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On the inadmissibility of non-evolutionary shocks

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Abstract. In recent years, numerical solutions of the equations of compressible magnetohydrodynamic (MHD) flows have been found to contain intermediate shocks for certain kinds of problems. Since these results would seem to be in conflict with the classical theory of MHD shocks, they have stimulated attempts to reexamine various aspects of this theory, in particular the role of dissipation. In this paper, we study the general relationship between the evolutionary conditions for discontinuous solutions of the dissipation-free system and the existence and uniqueness of steady dissipative shock structures for systems of quasilinear conservation laws with a concave entropy function. Our results confirm the classical theory. We also show that the appearance of intermediate shocks in numerical simulations can be understood in terms of the properties of the equations of planar MHD, for which some of these shocks turn out to be evolutionary. Finally, we discuss ways in which numerical schemes can be modified in order to avoid the appearance of intermediate shocks in simulations with such symmetry.

1. Introduction

It is well known that not all discontinuous solutions of hyperbolic conservation laws are admissible. Some of these can be excluded on physical grounds. For example, expansion shocks in gas dynamics must be discarded, since they do not satisfy the second law of thermodynamics. Others can be excluded for purely mathematical reasons, such as the fact that they do not satisfy uniqueness and existence conditions or are structurally unstable with respect to small perturbations of the initial data. These mathematical conditions are usually called evolutionary conditions. For example, intermediate shocks in magnetohydrodynamics (MHD) satisfy the second law but are not evolutionary.

This subject was extensively studied between the late 1940s and early 1960s (see e.g. Courant and Friedrichs 1948; Lax 1957; Akhiezer et al. 1959; Germain 1960; Polovin 1961; Gel'fand 1963), and a full account can be found in numerous textbooks (see e.g. Jeffrey and Taniuti 1964; Cabannes 1970; Somov 1994). Until recently, there was general agreement that admissible shocks must both satisfy the evolutionary condition and possess a steady dissipative shock structure, although the relation between these conditions was not entirely clear. There the matter rested until time-dependent numerical solutions of the dissipative MHD equations showed that certain types of intermediate shocks can arise from smooth initial data (Wu

1987). Shortly thereafter, Brio and Wu (1988) found intermediate shocks in their numerical solution for a particular MHD Riemann problem. More recently, intermediate shocks have been also been found in two-dimensional simulations (De Sterck et al. 1998). Furthermore, Chao et al. (1993) have reported the detection of an interplanetary intermediate shock in the Voyager 1 data. All this has caused some authors to reject the classical theory and to suggest that the evolutionary condition is not relevant to dissipative MHD (Wu 1987, 1988a,b, 1990; Kennel et al. 1990; Hada 1994; Myong & Roe 1997a,b), and has led to a reexamination of the whole question of the existence, or otherwise, of non-classical shocks (see Glimm 1988; Freistuhler and Liu 1993; Myong & Roe 1997a; and references therein). There are, however, others who argue that there is nothing wrong with the classical theory (see e.g. Barmin et al. 1996; Falle and Komissarov 1997; Markovskii 1998a,b).

The matter clearly needs to be resolved, particularly since the existence, or otherwise, of intermediate shocks not only is of crucial importance for fundamental MHD processes such as reconnection (Wu 1995), but also is relevant to many other astrophysical applications. The purpose of this paper is to try and clear the matter up by showing that there is neither a real conflict between the classical shock theory and the results of numerical calculations nor any incompatibility between ideal and dissipative MHD. Furthermore, we show that the classical theory is of great utility in analysing the results of numerical calculations in order to determine whether the numerical solutions are physically correct. In order to make the discussion complete, we have put together and extended a number of results from the literature that have tended be ignored or misunderstood.

This paper is organised as follows. In Sec. 2, we briefly review the classical shock theory and the evolutionary conditions. In Sec. 3, we study the relationship between these conditions and the uniqueness and existence of steady dissipative shock structures for systems with a concave entropy function. In Sec. 4, we apply these results to the full system of MHD equations and to the reduced system of planar MHD. In Sec. 5, we present the results of numerical calculations that show that, for both these systems, the behaviour of the shocks is entirely consistent with the predictions of the classical shock theory. In Sec. 6, we consider various aspects of the problem of intermediate shocks, and discuss ways in which to avoid their appearance in MHD simulations with planar symmetry. In particular, we present the results of one dimensional simulations using a modified Glimm scheme (Glimm 1965) in which these shocks do not appear.

2. General theory of shocks

In this section, we give a brief review of the classical theory of discontinuous solutions of hyperbolic conservation laws. For our purposes, it is sufficient to consider only the one-dimensional equations of the form

$$\frac{\partial \mathbf{u}}{\partial t} + \frac{\partial \mathbf{f}}{\partial x} = 0, \tag{2.1}$$

where $\mathbf{u} \in \mathbb{R}^n$ is a vector of conserved variables and $\mathbf{f}(\mathbf{u}) \in \mathbb{R}^n$ is a vector of the corresponding fluxes.

As is well known, the system (2.1) is called hyperbolic if the Jacobian matrix

$$\mathsf{A} = \frac{\partial \mathbf{f}}{\partial \mathbf{u}}.$$

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has n real eigenvalues, λ_k ($k = 1, \dots, n$) corresponding to n linearly independent right-eigenvectors $\mathbf{r}_k$ and is called strictly hyperbolic if all the λ_k are different. The physical significance of the λ_k is that they are the speeds of small-amplitude waves.

Waves are classified as linear or nonlinear according to the behaviour of

$$C_k(\mathbf{u}) \equiv \mathbf{r}_k(\mathbf{u}) \cdot \boldsymbol{\nabla}_u \lambda_k(\mathbf{u}).$$

If $C_k(\mathbf{u}) = 0$ for all $\mathbf{u}$, then the k-wave is called linear, whereas if the dimension of the surface defined by $C_k(\mathbf{u}) = 0$ is less then n, then it is called nonlinear or genuinely nonlinear.

The states $\mathbf{u}_l$ and $\mathbf{u}_r$ on either side of a discontinuity travelling with speed s must satisfy the shock equations

$$s(\mathbf{u}_l - \mathbf{u}_r) = \mathbf{f}_l - \mathbf{f}_r. \tag{2.2}$$

The number n_s of independent shock equations can be less than n. For example, a contact discontinuity in gas dynamics has $n_s = 3$, whereas $n = 5$. Since $\mathbf{A}$ is the Jacobian, we clearly have $s \to \lambda_k$ for some k as $\mathbf{u}_l \to \mathbf{u}_r$, which means that one can associate each discontinuity that allows this limit with one of the waves of the system. A discontinuity is called linear if the corresponding characteristic speed does not change across it; otherwise it is called nonlinear. The mere fact that a discontinuity satisfies (2.2) does not necessarily imply that it is either stable or that it can arise from continuous initial data.

For some hyperbolic systems, (2.2) allow nonlinear shocks that propagate with a characteristic speed associated with a nonlinear wave, which means that they can be attached to such a wave to form compound waves. Systems with such shock solutions are called non-convex. Compound waves may arise from continuous initial data if the system allows single simple waves in which $C_k(\mathbf{u})$ changes sign along the phase curve of a simple wave. This condition is therefore often used as an alternative definition of non-convexity. Although these definitions are equivalent for a single conservation law, they are not necessarily so for systems.

