Journal of the Inst. of Math. Jussieu (2004) **3**(2), 165–183 © Cambridge University Press 165
DOI:10.1017/S1474748004000064 Printed in the United Kingdom

A CONVERSE TO MAZUR'S INEQUALITY FOR SPLIT CLASSICAL GROUPS

CATHERINE LUCARELLI

Department of Mathematics, University of Chicago, 5734 South University Avenue, Chicago, IL 60637, USA (cleigh@math.uchicago.edu)

(Received 27 February 2003; accepted 11 June 2003)

Abstract Given a lattice in an isocrystal, Mazur's inequality states that the Newton point of the isocrystal is less than or equal to the invariant measuring the relative position of the lattice and its transform under Frobenius. Conversely, it is known that any potential invariant allowed by Mazur's inequality actually arises from some lattice. These can be regarded as statements about the group GL_n. This paper proves an analogous converse theorem for all split classical groups.

Keywords: Mazur's inequality; isocrystal; Newton polygon

AMS 2000 *Mathematics subject classification:* Primary 14L05
Secondary 11S25; 14F30; 20G25

0. Introduction

Our goal is to prove a converse to Mazur's inequality for split classical groups. This work stems from results in [**5**], whose notation we will follow. Let F be a finite extension of $\mathbb{Q}_p$ and let L be the completion of the maximal unramified extension of F. Denote the ring of integers in F (respectively, L) by $\mathcal{O}_F$ (respectively, $\mathcal{O}_L$). Let $\pi \in \mathcal{O}_F$ be a uniformizer and let σ be the relative Frobenius automorphism of L/F. Let G be a split connected reductive group with Borel subgroup B, both defined over $\mathcal{O}_F$, and let T be a maximal torus of G over $\mathcal{O}_F$ contained in B. We abbreviate X_*T to X. We denote by $\mathfrak{a}$ the real vector space $X \otimes_{\mathbb{Z}} \mathbb{R}$, and by $\mathfrak{a}_{\mathrm{dom}}$ the cone of dominant elements in $\mathfrak{a}$. For $x, y \in \mathfrak{a}$, we say $x \leqslant y$ if $\langle x, \omega\rangle \leqslant \langle y, \omega\rangle$ for all fundamental weights ω and $y - x$ is in the linear span of the co-roots of G. We write X_G for the quotient of X by the co-root lattice for G, and $\varphi_G : X \to X_G$ for the natural projection map.

Now let $\mu \in X$ be G-dominant. The Weyl group W acts on X. We define the subset $P_\mu \subset X$ by

$$P_\mu := \{\nu \in X : \text{(i) } \varphi_G(\nu) = \varphi_G(\mu);\ \text{and (ii) } \nu \in \mathrm{Conv}(W\mu)\},$$

where $W\mu = \{w(\mu) : w \in W\}$ and $\mathrm{Conv}(W\mu)$ denotes the convex hull of $W\mu$ in $\mathfrak{a}$.

Let P be a parabolic subgroup of G which contains B and let M be the unique Levi subgroup of P containing T. Thus T is also a maximal torus in M. We write X_M for the quotient of X by the co-root lattice for M, and let $\varphi_M : X \to X_M$ be the natural projection map. Since the co-root lattice for M is a subgroup of the co-root lattice for G, the map φ_G factors through X_M via φ_M.

Recall that $b_1, b_2 \in G(L)$ are said to be σ-conjugate if there exists $g \in G(L)$ such that $b_2 = g^{-1} b_1 \sigma(g)$ and that $B(G)$ denotes the set of σ-conjugacy classes in $G(L)$. Finally, as in [**3**, § 6], define

$$B(G, \mu) := \{b \in B(G) : \kappa_G(b) = \varphi_G(\mu) \text{ and } \bar{\nu}(b) \leqslant \mu\},$$

where $\bar{\nu}(b) \in \mathfrak{a}_{\mathrm{dom}}$ is the image of b under the Newton map (cf. [**3**, **7**]) and κ_G is the map (4.9.1) of [**3**]. We can now state our main theorem. Set $\tilde{K} = G(\mathcal{O}_L)$ and $\pi^\mu = \mu(\pi)$.

Converse to Mazur's inequality. *Let G be a split connected reductive group over F which is weakly classical, in the sense that each irreducible component of its Dynkin diagram is of type A_n, B_n, C_n, or D_n. Let $\mu \in X$ be G-dominant and let $b \in B(G, \mu)$. Then the σ-conjugacy class of b in $G(L)$ meets $\tilde{K}\pi^\mu\tilde{K}$.*

The converse to Mazur's inequality is proven in [**5**] (respectively, [**2**]), for $G = GL_n$ and $G = GSp_{2n}$ (respectively, $G = GL_n$). For a discussion of different formulations of the converse to Mazur's inequality, the reader should see [**4**, § 4.3]. The reader who would like to know how the theorem relates to the reduction modulo p of Shimura varieties should consult the survey article [**6**].

To prove the converse for all split classical groups, we use the following result of [**5**]. Note that the result holds for any split reductive group G whether or not the derived group of G is simply connected.

Proposition 0.1 (cf. Proposition 4.6 of [5]). *Let $b \in M(L)$ be basic, and let $\mu \in X$. The σ-conjugacy class of b in $G(L)$ meets $\tilde{K}\pi^\mu\tilde{K}$ if and only if $\kappa_M(b) \in \varphi_M(P_\mu)$.*

We will show that $\varphi_M(P_\mu)$ is equal to a certain subset of X_M, described below. This is the method used in [**5**] for $G = GL_n$ and $G = GSp_{2n}$. We will provide proofs for $G = SO_{2n+1}$ and $G = SO_{2n}/\{\pm 1\}$ as well as an alternate proof for $G = GL_n$. While we have not yet proven the result for exceptional groups, we suspect that it is true.

Now we describe the subset of X_M which we will prove is equal to $\varphi_M(P_\mu)$. Recall that $\mathfrak{a} = X \otimes_{\mathbb{Z}} \mathbb{R}$ and define $\mathfrak{a}_M = X_M \otimes_{\mathbb{Z}} \mathbb{R}$. There is a natural projection map $\mathrm{pr}_M : \mathfrak{a} \to \mathfrak{a}_M$ induced by φ_M. Note that W acts on $\mathfrak{a}$ since it acts on X. Therefore, the Weyl group of M, denoted by W_M, acts on $\mathfrak{a}$. The restriction of pr_M to $\mathfrak{a}^{W_M}$ identifies $\mathfrak{a}^{W_M}$ with $\mathfrak{a}_M$. With this identification, it is easy to see that $\mathrm{Conv}(W\mu) \cap \mathfrak{a}^{W_M} = \mathrm{pr}_M(\mathrm{Conv}(W\mu))$. We can now formulate the theorem. For a detailed discussion of how the theorem implies the converse to Mazur's inequality, the reader should see [**4**, § 4.3, Proposition 4.10].

Theorem 0.2. *Let G be a split connected reductive group over F which is weakly classical in the above sense and let M and μ be as above. Then*

$$\varphi_M(P_\mu) = \{\nu_1 \in X_M : \text{(i) } \nu_1, \mu \textit{ have the same image in } X_G; \textit{ and } \\ \text{(ii) } \textit{the image of } \nu_1 \textit{ in } \mathfrak{a}_M \textit{ lies in } \mathrm{Conv}(W\mu) \cap \mathfrak{a}^{W_M}\}.$$

https://doi.org/10.1017/S1474748004000064 Published online by Cambridge University Press

For all G we can easily check that the left-hand side is a subset of the right-hand side by verifying that for $\nu \in P_\mu$, the image $\varphi_M(\nu)$ satisfies conditions (i) and (ii). First, ν and μ have the same image in X_G, hence $\varphi_M(\nu)$ and μ also have the same image in X_G. Second, the image of $\varphi_M(\nu)$ in $\mathfrak{a}_M$ is $\mathrm{pr}_M(\nu)$, which lies in $\mathrm{pr}_M(\mathrm{Conv}(W\mu))$ since $\nu \in \mathrm{Conv}(W\mu)$.

To prove that the right-hand side is a subset of the left-hand side, we must show that given $\nu_1 \in X_M$ satisfying conditions (i) and (ii), we can find $\nu \in X$ such that $\varphi_M(\nu) = \nu_1$ and $\nu \in \mathrm{Conv}(W\mu)$. Note that if $\varphi_M(\nu) = \nu_1$, then ν and μ will have the same image in X_G by condition (i). To show that we can find such a ν for classical groups, we will examine each type separately. Before doing so, we will make several reductions.

We claim that we may assume without loss of generality that ν_1 is G-dominant. Indeed, in the theorem only the Weyl group orbit of μ plays a role, and therefore we are free to choose the Borel subgroup B such that ν_1 is G-dominant. We now do so, and then (cf. [**7**, Lemma 2.2 (ii)]) the condition that $\nu_1 \in \mathrm{Conv}(W\mu)$ is equivalent to $\nu_1 \leqslant \mu$.

