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Convergence in Capacity

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Abstract. In this note we study the convergence of sequences of Monge–Ampère measures $\{(dd^c u_s)^n\}$, where $\{u_s\}$ is a given sequence of plurisubharmonic functions, converging in capacity.

1 Introduction

It is well known that the complex Monge–Ampère operator is continuous under monotone limits, but not continuous in the $L^1_{\rm loc}$-topology [3]. Therefore it is important to find conditions on sequences of plurisubharmonic functions so that the sequence converges to a function having Monge–Ampère measure equal to the weak limit of the Monge–Ampère measures of the functions in the sequence. Convergence in capacity is such a condition and is very useful in pluripotential theory, see [2,4,11].

With notations introduced in the next section, the purpose of this paper is to prove the following theorem.

Theorem *Assume that $u_0 \in \mathcal{E}$ and that $\{u_s\} \subset \mathcal{E}$ is a sequence with $u_0 \le u_s$ for all $s \in \mathbb{N}$. If $\{u_s\}$ converges to a plurisubharmonic function u in C_{n-1}-capacity, then the sequence of measures $\{(dd^c u_s)^n\}$ converges to $(dd^c u)^n$ in the weak*-topology as s tends to $+\infty$.*

This theorem is a generalization of Theorem 1.1 in [5], where the assumption was that $\{u_s\}$ converges to u in C_n-capacity as s tends to $+\infty$. The theorem also generalizes [1, Theorem 5.3], [12, Theorem 1], and [13, Theorem 5] and is quite sharp, as shown in [12, Theorem 2(ii)].

The sequence $\{\max(\frac{1}{s}\log|z|, s\log|w|)\}$ shows that the theorem would be false without the assumption of a common minorizing function $u_0 \in \mathcal{E}$.

2 Preliminaries

Recall that $\Omega \subseteq \mathbb{C}^n$, $n \ge 1$ is a *bounded hyperconvex domain* if it is a bounded, connected, and open set such that there exists a bounded plurisubharmonic function $\varphi\colon \Omega \to (-\infty, 0)$ such that the closure of the set $\{z \in \Omega : \varphi(z) < c\}$ is compact in Ω, for every $c \in (-\infty, 0)$. We denote by $\mathcal{PSH}(\Omega)$ the family of plurisubharmonic functions defined on Ω

We say that a bounded plurisubharmonic function φ defined on Ω belongs to $\mathcal{E}_0$ if $\lim_{z\to\xi}\varphi(z) = 0$, for every $\xi \in \partial\Omega$, and $\int_\Omega (dd^c\varphi)^n < +\infty$. See [6,9] for details.

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Let $\mathcal{E}$ be the family of plurisubharmonic functions φ defined on Ω, such that for each $z_0 \in \Omega$ there exists a neighborhood ω of z_0 in Ω and a decreasing sequence $\{\varphi_s\} \subset \mathcal{E}_0$ that converges pointwise to φ on ω as $s \to +\infty$ and

$$\sup_s \int_\Omega (dd^c \varphi_s)^n < +\infty.$$

Furthermore, let $\mathcal{F}(\subset \mathcal{E})$ denote those functions for which we can take $\omega = \Omega$.

For $v \in \mathcal{PSH}(\Omega)$, $-1 \leq v < 0$, fixed, we define the C^v_{n-1}-capacity by

$$\begin{aligned} C^v_{n-1}(E) &= C^v(E) \\ &= \sup\Big\{ \int_E dd^c w_1 \wedge \cdots \wedge dd^c w_{n-1} \wedge dd^c v : \\ &\qquad -1 \leq w_j \leq 0,\ w_j \in \mathcal{PSH}(\Omega),\ 1 \leq j \leq n-1 \Big\}. \end{aligned}$$

Following [12] we define for $E \subset \Omega$, the C_{n-1}-capacity as C^v_{n-1} in the case when $v \in \mathcal{E}_0 \cap C^\infty(\Omega)$, $-1 \leq v \leq 0$ is a strictly plurisubharmonic function. By [8] such a function always exists.