The evolutionary condition is directly related to the question of existence and uniqueness of discontinuous solutions. It is well known that, for hyperbolic equations, there is a general way of deciding this question, which is to use the compatibility conditions that must be satisfied along the characteristics (Friedrichs 1955). If a characteristic with wave speed λ_k enters one side of a discontinuity then the state on that side must satisfy the compatibility relation associated with that characteristic,

$$\mathbf{l}_k(\mathbf{u}) \cdot d\mathbf{u} = 0,$$

where $\mathbf{l}_k(\mathbf{u})$ is the left-eigenvector of $\mathbf{A}$ corresponding to that characteristic. These equations are independent provided that the $\mathbf{l}_k$ are linearly independent, i.e. for all hyperbolic systems. If the wave speeds on either side of the discontinuity are such that m_i compatibility relations have to be satisfied, then there are $n_s + m_i$ equations relating the $2n + 1$ unknowns associated with the discontinuity, $\mathbf{u}_l$, $\mathbf{u}_r$, and the shock speed s. A discontinuous solution can therefore only exist and be unique if

$$m_i = 2n - n_s + 1. \tag{2.3}$$

Obviously, when $n_s = n$, (2.3) reduces to

$$m_i = n + 1. \tag{2.4}$$

It is clear from this that if a characteristic is parallel to the shock curve, then it is counted as incoming, since the corresponding compatibility relation must be satisfied (Gel'fand 1963).

If $m_i > 2n - n_s + 1$, then the system is overdetermined and there is no solution except for certain special initial conditions. There will therefore always be arbitrarily small perturbations of this data that will destroy such a discontinuity by splitting it into a number of waves, just as an arbitrary initial dicontinuity splits in a Riemann problem. If $m_i < 2n - n_s + 1$, then the solution exists, but is not unique, and one might hope that this non-uniqueness can be removed by including dissipative terms. In the following, we shall call the condition (2.3) the strong evolutionary condition and the condition

$$m_i \leqslant 2n - n_s + 1,$$

which allows non-unique solutions, a relaxed evolutionary condition.

An equivalent way of obtaining (2.3) is by a linear stability analysis of shock solutions (see e.g. Landau and Lifshitz 1959; Jeffrey and Taniuti 1964). A discontinuity that is exposed to a small-amplitude incident wave will only survive if it can respond by changing its speed and emitting small-amplitude waves. Each such wave is described by one parameter, and we also have the perturbation in the shock speed, which means that there are $m_o + 1$ unknowns in this problem, where m_o is the number of outgoing characteristics. Since these are related to the amplitude of the incoming wave by the n_s shock relations, the discontinuity can only have a unique response if

$$m_o = n_s - 1. \tag{2.5}$$

It is worth pointing out that, contrary to what is claimed in Myong and Roe (1997a), this analysis does not assume that the discontinuity is weak. This suggests that non-unique discontinuous solutions should spontaneously self-destruct by emitting waves even if they are not perturbed (Anderson 1963). However, since the perturbations are incident on a discontinuity, this analysis is only valid for perturbations whose wavelength is large compared with the width of the structure. Roĭkhvarger and Syrovatskii (1974) and Markovskii (1998*a*, *b*) have considered the much more difficult problem of the interaction of MHD shocks with perturbations whose wavelength is short compared with the shock width. They find that, in this case also, evolutionary shocks are stable, whereas non-evolutionary ones are not.

Although the conditions (2.3) and (2.5) appear to be different, the fact that $m_o + m_i = 2n$ means that they are entirely equivalent (Gel'fand 1963). Note that, if the system of shock and compatibility equations splits into independent subsets, then the discontinuity is only evolutionary if each of these subsets has the same number of equations as variables (Jeffrey and Tanuiti 1964).

Finally, as far as the evolutionary conditions are concerned, it does not matter whether or not the system (2.1) is strictly hyperbolic and convex, since these properties are not used in the derivation of (2.3) and (2.5). However, it is only in the case of strictly hyperbolic systems that these conditions reduce to the Lax conditions (Lax 1957)

$$\lambda_{k-1}(\mathbf{u}_l) < s < \lambda_k(\mathbf{u}_l),$$
$$\lambda_k(\mathbf{u}_r) < s < \lambda_{k+1}(\mathbf{u}_r)$$

for a nonlinear discontinuity associated with the kth characteristic (here we have assumed that $\lambda_1 < \lambda_2 < \ldots < \lambda_n$).

3. Evolutionary conditions and dissipative shock structure

In order to assess recent claims that non-evolutionary shocks become admissible if dissipative terms are included, we need to look at the general relationship between the evolutionary conditions and the uniqueness and existence of steady dissipative shock structures. Godunov (1961) has shown that it is much easier to explore this question if the equations can be transformed into a symmetric form. Although this is not possible for arbitrary hyperbolic systems of conservation laws, it can certainly be done for gas dynamics, MHD, and the shallow-water equations, and probably for any system that can arise in nature.

3.1. Symmetric form of the ideal equations

We start by summarizing some of the results described by Friedrichs (1954), Friedrichs and Lax (1971), and Boillat (1974, 1982). As before, it is only necessary to consider the one-dimensional case.

Consider a dissipation-free system of conservation laws described by (2.1). Suppose now that there exists a quantity $h(\mathbf{u})$, which is also conserved as long as the solution to this system is continuous. For example, $h(\mathbf{u})$ is the entropy in gas dynamics or MHD, whereas it is the total energy for the shallow-water equations. If such a quantity exists, then there must exist a flux function $g(\mathbf{u})$ such that

$$\frac{\partial h}{\partial t} + \frac{\partial g}{\partial x} = 0, \tag{3.1}$$

Equations (2.1) and (3.1) can only be consistent if

$$\frac{\partial h}{\partial u_i}\frac{\partial f_i}{\partial u_j} = \frac{\partial g}{\partial u_j}, \tag{3.2}$$

(the summation convention is assumed), since then

$$\frac{\partial h}{\partial t} + \frac{\partial g}{\partial x} = \frac{\partial h}{\partial u_i}\left(\frac{\partial u_i}{\partial t} + \frac{\partial f_i}{\partial x}\right) = 0$$

for any C^1 solution satisfying (2.1).

If we now use h to define the Legendre transformation

$$u'_i = -\frac{\partial h}{\partial u_i}, \tag{3.3}$$

$$u_i = \frac{\partial h'}{\partial u'_i}, \tag{3.4}$$

$$h' = h + {u'}_i u_i, \tag{3.5}$$

then (3.2) allows us to write the fluxes as

$$f_i = \frac{\partial g'}{\partial u'_i},$$

where

$$g' = g + {u'}_i f_i.$$

In terms of the variables $\mathbf{u}'$, (2.1) becomes a symmetric system

$$\mathsf{P}\frac{\partial \mathbf{u}'}{\partial t} + \mathsf{Q}\frac{\partial \mathbf{u}'}{\partial x} = 0, \tag{3.6}$$

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where the symmetric matrices $\mathbf{P}$ and $\mathbf{Q}$ are given by

$$P_{ij} = \frac{\partial u_i}{\partial u'_j} = \frac{\partial^2 h'}{\partial u'_i\,\partial u'_j} = -\frac{\partial^2 h}{\partial u_i\,\partial u_j},$$
$$Q_{ij} = \frac{\partial f_i}{\partial u'_j} = \frac{\partial^2 g'}{\partial u'_i\,\partial u'_j}. \tag{3.7}$$

Note that h is usually a strictly concave function, in which case (3.7) ensures that $\mathbf{P}$ is positive-definite and the transformation is non-singular. In ordinary gas dynamics or MHD, h is the entropy per unit volume, and is therefore guaranteed to be concave by the second law of thermodynamics. For the shallow-water equations, $h = -e$, where e is the sum of the kinetic and potential energies, and dissipation ensures that this is also concave.