A co-weight $\nu \in X$ is said to be G-minuscule if for every root α of G, we have $\langle \alpha, \nu \rangle \in \{-1, 0, 1\}$. We define M-minuscule analogously. Bourbaki (cf. [**1**, Chapter VIII, § 7, Proposition 8]) shows that given $\nu_1 \in X_M$ there exists a unique M-dominant, M-minuscule co-weight $\nu \in X$ such that $\varphi_M(\nu) = \nu_1$. We will show that this choice of ν satisfies the desired condition, i.e. that $\nu \in \mathrm{Conv}(W\mu)$. Thus the theorem is proven by the following proposition.

Proposition 0.3. *Denote $\mathrm{pr}_M(\nu)$ by ν_M. Let $\nu, \mu \in X$ be such that μ is G-dominant, ν is M-dominant and M-minuscule, and $\varphi_G(\nu) = \varphi_G(\mu)$. Then if $\nu_M \in \mathfrak{a}^{W_M}$ is G-dominant and $\nu_M \leqslant \mu$, it follows that $\nu \in \mathrm{Conv}(W\mu)$.*

As we have already stated, Kottwitz and Rapoport proved the theorem for $G = GL_n$ and $G = GSp_{2n}$. Via Proposition 0.3, we will prove the theorem for $G = SO_{2n+1}$ and $G = SO_{2n}/\{\pm 1\}$ as well as giving an alternate proof for GL_n. We claim that to prove the theorem for all split connected reductive groups which are weakly classical, it is enough to do so for the above groups G. We will need the following four facts. Fact 3 is standard. Fact 4 is obvious. For a sketch of the proofs of Facts 1 and 2, the reader should see § 6.

Fact 1. If the theorem holds for G and the centre of G is connected, then the theorem holds for the adjoint group of G.

Fact 2. If the theorem holds for the adjoint group of G, then it holds for G.

Fact 3. The adjoint group of G factors as a product of simple groups.

Fact 4. The theorem holds for $G_1 \times G_2$ if and only if it holds for G_1 and G_2.

These facts show that if the theorem holds for $G = GL_n$, $G = GSp_{2n}$, $G = SO_{2n+1}$, and $G = SO_{2n}/\{\pm 1\}$, then it will also hold for any group whose adjoint group can be written as a product of the adjoint groups of these groups; therefore the theorem will hold for all split connected reductive groups which are weakly classical.

The strategy of the proof of Proposition 0.3 is as follows. For each of the above groups G, we modify ν by a certain procedure to obtain a co-weight $\eta \in W\nu$. We then show that η is G-dominant, hence $\eta = \nu_{\mathrm{dom}}$, and moreover we find a Levi subgroup $L \supseteq M$ for which η satisfies all of the hypotheses on ν. (In fact, η satisfies all of the hypotheses on ν for M, but it is easier to work with L.) Recall that $\nu \in \mathrm{Conv}(W\mu)$ if and only if $\nu_{\mathrm{dom}} \leqslant \mu$. Since η is G-dominant, we have $\eta \in \mathrm{Conv}(W\mu)$ if and only if $\eta \leqslant \mu$, i.e. if and only if $\nu_{\mathrm{dom}} \leqslant \mu$. Hence $\eta \in \mathrm{Conv}(W\mu)$ if and only if $\nu \in \mathrm{Conv}(W\mu)$. Thus the problem is reduced to proving the proposition with the additional hypothesis that ν is G-dominant. We complete the proof by showing that $\nu \leqslant \mu$.

1. A useful lemma

Let G, B, T, and $\mathfrak{a}$ be as above. Let P be a parabolic subgroup of G containing B. Let N be the unipotent radical of P and let M be the unique Levi subgroup of P containing T; thus $P = MN$. As usual, let W_M be the Weyl group for (M, T). Let $\{\alpha_i\}_{i \in I}$ be the set of simple roots for G. Then we can write $I = I_M \sqcup I_N$, where I_M (respectively, I_N) is the set of indices for the simple roots in M (respectively, N).

Lemma 1.1. *Let $\mu \in \mathfrak{a}$ be G-dominant and let $\beta \in \mathfrak{a}^{W_M}$. Assume further that $\mu - \beta$ is in the linear span of the co-roots of G. If $\langle \beta, \omega_i \rangle \leqslant \langle \mu, \omega_i \rangle$ for all fundamental weights ω_i such that $i \in I_N$, then $\beta \leqslant \mu$.*

Proof. Say

$$\mu - \beta = \sum_{i \in I} r_i \alpha_i^\vee.$$

Since $I = I_M \sqcup I_N$, we can rewrite this as

$$\mu - \beta = \sum_{i \in I_M} r_i \alpha_i^\vee + \sum_{i \in I_N} r_i \alpha_i^\vee. \tag{1.1}$$

By hypothesis, we have $r_i \geqslant 0$ for all $i \in I_N$. It remains to show that $r_i \geqslant 0$ for all $i \in I_M$. To do so, it is enough to show that $\sum_{i \in I_M} r_i \alpha_i^\vee$ is M-dominant (cf. [**1**, Chapter VI, §§ 1, 10]), i.e. that $\langle \sum_{i \in I_M} r_i \alpha_i^\vee, \alpha_j \rangle \geqslant 0$ for all $j \in I_M$. By Equation (1.1), this is equivalent to showing that

$$\left\langle \mu - \beta - \sum_{i \in I_N} r_i \alpha_i^\vee, \alpha_j \right\rangle \geqslant 0. \tag{1.2}$$

For all $j \in I_M$, we have $\langle \mu, \alpha_j \rangle \geqslant 0$ since μ is G-dominant, and we have $\langle \beta, \alpha_j \rangle = 0$ since $\beta \in \mathfrak{a}^{W_M}$. Finally, $\langle \sum_{i \in I_N} r_i \alpha_i^\vee, \alpha_j \rangle \leqslant 0$ since $\langle \alpha_i^\vee, \alpha_j \rangle \leqslant 0$ and $r_i \geqslant 0$ for all $i \in I_N$ and $j \in I_M$. Therefore, inequality (1.2) holds for all $j \in I_M$. □

2. The proof of Proposition 0.3 for $G = GL_n$

Let $G = GL_n$ and let M be the Levi subgroup $GL_{n_1} \times GL_{n_2} \times \cdots \times GL_{n_r}$ of G where $n_1 + n_2 + \cdots + n_r = n$. Let T be diagonal matrices and B upper triangular matrices. Then

$X = \mathbb{Z}^n$. In this case, $x \in X$ is G-dominant if $x_1 \geqslant x_2 \geqslant \cdots \geqslant x_n$. Thus the G-dominant, G-minuscule elements of X are of the form

$$(\underbrace{1, \dots, 1}_{t}, \underbrace{0, \dots, 0}_{n-t}) + k(1, \dots, 1),$$

where $0 \leqslant t \leqslant n$ and k is an integer. Also $x \leqslant \mu$ if the following conditions hold for $S_i(x) = x_1 + x_2 + \cdots + x_i$:

$$S_i(x) \leqslant S_i(\mu) \quad \text{for } 1 \leqslant i < n, \tag{2.1}$$

$$S_n(x) = S_n(\mu). \tag{2.2}$$

The vector ν_M is obtained from ν by averaging the entries of ν over batches where the first n_1 entries of ν constitute the first batch, the next n_2 the second batch, and so on. Thus we have

$$\nu_M = (\underbrace{\bar{\nu}_1, \dots, \bar{\nu}_1}_{n_1}, \underbrace{\bar{\nu}_2, \dots, \bar{\nu}_2}_{n_2}, \dots, \underbrace{\bar{\nu}_r, \dots, \bar{\nu}_r}_{n_r}),$$

where $\bar{\nu}_k$ denotes the average of the entries of the kth batch of ν. We define $\sigma(k) = n_1 + n_2 + \cdots + n_k$.

To prove the proposition, we will reorder the entries of ν to form a new co-weight η and show that, for the proper choice of Levi subgroup, η satisfies all of the hypotheses on ν as well as being G-dominant. This reduces the problem to proving the proposition with the additional hypothesis that ν is G-dominant. We first show that the batches of ν satisfy a nice order property.

Lemma 2.1. *Let $f_k(\nu)$ denote the first entry of the kth batch of ν. Then $f_1(\nu) \geqslant f_2(\nu) \geqslant \cdots \geqslant f_r(\nu)$.*

Proof. Suppose that there exists $k < r$ such that $f_{k+1}(\nu) > f_k(\nu)$. Both are integers, so

$$f_{k+1}(\nu) - 1 \geqslant f_k(\nu). \tag{2.3}$$

Since ν is M-minuscule and M-dominant and $\bar{\nu}_k$ is the average of the kth batch of ν, we have $f_k(\nu) \geqslant \bar{\nu}_k > f_k(\nu) - 1$. Similarly,

$$f_{k+1}(\nu) \geqslant \bar{\nu}_{k+1} > f_{k+1}(\nu) - 1.$$

Combining these with (2.3) gives $\bar{\nu}_{k+1} > f_{k+1}(\nu) - 1 \geqslant f_k(\nu) \geqslant \bar{\nu}_k$, which contradicts the G-dominance of ν_M. □

We will now create the new co-weight η by reordering the entries of ν in such a way that the inequalities in Lemma 2.1 are strict for η. We will then show that η still satisfies all of the hypothesis on ν, but for a different Levi subgroup. To simplify doing so, we first prove the following lemma.