Let u, u_s, $s \in \mathbb{N}$, be real-valued, Borel measurable, functions defined on Ω. Then we say that $\{u_s\}$ *converges to u in C^v–capacity as s tends to $+\infty$* if for every compact subset K of Ω and every $\varepsilon > 0$ it holds that

$$\lim_{s\to+\infty} C^v(\{z \in K : |u_s(z) - u(z)| > \varepsilon\}) = 0.$$

Furthermore, for $v \in \mathcal{E}_0, u_0 \in \mathcal{F}$ we define

$$\begin{aligned} &C^{v,u_0}_{n-1}(E) = C^{v,u_0}(E) = \\ &\quad \sup\Big\{ \int_E dd^c w_1 \wedge \cdots \wedge dd^c w_{n-1} \wedge dd^c v : u_0 + v \leq w_j \in \mathcal{F},\ 1 \leq j \leq n-1 \Big\}, \end{aligned}$$

and we say that $\{u_s\}$ *converges to u in C^{v,u_0}_{n-1}-capacity as s tends to $+\infty$* if for every compact subset K of Ω and for every $\varepsilon > 0$ it holds that

$$\lim_{s\to+\infty} C^{v,u_0}(\{z \in K : \{|u_s(z) - u(z)| > \varepsilon\}) = 0.$$

Lemma 2.1 *Assume that u, u_s, $s \in \mathbb{N}$, are real-valued, Borel measurable, functions. Then the following two assertions are equivalent:*

(i) *the sequence $\{u_s\}$ converges to u in C^v-capacity,*
(ii) *the sequence $\{u_s\}$ converges to u in C^{v,u_0}-capacity.*

Proof For every $K \Subset \Omega$, there exists a constant $A_K > 0$ such that $-u_0 \geq A_K$ on K. Therefore, $C^v(E \cap K) A_K^{n-1} \leq C^{v,u_0}(E)$.

On the other hand, for $u_0 + v \le w_j \in \mathcal{F}$, $1 \le j \le n-1$, it follows from [10, Theorem 4.1] that for each $m > 0$ it holds that

$$\chi_{\{w_1 > -m\}} \cdots \chi_{\{w_{n-1} > -m\}} dd^c w_1 \wedge \cdots \wedge dd^c w_{n-1} \wedge dd^c v =$$
$$\chi_{\{w_1 > -m\}} \cdots \chi_{\{w_{n-1} > -m\}} dd^c \max(w_1, -m) \wedge \cdots \wedge dd^c \max(w_{n-1}, -m) \wedge dd^c v.$$

Hence,

$$\chi_{\{u_0 + v > -m\}} dd^c w_1 \wedge \cdots \wedge dd^c w_{n-1} \wedge dd^c v = \chi_{\{u_0 + v > -m\}} dd^c \max(w_1, -m) \wedge \cdots \wedge dd^c \max(w_{n-1}, -m) \wedge dd^c v,$$

and therefore we have that

$$\begin{aligned}\int_{E\cap K} & dd^c w_1 \wedge \cdots \wedge dd^c w_{n-1} \wedge dd^c v \\ &= \int_{E\cap\{u_0+v>-m\}\cap K} dd^c w_1 \wedge \cdots \wedge dd^c w_{n-1} \wedge dd^c v \\ &\quad + \int_{E\cap\{u_0+v\le -m\}\cap K} dd^c w_1 \wedge \cdots \wedge dd^c w_{n-1} \wedge dd^c v \\ &\le m^{n-1} C^v(E\cap K) + \frac{1}{m}\int_\Omega -(u_0+v)\, dd^c w_1 \wedge \cdots \wedge dd^c w_{n-1} \wedge dd^c v \\ &\le m^{n-1} C^v(E\cap K) + \frac{1}{m}\int_\Omega (dd^c(u_0+v))^n. \end{aligned}$$

