q) < \infty$. *If* $ab \geq 1$ *then* $f = 0$ *almost everywhere. If* $ab < 1$ *then there exist infinitely many linearly independent functions satisfying* (i) *and* (ii).

THEOREM 3 (Morgan). *Let* $f : \mathbb{R} \to \mathbb{C}$ *be measurable and for all* x, y

(i) $|f(x)| \leq Ce^{-a\pi|x|^p}$,
(ii) $|\hat{f}(y)| \leq Ce^{-(A(a)+\epsilon)\pi|y|^q}$,

where $p > 2$, $p^{-1} + q^{-1} = 1$, $a, \epsilon > 0$ *and* $A(a) = 2^q/[\sin\alpha(q(pa)^{q-1})]$ *with* $\alpha = \pi(q-1)/2$. *Then* $f = 0$ *almost everywhere.*

THEOREM 4 (Beurling). *For* $f \in L^1(\mathbb{R})$,

$$\int_{\mathbb{R}}\int_{\mathbb{R}} |f(x)||\hat{f}(y)|e^{2\pi|xy|}dxdy < \infty$$

implies $f = 0$ *almost everywhere.*

For the proofs of the above theorems see [1, 5, 6].

Barring the case $ab = 1$ it is clear that the theorem of Cowling and Price implies the theorem of Hardy. Also the theorem of Beurling implies that of Cowling and Price for $ab > 1$. From Beurling's theorem we get yet another result which is somewhat stronger than Morgan's theorem (see [6]).

THEOREM 5. *Let* $f : \mathbb{R} \to \mathbb{C}$ *be measurable and for all* x, y

(i) $|f(x)| \leq Ce^{-a\pi|x|^p}$,
(ii) $|\hat{f}(y)| \leq Ce^{-b\pi|y|^q}$,

where $p^{-1} + q^{-1} = 1$. *If* $(ap)^{1/p}(bq)^{1/q} > 2$, *then* $f = 0$ *almost everywhere.*

NOTE. Clearly $(ap)^{1/p}\{(A(a) + \epsilon)q\}^{1/q} > 2$. Hence Morgan's theorem follows from Theorem 5.

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One of our results in this paper shows that Hardy's theorem implies the case $ab > 1$ of the theorem of Cowling and Price, although in both the theorems the case $a = 1 = b$ is a key point.

Recently Hardy's theorem has been extended to Euclidean spaces and to some non-commutative groups (see [10, 11, 13]). Our purpose in this paper is to extend the theorem of Cowling and Price to the following groups: $\mathbb{R}^n$, H_n and $M(2)$. Apart from this we will point out an analogue of Beurling's theorem on $\mathbb{R}^n$.

The paper is organized as follows: In Section 1 we consider the extensions of the above theorems to $\mathbb{R}^n$. In Section 2 we take up the theorem of Cowling and Price and also Theorem 5 for the Heisenberg groups H_n. We end this section with our proof that *Theorem 1 implies Theorem 2 when $ab > 1$*, for the real line. We do so since our approach to the theorem of Cowling and Price on H_n relies on the idea of this proof. In Section 3 we take up $M(2)$, the Euclidean motion group of the plane and we make some comment about the analogue of Theorem 2 on the oscillator group.

Our results in Sections 1 and 3 exploit the easily available complexification of lines in the unitary dual of the group. The Heisenberg groups treated in Section 2 do not admit such complexification and hence need a different treatment.

1. Euclidean spaces

The proof of Theorem 2 depends on the following result for entire functions.

LEMMA 1.1. *If $g : \mathbb{C} \to \mathbb{C}$ is entire and for $1 \leq p < \infty$*

(i) $|g(x+iy)| \leq Ae^{\pi x^2}$,

(ii) $\left(\int_{\mathbb{R}} |g(x)|^p dx\right)^{1/p} < \infty$,

then $g = 0$.

Lemma 1.1 which was proved in [1], uses an L^p-analogue of Phragmen-Lindelöff Theorem. We use it to prove an extension of Theorem 2 on $\mathbb{R}^n$.

NOTATION. In what follows, $(x_1, \ldots, \hat{x}_k, \ldots, x_n)$ stands for the vector $(x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_n) \in \mathbb{R}^{n-1}$.

THEOREM 1.1. *Let $f : \mathbb{R}^n \to \mathbb{C}$ be measurable. Suppose for some k, $1 \leq k \leq n$,*

(i) $\displaystyle\int_{\mathbb{R}^n} e^{pa\pi x_k^2} |g(x_1, \ldots, \hat{x}_k, \ldots, x_n)|^p |f(x_1, \ldots, x_n)|^p dx_1 \cdots dx_n < \infty$,

(ii) $\displaystyle\int_{\mathbb{R}^n} e^{qb\pi y_k^2} |h(y_1, \ldots, \hat{y}_k, \ldots, y_n)|^q |\hat{f}(y_1, \ldots, y_n)|^q dy_1 \cdots dy_n < \infty$,

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where $a, b > 0$, g, $h : \mathbb{R}^{n-1} \to \mathbb{C}$ are measurable with $g \geq \alpha > 0$, $h \geq \beta > 0$, where α, β are constants, $1/g \in L^{p'}(\mathbb{R}^{n-1})$, $p^{-1} + p'^{-1} = 1$, $1/h \in L^{q'}(\mathbb{R}^{n-1})$, $q^{-1} + q'^{-1} = 1$. If $ab \geq 1$ then $f = 0$ almost everywhere.