Lemma 2.2. *Let $\beta \in \mathfrak{a}^{W_M}$ be of the form*

$$\beta = (\underbrace{\beta_1, \dots, \beta_1}_{n_1}, \underbrace{\beta_2, \dots, \beta_2}_{n_2}, \dots, \underbrace{\beta_r, \dots, \beta_r}_{n_r}).$$

To show that $\beta \leqslant \mu$, it is enough to show that the inequalities corresponding to the end of each batch are satisfied, i.e. that inequality (2.1) holds for $i = \sigma(k)$ for all $k < r$ and that condition (2.2) holds.

Proof. Follows from Lemma 1.1. □

Now we form the co-weight η by combining all batches of ν which have the same first entry into one batch and reordering each new batch in non-increasing order. Let L be the Levi subgroup corresponding to these new batches. Let η_L be the vector obtained by averaging η over the batches of L and denote its entries by $\bar{\eta}_k$. We now check that η is G-dominant and satisfies all of the hypotheses on ν, but for the Levi subgroup L.

Lemma 2.3. *The co-weight η is L-minuscule and $\eta_L \leqslant \mu$. Moreover, η is G-dominant, therefore η is L-dominant and η_L is G-dominant.*

Proof. By construction, η is L-minuscule since ν is M-minuscule. Also, the entries of η are non-increasing, so η is G-dominant. Therefore, η is L-dominant and η_L is G-dominant.

Finally, for every k there exists a j so that the sum of the first k batches of η is equal to the sum of the first j batches of ν. Therefore, since $\nu_M \leqslant \mu$, the inequalities for $\eta_L \leqslant \mu$ are satisfied at the end of each batch and by Lemma 2.2, we have $\eta_L \leqslant \mu$. □

We have shown that η satisfies all of the hypotheses on ν for the Levi subgroup L and that η is G-dominant. Since η is a permutation of ν, it is clear that $\varphi_G(\eta) = \varphi_G(\nu)$. By hypothesis $\varphi_G(\nu) = \varphi_G(\mu)$ so we have $\varphi_G(\eta) = \varphi_G(\mu)$. Moreover, by its construction, $\eta \in W\nu$ so it is enough to prove the proposition for (L, η) instead of (M, ν). Thus it is enough to prove the proposition with the additional hypothesis that ν is G-dominant. We can now prove that $\nu \in \mathrm{Conv}(W\mu)$ by proving that $\nu \leqslant \mu$.

Theorem 2.4. $\nu \in \mathrm{Conv}(W\mu)$.

Proof. We will suppose that $\nu \not\leqslant \mu$ and obtain a contradiction. If $\nu \not\leqslant \mu$, then there exists an i such that

$$\nu_1 + \nu_2 + \cdots + \nu_i > \mu_1 + \mu_2 + \cdots + \mu_i. \tag{2.4}$$

Choose the smallest such i. Then $\nu_i > \mu_i$ and both are integers, so $\nu_i - 1 \geqslant \mu_i$.

Suppose ν_i is in the kth batch of ν. We consider the $(i+1)$th to $\sigma(k)$th entries of ν and μ. Since ν is M-dominant and M-minuscule, $\nu_{i+1}, \dots, \nu_{\sigma(k)} \in \{\nu_i, \nu_i - 1\}$. Thus $\nu_{i+1} + \cdots + \nu_{\sigma(k)} \geqslant (\sigma(k) - i)(\nu_i - 1)$. Also, since μ is G-dominant and $\mu_i \leqslant \nu_i - 1$, it follows that $\mu_{i+1} + \cdots + \mu_{\sigma(k)} \leqslant (\sigma(k) - i)(\nu_i - 1)$. Thus

$$\mu_{i+1} + \cdots + \mu_{\sigma(k)} \leqslant \nu_{i+1} + \cdots + \nu_{\sigma(k)}.$$

Combining this with inequality (2.4) yields

$$\mu_1 + \cdots + \mu_{\sigma(k)} < \nu_1 + \cdots + \nu_{\sigma(k)},$$

which contradicts the hypothesis that $\nu_M \leqslant \mu$ since $\nu_1 + \cdots + \nu_{\sigma(k)} = n_1\bar{\nu}_1 + \cdots + n_r\bar{\nu}_r$. □

3. The proof of Proposition 0.3 for $G = SO_{2n+1}$

Let $G = SO_{2n+1}$ and let M be the Levi subgroup $GL_{n_1} \times GL_{n_2} \times \cdots \times GL_{n_r} \times SO_{2j+1}$ of G where $n_1 + n_2 + \cdots + n_r + j = n$. Let T be diagonal matrices and B upper triangular matrices. Then

$$X = \{(a_1, a_2, \ldots, a_n, 0, -a_n, \ldots, -a_2, -a_1) : a_i \in \mathbb{Z}\} \cong \mathbb{Z}^n.$$

In this case, $x \in X$ is G-dominant if $x_1 \geqslant x_2 \geqslant \cdots \geqslant x_n \geqslant 0$. Thus the G-dominant, G-minuscule elements of X are $(1, 0, \ldots, 0)$ and $(0, \ldots, 0)$. Also $x \leqslant \mu$ if the following condition holds for $S_i(x) = x_1 + x_2 + \cdots + x_i$:

$$S_i(x) \leqslant S_i(\mu) \quad \text{for } 1 \leqslant i \leqslant n. \tag{3.1}$$

The hypothesis that $\varphi_G(x) = \varphi_G(\mu)$ is equivalent to

$$S_n(\mu) - S_n(x) \in 2\mathbb{Z}. \tag{3.2}$$

The vector ν_M is obtained from ν by averaging the entries of ν over batches where the first n_1 entries of ν constitute the first batch, the next n_2 the second batch, and so on until the final batch. The value for the entries of the final batch of ν_M is obtained by averaging over the middle $2j + 1$ entries of ν. Thus we have

$$\nu_M = (\underbrace{\bar{\nu}_1, \ldots, \bar{\nu}_1}_{n_1}, \underbrace{\bar{\nu}_2, \ldots, \bar{\nu}_2}_{n_2}, \ldots, \underbrace{\bar{\nu}_r, \ldots, \bar{\nu}_r}_{n_r}, \underbrace{0, \ldots, 0}_{j}),$$

where $\bar{\nu}_k$ denotes the average of the entries of the kth batch of ν. We define $\sigma(k) = n_1 + n_2 + \cdots + n_k$ for $k \leqslant r$ and $\sigma(r + 1) = n$.

To prove the proposition, we will reorder the entries of ν to form a new co-weight η and show that, for the proper choice of Levi subgroup, η satisfies all of the hypotheses on ν as well as being G-dominant. This reduces the problem to proving the proposition with the additional hypothesis that ν is G-dominant. We first show that all of the entries of ν are non-negative.

Lemma 3.1. *$\nu_i \geqslant 0$ for all i.*

Proof. Since ν is M-dominant, $\nu_i \geqslant 0$ for $i > n - j$. Suppose $\nu_i < 0$ for some $i \leqslant n - j$ and that ν_i is in the kth batch of ν. For ν_m in the kth batch, we have $\nu_m \leqslant \nu_i + 1$ since ν is M-minuscule, and therefore $\nu_m \leqslant 0$ since ν_i is an negative integer. Thus $\bar{\nu}_k < 0$. This contradicts the G-dominance of ν_M. □

https://doi.org/10.1017/S1474748004000064 Published online by Cambridge University Press

Next we show that the batches of ν satisfy a nice order property.

Lemma 3.2. *Let $f_k(\nu)$ denote the first entry of the kth batch of ν. Then $f_1(\nu) \geqslant f_2(\nu) \geqslant \cdots \geqslant f_r(\nu)$.*

Proof. The proof proceeds exactly as in Lemma 2.1. □

We will now create a new co-weight η' by reordering the entries of ν in such a way that the inequalities in Lemma 3.2 are strict for η'. We will then modify η' slightly to form the co-weight η and show that η still satisfies all of the hypotheses on ν, but for a different Levi subgroup. To simplify doing so, we first prove the following lemma.

Lemma 3.3. *Let $\beta \in \mathfrak{a}^{W_M}$ be of the form*

$$\beta = (\underbrace{\beta_1, \ldots, \beta_1}_{n_1}, \underbrace{\beta_2, \ldots, \beta_2}_{n_2}, \ldots, \underbrace{\beta_r, \ldots, \beta_r}_{n_r}, \underbrace{\beta_{r+1}, \ldots, \beta_{r+1}}_{j}),$$

where $\beta_{r+1} = 0$. To show that $\beta \leqslant \mu$, it is enough to show that the inequalities corresponding to the end of each batch are satisfied, i.e. that inequality (3.1) holds for $i = \sigma(k)$ for all $k \leqslant r$.