■

3 Convergence in Capacity

Lemma 3.1 *Assume that μ is a positive measure defined on Ω that vanishes on all pluripolar sets, $u_0 \in \mathcal{E}$ and $\mu(\Omega) - \int_\Omega u_0 d\mu < +\infty$. Assume that $\{u_s\} \subset \mathcal{E}$ is a sequence with $u_0 \le u_s$ for all $s \in \mathbb{N}$. If $\{u_s\}$ converges in the sense of distributions to a function u, then*

$$\lim_{s\to+\infty} \int_\Omega u_s\, d\mu = \int_\Omega u\, d\mu.$$

Proof Without loss of generality we can assume that $u_0 \in \mathcal{F}$ and $\{u_s\} \subset \mathcal{F}$. Let $d\lambda$ be the Lebesgue measure, and use [6, Theorem 2.1] to choose $\tilde{u}_s \in \mathcal{E}_0 \cap C(\bar{\Omega})$, $\tilde{u}_s \ge u_s$, such that

$$\int_\Omega (\tilde{u}_s - u_s)(d\mu + d\lambda) < \frac{1}{s}.$$

Then $\{\tilde{u}_s\}$ converges in the sense of distributions to a function u, and

$$\lim_{s\to+\infty} \Big(\int_\Omega u_s\, d\mu - \int_\Omega \tilde{u}_s\, d\mu \Big) = 0.$$

Therefore, it is enough to prove that

$$\lim_{s\to+\infty}\int_\Omega \tilde{u}_s\,d\mu=\int_\Omega u\,d\mu.$$

To simplify the notation we let $\tilde{u}_s$ be denoted by u_s, and therefore in the rest of this proof $\{u_s\}\subset\mathcal{E}_0\cap C(\bar\Omega)$. Theorem 6.3 in [4] implies that there are functions $\psi\in\mathcal{E}_0$, $f\in L^1((dd^c\psi)^n)$ with $\mu=f(dd^c\psi)^n$, and by [4, Lemma 5.2] we have that for every $p<+\infty$ it holds that

$$\lim_{s\to+\infty}\int_\Omega u_s\,d\mu_p=\int u\,d\mu_p,$$

where $\mu_p=\min(f,p)(dd^c\psi)^n$. The monotone convergence theorem now gives us that

$$\begin{aligned}\lim_{s\to+\infty}\int_\Omega u_s\,d\mu&=\lim_{s\to+\infty}\int_\Omega u_s\,d\mu_p+\lim_{s\to+\infty}\int_\Omega u_s(f-\min(f,p))(dd^c\psi)^n\\&\geq\int_\Omega u\,d\mu_p+\int_\Omega u_0(f-\min(f,p))(dd^c\psi)^n\xrightarrow[p\to+\infty]{}\int_\Omega u\,d\mu.\end{aligned}$$

On the other hand, by Fatou's lemma,

$$\limsup_{s\to+\infty}\int_\Omega u_s\,d\mu\leq\int_\Omega u\,d\mu,$$

which yields the desired conclusion. ■

Lemma 3.2 *Let $v\in\mathcal{E}_0(\Omega)$. Assume that $u_0\in\mathcal{F}$, and that $\{u_s\}\subset\mathcal{F}$ is a sequence with $u_0\leq u_s$ for all $s\in\mathbb{N}$. If $\{u_s\}$ converges to a function u in C^{v,u_0}-capacity, then*

$$\begin{aligned}\lim_{s\to+\infty}\int_\Omega w_1(dd^cu_s)^j\,dd^cw_2\wedge\cdots\wedge dd^cw_{n-j}\wedge dd^cv=\\\int_\Omega w_1(dd^cu)^j\,dd^cw_2\wedge\cdots\wedge dd^cw_{n-j}\wedge dd^cv\quad 1\leq j\leq n-1,\end{aligned}$$

for all $w_j\in\mathcal{F}$, $u_0+v\leq w_j$, $j=1,\dots,n-j$.