PROOF. By (i) and (ii) it follows that $f, \hat{f} \in L^1(\mathbb{R}^n)$. As in the real line case it is enough to prove the case $a = 1 = b$, otherwise we use dilation. Now if we fix $(y_1, \ldots, \hat{y}_k, \ldots, y_n) \in \mathbb{R}^{n-1}$, then for all $\omega = u + iv \in \mathbb{C}$

$$\begin{aligned}|\hat{f}(y_1, \ldots, y_{k-1}, \omega, y_{k+1}, \ldots, y_n)| \\ &\leq \int_{\mathbb{R}^n} |f(x_1, \ldots, x_n)| e^{2\pi v x_k} dx_1 \cdots dx_n \\ &\leq A e^{\pi v^2},\end{aligned}$$

where A is a constant and the last inequality follows from Holder's inequality and (i). Then by a standard argument using Lebesgue's dominated convergence theorem, Fubini's theorem and Morera's theorem it follows that for fixed $(y_1, \ldots, y_{k-1}, y_{k+1}, \ldots, y_n)$, $\hat{f}$ is an entire function in ω. We define

$$g(\omega) = e^{\pi\omega^2} \hat{f}(y_1, \ldots, y_{k-1}, \omega, y_{k+1}, \ldots, y_n).$$

So for almost every $(y_1, \ldots, y_{k-1}, y_{k+1}, \ldots, y_n)$, g satisfies conditions of Lemma 1.1. Hence $\hat{f} = 0$ almost everywhere. By the inversion formula $f = 0$ almost everywhere. □

Now we take up the theorem of Beurling.

THEOREM 1.2. *Let $f \in L^1(\mathbb{R}^n)$ and for some k, $1 \leq k \leq n$,*

$$\int_{\mathbb{R}^{2n}} |f(x_1, \ldots, x_n)||\hat{f}(y_1, \ldots, y_n)| e^{2\pi|x_k y_k|} dx_1 \cdots dx_n dy_1 \cdots dy_n < \infty.$$

Then $f = 0$ almost everywhere.

PROOF. We fix $\mathbf{y} = (y_1, \ldots, y_{k-1}, y_{k+1}, \ldots, y_n)$. We define

$$g_{\mathbf{y}}(x) = \tilde{\mathscr{F}}_k f(y_1, \ldots, y_{k-1}, x, y_{k+1}, \ldots, y_n), \quad x \in \mathbb{R},$$

where

$$\begin{aligned}\tilde{\mathscr{F}}_k f(y_1, \ldots, y_{k-1}, x, y_{k+1}, \ldots, y_n) \\ &= \int_{\mathbb{R}^{n-1}} f(x_1, \ldots, x_{k-1}, x, x_{k+1}, \ldots, x_n) \\ &\quad \times e^{-2i\pi\langle (x_1, \ldots, \hat{x}_k, \ldots, x_n), (y_1, \ldots, \hat{y}_k, \ldots, y_n)\rangle} dx_1 \cdots dx_{k-1} dx_{k+1} \cdots dx_n.\end{aligned}$$

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Then $\hat{g}_y(y) = \hat{f}(y_1, \ldots, y_{k-1}, y, y_{k+1}, \ldots, y_n)$. Now

$$\begin{aligned}
&\int_{\mathbb{R}} \int_{\mathbb{R}} |g_y(x)||\hat{g}_y(y)|e^{2\pi|xy|}dxdy \\
&\quad \leq \int_{\mathbb{R}^n} \int_{\mathbb{R}} |f(x_1, \ldots, x_{k-1}, x, x_{k+1}, \ldots, x_n)||\hat{f}(y_1, \ldots, y_{k-1}, y, y_{k+1}, \ldots, y_n)| \\
&\quad\quad \times e^{2\pi|xy|}dx_1 \cdots dx_{k-1}dx_{k+1} \cdots dx_n dxdy \\
&\quad < \infty
\end{aligned}$$

for almost every $(y_1, \ldots, y_{k-1}, y_{k+1}, \ldots, y_n)$. So by Beurling's theorem on $\mathbb{R}$ for almost every $(y_1, \ldots, y_{k-1}, y_{k+1}, \ldots, y_n)$,

$$\tilde{\mathscr{F}}_k f(y_1, \ldots, y_{k-1}, x, y_{k+1}, \ldots, y_n) = 0$$

for almost every x. Hence by Fubini's theorem and the inversion formula $f = 0$ almost everywhere. □

COROLLARY 1. *Let* $f \in L^1(\mathbb{R}^n)$.