3.2. *Dissipative equations*

If we now assume that the dissipative fluxes are proportional to the spatial gradients of the dependent variables, then the dissipative version of (3.6) is

$$\frac{\partial \mathbf{u}}{\partial t} + \frac{\partial \mathbf{f}}{\partial x} = \mathbf{P}\frac{\partial \mathbf{u}'}{\partial t} + \mathbf{Q}\frac{\partial \mathbf{u}'}{\partial x} = \frac{\partial}{\partial x}\mathbf{D}\frac{\partial \mathbf{u}'}{\partial x} \tag{3.8}$$

where $\mathbf{D}$ is a matrix of dissipation coefficients. Multiplying this on the left by $\mathbf{u}'^{\mathrm{t}}$ (the superscript t denotes the transpose) and using (3.1)–(3.3) gives the evolution equation for h:

$$\frac{\partial h}{\partial t} + \frac{\partial g}{\partial x} = -\mathbf{u}'^{\mathrm{t}}\frac{\partial}{\partial x}\mathbf{D}\frac{\partial \mathbf{u}'}{\partial x},$$

Integrating this over an arbitrary fixed interval $[a, b]$ and integrating the dissipative term by parts gives

$$\frac{d}{dt}\int_a^b h\,dx + \left[g + \mathbf{u}'^{\mathrm{t}}\mathbf{D}\frac{\partial \mathbf{u}'}{\partial x}\right]_a^b = \int_a^b \frac{\partial \mathbf{u}'^{\mathrm{t}}}{\partial x}\mathbf{D}\frac{\partial \mathbf{u}'}{\partial x}\,dx.$$

Since the term on the right-hand side of this equation represents a source term for h and the second law of thermodynamic requires that this be positive if h is the entropy per unit volume, the matrix $\mathbf{D}$ must be positive-definite for gas dynamics and MHD. The dissipative shallow-water equations must also satisfy this condition if we set $h = -e$, where e is the total energy.

One can also show that all linear waves decay if $\mathbf{D}$ is positive-definite and h is a strictly concave. The linear version of (3.8) is simply

$$\mathbf{P}\frac{\partial \mathbf{u}'}{\partial t} + \mathbf{Q}\frac{\partial \mathbf{u}'}{\partial x} = \mathbf{D}\frac{\partial^2 \mathbf{u}'}{\partial x^2}$$

where $\mathbf{P}$, $\mathbf{Q}$, and $\mathbf{D}$ are now constant matrices. Multiplying this by $\mathbf{u}'^{\mathrm{t}}$ and integrating over $[a, b]$ gives

$$\frac{d}{dt}\int_a^b \mathbf{u}'^{\mathrm{t}}\mathbf{P}\mathbf{u}'\,dx + \left[\mathbf{u}'^{\mathrm{t}}\mathbf{Q}\mathbf{u}' - 2\mathbf{u}'^{\mathrm{t}}\mathbf{D}\frac{\partial \mathbf{u}'}{\partial x}\right]_a^b = -2\int_a^b \frac{\partial \mathbf{u}'^{\mathrm{t}}}{\partial x}\mathbf{D}\frac{\partial \mathbf{u}'}{\partial x}\,dx,$$

after integrating the dissipative term by parts. Since $\mathbf{P}$ is positive-definite if h is strictly concave, the term on the right-hand side ensures that all linear waves decay if $\mathbf{D}$ is positive-definite.

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3.3. Steady shock structures

Now consider a solution of the steady version of (3.8),

$$\frac{d}{dx}\mathbf{f} = \frac{d}{dx}\mathsf{D}\frac{d}{dx}\mathbf{u}', \tag{3.9}$$

with the boundary conditions

$$\mathbf{u}' \to \begin{cases} \mathbf{u}'_l & (x \to -\infty), \\ \mathbf{u}'_r & (x \to +\infty). \end{cases} \tag{3.10}$$

If this represents a shock structure, then $\mathbf{u}'_l$ and $\mathbf{u}'_r$ must satisfy the shock relations in the shock frame

$$\mathbf{f}(\mathbf{u}'_l) = \mathbf{f}(\mathbf{u}'_r). \tag{3.11}$$

Integrating (3.9) and applying the boundary conditions (3.10) gives

$$\mathsf{D}\frac{d\mathbf{u}'}{dx} = \mathbf{f}(\mathbf{u}') - \mathbf{f}(\mathbf{u}'_l) = \mathbf{f}(\mathbf{u}') - \mathbf{f}(\mathbf{u}'_r). \tag{3.12}$$

A steady shock structure therefore corresponds to a solution of (3.12) that connects the equilibrium points $\mathbf{u}'_l$ and $\mathbf{u}'_r$. We now show that there is no guarantee that this solution is unique and structurally stable unless the corresponding discontinuous solution of the ideal system satisfies the evolutionary conditions (2.3).

Let L_u be the unstable manifold of the point $\mathbf{u}'_l$ and R_s the stable manifold of the point $\mathbf{u}'_r$. Then the trajectories in L_u and R_s are described by $\dim(L_u) - 1$ and $\dim(R_s) - 1$ parameters respectively. Since any trajectory that lies in both has to satisfy $n - 1$ matching conditions, this means that, in general, there will only be a unique trajectory connecting $\mathbf{u}'_l$ and $\mathbf{u}_r$ if $\dim(L_u) + \dim(R_s) = n + 1$. If $\dim(L_u) + \dim(R_s) > n + 1$, then the trajectory may not be unique, whereas if $\dim(L_u) + \dim(R_s) < n + 1$, then any trajectory that does exist can be destroyed by perturbations of $\mathbf{u}'_l$ and $\mathbf{u}'_r$, i.e. it is not structurally stable.

The following theorem relates $\dim(L_u)$ and $\dim(R_s)$ to the number of characteristics entering the shock:

Theorem 3.1. *If $\mathbf{u}'_e$ is an equilibrium point of the dissipative shock equations (3.12) at which none of the characteristic speeds vanish, then the equilibrium point is hyperbolic and the dimension of its stable (unstable) manifold is given by the number of positive (negative) characteristic speeds in the state $\mathbf{u}'_e$.*

Proof. Suppose that $\mathbf{u}'_e = \mathbf{u}'_l$ (the proof for $\mathbf{u}'_r$ is identical). Then linearizing (3.12) in the neighbourhood of $\mathbf{u}'_l$ gives

$$\mathsf{D}_l\frac{d\mathbf{v}}{dx} = \mathsf{Q}_l\mathbf{v},$$

where $\mathbf{v} = \mathbf{u}' - \mathbf{u}'_l$, $\mathsf{Q}_l = \mathsf{Q}(\mathbf{u}'_l)$ and $\mathsf{D}_l = \mathsf{D}(\mathbf{u}'_l)$. If this equilibrium point is hyperbolic, then the dimension of its stable (unstable) manifold is given by the numbers of eigenvalues μ_k satisfying

$$|\mathsf{Q}_l - \mu\mathsf{D}_l| = 0 \tag{3.13}$$

and with positive (negative) real parts.

On the other hand, the characteristic speeds for the system (3.6), λ_k, in the state $\mathbf{u}'_l$ are given by

$$|\mathsf{Q}_l - \lambda\mathsf{P}_l| = 0. \tag{3.14}$$

A standard result (see e.g. Gantmacher 1959) tells us that, since $\mathbf{P}_l$ and $\mathbf{Q}_l$ are symmetric and $\mathbf{P}_l$ is positive-definite, $\mathbf{Q}_l$ has the same number of positive, negative and zero eigenvalues as the set λ_k. If, like Godunov (1961), we assumed that $\mathbf{D}_l$ is symmetric as well as positive-definite, then the theorem would follow immediately from (3.13) and (3.14). However, the following lemma shows that this is an unnecessary restriction.

Lemma 1. *Let* $\mathbf{Q}$ *be a non-singular symmetric matrix,* $\mathbf{D}$ *a positive-definite matrix, and* μ_k *the solutions of*

$$|\mathbf{Q} - \mu\mathbf{D}| = 0.$$

Then the number of μ_k *with positive (negative) real parts is the same as the number of positive (negative) eigenvalues of* $\mathbf{Q}$.