Proof. Follows from Lemma 1.1. □

Now we form the co-weight η in two steps. First, we form the co-weight η' by considering the first r batches of ν; we combine all batches which have the same first entry into one batch and reorder each new batch in non-increasing order. We take the final batch of ν as the final batch of η'. Let $L = GL_{m_1} \times \cdots \times GL_{m_s} \times SO_{2j+1}$ be the Levi subgroup corresponding to these new batches. By construction, η' is L-dominant, and is L-minuscule since ν is M-minuscule. Moreover, η' is $GL_{n-j} \times SO_{2j+1}$-dominant as in Lemma 2.3. To form η we modify η' by considering its final batch. There are two cases.

First, if the final batch is of the form $1, 0, \ldots, 0$ (so, in particular, $j \neq 0$) and the last entry of the sth batch of η' is zero, then we form η by swapping the one at the beginning of the final batch of η' with the left most zero entry in η'. For example, let $G = SO_{13}$, let $M = GL_2 \times GL_1 \times GL_1 \times SO_5$, and let $\nu = (2, 1, 2, 0, 1, 0)$. Then $\eta' = (2, 2, 1, 0, 1, 0)$, the Levi subgroup $L = GL_3 \times GL_1 \times SO_5$, and $\eta = (2, 2, 1, 1, 0, 0)$. Note that the one will move into a batch consisting only of zeros and ones since η' is L-minuscule and has no negative entries; hence η is L-minuscule.

Otherwise, we set $\eta = \eta'$.

We obtain η_L (respectively, η'_L) from η (respectively, η') in the same manner in which we obtained ν_M from ν, and denote its entries by $\bar{\eta}_k$ (respectively, $\bar{\eta}'_k$). We now check that η is G-dominant and satisfies all of the hypotheses on ν, but for the Levi subgroup L.

Lemma 3.4. *The co-weight η is L-minuscule and $\eta_L \leqslant \mu$. Moreover, η is G-dominant, hence η is L-dominant and η_L is G-dominant.*

Proof. We have already shown that η is L-minuscule. By construction, the entries of η are non-increasing and, by Lemma 3.1, they are all non-negative, so η is G-dominant. Therefore, η is L-dominant and η_L is G-dominant.

It remains to show that $\eta_L \leqslant \mu$. By Lemma 3.3, it is enough to check that inequality (3.1) holds for $i = \tilde{\sigma}(k)$ for all k, where $\tilde{\sigma}(k) = m_1 + \cdots + m_k$ for $k \leqslant s$ and $\tilde{\sigma}(s+1) = n$. We see that $\eta'_L \leqslant \mu$ as Lemma 2.3; therefore, if $\eta = \eta'$, we have $\eta_L \leqslant \mu$. Otherwise, suppose that the left most zero entry of η' is in its lth batch, i.e. that we swapped the one from the final batch with a zero from the lth batch. It follows that inequality (3.1) holds for $k < l$ as Lemma 2.3. Moreover, since η is G-dominant and all of its entries are non-negative, all of the entries to the right of the one must be zero. Hence, since $\mu_i \geqslant 0$ for all i as μ is G-dominant, it is enough to check inequality (3.1) for $i = \tilde{\sigma}(l)$.

Since $\eta'_L \leqslant \mu$, we have $S_{\tilde{\sigma}(l)}(\eta'_L) \leqslant S_{\tilde{\sigma}(l)}(\mu)$. If $S_{\tilde{\sigma}(l)}(\eta'_L) < S_{\tilde{\sigma}(l)}(\mu)$, then $S_{\tilde{\sigma}(l)}(\eta_L) \leqslant S_{\tilde{\sigma}(l)}(\mu)$ since both sides are integral and $S_{\tilde{\sigma}(l)}(\eta_L) = S_{\tilde{\sigma}(l)}(\eta'_L) + 1$. Otherwise,

$$S_{\tilde{\sigma}(l)}(\eta'_L) = S_{\tilde{\sigma}(l)}(\mu). \tag{3.3}$$

Since $\eta'_L \leqslant \mu$, we have

$$S_{\tilde{\sigma}(l)-1}(\eta'_L) \leqslant S_{\tilde{\sigma}(l)-1}(\mu). \tag{3.4}$$

Combining this with equality (3.3) yields

$$\bar{\eta}'_l \geqslant \mu_{\tilde{\sigma}(l)}. \tag{3.5}$$

Since the lth batch of η' consists only of zeros and ones and contains at least one zero, we have $\bar{\eta}'_l < 1$. We will obtain a contradiction by showing that $\mu_{\tilde{\sigma}(l)} \geqslant 1$. Now, since $\eta'_L \leqslant \mu$, we have

$$S_{\tilde{\sigma}(s)}(\eta'_L) + 1 \leqslant S_{\tilde{\sigma}(s)+1}(\mu). \tag{3.6}$$

Therefore, $1 \leqslant \mu_{\tilde{\sigma}(l)+1} + \cdots + \mu_{\tilde{\sigma}(s)+1}$ by inequalities (3.3) and (3.6) and since $\bar{\eta}'_k = 0$ for all $k > l$. Hence, since μ is G-dominant, it follows that $\mu_{\tilde{\sigma}(l)} \geqslant \mu_{\tilde{\sigma}(l)+1} \geqslant 1$, giving the desired contradiction. Thus $S_{\tilde{\sigma}(l)}(\eta'_L) < S_{\tilde{\sigma}(l)}(\mu)$ and $\eta_L \leqslant \mu$. □

We have shown that η satisfies all of the hypotheses on ν for the Levi subgroup L and that η is G-dominant. Moreover, by its construction, $\eta \in W\nu$ so it is enough to prove the proposition for (L, η) instead of (M, ν). Thus it is enough to prove the proposition with the additional hypothesis that ν is G-dominant. We can now prove that $\nu \in \mathrm{Conv}(W\mu)$ by proving that $\nu \leqslant \mu$.

Theorem 3.5. $\nu \in \mathrm{Conv}(W\mu)$.

Proof. We will suppose $\nu \nleqslant \mu$ and obtain a contradiction. If $\nu \nleqslant \mu$, then there exists an i such that

$$\nu_1 + \nu_2 + \cdots + \nu_i > \mu_1 + \mu_2 + \cdots + \mu_i. \tag{3.7}$$

https://doi.org/10.1017/S1474748004000064 Published online by Cambridge University Press

Choose the smallest such i and suppose that ν_i is in the kth batch of ν. As in Theorem 2.4, we obtain

$$\mu_1 + \cdots + \mu_{\sigma(k)} < \nu_1 + \cdots + \nu_{\sigma(k)}, \tag{3.8}$$

which produces a contradiction as follows. If $\sigma(k) = n$ and the first entry of the final batch is one, then $\nu_1 + \cdots + \nu_{\sigma(k)} = n_1\bar{\nu}_1 + \cdots + n_k\bar{\nu}_k + 1$. Thus, since both values are integers, $\mu_1 + \cdots + \mu_{\sigma(k)} \leqslant n_1\bar{\nu}_1 + \cdots + n_k\bar{\nu}_k$. Since $\nu_M \leqslant \mu$, we must have equality. This contradicts condition (3.2). Otherwise $\nu_1 + \cdots + \nu_{\sigma(k)} = n_1\bar{\nu}_1 + \cdots + n_k\bar{\nu}_k$, so inequality (3.8) contradicts the hypothesis that $\nu_M \leqslant \mu$. □

4. The proof of Proposition 0.3 for $G = SO_{2n}/\{\pm 1\}$ (integers)

Let $G = SO_{2n}/\{\pm 1\}$ and let M be the Levi subgroup $GL_{n_1} \times GL_{n_2} \times \cdots \times GL_{n_r} \times SO_{2j}$ of G where $n_1 + n_2 + \cdots + n_r + j = n$. We may assume that $j \neq 1$ since the case $j = 1$ is the same as the one in which $j = 0$ and $n_r = 1$. There are other Levi subgroups in G, but it is not necessary to consider them since there exists an outer automorphism of SO_{2n} which takes each of these to a Levi subgroup which we have considered. Let T be diagonal matrices and B upper triangular matrices. Then

$$X = \{(a_1, a_2, \ldots, a_n, -a_n, \ldots, -a_2, -a_1) : \text{either } a_i \in \mathbb{Z}\ \forall i \text{ or } a_i \in \tfrac{1}{2}(\mathbb{Z} \setminus 2\mathbb{Z})\ \forall i\}.$$

We first consider the case in which $a_i \in \mathbb{Z}$ for all i; we will consider the other case in § 5. In this case, $x \in X$ is G-dominant if

$$x_1 \geqslant x_2 \geqslant \cdots \geqslant x_{n-1} \geqslant x_n \quad \text{and} \quad x_{n-1} + x_n \geqslant 0.$$