(a) *If* $\int_{\mathbb{R}^{2n}} |f(x)||\hat{f}(y)|e^{2\pi\|x\|\|y\|}dxdy < \infty$ *then* $f = 0$ *almost everywhere.*

(b) *If* $\int_{\mathbb{R}^{2n}} |f(x_1, \ldots, x_n)||\hat{f}(y_1, \ldots, y_n)|e^{2\pi\sum_j|x_jy_j|}dx_1 \cdots dx_n dy_1 \cdots dy_n < \infty$ *then* $f = 0$ *almost everywhere.*

(c) *Suppose for some* k, $1 \leq k \leq n$, f *and* $\hat{f}$ *satisfy*

(i) $|f(x_1, \ldots, x_n)| \leq Cg(x_1, \ldots, \hat{x}_k, \ldots, x_n)e^{-a\pi|x_k|^p}$,

(ii) $|\hat{f}(y_1, \ldots, y_n)| \leq Ch(y_1, \ldots, \hat{y}_k, \ldots, y_n)e^{-b\pi|y_k|^{p'}}$,

where $p^{-1} + p'^{-1} = 1$, $g, h(\geq 0) \in L^1(\mathbb{R}^{n-1})$. *If* $(ap)^{1/p}(bq)^{1/q} > 2$ *then* $f = 0$ *almost everywhere.*

REMARK 1.1. For $\mathbb{R}^n$, decay in the k-th coordinate of f and $\hat{f}$ is enough to conclude that the function is zero. What matters is the fact that $\mathbb{R}^n$ is a direct product of copies of $\mathbb{R}$. We may also remark that the Fubini argument in Theorem 1.1 appears to be more effective than using an n dimensional version of Lemma 1.1 which would not yield Theorem 1.1 in the case $ab = 1$.

2. Heisenberg groups

The main result of this section, Theorem 2.3, proves an analogue of the theorem of Cowling and Price for the Heisenberg groups.

We recall some facts about the Heisenberg groups. The n dimensional Heisenberg group, denoted by H_n, as a set is $\mathbb{R}^{2n+1}$ with the group multiplication

$$(x, \xi, t)(x_1, \xi_1, t_1) = \left(x + x_1, \xi + \xi_1, t + t_1 + \frac{1}{2}[\langle x, \xi_1\rangle - \langle x_1, \xi\rangle]\right)$$

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where $x, \xi, x_1, \xi_1 \in \mathbb{R}^n$, $t, t_1 \in \mathbb{R}$, and $\langle ., . \rangle$ denotes the usual Euclidean inner product on $\mathbb{R}^n$. With this multiplication H_n is a unimodular, connected and simply connected, step two nilpotent Lie group whose Haar measure is $dx d\xi dt$. The reduced dual of H_n is parametrized by $\lambda \in \mathbb{R} \setminus \{0\}$ and is given by

$$\Pi_\lambda : H_n \to \mathscr{U}(L^2(\mathbb{R}^n)), \quad \Pi_\lambda(x, \xi, t) f(y) = e^{2\pi i \lambda [t + \langle \xi, x \rangle/2 - \langle \xi, y \rangle]} f(y - x).$$

Given $f \in L^1(H_n) \cap L^2(H_n)$, the group Fourier transform on reduced dual is given by

$$\hat{f}(\Pi_\lambda) = \int_{H_n} f(x, \xi, t) \Pi_\lambda((x, \xi, t)^{-1}) dx d\xi dt,$$

the integral being interpreted in the weak sense.

If we think of $\hat{f}$ as an operator-valued function on $\mathbb{R} \setminus \{0\}$ then it can be shown that $\hat{f}(\lambda)$ (by which we mean $\hat{f}(\Pi_\lambda)$) is an integral operator on $L^2(\mathbb{R}^n)$ with a kernel given by

$$K_\lambda^f(y, x) = \mathscr{F}_2 \mathscr{F}_3(x - y, -\lambda(x + y)/2, \lambda), \quad x, y \in \mathbb{R}^n,$$

where $\mathscr{F}_2$ and $\mathscr{F}_3$ mean the Euclidean Fourier transforms of f with respect to its second and third argument. It follows that

$$\|\hat{f}(\lambda)\|_{HS}^2 = |\lambda|^{-n} \int_{\mathbb{R}^{2n}} |\mathscr{F}_3 f(x, y, \lambda)|^2 dx dy \tag{2.1}$$

where $\| \cdot \|_{HS}$ denotes the Hilbert-Schmidt norm. Then by the (Euclidean) Parseval formula

$$\int_{\mathbb{R}} \|\hat{f}(\lambda)\|_{HS}^2 |\lambda|^n d\lambda = \|f\|_{L^2(H_n)}^2.$$

So $|\lambda|^n d\lambda$ is the Plancherel measure for H_n (see [2]).

Now we prove an analogue of Theorem 2 on H_n.

THEOREM 2.1. *Let $f \in L^1(H_n) \cap L^2(H_n)$. Suppose that for $a, b > 0$ and* $\min(p, q) < \infty$

(i) $\int_{H_n} e^{pa\pi \|(x, \xi, t)\|^2} |f(x, \xi, t)|^p dx d\xi dt < \infty$,

(ii) $\int_{\mathbb{R}} e^{qb\pi\lambda^2} \|\hat{f}(\lambda)\|_{HS}^q |\lambda|^n d\lambda < \infty$.

(a) *If $q \geq 2$, then $f = 0$ if $ab > 1$.*

(b) *If $1 \leq q < 2$, then for $p = \infty$, $f = 0$ if $ab \geq 2$ and for $p < \infty$, $f = 0$ if $ab > 2$.*

PROOF. We consider two cases separately.

