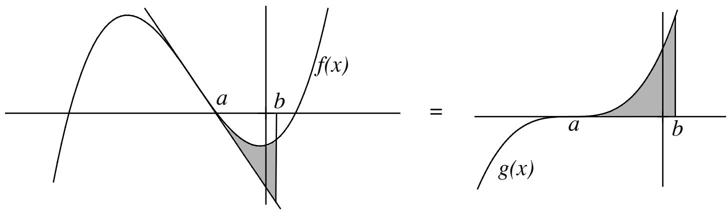

FIGURE 4: for $a_0 > 0, a < b$

Polynomials of degree 4

Let $f(x) = a_0x^4 + a_1x^3 + a_2x^2 + a_3x + a_4$, with $a_0 \neq 0$, and $T_f(x)$ the tangent line to the graph of $f(x)$ at an inflection point $x = a$. Then $f''(a) = 0, f'''(a) = 6a_1 + 24aa_0 \neq 0$ and $f^{(4)}(a) = 24a_0$. Therefore, by (2) we have,

$$f(x) - T_f(x) = (x-a)^3\left(\frac{6a_1 + 24aa_0}{6} + \frac{24a_0}{24}(x-a)\right) = a_0(x-a)^3(x-b)$$

for $b = -3a - \dfrac{a_1}{a_0}$. (We note that $b \neq a$ because $f'''(a) \neq 0$.)

Consequently the area between the graphs of $f(x)$ and $T_f(x)$ is

$$\left|\int_a^b a_0(x-a)^3(x-b)\,dx\right| = \left|\frac{1}{20}a_0(b-a)^5\right| = \left|\frac{1}{20}a_0\left(4a + \frac{a_1}{a_0}\right)^5\right|.$$

This value is the same as the area between the graph of $g(x) = a_0(x-a)^3(x-b)$ and the x-axis (Figure 5).

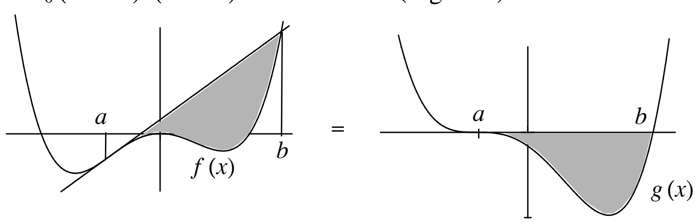

FIGURE 5: for $a_0 > 0, a < b$

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The full story of invariant lines

The topics of invariant points and lines (for 2×2 matrix transformations) and eigenvalues and eigenvectors appear on some of the current AS/A level Further Mathematics specifications. These topics are treated separately in the textbooks. The aim of this short Note is to describe how the latter enables a full and succinct treatment of the former which also explains the rather limited range of examples.

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Let **A** be a 2×2 matrix with real entries. Then **v**, an invariant point for **A**, satisfies $\mathbf{Av} = \mathbf{v}$ so corresponds to an eigenvector of **A** with eigenvalue 1, together with **0**.

Suppose now that $\mathbf{r} = \mathbf{a} + t\mathbf{d}$ $(\mathbf{d} \neq \mathbf{0}, t \in \mathbb{R})$ is an invariant line for **A**. When $t = 0$, we have $\mathbf{Aa} = \mathbf{a} + t_1\mathbf{d}$ and when $t = 1$ we have $\mathbf{A}(\mathbf{a} + \mathbf{d}) = \mathbf{a} + t_2\mathbf{d}$. From these we deduce that $\mathbf{Ad} = \mathbf{A}(\mathbf{a} + \mathbf{d}) - \mathbf{Aa} = (t_2 - t_1)\mathbf{d}$, so that **d** is an eigenvector of **A** with (real) eigenvalue $t_2 - t_1$. Thus the direction of any invariant line is parallel to an eigenvector of **A**. In particular, if **A** has no real eigenvalues then it has no invariant lines: a typical example of this is a rotation, centre O. If **A** has real eigenvalues, then lines through O in the directions of the eigenvectors are invariant. An example of this is a matrix describing a pair of one-way stretches such as $\begin{pmatrix} 7 & 6 \\ -4 & 18 \end{pmatrix}$ (with eigenvalues 10, 15).

Could there be other invariant lines not containing O?

Suppose first that **A** has two linearly independent eigenvectors **c**, **d** with $\mathbf{Ac} = \lambda\mathbf{c}$ and $\mathbf{Ad} = \mu\mathbf{d}$: this necessarily happens if its two (real) eigenvalues are different. An invariant line not containing O can then be written as $\mathbf{r} = k\mathbf{c} + t\mathbf{d}$ for some fixed $k \neq 0$. When $t = 0$ invariance means that $\mathbf{A}(k\mathbf{c}) = k\mathbf{c} + t_1\mathbf{d}$. But $\mathbf{A}(k\mathbf{c}) = k\lambda\mathbf{c}$, so that $k\lambda\mathbf{c} = k\mathbf{c} + t_1\mathbf{d}$ from which $\lambda = 1$, by linear independence, so that **A** has a line of invariant points. Reflection in a line through O provides a typical example for this case, with eigenvalues 1 and -1.

The one remaining case is where **A** has a repeated eigenvalue λ with just one linearly independent eigenvector **d**. If the line $\mathbf{r} = \mathbf{a} + t\mathbf{d}$ is invariant with $\mathbf{a} \neq 0$ linearly independent of **d**, then $\mathbf{Aa} = \mathbf{a} + t_1\mathbf{d}$. Hence

$$\mathbf{A}^2\mathbf{a} = \mathbf{A}(\mathbf{a} + t_1\mathbf{d}) = \mathbf{a} + t_1\mathbf{d} + t_1\lambda\mathbf{d} = \mathbf{a} + t_1(1 + \lambda)\mathbf{d}.$$

On the other hand, from the characteristic equation, $(\mathbf{A} - \lambda\mathbf{I})^2 = 0$ or $\mathbf{A}^2 = 2\lambda\mathbf{A} - \lambda^2\mathbf{I}$, so that

$$\mathbf{A}^2\mathbf{a} = (2\lambda\mathbf{A} - \lambda^2\mathbf{I})\mathbf{a} = 2\lambda\mathbf{a} + 2\lambda t_1\mathbf{d} - \lambda^2\mathbf{a} = (2\lambda - \lambda^2)\mathbf{a} + 2\lambda t_1\mathbf{d}.$$

Comparing the two expressions for $\mathbf{A}^2\mathbf{a}$, we deduce that $1 = 2\lambda - \lambda^2$ so that $\lambda = 1$ again. A shear parallel to a line through O is a typical example of this case.

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