It follows that if x is G-dominant, then $x_i \geqslant 0$ for all $i \leqslant n-1$. The G-dominant, G-minuscule elements of X are $(1, 0, \ldots, 0)$ and $(0, \ldots, 0)$. Also $x \leqslant \mu$ if the following conditions hold for $S_i(x) = x_1 + x_2 + \cdots + x_i$:

$$S_i(x) \leqslant S_i(\mu) \quad \text{for } 1 \leqslant i \leqslant n-2, \tag{4.1}$$

$$S_{n-1}(x) - x_n \leqslant S_{n-1}(\mu) - \mu_n, \tag{4.2}$$

$$S_n(x) \leqslant S_n(\mu). \tag{4.3}$$

The hypothesis that $\varphi_G(x) = \varphi_G(\mu)$ is equivalent to

$$S_n(\mu) - S_n(x) \in 2\mathbb{Z}. \tag{4.4}$$

The vector ν_M is obtained from ν by averaging the entries of ν over batches where the first n_1 entries of ν constitute the first batch, the next n_2 the second batch, and so on until the final batch. The value for the entries of the final batch of ν_M is obtained by averaging over the middle $2j$ entries of ν. Thus we have

$$\nu_M = (\underbrace{\bar{\nu}_1, \ldots, \bar{\nu}_1}_{n_1}, \underbrace{\bar{\nu}_2, \ldots, \bar{\nu}_2}_{n_2}, \ldots, \underbrace{\bar{\nu}_r, \ldots, \bar{\nu}_r}_{n_r}, \underbrace{0, \ldots, 0}_{j}),$$

where $\bar{\nu}_k$ denotes the average of the entries of the kth batch of ν. We define $\sigma(k) = n_1 + n_2 + \cdots + n_k$ for $k \leqslant r$ and $\sigma(r+1) = n$.

To prove the theorem, we will reorder the entries of ν to form a new co-weight η and show that, for the proper choice of Levi subgroup, η satisfies all of the hypotheses on ν as well as being G-dominant. This reduces the problem to proving the proposition with the additional hypothesis that ν is G-dominant. We first show that at most one of the entries of ν is negative.

Lemma 4.1. *$\nu_i \geqslant 0$ for all i, unless $j = 0$ and $n_r = 1$, in which case, $\nu_i \geqslant 0$ for all $i \leqslant n-1$.*

Proof. Suppose that $\nu_i < 0$ and that ν_i is in the kth batch of ν. As in Lemma 3.1, we see that $k \neq r+1$ and that $\bar{\nu}_k < 0$. Since ν_M is G-dominant, $\bar{\nu}_k < 0$ can occur only if $j = 0$, $n_r = 1$, and $k = r$, in which case $i = n$. □

Next we show that the batches of ν satisfy a nice order property.

Lemma 4.2. *Let $f_k(\nu)$ denote the first entry of the kth batch of ν. Then $f_1(\nu) \geqslant f_2(\nu) \geqslant \cdots \geqslant f_r(\nu)$.*

Proof. The proof proceeds exactly as Lemma 2.1. □

We will now create a co-weight η' by reordering the entries of ν in such a way that the inequalities in Lemma 4.2 are strict for η'. We will then modify η' slightly to form the co-weight η and show that η still satisfies all of the hypothesis on ν, but for a different Levi subgroup. To simplify doing so, we first prove the following lemma.

Lemma 4.3. *Let $\beta \in \mathfrak{a}^{W_M}$ be of the form*

$$\beta = (\underbrace{\beta_1, \ldots, \beta_1}_{n_1}, \underbrace{\beta_2, \ldots, \beta_2}_{n_2}, \ldots, \underbrace{\beta_r, \ldots, \beta_r}_{n_r}, \underbrace{\beta_{r+1}, \ldots, \beta_{r+1}}_{j}),$$

where $\beta_{r+1} = 0$. To show that $\beta \leqslant \mu$, it is enough to show that the inequalities corresponding to the end of each batch are satisfied, i.e. that inequality (4.1) holds for $i = \sigma(k)$ for all k such that $\sigma(k) \leqslant n-2$, that inequality (4.3) holds, and, if $j = 0$ and $n_r = 1$, that inequality (4.2) holds.

Proof. Follows from Lemma 1.1. □

Now we form the co-weights η' and η as for $G = SO_{2n+1}$. Again, we let $L = GL_{m_1} \times \cdots \times GL_{m_s} \times SO_{2j}$ be the Levi subgroup corresponding to the new batches, We obtain η_L (respectively, η'_L) from η (respectively, η') in the same manner in which we obtained ν_M from ν, and denote its entries by $\bar{\eta}_k$ (respectively, $\bar{\eta}'_k$).

We now check that η is G-dominant and satisfies all of the hypotheses on ν, but for the Levi subgroup L.

Lemma 4.4. *The co-weight η is L-minuscule and $\eta_L \leqslant \mu$. Moreover, η is G-dominant, hence η is L-dominant and η_L is G-dominant.*

Proof. As in the SO_{2n+1} case, we see that η and η' are L-minuscule. We claim that η is G-dominant. As before, the inequalities $\eta_1 \geqslant \cdots \geqslant \eta_n$ follow from the way η was constructed, so we need only check that $\eta_{n-1} + \eta_n \geqslant 0$. This is automatic unless some entry of ν is strictly negative, so by Lemma 4.1 we may now assume that $j = 0$ and $n_r = 1$. Thus we have $\eta_n = \bar{\nu}_r$. We know that $\bar{\nu}_{r-1} + \bar{\nu}_r \geqslant 0$ since ν_M is G-dominant. Hence $\bar{\eta}_{s-1} + \eta_n \geqslant 0$ since $\bar{\eta}_{s-1} \geqslant \bar{\nu}_{r-1}$. Finally, since η is L-minuscule, η_{n-1} is the greatest integer less than or equal to $\bar{\eta}_{s-1}$, so $\eta_{n-1} + \eta_n \geqslant 0$. Thus η is G-dominant. Therefore, η is L-dominant and η_L is G-dominant.

It only remains to show that $\eta_L \leqslant \mu$. To do so we apply Lemma 4.3. Define $\tilde{\sigma}(k) = m_1 + \cdots + m_k$ for $k \leqslant s$ and $\tilde{\sigma}(s+1) = n$. The method used in Lemma 3.4 establishes inequality (4.1) for $i = \tilde{\sigma}(k)$ for k such that $\tilde{\sigma}(k) \leqslant n-2$, as well as inequality (4.3). It remains to verify inequality (4.2) under the assumption that $j = 0$ and $m_s = 1$. In this case, we have $\eta = \eta'$, and we must have $n_r = 1$. Thus $\bar{\eta}_s = \bar{\nu}_r$ and $S_{\tilde{\sigma}(s-1)}(\eta_L) = S_{\sigma(r-1)}(\nu_M)$. Therefore, since $\nu_M \leqslant \mu$, inequality (4.2) holds. □

We have shown that η satisfies all of the hypotheses on ν for the Levi subgroup L and that η is G-dominant. Moreover, by its construction, $\eta \in W\nu$ so it is enough to prove the proposition for (L, η) instead of (M, ν). Thus it is enough to prove the proposition with the additional hypothesis that ν is G-dominant. We can now prove that $\nu \in \mathrm{Conv}(W\mu)$ by proving that $\nu \leqslant \mu$.

Theorem 4.5. $\nu \in \mathrm{Conv}(W\mu)$.

Proof. We will suppose $\nu \nleqslant \mu$ and obtain a contradiction. If $\nu \nleqslant \mu$, then either there exists an $i \neq n-1$ such that

$$\nu_1 + \nu_2 + \cdots + \nu_i > \mu_1 + \mu_2 + \cdots + \mu_i \tag{4.5}$$

or inequality (4.2) fails.

If inequality (4.5) holds for $i = n$, then we will argue as in Theorem 3.5. If the first entry of the final batch is zero, then $\nu_1 + \cdots + \nu_n = n_1\bar{\nu}_1 + \cdots + n_k\bar{\nu}_k$, so inequality (4.5) contradicts the hypothesis that $\nu_M \leqslant \mu$. If the first entry of the final batch is one, then $\nu_1 + \cdots + \nu_n = n_1\bar{\nu}_1 + \cdots + n_k\bar{\nu}_k + 1$. Combining this with inequality (4.5) yields $\mu_1 + \cdots + \mu_n \leqslant n_1\bar{\nu}_1 + \cdots + n_k\bar{\nu}_k$ since both values are integers. Since $\nu_M \leqslant \mu$, we must have equality. This contradicts condition (4.4).

Now suppose that there exists an $i \leqslant n-2$ for which inequality (4.5) holds. Choose the smallest such i and suppose that ν_i is in the kth batch of ν. As in Lemma 2.4, we obtain

$$\mu_1 + \cdots + \mu_{\sigma(k)} < \nu_1 + \cdots + \nu_{\sigma(k)}. \tag{4.6}$$

If $\sigma(k) \neq n-1$, then we obtain a contradiction as for SO_{2n+1}.