CASE 1. ($p = \infty$) In this case the hypothesis (i) reduces to,

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(iii) $|f(x,\xi,t)| \le Ce^{-a\pi\|(x,\xi,t)\|^2}$ for all $(x,\xi,t) \in H_n$,

Let us define $f_{(x,\xi)}(t) = f(x,\xi,t)$, $f^*_{(x,\xi)}(t) = \overline{f(x,\xi,-t)}$ and

$$h(t) = \int_{\mathbb{R}^{2n}} (f_{(x,\xi)} * f^*_{(x,\xi)})(t)\,dx\,d\xi.$$

Then it follows that

$$\hat{h}(\lambda) = |\lambda|^n \|\hat{f}(\lambda)\|^2_{HS} \quad \text{(because of (2.1))}, \tag{2.2}$$

$$|h(t)| \le Ce^{-\frac{a}{2}\pi t^2} \quad \text{(by (iii))}. \tag{2.3}$$

First let us assume $q \ge 2$. Let $\epsilon > 0$ be such that $a(b-\epsilon) > 1$. Then, with $b' = b - \epsilon$

$$\begin{aligned}
\|e_{2b'\pi}\hat{h}\|^{q/2}_{q/2} &= \int_{\mathbb{R}} e^{qb'\pi\lambda^2} |\hat{h}(\lambda)|^{\frac{q}{2}} d\lambda \\
&= \int_{\mathbb{R}} e^{b'q\pi\lambda^2} |\lambda|^{\frac{nq}{2}} \|\hat{f}(\lambda)\|^q_{HS} d\lambda \qquad \text{(by 2.2)} \\
&= \int_{\mathbb{R}} e^{bq\pi\lambda^2} \|\hat{f}(\lambda)\|^q_{HS} |\lambda|^n \left(|\lambda|^{n(\frac{q}{2}-1)} e^{-\epsilon q\pi\lambda^2}\right) d\lambda \\
&\le K \int_{\mathbb{R}} e^{bq\pi\lambda^2} \|\hat{f}(\lambda)\|^q_{HS} |\lambda|^n d\lambda < \infty \qquad \text{(by (ii))}, \qquad (2.4)
\end{aligned}$$

where K is a constant. It follows from (2.3) and (2.4) that h satisfies conditions of Theorem 2 for $p = \infty$ and $q/2$ and hence $h = 0$ almost everywhere. So $\|\hat{f}(\lambda)\|_{HS} = 0$ for almost every λ which implies $f = 0$ almost everywhere by the Plancherel theorem. Now we assume that $q < 2$. By (ii) we have

$$\begin{aligned}
\infty &> \int_{\mathbb{R}} e^{qb\pi\lambda^2} \|\hat{f}(\lambda)\|^q_{HS} |\lambda|^n d\lambda \\
&= \int_{\mathbb{R}} e^{qb\pi\lambda^2} \hat{h}(\lambda)^{q/2} |\lambda|^{-nq/2} |\lambda|^n d\lambda \\
&\ge \int_{|\lambda|>1} e^{qb\pi\lambda^2} \hat{h}(\lambda)^{q/2} d\lambda.
\end{aligned}$$

As the integrand is a continuous function of λ we have

$$\int_{\mathbb{R}} e^{qb\pi\lambda^2} \hat{h}(\lambda)^{q/2} d\lambda < \infty.$$

So

$$\begin{aligned}
\int_{\mathbb{R}} e^{qb\pi\lambda^2} \hat{h}(\lambda)^q d\lambda &= \int_{\mathbb{R}} e^{qb\pi\lambda^2} \hat{h}(\lambda)^{q/2} \hat{h}(\lambda)^{q/2} d\lambda \\
&\le \|h\|^{q/2}_{L^1(\mathbb{R})} \int_{\mathbb{R}} e^{qb\pi\lambda^2} \hat{h}(\lambda)^{q/2} d\lambda < \infty.
\end{aligned}$$

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By (2.3), the above inequality and Theorem 2 we get $h = 0$ if $ab \geq 2$. Hence $f = 0$.

CASE 2. ($p < \infty$) If $\|\cdot\|$ denotes the Euclidean norm on $\mathbb{R}^{2n+1}$, then for (x, ξ, t), $(x_1, \xi_1, t_1)^{-1} (= (-x_1, -\xi_1, -t_1)) \in H_n$

$$\begin{aligned} &\|(x, \xi, t)(-x_1, -\xi_1, -t_1)\| \\ &\quad = \|(x - x_1, \xi - \xi_1, t - t_1 + \tfrac{1}{2}[\langle x_1, \xi\rangle - \langle x, \xi_1\rangle])\| \\ &\quad \geq \|(x, \xi, t)\| - \|(x_1, \xi_1, t_1)\| - \tfrac{1}{2}\|(x_1, \xi_1, t_1)\|\|(x, \xi, t)\|. \end{aligned} \tag{2.5}$$

Let $g \in C_c(H_n)$ with $\operatorname{supp} g \subset \{(x_1, \xi_1, t_1) : \|(x_1, \xi_1, t_1)\| \leq (1/m)\}$. Now let $(x, \xi, t) \in H_n$ be such that $\|(x, \xi, t)\| > 1$. Then by (2.5) we get

$$\|(x, \xi, t)(x_1, \xi_1, t_1)^{-1}\| \geq \|(x, \xi, t)\| \left(1 - \frac{3}{2m}\right)$$

for (x_1, ξ_1, t_1) in $\operatorname{supp} g$. If $e_a(x, \xi, t) = e^{a\pi\|(x,\xi,t)\|^2}$ we further get

$$((e_a|f|) * |g|)(x, \xi, t) \geq e_{a(1-\frac{3}{2m})^2}(x, \xi, t)(|f| * |g|)(x, \xi, t).$$