Proof. Define

$$\mathbf{D}_\epsilon = \mathbf{D}_s + \epsilon\mathbf{D}_a,$$

where $\epsilon \in [0, 1]$ and

$$\mathbf{D}_s = \tfrac{1}{2}(\mathbf{D} + \mathbf{D}^{\mathrm{t}}), \qquad \mathbf{D}_a = \tfrac{1}{2}(\mathbf{D} - \mathbf{D}^{\mathrm{t}}).$$

It easy to see that $\mathbf{D}_\epsilon$ is also positive-definite.

Now consider the eigenvalue problem

$$|\mathbf{Q} - \mu(\epsilon)\mathbf{D}_\epsilon| = 0.$$

The conclusion of the lemma is certainly true for $\epsilon = 0$, since then $\mathbf{D}_\epsilon$ is symmetric. If we can show that the $\mu_k(\epsilon)$ are continuous functions of ϵ and that $\Re\{\mu_k(\epsilon)\} \neq 0 \;\forall k$ for $\epsilon \in [0, 1]$, then it will also be true for $\epsilon = 1$.

The $\mu_k(\epsilon)$ are the roots of a polynomial of degree n whose coefficients are polynomials in ϵ. A root can therefore only change discontinuously by going to infinity, which can only occur if the coefficient $|D_\epsilon|$ of the highest power of μ vanishes. However, this cannot happen, since D_ϵ is positive-definite for $\epsilon \in [0, 1]$. The $\mu_k(\epsilon)$ must therefore be continuous functions of ϵ for $\epsilon \in [0, 1]$.

In order to prove that the μ_k cannot cross the imaginary axis, suppose that for some k, $\mu_k(\epsilon) = i\eta$, where η is real. If $\mathbf{a} + i\mathbf{b}$ is the corresponding eigenvector, we have

$$\begin{aligned} \mathbf{Q}\mathbf{a} + \eta\mathbf{D}_\epsilon\mathbf{b} &= 0, \\ \mathbf{Q}\mathbf{b} - \eta\mathbf{D}_\epsilon\mathbf{a} &= 0. \end{aligned}$$

Multiplying the first of these by $\mathbf{b}^{\mathrm{t}}$ and the second by $\mathbf{a}^{\mathrm{t}}$, and subtracting gives

$$\eta(\mathbf{b}^{\mathrm{t}}\mathbf{D}_\epsilon\mathbf{b} + \mathbf{a}^{\mathrm{t}}\mathbf{D}_\epsilon\mathbf{a}) = 0.$$

Since $\mathbf{D}_\epsilon$ is positive-definite, this requires $\eta = 0$ and hence $\mu_k = 0$, which cannot be true if the eigenvalues of $\mathbf{Q}$ are non-zero. This completes the proof of the lemma. □

Equations (3.13) and (3.14) and Lemma 1 show that the theorem is true even if $\mathbf{D}$ is not symmetric. □

This is a somewhat more direct proof of a result that has also been obtained by Kulikovsky and Lyubimov (1965). In their analysis of viscous shock structures, Myong and Roe (1997a) assumed that Theorem 3.1 holds for MHD, but did not give a proof.

This analysis tells us that if the shock relations (3.11) have a solution such that none of the characteristic speeds given by (3.14) vanish in both the left and the right states and m_i is the number of characteristics entering the shock, then

(a) for $m_i = n + 1$, the shock can have a unique structurally stable dissipative structure;

(b) for $m_i > n + 1$, the dissipative structure is not guaranteed to be unique;

(c) for $m_i < n + 1$, there might be a unique dissipative structure, but it cannot be structurally stable.

These conditions are not only compatible with the evolutionary conditions, they are complementary to them. Shocks for which $m_i > n + 1$ have a dissipative shock structure and could therefore be regarded as admissible on these grounds. However, the left and right states of such shocks must be carefully tuned, since they cannot adjust themselves to an arbitrary small perturbations of their left and right states. Shocks that satisfy the relaxed evolutionary condition $m_i < n + 1$ are apparently permitted by the ideal equations, but cannot establish a dissipative structure and must spontaneously self-destruct. It is therefore clear that the only physically admissible shocks are those those that satisfy the strong evolutionary conditions (2.3) or (2.4).

Theorem 3.1 gives us no information in those cases for which the shock speed coincides with at least one of the characteristic speeds. The corresponding critical point is then no longer hyperbolic, and its type depends on the details of the particular system.

4. Application to magnetohydrodynamics

As we shall see, the mathematical properties of the full system of MHD and the reduced planar system of MHD are somewhat different, and this has to be clearly understood when the evolutionary conditions are applied. We therefore discuss these systems separately.

4.1. Full system of MHD

It is well known that the one-dimensional equations of MHD can be written in the form (2.1) (see e.g. Brio and Wu 1988). The conserved quantities $\mathbf{u}$ and the corresponding fluxes $\mathbf{f}$ are

$$\mathbf{u} = \begin{bmatrix} \rho \\ \rho v_x \\ \rho v_y \\ \rho v_z \\ e \\ B_y \\ B_z \end{bmatrix}, \qquad \mathbf{f} = \begin{bmatrix} \rho v_x \\ \rho v_x^2 + p_g + \frac{1}{2}B^2 - B_x^2 \\ \rho v_x v_y - B_x B_y \\ \rho v_x v_z - B_x B_z \\ (e + p_g + \frac{1}{2}B^2)v_x - B_x(\mathbf{v} \cdot \mathbf{B}) \\ v_x B_y - v_y B_x \\ v_x B_z - v_z B_x \end{bmatrix}.$$

Here p_g is the gas pressure,

$$e = i + \tfrac{1}{2}B^2 + \tfrac{1}{2}\rho v^2$$

is the total energy per unit volume, and i is the enthalpy per unit volume. Here we use units such that the velocity of light and the factor 4π do not appear.

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As we have already discussed, ideal MHD has a supplementary conservation law representing the conservation of thermodynamic entropy. The second law of thermodynamics guarantees that the function $h = \rho S$, where S is the entropy per unit mass, is strictly concave (see e.g. ter Haar and Wergeland 1966), and hence that the matrix **P** defined by (3.7) is positive-definite. The system of MHD equations can therefore be written in the symmetric form (3.6) and is hyperbolic. Although this has been demonstrated for relativistic MHD by Ruggeri and Strumia (1981), we have been unable to find an account of the corresponding analysis for classical MHD in the literature. However, since the derivations are similar to those for the relativistic case, we shall simply give the symmetric variables. They are

$$u_1' = \frac{1}{T}\left(\frac{w}{\rho} - \frac{1}{2}v^2\right), \qquad u_2' = \frac{v_x}{T}, \qquad u_3' = \frac{v_y}{T}, \qquad u_4' = \frac{v_z}{T},$$
$$u_5' = -\frac{1}{T}, \qquad u_6' = \frac{B_y}{T}, \qquad u_7' = \frac{B_z}{T}.$$

There is no need to verify that the matrix **D** of dissipation coefficients is positive-definite, since this must be true for any system that obeys the second law of thermodynamics. Indeed, this condition is used to derive the dissipative equations in the first place (see e.g. Landau and Lifshitz 1960). The exact form of symmetrized equations is also of no importance for our purposes. Their existence, does, however, allow us to apply the conclusions of the general theory described in Secs 2 and 3 to dissipative MHD.