If $\sigma(k) = n-1$, then we have $j = 0$ and $n_r = 1$. We will first show that inequality (4.2) holds in this case. Since $\nu_M \leqslant \mu$, we have

$$S_{\sigma(r-1)}(\nu_M) - \bar{\nu}_r \leqslant S_{n-1}(\mu) - \mu_n. \tag{4.7}$$

By the definition of ν_M, the partial sums are equal at the end of each batch and we are assuming that $j = 0$ and $n_r = 1$, so $\bar{\nu}_r = \nu_n$ and $S_{\sigma(r-1)}(\nu_M) = S_{n-1}(\nu)$. Substituting these values into inequality (4.7) gives inequality (4.2) as desired. Adding inequality (4.2) to inequality (4.3) yields $S_{n-1}(\nu) \leqslant S_{n-1}(\mu)$, contradicting inequality (4.6).

Now suppose that inequality (4.2) fails. Thus we have

$$S_{n-1}(\nu) - \nu_n > S_{n-1}(\mu) - \mu_n. \tag{4.8}$$

By what we have already shown, we have

$$S_{n-2}(\nu) \leqslant S_{n-2}(\mu). \tag{4.9}$$

We claim that

$$\nu_{n-1} - \nu_n > \mu_{n-1} - \mu_n \geqslant 0. \tag{4.10}$$

The first inequality follows from inequalities (4.8) and (4.9), and the second holds since μ is G-dominant. We have shown that inequality (4.2) holds if $j = 0$ and $n_r = 1$ so we may assume that this is not the case. Thus ν_{n-1} and ν_n are in the same batch of ν, so, since ν is M-minuscule, $\nu_{n-1} - \nu_n \in \{0, 1\}$. Combining this with inequality (4.10) gives

$$\nu_{n-1} - \nu_n = 1 \quad \text{and} \quad \mu_{n-1} - \mu_n = 0. \tag{4.11}$$

Substituting these values into inequality (4.8) and combining the result with inequality (4.9) gives

$$S_{n-2}(\nu) + 1 > S_{n-2}(\mu) \geqslant S_{n-2}(\nu).$$

Both are integers, so $S_{n-2}(\mu) = S_{n-2}(\nu)$. This contradicts condition (4.4) since it follows from the equalities in (4.11) that $S_n(\mu)$ has the same parity as $S_{n-2}(\mu)$ and $S_n(\nu)$ has the opposite parity from $S_{n-2}(\nu)$. Thus $\nu \leqslant \mu$. □

5. The proof of Proposition 0.3 for $G = SO_{2n}/\{\pm 1\}$ (half integers)

Recall from § 4 that for $G = SO_{2n}/\{\pm 1\}$, we have

$$X = \{(a_1, a_2, \dots, a_n, -a_n, \dots, -a_2, -a_1) : \text{either } a_i \in \mathbb{Z}\ \forall i \text{ or } a_i \in \tfrac{1}{2}(\mathbb{Z} \setminus 2\mathbb{Z})\ \forall i\}.$$

We now consider the case in which $a_i \in \frac{1}{2}(\mathbb{Z} \setminus 2\mathbb{Z})$ for all i. As in § 4, we let M be the Levi subgroup $GL_{n_1} \times GL_{n_2} \times \cdots \times GL_{n_r} \times SO_{2j}$ of G where $n_1 + n_2 + \cdots + n_r + j = n$ and assume that $j \neq 1$. Rather than consider X, we will double the entries of X and adjust the notion of minuscule and the definition of the partial order accordingly.

First we note that all of the elements of X that we are considering in this case are now $2n$-tuples of odd integers. As before, $x \in X$ is G-dominant if

$$x_1 \geqslant x_2 \geqslant \cdots \geqslant x_{n-1} \geqslant x_n \quad \text{and} \quad x_{n-1} + x_n \geqslant 0.$$

https://doi.org/10.1017/S1474748004000064 Published online by Cambridge University Press

It follows that if x is G-dominant, then $x_i \geqslant 1$ for all $i \leqslant n-1$. Since we have multiplied X by two, we will now be concerned with elements whose pairing with any root yields two, zero, or negative two. We will refer to these elements as 2-minuscule. The G-dominant, G-2-minuscule elements of X are $(1,1,\dots,1)$ and $(1,\dots 1,-1)$. Also $x \leqslant \mu$ if the following conditions hold for $S_i(x) = x_1 + x_2 + \cdots + x_i$:

$$S_i(x) \leqslant S_i(\mu) \quad \text{for } 1 \leqslant i \leqslant n-2, \tag{5.1}$$

$$S_{n-1}(x) - x_n \leqslant S_{n-1}(\mu) - \mu_n, \tag{5.2}$$

$$S_n(x) \leqslant S_n(\mu). \tag{5.3}$$

The hypothesis that $\varphi_G(x) = \varphi_G(\mu)$ is equivalent to

$$S_n(\mu) - S_n(x) \in 4\mathbb{Z}. \tag{5.4}$$

As in the previous sections, the first n_1 entries of ν constitute the first batch, the next n_2 the second batch, and so on. The M-dominant, M-2-minuscule elements of X will now be those such that the entries of each batch are non-increasing and differ by either zero or two and such that the $(r+1)$th batch is of the form $(1,1,\dots,1)$ or $(1,\dots 1,-1)$.

The vector ν_M is obtained from ν by averaging the entries of ν over batches with the exception that the value for the entries of the final batch of ν_M is obtained by averaging over the middle $2j$ entries of ν. Thus we have

$$\nu_M = (\underbrace{\bar{\nu}_1,\dots,\bar{\nu}_1}_{n_1}, \underbrace{\bar{\nu}_2,\dots,\bar{\nu}_2}_{n_2}, \dots, \underbrace{\bar{\nu}_r,\dots,\bar{\nu}_r}_{n_r}, \underbrace{0,\dots,0}_{j}),$$

where $\bar{\nu}_k$ denotes the average of the entries of the kth batch of ν. We define $\sigma(k) = n_1 + n_2 + \cdots + n_k$ for $k \leqslant r$ and $\sigma(r+1) = n$.

To prove the theorem, we will reorder the entries of ν to form a new co-weight η and show that, for the proper choice of Levi subgroup, η satisfies all of the hypotheses on ν as well as being G-dominant. This reduces the problem to proving the proposition with the additional hypothesis that ν is G-dominant. We first show that at most one of the entries of ν is less than -1.

Lemma 5.1. *$\nu_i \geqslant -1$ for all i, unless $j = 0$ and $n_r = 1$, in which case, $\nu_i \geqslant -1$ for all $i \leqslant n-1$.*

Proof. Since ν is M-dominant, we have $\nu_i \geqslant -1$ for $i > n-j$. Suppose that $\nu_i < -1$ for some $i \leqslant n-j$ and that ν_i is in the kth batch of ν. For ν_m in the kth batch, we have $\nu_m \leqslant \nu_i + 2$ since ν is M-2-minuscule, and therefore $\nu_m \leqslant -1$ since $\nu_i < -1$ is an odd integer. Thus $\bar{\nu}_k < -1$. Since ν_M is G-dominant, $\bar{\nu}_k < 0$ can only occur if $j = 0$, $n_r = 1$, and $k = r$, in which case $i = n$. □

Next we show that the batches of ν satisfy a nice order property.

Lemma 5.2. *Let $f_k(\nu)$ denote the first entry of the kth batch of ν. Then $f_1(\nu) \geqslant f_2(\nu) \geqslant \cdots \geqslant f_r(\nu)$.*

Proof. Suppose that there exists $k < r$ such that $f_{k+1}(\nu) > f_k(\nu)$. Both are odd integers, so

$$f_{k+1}(\nu) - 2 \geqslant f_k(\nu). \tag{5.5}$$

Since ν is M-2-minuscule and M-dominant and $\bar{\nu}_k$ is the average of the kth batch of ν, we have $f_k(\nu) \geqslant \bar{\nu}_k > f_k(\nu) - 2$. Similarly, $f_{k+1}(\nu) \geqslant \bar{\nu}_{k+1} > f_{k+1}(\nu) - 2$. Combining these with inequality (5.5) gives $\bar{\nu}_{k+1} > f_{k+1}(\nu) - 2 \geqslant f_k(\nu) \geqslant \bar{\nu}_k$, which contradicts the G-dominance of ν_M. □

We will now create a co-weight η' by reordering the entries of ν in such a way that the inequalities in Lemma 5.2 are strict for η'. We will then modify η' slightly to form the co-weight η and show that η still satisfies all of the hypothesis on ν, but for a different Levi subgroup. To simplify doing so, we first prove the following lemma.

Lemma 5.3. *Let $\beta \in \mathfrak{a}^{W_M}$ be of the form*

$$\beta = (\underbrace{\beta_1, \dots, \beta_1}_{n_1}, \underbrace{\beta_2, \dots, \beta_2}_{n_2}, \dots, \underbrace{\beta_r, \dots, \beta_r}_{n_r}, \underbrace{\beta_{r+1}, \dots, \beta_{r+1}}_{j}),$$

where $\beta_{r+1} = 0$. To show that $\beta \leqslant \mu$, it is enough to show that the inequalities corresponding to the end of each batch are satisfied, i.e. that inequality (5.1) holds for $i = \sigma(k)$ for all $\sigma(k) \leqslant n - 2$, that inequality (5.3) holds, and, if $j = 0$ and $n_r = 1$, that inequality (5.2) holds.