By (i) $e_a|f|$ is an L^p function and g is an $L^{p'}$ function ($p^{-1} + p'^{-1} = 1$). So $(e_a|f|) * |g|$ is an L^∞ function and let C be the L^∞ norm. Then

$$|(f * g)(x, \xi, t)| \leq (|f| * |g|)(x, \xi, t) \leq Ce^{-a(1-\frac{3}{2m})^2\pi\|(x,\xi,t)\|^2}$$

for all (x, ξ, t) with $\|(x, \xi, t)\| > 1$. Using continuity of $f * g$ ($f \in L^1(\mathbb{R}^{2n+1})$, $g \in L^\infty(\mathbb{R}^{2n+1})$) we get (changing the constant if needed)

$$|(f * g)(x, \xi, t)| \leq Ce^{-a(1-\frac{3}{2m})^2\pi\|(x,\xi,t)\|^2} \qquad \text{for all } (x, \xi, t) \in H_n. \tag{2.6}$$

Since $\widehat{(f * g)}(\lambda) = \hat{f}(\lambda) \circ \hat{g}(\lambda)$ and $\hat{g}(\lambda)$ is a bounded linear operator on $L^2(\mathbb{R}^n)$ we have

$$\|\widehat{(f * g)}(\lambda)\|_{HS} \leq \|\hat{g}(\lambda)\|_{op}\|\hat{f}(\lambda)\|_{HS} \leq \|g\|_{L^1(H_n)}\|\hat{f}(\lambda)\|_{HS}.$$

So

$$\begin{aligned} &\int_{\mathbb{R}} e^{qb\pi\lambda^2}\|\widehat{(f * g)}(\lambda)\|^q_{HS}|\lambda|^n d\lambda \\ &\quad \leq \|g\|^q_{L^1(H_n)} \int_{\mathbb{R}} e^{qb\pi\lambda^2}\|\hat{f}(\lambda)\|^q_{HS}|\lambda|^n d\lambda < \infty \quad (\text{ by (ii) }). \end{aligned} \tag{2.7}$$

We can choose m so large that $ab(1 - (3/2m))^2 > 1$ (respectively > 2), given $ab > 1$ (respectively > 2). So by (2.6) and (2.7) we are reduced to Case 1 and hence $f * g = 0$. By running g over an approximate identity we get $f = 0$ almost everywhere. This completes the proof. □

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Next we prove an analogue of Theorem 5 on Heisenberg groups.

THEOREM 2.2. *Let $f : H_n \to \mathbb{C}$ be measurable and*

(i) $|f(x,\xi,t)| \leq Cg(x,\xi)e^{-a\pi|t|^p}$ *for all* $(x,\xi,t) \in H_n$,

(ii) $\|\hat{f}(\lambda)\|_{HS} \leq Ce^{-b\pi|\lambda|^q}$ *for all* $\lambda \in \mathbb{R} \setminus \{0\}$,

where $a, b, C > 0$, $g \in L^1(\mathbb{R}^{2n}) \cap L^2(\mathbb{R}^{2n})$, $p \geq 2$, $p^{-1} + q^{-1} = 1$. *If* $(ap)^{1/p}(bq)^{1/q} > 2$, *then* $f = 0$.

PROOF. We define h as in the previous theorem. Now

$$\begin{aligned}
|h(t)| &= \int_{\mathbb{R}^{2n-1}} |f(x,\xi,t-s)||f(x,\xi,-s)|dx\,d\xi\,ds \\
&\leq C' \int_{\mathbb{R}} e^{-a\pi\{|t-s|^p+|s|^p\}}ds \\
&\leq C' \int_{\mathbb{R}} e^{-a\pi 2^{1-p/2}\{(t-s)^2+s^2\}^{p/2}}ds \\
&\leq Ae^{-a\pi 2^{1-p}|t|^p}.
\end{aligned}$$

By (2.2) and (ii) we have

$$|\hat{h}(\lambda)| \leq C|\lambda|^n e^{-2b\pi|\lambda|^q}.$$

We choose $b' < b$ such that $(ap)^{1/p}(b'q)^{1/q} > 2$ when $(ap)^{1/p}(bq)^{1/q} > 2$. Hence

$$|\hat{h}(\lambda)| \leq Be^{-2b'\pi|\lambda|^q}.$$

Since $(a2^{1-p}p)^{1/p}(2b'q)^{1/q} = (ap)^{1/p}(b'q)^{1/q}2^{(1-p)/p+1/q} > 2$, by Theorem 5, $\hat{h}(\lambda) = 0$ for almost every λ and hence $\|\hat{f}(\lambda)\|_{HS} = 0$ for almost every λ and then by the Plancherel theorem $f = 0$ almost everywhere. This completes the proof. □

Going back to Theorem 2.3 we notice that case 2 reduces the integral decay condition (i) to the pointwise decay condition (iii). Exploiting this idea on the real line we are led to a somewhat surprising result. We introduce some notation. Let $e_k(x) = e^{k\pi x^2}$ where $x \in \mathbb{R}$ and $k > 0$ and

$$E_{p,q}(a,b) = \{f : \mathbb{R} \to \mathbb{C}/f \text{ is measurable and } \|e_a f\|_p < \infty, \|e_b\hat{f}\|_q < \infty\}.$$

THEOREM 2.3. *The following are equivalent.*

(i) *If* $ab > 1$ *then* $E_{\infty,\infty}(a,b) = 0$.