Proof By [6, Theorem 5.5], $\mu=dd^cw_1\wedge\cdots\wedge dd^cw_{n-1}\wedge dd^cv$ satifies the conditions of Lemma 3.1. Integration by parts shows the statement in this lemma is true for $j=1$. Assume now that the lemma is true for $j<n-1$. We shall prove it is true for

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$j+1$. Let $w_1, \dots, w_{n-j-1}$ be as in the statement, and let $\varepsilon > 0$ be given. Then

$$\begin{aligned}
&\int_\Omega w_1 (dd^c u_s)^{j+1} \wedge dd^c w_2 \wedge \cdots \wedge dd^c w_{n-j-1} \wedge dd^c v \\
&\qquad - \int_\Omega w_1 (dd^c u_s)^j \wedge dd^c u \wedge dd^c w_2 \wedge \cdots \wedge dd^c w_{n-j-1} \wedge dd^c v \\
&= \int_\Omega (u_s - u)\,(dd^c u_s)^j \wedge dd^c w_1 \wedge \cdots \wedge dd^c w_{n-j-1} \wedge dd^c v \\
&= \int_{\{|u_s-u|>\varepsilon\}} (u_s - u)\,(dd^c u_s)^j \wedge dd^c w_1 \wedge \cdots \wedge dd^c w_{n-j-1} \wedge dd^c v \\
&\qquad + \int_{\{|u_s-u|\le\varepsilon\}} (u_s - u)\,(dd^c u_s)^j \wedge dd^c w_1 \wedge \cdots \wedge dd^c w_{n-j-1} \wedge dd^c v \\
&= \mathrm{I}_s + \mathrm{II}_s\,.
\end{aligned}$$

By the induction hypotheses we have that

$$\begin{aligned}
\lim_{s\to+\infty} \int_\Omega w_1 (dd^c u_s)^j \wedge dd^c u \wedge\; dd^c w_2 \wedge \cdots \wedge dd^c w_{n-j-1} \wedge dd^c v = \\
\int_\Omega w_1 (dd^c u)^{j+1} \wedge\; dd^c w_2 \wedge \cdots \wedge dd^c w_{n-j-1} \wedge dd^c v \quad 1 \le j < n-1.
\end{aligned}$$

Hence, it remains to estimate $\mathrm{I}_s + \mathrm{II}_s$. We have that

$$\begin{aligned}
|\,\mathrm{II}_s\,| &\le \varepsilon \int_\Omega (dd^c(u_0 + v))^{n-1} \wedge dd^c v, \\
|\,\mathrm{I}_s\,| &\le \int_{\{|u_s-u|>\varepsilon\}} -2u_0 (dd^c u_s)^j \wedge dd^c w_1 \wedge \cdots \wedge dd^c w_{n-j-1} \wedge dd^c v \\
&\le 2 \int_\Omega (-u_0 + \max(u_0, -N))(dd^c u_s)^j \wedge dd^c w_1 \wedge \cdots \wedge dd^c w_{n-j-1} \wedge dd^c v \\
&\qquad - 2 \int_{\{|u_s-u|>\varepsilon\}} \max(u_0, -N)(dd^c u_s)^j \wedge dd^c w_1 \wedge \cdots \wedge dd^c w_{n-j-1} \wedge dd^c v \\
&\le 2 \int_\Omega (-u_0 + \max(u_0, -N))(dd^c u_s)^j \wedge dd^c w_1 \wedge \cdots \wedge dd^c w_{n-j-1} \wedge dd^c v \\
&\qquad - 2 \int_{\{|u_s-u|>\varepsilon\}\cap\{v>-\varepsilon\}} u_0 (dd^c u_s)^j \wedge dd^c w_1 \wedge \cdots \wedge dd^c w_{n-j-1} \wedge dd^c v \\
&\qquad + 2N \int_{\{|u_s-u|>\varepsilon\}\cap\{v\le-\varepsilon\}} (dd^c u_s)^j \wedge dd^c w_1 \wedge \cdots \wedge dd^c w_{n-j-1} \wedge dd^c v.
\end{aligned}$$