Proof. Follows from Lemma 1.1. □

Now we form the co-weight η in three steps. First, we form the co-weight η' as in the SO_{2n+1} case. Again, we let $L = GL_{m_1} \times \dots \times GL_{m_s} \times SO_{2j}$ be the Levi subgroup corresponding to the new batches. By construction, η' is L-2-minuscule since ν is M-2-minuscule. It follows from Lemma 5.1 and the construction of η' that $\eta'_i \geqslant -1$ for all $i \leqslant n - 1$.

Moreover, we claim that any negative ones in η' will be in its sth or $(s+1)$th batch. Suppose that η' contains a negative one prior to the sth batch. Since the entries of η' are non-increasing, this implies that all of the entries of the sth batch will be at most negative one. The sth batch of η' was formed by combining batches of ν, so this yields that all of the entries of the rth batch of ν will also be at most negative one. Thus $\bar{\nu}_r < 0$ which contradicts the G-dominance of ν_M unless $j = 0$ and $n_r = 1$. In this case, we have $\bar{\nu}_r = \nu_n$ and since ν_M is G-dominant, $\bar{\nu}_{r-1} + \bar{\nu}_r \geqslant 0$. Therefore, $\bar{\nu}_{r-1} \geqslant 1$, so since ν_M is G-dominant, $\bar{\nu}_k \geqslant 1$ for all $k \leqslant r - 1$. Thus, since ν is M-2-minuscule, $\nu_i \geqslant 1$ for all $i \leqslant n-1$ and it follows from the construction of η' that $\eta'_i \geqslant 1$ for all $i \leqslant n-1$. Therefore, η' cannot contain a negative one prior to the sth batch.

Next we form a co-weight η'' by replacing every negative one in the sth batch of η' with a positive one. Finally, if we made an odd number of sign changes in the previous step, then we change the sign of the final entry of η'' to form η; otherwise, $\eta = \eta''$.

https://doi.org/10.1017/S1474748004000064 Published online by Cambridge University Press

We obtain η_L (respectively, η'_L) from η (respectively, η') in the same manner as we obtained ν_M from ν, and denote its entries by $\bar{\eta}_k$ (respectively, $\bar{\eta}'_k$). We now check that η is G-dominant and satisfies all of the hypotheses on ν, but for the Levi subgroup L.

Lemma 5.4. *The co-weight η is L-2-minuscule and $\eta_L \leqslant \mu$. Moreover, η is G-dominant, hence η is L-dominant and η_L is G-dominant.*

Proof. We begin by showing that η is L-2-minuscule and G-dominant. It follows that η is L-dominant and η_L is G-dominant.

First, if $j = 0$ and $n_r = 1$, then $\eta = \eta'$, so we have shown that η is L-2-minuscule. We claim that η is G-dominant. The inequalities $\eta_1 \geqslant \cdots \geqslant \eta_n$ follow from the way η was constructed, so we need only check that $\eta_{n-1} + \eta_n \geqslant 0$. As in § 4, we obtain $\bar{\eta}_{s-1} + \eta_n \geqslant 0$. Since η is L-2-minuscule, η_{n-1} is the greatest odd integer less than or equal to $\bar{\eta}_{s-1}$; therefore we have $\eta_{n-1} + \eta_n \geqslant 0$ since η_n is an odd integer.

Otherwise, it is clear that the $(s+1)$th batch of η is still L-2-minuscule. If the sth batch of η' contains any negative ones, then, since we have shown that η' is L-2-minuscule and by Lemma 5.1, all other entries of the batch must be either positive or negative one. Therefore, since we only changed the signs of positive and negative ones, η is L-2-minuscule. Moreover, by construction, $\eta_i \geqslant 1$ for all $i < n$, the entries of η are non-increasing, and, by Lemma 5.1, $\eta_n \geqslant -1$; hence it is clear that η is G-dominant.

It only remains to show that $\eta_L \leqslant \mu$. To do so, we apply Lemma 5.3. Define $\tilde{\sigma}(k) = m_1 + \cdots + m_k$ for $k \leqslant s$ and $\tilde{\sigma}(s+1) = n$.

First, we show that $\eta'_L \leqslant \mu$. The method used in Lemma 2.3 establishes inequality (5.1) for $i = \tilde{\sigma}(k)$ for k such that $\tilde{\sigma}(k) \leqslant n-2$, as well as inequality (5.3). It remains to verify inequality (5.2) under the assumption that $j = 0$ and $m_s = 1$; the proof proceeds exactly as in Lemma 4.4. Thus if $\eta = \eta'$, then we have $\eta_L \leqslant \mu$.

Now we consider the case in which $\eta \neq \eta'$. (In particular, it is not the case that $j = 0$ and $m_s = 1$.) Since η and η' do not differ prior to the sth batch and $\eta'_L \leqslant \mu$, inequality (5.1) is satisfied for $i = \tilde{\sigma}(k)$ for all $k < s$. Moreover, since either the sth batch is the final batch (if $j = 0$) or $\bar{\eta}_{s+1} = 0$ (if $j \neq 0$), and since μ is G-dominant, it is enough to check that inequality (5.1) holds for $i = \tilde{\sigma}(s)$.

We have shown that $S_{\tilde{\sigma}(s-1)}(\eta_L) \leqslant S_{\tilde{\sigma}(s-1)}(\mu)$. Since the last entry of the sth batch of η is either positive or negative one and η is L-2-minuscule, we also have that $\bar{\eta}_s \leqslant 1$. Moreover, since μ is G-dominant, we have $\mu_i \geqslant 1$ for all $i < n$. It follows that $S_{\tilde{\sigma}(s)}(\eta_L) \leqslant S_{\tilde{\sigma}(s)}(\mu)$ unless $\tilde{\sigma}(s) = n$. If $\tilde{\sigma}(s) = n$ (so, in particular, $j = 0$), then

$$\mu_{\tilde{\sigma}(s-1)+1} + \cdots + \mu_n \geqslant m_s - 2,$$

since μ is G-dominant. Thus $S_n(\eta_L) \leqslant S_n(\mu)$ unless $S_{\tilde{\sigma}(s-1)}(\eta_L) = S_{\tilde{\sigma}(s-1)}(\mu)$, $\bar{\eta}_s = 1$, and $\mu_{\tilde{\sigma}(s-1)+1} + \cdots + \mu_n = m_s - 2$. In this case, we have $S_n(\eta_L) = S_n(\mu) + 2$. This is a contradiction to condition (5.4) since $S_n(\eta_L) = S_n(\eta)$ and we see that $S_n(\eta)$ and $S_n(\nu)$ are congruent modulo four since the difference in the two sums comes from changing an even number of negative ones to positive ones. □

We have shown that η satisfies all of the hypotheses on ν for the Levi subgroup L and that η is G-dominant. Moreover, by its construction, $\eta \in W\nu$ so it is enough to prove the proposition for (L,η) instead of (M,ν). Thus it is enough to prove the proposition with the additional hypothesis that ν is G-dominant. We can now prove that $\nu \in \mathrm{Conv}(W\mu)$ by proving that $\nu \leqslant \mu$.

Theorem 5.5. $\nu \in \mathrm{Conv}(W\mu)$.

Proof. We will suppose $\nu \nleqslant \mu$ and obtain a contradiction. If $\nu \nleqslant \mu$, then either there exists an $i \neq n-1$ such that

$$\nu_1 + \nu_2 + \cdots + \nu_i > \mu_1 + \mu_2 + \cdots + \mu_i, \tag{5.6}$$

or inequality (5.2) fails.

First note that if $j = 0$, then $S_n(\nu) = S_n(\nu_M)$, so inequality (5.6) cannot hold for $i = n$ since $\nu_M \leqslant \mu$.

Now suppose that there exists an $i \leqslant n-2$ such that inequality (5.6) holds. Choose the smallest such i. Then $\nu_i > \mu_i$, and both are odd integers, so $\nu_i - 2 \geqslant \mu_i$. Suppose ν_i is in the kth batch of ν.

We consider the $(i+1)$th to $\sigma(k)$th entries of ν and μ. Since ν is M-dominant and M-2-minuscule, $\nu_{i+1}, \dots, \nu_{\sigma(k)} \in \{\nu_i, \nu_i - 2\}$. Thus

$$\nu_{i+1} + \cdots + \nu_{\sigma(k)} \geqslant (\sigma(k) - i)(\nu_i - 2).$$

Also, since μ is G-dominant and $\mu_i \leqslant \nu_i - 2$, it follows that

$$\mu_{i+1} + \cdots + \mu_{\sigma(k)} \leqslant (\sigma(k) - i)(\nu_i - 2).$$

Thus

$$\mu_{i+1} + \cdots + \mu_{\sigma(k)} \leqslant \nu_{i+1} + \cdots + \nu_{\sigma(k)}.$$

Combining this with inequality (5.6) yields

$$\mu_1 + \cdots + \mu_{\sigma(k)} < \nu_1 + \cdots + \nu_{\sigma(k)}. \tag{5.7}$$

If $j \geqslant 2$ and ν_i is in the final batch, then we have $\mu_i < \nu_i = \pm 1$, so $\mu_i \leqslant -1$. Since μ is G-dominant, it follows that $i = n$. This contradicts our assumption that $i \leqslant n-2$. If $j \geqslant 2$ and ν_i is not in the final batch, then $\sigma(k) \leqslant \sigma(r) \leqslant n-2$ and inequality (5.7) contradicts $\nu_M \leqslant \mu$.