(ii) *If* $ab > 1$ *and* $\min(p,q) < \infty$ *then* $E_{p,q}(a,b) = 0$.

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PROOF. It is easy to see that (i) follows from (ii) by Holder's inequality. Now we show that (i) implies (ii). Without loss of generality we assume $p < \infty$. Let $g \in C_c(\mathbb{R})$ be such that $\operatorname{supp} g \subset \{y : |y| < \delta\}$. We choose an $\epsilon > 0$, which is to be specified later. We choose x such that $|x| > \delta/\epsilon$. Then for all $y \in \operatorname{supp} g$ we have

$$|x-y| \geq |x| - |y| > |x| - \delta > |x|(1-\epsilon),$$

then by (ii) and the fact that $g \in L^p(\mathbb{R})$ for all p, we get, by Holder's inequality, for a constant C,

$$\begin{aligned} C &\geq \int_{\mathbb{R}} e^{a\pi|x-y|^2} |f(x-y)||g(y)|dy \\ &\geq e^{a\pi|x|^2(1-\epsilon)^2}(|f| * |g|)(x). \end{aligned}$$

So, $|f * g(x)| \leq (|f| * |g|)(x) \leq Ce^{-a\pi|x|^2(1-\epsilon)^2}$, for all x such that $|x| > \delta/\epsilon$. Since $f * g$ is a continuous function we have

$$|(f * g)(x)| \leq Ce^{-a(1-\epsilon)^2\pi|x|^2} \quad \text{for all } x, \tag{2.8}$$

$$\|e_b\widehat{(f * g)}\|_q \leq \|\hat{g}\|_\infty \|e_b\hat{f}\|_q < \infty \quad (\text{ by (ii) }). \tag{2.9}$$

Starting from (2.8) and (2.9) if we can show that $f * g = 0$ (with a condition on ϵ), then by running g over an approximate identity we get $f = 0$. So we prove the following:

Let $f : \mathbb{R} \to \mathbb{C}$ be measurable and

$$|f(x)| \leq Ce^{-a(1-\epsilon)^2\pi|x|^2} \quad \text{for all } x \in \mathbb{R}, \quad \|e_b\hat{f}\|_q < \infty,$$

then $f = 0$. Let $h \in C_c(\mathbb{R})$ be such that $\operatorname{supp} h \subset \{x : |x| < \delta_1\}$. We choose an $\epsilon_1 > 0$, to be specified later and do the same thing as above to get

$$|\hat{f} * h(y)| \leq Ce^{-b(1-\epsilon_1)^2\pi|y|^2} \qquad \text{for all } y. \tag{2.10}$$

If $\mathscr{F}^{-1}f$ denotes the inverse Fourier transform of f then

$$|\mathscr{F}^{-1}(\hat{f} * h)(x)| \leq C_1 e^{-a(1-\epsilon)^2\pi|x|^2}, \tag{2.11}$$

by the condition on f and the fact that $\mathscr{F}^{-1}h \in L^\infty(\mathbb{R})$. We choose our ϵ and ϵ_1 such that $ab(1-\epsilon)^2(1-\epsilon_1)^2 > 1$ whenever $ab > 1$. Then by (2.10) and (2.11) we get that $\hat{f} * h \in E_{\infty,\infty}(b(1-\epsilon_1)^2, a(1-\epsilon)^2)$ and hence by (i), $\hat{f} * h = 0$. By running h over an approximate identity we get $\hat{f} = 0$. Thus $f = 0$. This completes the proof. $\square$

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3. The Euclidean motion group of the plane

The Euclidean motion group of the plane, denoted by $M(2)$, is the semidirect product of $\mathbb{R}^2$ and $SO(2)$ with respect to the obvious action of $SO(2)$ on $\mathbb{R}^2$. This is a connected, unimodular, solvable Lie group. If we denote elements of $M(2)$ by (z, β), where $z \in \mathbb{C}$ (identified with $\mathbb{R}^2$) and $\beta \in \mathbb{T}$ (identified with $SO(2)$) then $dzd\beta$ is a Haar measure. The irreducible, unitary, infinite dimensional representations of $M(2)$ are realized on $L^2(\mathbb{T})$ and the equivalence classes of them are parametrized by $\{r \in \mathbb{R} : r \in \mathbb{R}^+\}$ and are given by

$$\Pi_r : M(2) \to \mathscr{U}(L^2(\mathbb{T}))$$
$$(\Pi_r(z, \beta)f)(\alpha) = e^{2\pi i \mathrm{Re}(r\bar{\alpha}z)} f(\bar{\beta}\alpha), \quad f \in L^2(\mathbb{T}), \quad \alpha \in \mathbb{T}.$$

Π_{-r} can be defined similarly, but Π_r and Π_{-r} are unitarily equivalent. The family $\{\Pi_r : r \in \mathbb{R}^+\}$ constitutes the support for the Plancherel measure and the measure is given by $crdr$, where c is a constant (see [12]).