4.1.1. Characteristic wave speeds. Since there are seven variables in this system, there are seven waves, whose speeds are

$$\begin{aligned} \lambda_{f\mp} &= v_x \mp c_f && \text{(fast waves)},\\ \lambda_{a\mp} &= v_x \mp c_a && \text{(Alfvén waves)},\\ \lambda_{s\mp} &= v_x \mp c_s && \text{(slow waves)},\\ \lambda_e &= v_x && \text{(entropy wave)}, \end{aligned}$$

where the Alfvén speed c_a and the slow and fast speeds c_s and c_f are given by

$$c_a = |B_x|\rho^{-1/2},$$

$$c_{s,f}^2 = \frac{1}{2}\left\{a^2 + \frac{B^2}{\rho} \mp \left[\left(a^2 + \frac{B^2}{\rho}\right)^2 - \frac{4a^2B_x^2}{\rho}\right]^{1/2}\right\},$$

where a is the adiabatic sound speed. Note that $0 \leqslant c_s \leqslant c_a \leqslant c_f$. If $B_x = 0$, then $c_s = c_a = 0$, whereas if the transverse component of the magnetic field, $\mathbf{B}_t$, vanishes then $c_f = c_a$ if $c_a > a$, $c_s = c_a$ if $c_a < a$, and $c_s = c_f = c_a$ if $c_a = a$. The MHD equations are therefore not strictly hyperbolic. Brio and Wu (1988) also argued that they are non-convex, but we shall postpone discussion of this until later.

4.1.2. Shock types. The MHD shock equations allow two linear solutions and several distinct types of nonlinear solutions that satisfy the entropy principle that the entropy of a fluid element always increases. A convenient way of classifying these is to use the jump in the transverse component of the magnetic field, $\mathbf{B}_t$. From the shock equations, one finds (Jeffrey and Taniuti 1964)

$$[\mathbf{B}_t(c_a^2 - v_x^2)]_l = [\mathbf{B}_t(c_a^2 - v_x^2)]_r, \tag{4.1}$$

where v_x is the velocity in the shock frame. Note that if $c_a^2 - v_x^2$ does not vanish, then $\mathbf{B}_t$ on one side of the discontinuity must be either parallel or antiparallel to that on the other.

The nonlinear solutions are as follows.

(a) Slow/fast shocks have non-zero $\mathbf{B}_t$ in the same direction on both sides. Equation (4.1) then implies that there is no change in the sign of $c_a^2 - v_x^2$. The magnitude of the magnetic field is larger on the downstream side for fast shocks and smaller downstream for slow shocks.

(b) Intermediate shocks also have non-zero $\mathbf{B}_t$, but in opposite directions on either side of the shock (Anderson 1963; Cabannes 1970). Equation (4.1) then implies that $c_a^2 - v_x^2$ changes sign.

(c) Switch-on shocks have vanishing $\mathbf{B}_t$ upstream. Equation (4.1) then implies that $v_x^2 = c_a^2$ on the downstream side.

(d) Switch-off shocks have vanishing $\mathbf{B}_t$ downstream. Equation (4.1) then implies that $v_x^2 = c_a^2$ on the upstream side.

The linear discontinuities are as follows.

(a) Alfvén discontinuities have $v_x^2 = c_a^2$ on both sides. Equation (4.1) then allows an arbitrary change in the direction of $\mathbf{B}_t$. However, the magnitude of $\mathbf{B}_t$ remains unchanged, which is why these are sometimes called rotational discontinuities.

(b) Contact discontinuities have the same value of v_x on both sides, but $v_x^2 \neq c_a^2$. Equation (4.1) then requires that $\mathbf{B}_t$ be continuous unless $B_x = 0$, and the other shock conditions require all other variables, except for the density, to be continuous.

We shall also find occasion to use the following classification of nonlinear MHD shocks, which is due to Germain (1960). The states in the shock frame are divided into four types:

(1) $|v_x| > c_f$;

(2) $c_f > |v_x| > c_a$;

(3) $c_a > |v_x| > c_s$;

(4) $c_s > |v_x|$.

A shock is defined to be of type $m \to n$ if the upstream and downstream states are of types m and n respectively. From the MHD shock equations, one finds that pressure and specific volume τ ($\tau = 1/\rho$) on each side of a nonlinear shock satisfy the following equations:

$$p + G^2\tau + \frac{1}{2}\frac{F_y^2}{(\tau - \tau_a)^2} = F_x,$$

$$w\tau + \frac{1}{2}G^2\tau^2 + \frac{\tau}{2\tau_a}\frac{F_y^2}{(\tau - \tau_a)^2} = H,$$

where G is the mass flux, F_x, F_y and H are shock invariants, and $\tau_a = B_x^2/G^2$. The analysis in Anderson (1963) can be used to show that the function $H(\tau)$ is as

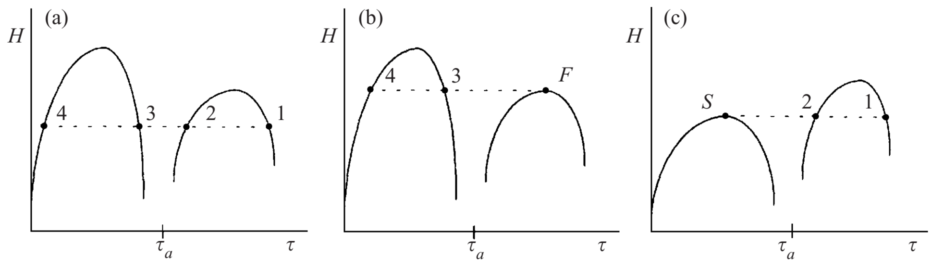

Figure 1. The shock invariant H as a function of specific volume τ for three different cases.

shown in Fig. 1. $\tau - \tau_i$ has the same sign as $v_x^2 - c_i^2$, where $i = s, a, f$. One can see that there are six different types of compressive shocks: fast shocks ($1 \to 2$), slow shocks ($3 \to 4$), and four intermediate shocks $1 \to 3$, $1 \to 4$, $2 \to 3$, and $2 \to 4$. Depending on the relative position of the maxima of H, there are also limit shocks that propagate with the fast speed relative to the upstream state and/or the slow speed relative to the downstream state (see Figs 1b,c). We shall denote such such shocks by $f \to n$ and $n \to s$ respectively. These shocks turn out not to be evolutionary, but if they were, then MHD would be a non-convex system.

4.1.3. Evolutionary conditions. When we apply the evolutionary conditions to MHD discontinuities, we have to take into account the fact that the system of shock and compatibility equations split into two independent subsets for all types of discontinuities, except the Alfvén discontinuity. If we choose a reference frame such that, on one side of a discontinuity, $B_z = 0$, and $v_z = 0$, then the system of shock equations contains two equations involving B_z and v_z. These are

$$B_{zl} = B_{zr}$$

and

$$v_{zl} = v_{zr}.$$

The compatibility relations along the Alfvén characteristics only involve B_z and v_z, and they are also the only ones that do so. An evolutionary discontinuity that is not an Alfvén discontinuity must therefore not only satisfy the general condition (2.3), but also have exactly two incoming, and hence two outgoing, Alfvén characteristics. These conditions also follow from the linear stability analysis (Syrovatskii 1959; Jeffrey and Tanuiti 1964).

In the rest of this subsection, we simply state the well-known results on the evolutionary properties of MHD discontinuities. We do, however, pay particular attention to those cases in which there are characteristics travelling with the same speed as the discontinuity. As we have pointed out in Sec. 2, such characteristics must be counted as incoming.

There is no dispute about the fact that fast and slow shocks are evolutionary, because they have eight incoming characteristics, two of which are Alfvén waves. Furthermore, since their speed can never be equal to a characteristic speed, Theorem 3.1 tells us that they also have a unique structurally stable dissipative structure.

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All intermediate shocks are super-Alfvénic with respect to the upstream state and sub-Alfvénic with respect to the downstream state, which means that they have too many (> 2) incoming Alfvén characteristics. They are therefore non-evolutionary, and can be destroyed by interactions with Alfvén waves.