If $j = 0$ and $\sigma(k) \neq n-1$, then inequality (5.7) contradicts $\nu_M \leqslant \mu$. If $j = 0$ and $\sigma(k) = n-1$, then we have $n_r = 1$. As in the proof of Theorem 4.5, we observe that inequality (5.2) holds, which when added to inequality (5.3) yields $S_{n-1}(\nu) \leqslant S_{n-1}(\mu)$, contradicting inequality (5.7).

Now suppose that inequality (5.2) fails. Thus we have

$$S_{n-1}(\nu) - \nu_n > S_{n-1}(\mu) - \mu_n. \tag{5.8}$$

By what we have already shown, we have

$$S_{n-2}(\nu) \leqslant S_{n-2}(\mu). \tag{5.9}$$

We claim that

$$\nu_{n-1} - \nu_n > \mu_{n-1} - \mu_n \geqslant 0. \tag{5.10}$$

The first inequality follows from inequalities (5.8) and (5.9), and the second holds since μ is G-dominant. We have shown that inequality (5.2) holds if $j = 0$ and $n_r = 1$ so we may assume that this is not the case. Thus ν_{n-1} and ν_n are in the same batch of ν, so, since ν is M-minuscule, $\nu_{n-1} - \nu_n \in \{0, 2\}$. Combining this with inequality (5.10) gives

$$\nu_{n-1} - \nu_n = 2 \quad \text{and} \quad \mu_{n-1} - \mu_n = 0. \tag{5.11}$$

Substituting these values into inequality (5.8) and combining the result with inequality (5.9) gives

$$S_{n-2}(\nu) + 2 > S_{n-2}(\mu) \geqslant S_{n-2}(\nu).$$

Both are integers of the same parity, so $S_{n-2}(\mu) = S_{n-2}(\nu)$. This contradicts condition (5.4) since it follows from the equalities in (5.11) (bearing in mind that all entries of μ and ν are odd) that $S_n(\mu)$ is not congruent modulo four to $S_{n-2}(\mu)$, and $S_n(\nu)$ is congruent modulo four to $S_{n-2}(\nu)$.

Finally, suppose that (5.6) holds for $i = n$. We have already handled the $j = 0$ case so we may assume that $j \geqslant 2$. Therefore, ν_{n-1} and ν_n will be in the $(r+1)$th batch, so since ν is M-2-minuscule, we have $\nu_{n-1} + \nu_n \in \{0, 2\}$. Also, since μ is G-dominant, we have $\mu_{n-1} + \mu_n \geqslant 0$. We have shown that $S_{n-2}(\nu) \leqslant S_{n-2}(\mu)$. It follows that $S_n(\nu) \leqslant S_n(\mu)$ unless $S_{n-2}(\nu) = S_{n-2}(\mu)$, $\nu_{n-1} + \nu_n = 2$, and $\mu_{n-1} + \mu_n = 0$. In this case, $S_n(\nu) = S_n(\mu) + 2$ which contradicts condition (5.4). Thus $\nu \leqslant \mu$. □

6. Sketch of the proofs of Facts 1 and 2

Let $Z = Z(G)$ be the centre of G and let $\tilde{T}$ be the image of T in $\mathrm{Ad}(G)$. If Z is connected, then the short exact sequence

$$1 \to Z \to T \to \tilde{T} \to 1$$

is a short exact sequence of tori. Hence we have a surjection $X_*(T) \twoheadrightarrow X_*(\tilde{T})$. Combining this with the following claim yields Facts 1 and 2.

Claim 6.1. *The theorem holds for G, M and μ if and only if it holds for $\mathrm{Ad}(G)$, $\tilde{M}$ and $\tilde{\mu}$, where $\tilde{M} = M/Z$ and $\tilde{\mu}$ is the image of μ in $X_*(\tilde{T})$.*

Proof. Let

$$S = \{\nu_1 \in X_M : \text{(i) } \nu_1,\ \mu \text{ have the same image in } X_G;\ \text{and} \\ \text{(ii) the image of } \nu_1 \text{ in } \mathfrak{a}_M \text{ lies in } \mathrm{Conv}(W\mu) \cap \mathfrak{a}^{W_M}\}.$$

Define $\tilde{\mathfrak{a}}$, $X_{\tilde{M}}$, $P_{\tilde{\mu}}$, $\varphi_{\tilde{M}}$, and $\tilde{S}$ analogously to $\mathfrak{a}$, X_M, P_μ, φ_M, and S, respectively. The claim states that $\varphi_M(P_\mu) = S$ if and only if $\varphi_{\tilde{M}}(P_{\tilde{\mu}}) = \tilde{S}$. We prove the claim by

https://doi.org/10.1017/S1474748004000064 Published online by Cambridge University Press

showing that $\varphi_M(P_\mu) \cong \varphi_{\tilde{M}}(P_{\tilde{\mu}})$ and that $S \cong \tilde{S}$. The key fact is that $P_\mu \cong P_{\tilde{\mu}}$. We omit the remainder of the proof as it is fairly straightforward using this fact and arguments similar to the one below.

Let $\beta : X_*(T) \to X_*(\tilde{T})$ be the obvious map and denote $\beta(\nu)$ by $\tilde{\nu}$ as above. We have that

$$\mathfrak{a} = \mathfrak{a}_{\mathrm{der}} \oplus \mathfrak{a}_G, \tag{6.1}$$

where $\mathfrak{a}_{\mathrm{der}}$ is the $\mathbb{R}$-linear span of the co-roots and $\mathfrak{a}_G = X_*(Z) \otimes_{\mathbb{Z}} \mathbb{R}$, and that $\mathfrak{a}_{\mathrm{der}} \cong \tilde{\mathfrak{a}}$. Thus

$$\mathrm{Conv}(W\mu) = (\mathrm{Conv}(W\tilde{\mu}), \xi) \tag{6.2}$$

for some $\xi \in \mathfrak{a}_G$ so it is clear that $\beta : P_\mu \to P_{\tilde{\mu}}$.

Now suppose that there exist $\nu_1, \nu_2 \in P_\mu$ such that $\tilde{\nu}_1 = \tilde{\nu}_2$. Then $\nu_2 = \nu_1 + \zeta$ for some $\zeta \in X_*(Z) = \mathrm{Hom}(\mathbb{G}_m, Z)$. Since $\nu_1, \nu_2 \in P_\mu$, we have $\varphi_G(\nu_1) = \varphi_G(\mu) = \varphi_G(\nu_2)$. Hence $\varphi_G(\zeta) = 0$, and since $\zeta \in X_*(Z)$, this implies that $\zeta = 0$. Thus $\nu_1 = \nu_2$ so $\beta : P_\mu \to P_{\tilde{\mu}}$ is injective.

Let $\tilde{\nu} \in P_{\tilde{\mu}}$. We can choose λ in the co-root lattice for G such that $\tilde{\nu} = \tilde{\mu} + \tilde{\lambda}$. Set $\nu = \mu + \lambda$. We must check that $\nu \in P_\mu$. It is clear that $\varphi_G(\nu) = \varphi_G(\mu)$. Using decomposition 6.1, we can write $\nu = (\tilde{\nu}, \xi)$ and $\mu = (\tilde{\mu}, \xi)$. Hence by Equation (6.2), we have $\nu \in \mathrm{Conv}(W\mu)$. Therefore, $\nu \in P_\mu$ and $\beta : P_\mu \to P_{\tilde{\mu}}$ is surjective. Thus $P_\mu \cong P_{\tilde{\mu}}$. □

Acknowledgements. I thank my advisor Robert Kottwitz for suggesting that I prove the converse to Mazur's inequality for all split classical groups.

References

1. N. Bourbaki, *Groupes et algèbres de Lie* (Paris, Masson, 1981).
2. J.-M. Fontaine and M. Rapoport, *Existence de filtrations admissibles sur des isocristaux*, preprint (2002).
3. R. Kottwitz, Isocrystals with additional structure, II, *Compositio Math.* **109** (1997), 255–339.
4. R. Kottwitz, On the Hodge–Newton decomposition for split groups, *Int. Math. Res. Not.* **26** (2003), 1433–1447.
5. R. Kottwitz and M. Rapoport, On the existence of F-isocrystals, *Comment. Math. Helv.* **78** (2003), 153–184.
6. M. Rapoport, *A guide to the reduction modulo p of Shimura varieties* (2002), preprint, arXiv:math.AG/0205022.
7. M. Rapoport and M. Richartz, On the classification and specialization of F-isocrystals with additional structure, *Compositio Math.* **103** (1996), 153–181.