We shall prove an analogue of Theorem 2 on $M(2)$. For $f \in L^1(M(2)) \cap L^2(M(2))$, the group Fourier transform is given by

$$\hat{f}(r) \overset{\text{def}}{=} \hat{f}(\Pi_r) = \int_{M(2)} f(z, \beta)\Pi_r((z, \beta)^{-1})dzd\beta$$

where the integral is interpreted in the weak sense, and then $\hat{f}(r)$ is a Hilbert-Schmidt operator on $L^2(\mathbb{T})$.

First, for our use here we state an equivalent version of Lemma 1.1.

LEMMA 3.1. *If $g : \mathbb{C} \to \mathbb{C}$ is entire and for $1 \le p < \infty$*

(i) $|g(x + iy)| \le Ae^{a\pi x^2}$, *where $a > 0$,*
(ii) $\left(\int_{\mathbb{R}} |g(x)|^p dx\right)^{1/p} < \infty$,

then $g = 0$.

Using this we prove

THEOREM 3.1. *Let $f \in L^1(M(2)) \cap L^2(M(2))$ and*

(i) $\int_{M(2)} e^{pa\pi|z|^2} |f(z, \alpha)|^p dzd\alpha < \infty$,
(ii) $\int_{\mathbb{R}^+} e^{qb\pi r^2} \|\hat{f}(r)\|_{HS}^q rdr < \infty$,

where $a, b > 0$, $1 \le q < \infty$, $1 \le p \le \infty$. If $ab > 1$ then $f = 0$ almost everywhere.

PROOF. Let $\{e_n : n \in \mathbb{Z}\}$ be the canonical orthonormal basis for $L^2(\mathbb{T})$. We define

$$\Phi^r_{m,n}(z, \beta) = \langle \Pi_r(z, \beta)e_m, e_n\rangle_{L^2(\mathbb{T})} \qquad ((z, \beta) \in M(2), \quad r > 0)$$
$$= \int_{\mathbb{T}} e^{2\pi i \mathrm{Re}(r\bar{\alpha}z)} e_m(\bar{\beta}\alpha)\overline{e_n(\alpha)}d\alpha.$$

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Now for $f \in L^2(\mathbb{T})$,

$$(\Pi_\omega(z,\beta)f)(\alpha) = e^{2\pi i\omega \mathrm{Re}(\bar{\alpha}z)} f(\bar{\beta}\alpha) \quad \omega = u + iv \in \mathbb{C}$$

continues to be a nonunitary representation of $M(2)$ and we get the complex extension of the function $r \to \Phi^r_{m,n}(z,\beta)$, for fixed (z,β), m, n. Further

$$|\Phi^\omega_{m,n}(z,\beta)| = |\langle \Pi_\omega(z,\beta)e_m, e_n\rangle| \le \int_{\mathbb{T}} e^{-2\pi v \mathrm{Re}(\bar{\alpha}z)} d\alpha, \tag{3.1}$$

for fixed $m, n, (z,\beta)$. From (3.1), $\omega \to \Phi^\omega_{m,n}(z,\beta)$ is an entire function by a standard argument. Also we have the estimate

$$\begin{aligned}
&|f(z,\beta)\Phi^\omega_{m,n}(-\bar{\beta}z,\bar{\beta})| \\
&\quad \le |f(z,\beta)| \int_{\mathbb{T}} |e^{2\pi i\omega \mathrm{Re}(-\bar{\alpha}\bar{\beta}z)} e_m(\beta\alpha)\overline{e_n(\alpha)}| d\alpha \\
&\quad \le |f(z,\beta)| \int_{\mathbb{T}} e^{2\pi v \mathrm{Re}(-\bar{\alpha}\bar{\beta}z)} d\alpha, \quad \text{where } \omega = u + iv \in \mathbb{C} \\
&\quad = |f(z,\beta)| e^{2\pi |v||z|}.
\end{aligned}$$

Hence

$$\begin{aligned}
&\int_{\mathbb{C}\times\mathbb{T}} |f(z,\beta)||\Phi^\omega_{m,n}(-\bar{\beta}z,\bar{\beta})| dz d\beta \\
&\quad \le \int_{\mathbb{C}\times\mathbb{T}} |f(z,\beta)| e^{2\pi|v||z|} dz d\beta \\
&\quad = e^{\pi|v|^2/a} \int_{\mathbb{C}\times\mathbb{T}} \left(|f(z,\beta)| e^{a\pi|z|^2}\right) \left(e^{-a\pi(|z|-|v|/a)^2}\right) dz d\beta \\
&\quad \le C_1 e^{\pi v^2/a}(A + B|v| + K|v|^2) \\
&\qquad \text{(by (i) and Holder's inequality, } A, B, K > 0\text{)} \\
&\quad \le C_1 e^{k\pi|v|^2},
\end{aligned} \tag{3.2}$$

for some k, such that $b > k > 1/a$. A routine argument now shows that the complex extension of the function $r \to \langle \hat{f}(r)e_m, e_n\rangle$, $r \in \mathbb{R}^+$, which we write as

$$\langle \hat{f}(\omega)e_m, e_n\rangle = \int_{\mathbb{C}\times\mathbb{T}} f(z,\beta)\Phi^\omega_{m,n}(-\bar{\beta}z,\bar{\beta}) dz d\beta,$$

is an entire function of the complex variable ω, for fixed m, n. We note further that $\langle \hat{f}(r)e_m, e_n\rangle = \langle \hat{f}(-r)e_m, e_n\rangle$ for $r \in \mathbb{R}^+$. Since $|\langle \hat{f}(r)e_m, e_n\rangle| \le \|\hat{f}(r)\|_{HS}$, we have from (ii)

$$\int_{\mathbb{R}} e^{qb\pi r^2} |\langle \hat{f}(r)e_m, e_n\rangle|^q r dr < \infty.$$