The same argument applies to switch-on and switch-off shocks, which also have too many (nine) incoming characteristics, three of which are Alfvén characteristics. However, these solutions are clearly limits of fast and slow shocks, and therefore have evolutionary solutions in their immediate neighbourhood, which is why Jeffrey and Taniuti (1964) call them weakly evolutionary. That they are not strictly evolutionary can also be understood from the following example. Consider a switch-on shock overtaking a weak switch-off fast rarefaction travelling in the same direction. Once these have merged, the shock is no longer propagating into a state with zero transverse magnetic field. Since the shock is superfast, it has no way of modifying its upstream state, and therefore cannot remain a switch-on shock. Instead, such an interaction leads to the appearance of a neighbouring fast-shock solution, together with some other waves, at least one of which must, in general, be an Alfvén wave.

If we count the two entropy characteristics as incoming on the grounds that they have the same speed as the discontinuity, then contact discontinuities have eight incoming characteristics, two of which are Alfvén characteristics. They are therefore evolutionary.

Alfvén discontinuities also have eight incoming characteristics if we include the two Alfvén characteristics that have the same speed as the discontinuity. The total number of incoming Alfvén characteristics is three, but this is allowed since the fact that the shock equations for these discontinuities couple the y and z components of velocity and magnetic field means that this is the one case for which the shock equations do not decompose into two sets.

Theorem 3.1 cannot be applied to contact and Alfvén discontinuities, since they propagate with a characteristic speed. However, they would in any case not possess a steady dissipative structure, simply because they are linear and therefore have no nonlinear steepening to balance the spreading due to dissipation. For this reason, Wu (1988b) considers them to be inadmissible, but since their width grows like $t^{1/2}$, whereas the separation between the waves in a Riemann problem grows like t, they must be regarded as admissible components of the solution for large times.

4.2. Reduced system of planar MHD

In this subsection, we discuss the system of equations that describes MHD in a world in which the plane defined by the velocity and the magnetic field is invariant. There are several reasons for doing this. First, it has some interesting properties. Secondly, we want to show that the general classical theory of shocks is as valid for this system as it is for the full system. Finally, the numerical simulations that gave rise to the current controversy surrounding intermediate shocks reflect the properties of this system.

When the z components of the magnetic field and velocity vanish, the equations reduce to a system of five variables with the following vectors of conserved quantities and fluxes:

$$\mathbf{u} = \begin{bmatrix} \rho \\ \rho v_x \\ \rho v_y \\ e \\ B_y \end{bmatrix}, \qquad \mathbf{f} = \begin{bmatrix} \rho v_x \\ \rho v_x^2 + p_g + B^2/2 - B_x^2 \\ \rho v_x v_y - B_x B_y \\ (e + p_g + \frac{1}{2}B^2)v_x - B_x(\mathbf{v}\cdot\mathbf{B}) \\ v_x B_y - v_y B_x \end{bmatrix}.$$

This is still a hyperbolic system, but it is fundamentally different from the full system of MHD, because it does not have Alfvén waves. However, the other characteristic fields are still present, with the same eigenvalues and with eigenvectors that are the same apart from the reduced number of components. Moreover, it has the same solutions of the shock equations, including the Alfvén discontinuity, except that these are now only allowed to change the direction of the transverse magnetic field by π. This follows from the remarkable property of the full system of MHD that there exists an inertial frame in which the variations of the transverse components of the magnetic field and velocity induced by all characteristic waves and shocks, except for Alfvén waves, are confined to single plane. Note that the Alfvén discontinuity still propagates with the Alfveń speed, but this is no longer one of the characteristic speeds. The Riemann problem for this system has been analysed in considerable detail by Myong and Roe (1997b), who came to the conclusion that the classical evolutionary conditions are inadequate for this system. However, we intend to show that this claim is based on a failure to recognize the essential difference between the reduced system and full MHD.

4.2.1. Evolutionary conditions. Since the number of equations is reduced by two and it is the Alfvén waves that are lost, we can conclude that all evolutionary discontinuities that have two incoming Alfvén characteristics in the full system remain evolutionary in the planar system. This implies that fast, slow, and contact discontinuities are evolutionary.

On the other hand, discontinuities that are evolutionary in the full system, but that do not have exactly two incoming Alfvén characteristics, must be non-evolutionary in the planar system. There is only one such discontinuity, the Alfvén discontinuity, which now only has five incoming characteristics and should therefore spontaneously self-destruct even if it is not perturbed.

Another interesting feature is that some of the shocks that are non-evolutionary in the full system become evolutionary in the reduced system. $1 \rightarrow 3$ shocks now satisfy the strong evolutionary condition; in fact, they have the same incoming and outgoing characteristics as fast and switch-on shocks. As far as the characteristic count is concerned, these three shocks are therefore indistinguishable, so that one can use a single name, *plane fast shock*, say, for all of them. Similarly, $2 \rightarrow 4$ shocks, switch-off shocks, and slow shocks become slightly different versions of evolutionary *plane slow shocks*.

However, $1 \rightarrow 4$ shocks remain non-evolutionary even in the plane system, since they have seven incoming characteristics. Such shocks, which have too many incoming characteristics, are often called *overcompressive* in the literature. As we

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have shown, although they do have a steady dissipative structure, it is not unique and it does not help them to survive interactions with external perturbations.

$2 \to 3$ shocks have only five incoming characteristics, and are therefore non-evolutionary. Such shocks, which have too few incoming characteristics, are often called *under-compressive*. Since they do not have a structurally stable steady dissipative structure, they should disintegrate spontaneously even without any external perturbation.

Now consider shocks that propagate at one of the characteristic speeds in either the upstream or downstream state. $1 \to s$, $f \to 4$, and $f \to s$ shocks are non-evolutionary, since they have seven incoming characteristics. On the other hand, $2 \to s$ and $f \to 3$ shocks have six incoming characteristics, and are therefore evolutionary. The planar system of MHD is therefore genuinely non-convex, and admits two evolutionary compound waves: a *slow compound wave* consisting of a $2 \to s$ shock with an attached slow rarefaction, and a *fast compound wave* consisting of a fast rarefaction with an attached $f \to 3$ shock.

Finally, we list the evolutionary shocks and compound waves of the planar system along with the notation used in Myong and Roe (1997b):

slow planar shock (S_1);
fast planar shock (S_2);
slow compound wave (C_1);
fast compound wave (C_2);
contact discontinuity (not considered here).

Myong and Roe (1997b) found that some Riemann problems only have a solution if non-evolutionary shocks are permitted. However, as we discuss in Sec. 6, these Riemann problems are confined to regions of parameter space with zero volume, which is exactly what is meant by the statement that non-evolutionary shocks are structurally unstable.

In the next section, we show that the results of numerical calculations are entirely consistent with these conclusions.

5. Numerical calculations

The numerical calculations were carried out using the scheme described in Falle et al. (1998). This is an upwind shock-capturing scheme that is capable of dealing with shocks of arbitrary strength even without the inclusion of any dissipation other than that introduced by the truncation errors. Careful test simulations have shown that this scheme provides accurate solutions for all types of MHD waves in all regimes. One can argue that if a numerical scheme works well, then its numerical dissipation must have the same qualitative properties as the physical dissipation. However, in order to remove any doubts, we modified our scheme so that it can now handle dissipative MHD, and all the calculations described here have a fully resolved dissipative shock structures (about 15 mesh points wide). For this, we used a simple scalar form for the dissipation for which (2.1) become

$$\frac{\partial \mathbf{u}}{\partial t} + \frac{\partial \mathbf{f}}{\partial x} = \frac{\partial \mathbf{g}}{\partial x},$$

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Table 1. Riemann problems for the numerical calculations.