By the induction hypotheses, we have that

$$2\int_\Omega(-u_0+\max(u_0,-N))\,(dd^cu_s)^j\wedge dd^cw_1\wedge\cdots\wedge dd^cw_{n-j-1}\wedge dd^cv$$
$$\xrightarrow[s\to+\infty]{} 2\int_\Omega(-u_0+\max(u_0,-N))\,(dd^cu)^j\wedge dd^cw_1\wedge\cdots\wedge dd^cw_{n-j-1}\wedge dd^cv,$$

which is small when N is big enough. For N fixed, then

$$2N\int_{\{|u_s-u|>\varepsilon\}\cap\{v\le-\varepsilon\}}(dd^cu_s)^j\wedge dd^cw_1\wedge\cdots\wedge dd^cw_{n-j-1}\wedge dd^cv\xrightarrow[s\to+\infty]{}0,$$

since $\{u_s\}$ tends to u in C^{v,u_0}-capacity. Also

$$\begin{aligned}-2\int_{\{|u_s-u|>\varepsilon\}\cap\{v>-\varepsilon\}}&u_0(dd^cu_s)^j\wedge dd^cw_1\wedge\cdots\wedge dd^cw_{n-j-1}\wedge dd^cv\\&\le-2\int_\Omega u_0(dd^cu_0+v)^{n-1}\wedge dd^c\max(v,-\varepsilon).\\&\le 2\varepsilon\int_\Omega(dd^c(u_0+v))^n.\end{aligned}$$

The proof is complete, since $\varepsilon>0$ was arbitrary. ■

Proof of the Theorem It is enough to prove this theorem for $u\in\mathcal{F}$. For if ψ is any negative plurisubharmonic function, then $\{\max(u_s,\psi)\}$ tends to $\max(u,\psi)$ in C^v-capacity, and since $u_0\in\mathcal{E}$, there is to every compact subset K of Ω a function $u^K\in\mathcal{F}(\Omega)$ such that $u_0\le u^K$ with equality near K. Also by Lemma 2.1 we can equally well work with C^{v,u_0}-capacity. We claim

$$\lim_{s\to+\infty}\int_\Omega v(dd^cu_s)^n=\int_\Omega v(dd^cu)^n, \tag{3.1}$$

We have

$$\begin{aligned}\int_\Omega v(dd^cu_s)^n&=\int_\Omega u_s(dd^cu_s)^{n-1}\wedge dd^cv\\&=\int_\Omega(u_s-u)(dd^cu_s)^{n-1}\wedge dd^cv+\int_\Omega u\,(dd^cu_s)^{n-1}\wedge dd^cv,\end{aligned}$$

and Lemma 3.2 yields that

$$\lim_{s\to+\infty}\int_\Omega u(dd^cu_s)^{n-1}\wedge dd^cv=\int_\Omega v(dd^cu)^n,$$

and from the proof of Lemma 3.2 it follows that

$$\lim_{s\to+\infty}\int_\Omega(u_s-u)(dd^cu_s)^{n-1}\wedge dd^cv=0.$$

Hence, we have that (3.1) is true.

Let now $v \in \mathcal{E}_0 \cap C^\infty(\Omega)$, $-1 \le v < 0$, be a strictly plurisubharmonic function. Equality (3.1) together with an application of [7, Lemma 2.1] completes the proof of the theorem. ■

The following corollary generalizes [14, Theorem 3.5].

Corollary 3.3 *Assume that $u_0 \in \mathcal{E}$ and that $\{u_s\} \subset \mathcal{E}$ is a sequence with $u_0 \le u_s$ for all $s \in \mathbb{N}$, $v \in \mathcal{E}_0 \cap C^\infty(\Omega)$, $-1 \le v \le 0$, is a strictly plurisubharmonic function and that f is a negative and locally bounded plurisubharmonic function. We can assume $v + f \ge -1$. If $\{u_s\}$ converges to a plurisubharmonic function u in C^{v+h}-capacity for every $h \in \mathcal{E}_0$, $f \le h$, then the sequence of measures $\{f(dd^c u_s)^n\}$ converges to $f(dd^c u)^n$ in the weak*-topology as s tends to $+\infty$.*

Proof We have already observed that we can assume that $u_0 \in \mathcal{F}$ so $\{u_s\} \subset \mathcal{F}$. If $f \in \mathcal{E}_0$, then the corollary follows from (3.1). To complete the proof we need only observe that every negative and locally bounded plurisubharmonic function is locally equal to a function in $\mathcal{E}_0(\Omega)$. ■

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