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Since $|\langle \hat{f}(r)e_m, e_n\rangle|$ is a continuous function of r,

$$\int_{\mathbb{R}} e^{qb\pi r^2}|\langle \hat{f}(r)e_m, e_n\rangle|^q dr < \infty. \tag{3.3}$$

From (3.2) we have

$$|\langle \hat{f}(\omega)e_m, e_n\rangle| \le C_1 e^{k\pi v^2} \quad \text{where } \omega = u + iv. \tag{3.4}$$

We define

$$g(\omega) = e^{k\pi\omega^2}\langle \hat{f}(\omega)e_m, e_n\rangle.$$

Then g is an entire function. From (3.4) and (3.3) it follows that

$$|g(u+iv)| \le C_1 e^{k\pi(u^2-v^2)}e^{k\pi v^2} = Ce^{k\pi u^2}, \tag{3.5}$$

$$\int_{\mathbb{R}} |g(r)|^q dr = \int_{\mathbb{R}} e^{qk\pi r^2}|\langle \hat{f}(r)e_m, e_n\rangle|^q dr < \infty \quad \text{as } k < b. \tag{3.6}$$

By (3.5) and (3.6) it follows that g satisfies conditions of Lemma 3.3 and hence $g = 0$. So $\langle \hat{f}(\omega)e_m, e_n\rangle = 0$. But m, n are arbitrary and hence $\|\hat{f}(r)\|_{HS} = 0$, which implies $f = 0$ by the Plancherel theorem. This completes the proof. □

We point out that a similar kind of technique works for another semidirect product namely the oscillator group.

The oscillator group is the semidirect product of H_1 (the one dimensional Heisenberg group) and $\mathbb{R}$ with respect to the homomorphism $\gamma : \mathbb{R} \to \mathrm{Aut}(H_1)$ given by

$$\gamma(r)(x, \xi, t) = (x\cos r + \xi\sin r, -x\sin r + \xi\cos r, t).$$

Since γ has cocompact kernel, $G = H_1 \ltimes_\gamma \mathbb{R}$ is a type 1, unimodular group with H_1 as a regularly embedded, closed normal subgroup (see [7, Theorem 3.1]). If we denote the elements of G by (x, ξ, t, r) where $(x, \xi, t) \in H_1$ and $r \in \mathbb{R}$ then $dx\,d\xi\,dt\,dr$ is a Haar measure.

To find $\widehat{G}$, we proceed by Mackey theory. For $\lambda \in \mathbb{R}\setminus\{0\}$, we consider $\Pi_\lambda \in \widehat{H_1}$. Then it is clear that $\Pi_\lambda|Z(H_1) = (\Pi_\lambda \circ \gamma(r))|Z(H_1)$ for all $r \in \mathbb{R}$ where $Z(H_1)$ is the center of H_1. Let $W(r)$ be the intertwining operator (which is unique up to a scalar). So $W(r) \in \mathscr{U}(L^2(\mathbb{R}))$ satisfies

$$\Pi_\lambda(\gamma(r)(x, \xi, t)) = W(r) \circ \Pi_\lambda(x, \xi, t) \circ W(r)^{-1},$$

for all $(x, \xi, t) \in H_1$. Since $\mathbb{R}$ has no nontrivial multiplier (see [9]), $r \to W(r)$ can be chosen to be a true unitary representation of $\mathbb{R}$. So by the little group method we get a family of irreducible unitary representations of G given by

$$\Pi_{\lambda,s} : G \to \mathscr{U}(L^2(\mathbb{R})), \quad \Pi_{\lambda,s}(x, \xi, t, r) = \chi_s(r)\Pi_\lambda(x, \xi, t) \circ W(r),$$

where $\chi_s(r) = e^{2\pi i s r}$. It follows from [7, Theorem 3.1] that the representations $\{\Pi_{\lambda,s} : \lambda \in \mathbb{R} \setminus \{0\}, s \in \mathbb{R}\}$ constitute the support for the Plancherel measure, and the Plancherel measure is given by $|\lambda| d\lambda ds$. Now by fixing λ and complexifying s if we concentrate on the matrix coefficients of the group Fourier transform $\hat{f}(\pi_{\lambda,s})$ and apply Lemma 3.3 we easily get the following theorem.

THEOREM 3.2. *Let* $f \in L^1(G) \cap L^2(G)$ *and*

(i) $\int_G e^{pa\pi \|(x,\xi,t,r)\|^2} |f(x,\xi,t,r)|^p dx d\xi dt dr < \infty$,

(ii) $\int_{\mathbb{R}} e^{qb\pi s^2} \|\hat{f}(\lambda, s)\|_{HS} ds < A_\lambda$,

where A_λ *is a constant depending on* λ *only, and* $a, b > 0$, $1 \le q < \infty$, $1 \le p \le \infty$. *If* $ab \ge 1$ *then* $f = 0$ *almost everywhere.*

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