$2 \to 3$ intermediate shock: Fig. 2 (left panels)
Left state: $\rho = 1,\ p_g = 1,\ \mathbf{v} = (-0.95, 0, 0),\ \mathbf{B} = (1, 0.5, 0)$
Right state: $\rho = 0.837,\ p_g = 0.705,\ \mathbf{v} = (-1.135, 1.266, 0),\ \mathbf{B} = (1, -0.7, 0)$

Alfvén shock: Fig. 2 (right panels)
Left state: $\rho = 1,\ p_g = 1,\ \mathbf{v} = (-1, 1, 0),\ \mathbf{B} = (1, 1, 0)$
Right state: $\rho = 1,\ p_g = 1,\ \mathbf{v} = (-1, 3, 0),\ \mathbf{B} = (1, -1, 0)$

$1 \to 3$ intermediate shock: Figs 3 (left panels), 5, 7 (left panels)
Left state: $\rho = 1,\ p_g = 1,\ \mathbf{v} = (-0.925, 0, 0),\ \mathbf{B} = (1, 0.5, 0)$
Right state: $\rho = 0.498,\ p_g = 0.258,\ \mathbf{v} = (-1.857, 0.648, 0),\ \mathbf{B} = (1, -0.1, 0)$

$2 \to 4$ intermediate shock: Figs 3 (right panels), 5, 7 (right panels)
Left state: $\rho = 1,\ p_g = 1,\ \mathbf{v} = (-0.4, 0, 0),\ \mathbf{B} = (0.5, 0.5, 0)$
Right state: $\rho = 0.561,\ p_g = 0.155,\ \mathbf{v} = (-0.714, 2.252, 0),\ \mathbf{B} = (0.5, -1.3, 0)$

$1 \to 4$ intermediate shock: Fig. 4
Left state: $\rho = 1,\ p_g = 1.2,\ \mathbf{v} = (-0.842, 0.0, 0.0),\ \mathbf{B} = (1.0, 0.4, 0)$
Right state: $\rho = 0.390,\ p_g = 0.161,\ \mathbf{v} = (-2.16, 0.644, 0),\ \mathbf{B} = (1.0, -0.142, 0)$

Brio and Wu Problem: Fig. 8
Left state: $\rho = 1,\ p_g = 1,\ \mathbf{v} = (0, 0, 0),\ \mathbf{B} = (0.75, 1, 0)$
Right state: $\rho = 0.125,\ p_g = 0.1,\ \mathbf{v} = (0, 0, 0),\ \mathbf{B} = (0.75, -1, 0)$

where the diffusive fluxes are

$$\mathbf{g} = \begin{bmatrix} 0 \\ \dfrac{4\mu}{3}\dfrac{\partial v_x}{\partial x} \\ \mu\dfrac{\partial v_y}{\partial x} \\ \mu\dfrac{\partial v_z}{\partial x} \\ \dfrac{4\mu v_x}{3}\dfrac{\partial v_x}{\partial x} + \mu v_y\dfrac{\partial v_y}{\partial x} + \mu v_z\dfrac{\partial v_z}{\partial x} + \nu_m\left(B_y\dfrac{\partial B_y}{\partial x} + B_z\dfrac{\partial B_z}{\partial x}\right) \\ \nu_m\dfrac{\partial B_y}{\partial x} \\ \nu_m\dfrac{\partial B_z}{\partial x} \end{bmatrix},$$

where μ is the dynamic viscosity, κ the thermal conductivity, and ν_m the resistivity.

The Riemann problems considered in the numerical calculations are summarized in Table 1, while other parameters used are listed in Table 2.

As expected, the outcomes of all the simulations presented here did not not depend on the size of dissipation, and were the same even when only numerical and/or artificial dissipation was present. The only effect of changing the dissipation was to alter the form and width of the shock structures.

First of all, we need to establish whether the behaviour of numerical MHD shocks agrees with the predictions of the evolutionary theory. In order to do this, we adopt the following procedure. First, we test whether a shock has a steady dissipative structure by setting up the relevant Riemann problem and running the calculation until a well-resolved steady dissipative shock structure is established,

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Table 2. Other parameters for the numerical calculations. n is the number of mesh points, μ is the kinematic viscosity, κ is the thermal conductivity, and ν_m is the resistivity.

Problem	Domain	n	μ/ρ	κ/ρ	ν_m
Fig. 2 (left panels)	$[-4, 1]$	250	0.02	0.01	0.01
Fig. 2 (right panels)	$[-2, 1]$	150	0.02	0.01	0.01
Fig. 3	$[-4, 1]$	250	0.02	0.01	0.01
Fig. 4	$[-4, 1]$	250	0.02	0.01	0.01
Fig. 5	$[-1, 1]$	200	0.01	0.005	0.005
Fig. 6	$[-2, 1]$	300	0.01	0.005	0.005
Fig. 7 (left panels)	$[-8, 2]$	500	0.02	0.01	0.01
Fig. 7 (right panels)	$[-14, 1]$	750	0.02	0.01	0.01
Fig. 8	$[2.5, 4.5]$	200	0.0	0.0	0.0

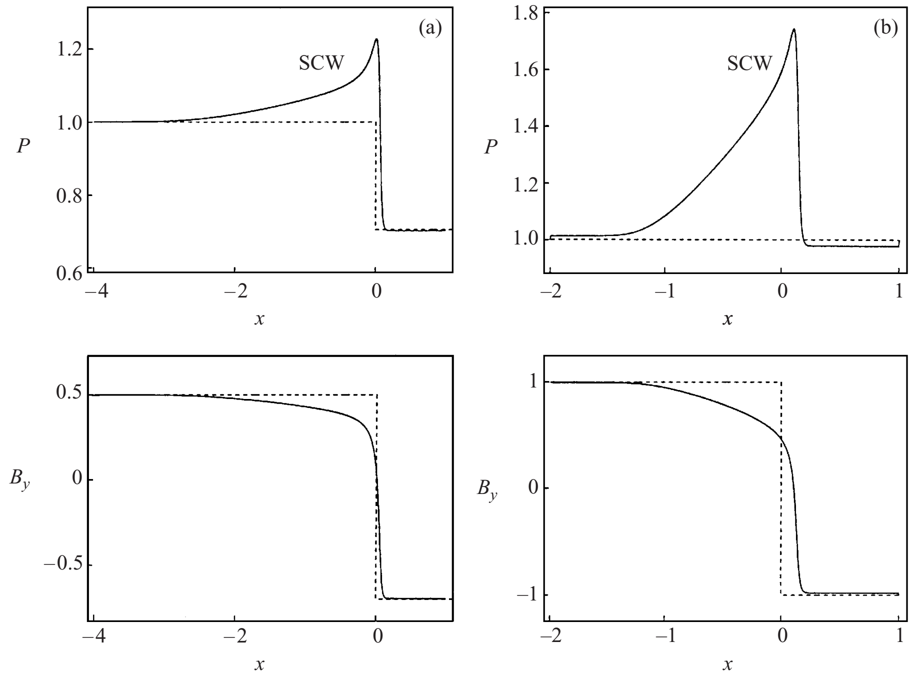

Figure 2. Planar simulations of shocks that should not have a steady dissipative structure in planar MHD: $2 \rightarrow 3$ shock (left panels); Alfvén shock (right panels). In both cases, the outcome is a slow compound wave (SCW). The dashed lines show the corresponding initial solutions. The continuous lines show the final solutions.

as expected for evolutionary and overdetermined shocks, or a completely different solution emerges, as expected for underdetermined shocks. If a steady structure exists, then we test to see whether it can survive small perturbations. This can be accomplished by considering a slightly different Riemann problem, as in Barmin et al. (1996) or, like Wu (1988a), allowing a small-amplitude wave to interact with the shock.

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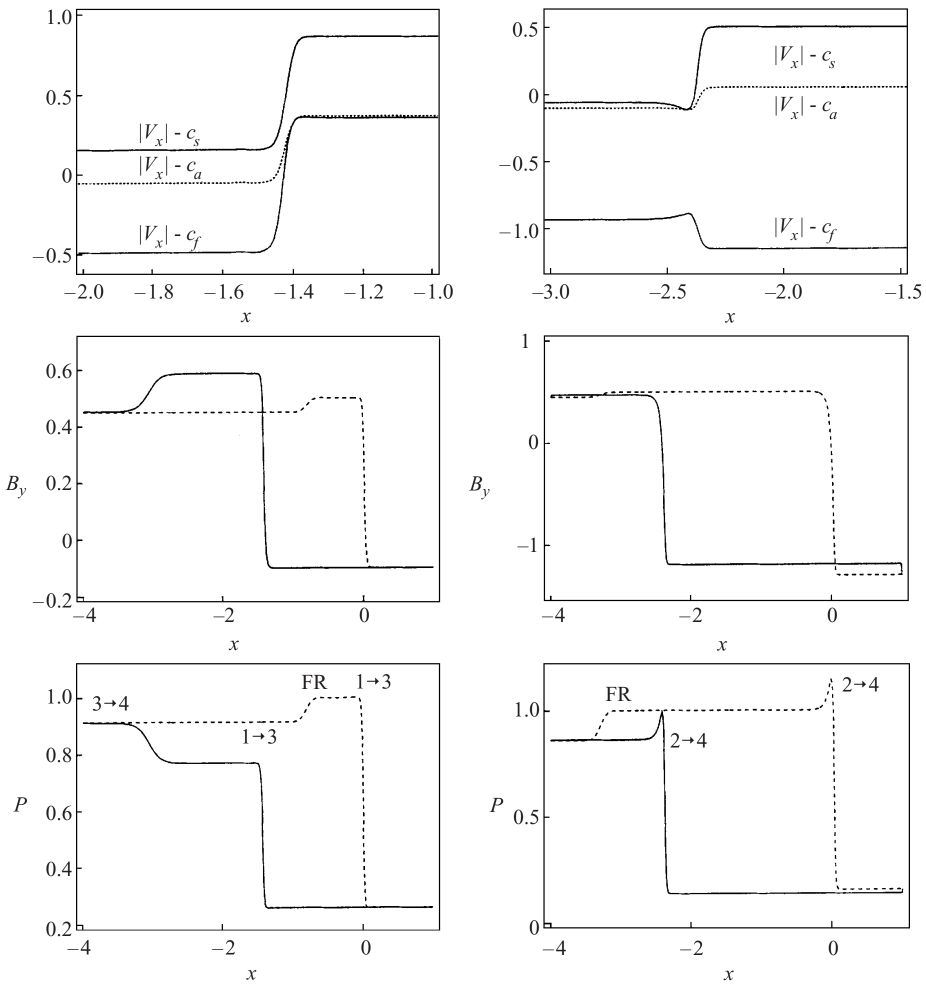

Figure 3. Planar simulations of the interaction between evolutionary shocks and small-amplitude fast rarefactions (δB_t = 10%): fast (1 $\longrightarrow$ 3) shock (left panels); slow (2 $\longrightarrow$ 4) shock (right panels). In both cases, the outcome is a shock of the same type, together with some other waves. Here FR denotes a fast rarefaction and V_x is the x component of velocity as measured in the shock frame. The dashed lines show the initial solutions. The continuous and dotted lines show the final solutions.

5.1. Planar MHD

We start by discussing the results of the planar simulations. They show that if the initial discontinuity corresponds to a slow planar shock, then a smooth steady shock structure connecting the initial left and right states finally develops, and it does not matter whether the shock is 3 $\longrightarrow$ 4 or 2 $\longrightarrow$ 4. The same thing happens for the fast planar shock and the overdetermined (overcompressive) 1 $\longrightarrow$ 4 shock. In contrast, Fig. 2 shows that 2 $\longrightarrow$ 3 shocks and Alfvén shocks always turn into a slow compound wave. All this is exactly as predicted by the theory described in Secs 3 and 4. Our simulations cannot be used to determine whether limit shocks (such 1 $\longrightarrow$ s and f $\longrightarrow$ 3) have a steady dissipative shock structure, simply because it is

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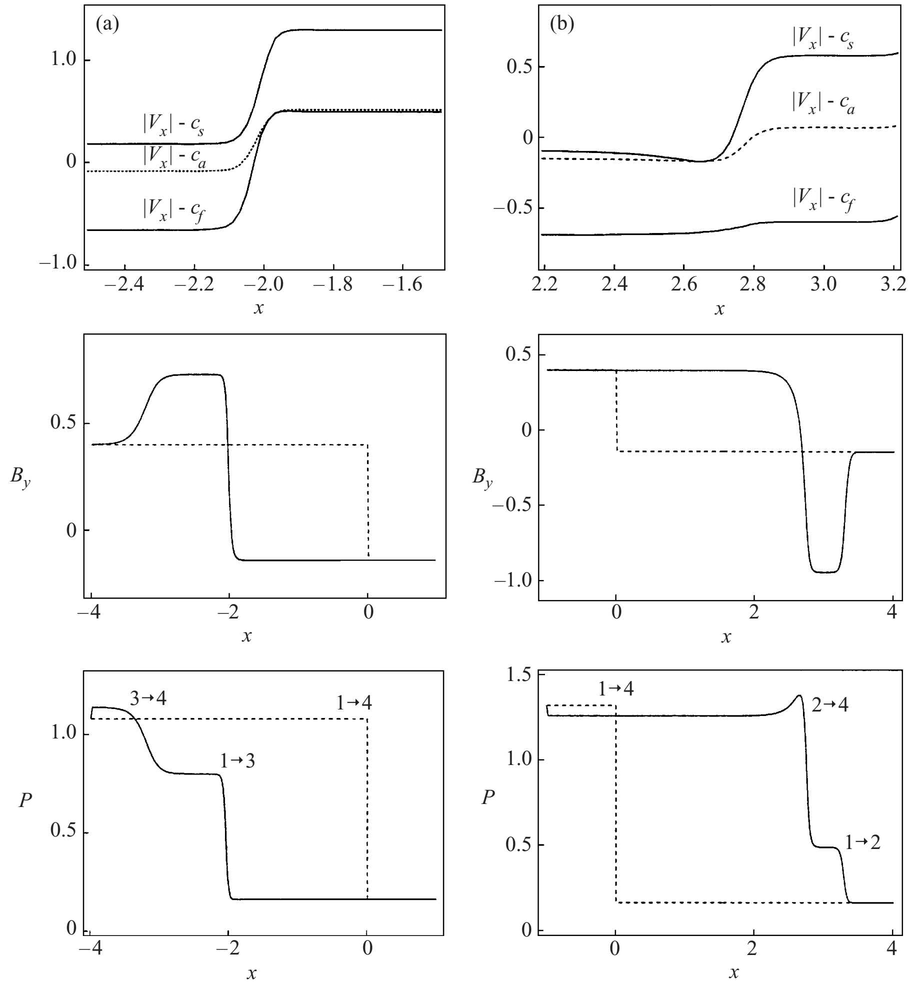

Figure 4. Planar simulations of a $1 \rightarrow 4$ shock subjected to a small variation of pressure ($\pm 10\%$) in the left state. This shock is non-evolutionary even in planar MHD, and splits as the result of the perturbation into two evolutionary shocks plus other small-amplitude waves The outcome is $1 \rightarrow 3$ and $3 \rightarrow 4$ shocks if $\delta p = -10\%$ (left panels) and $1 \rightarrow 2$ and $2 \rightarrow 4$ shocks if $\delta p = +10\%$ (right panels). The dashed lines show the initial solutions. The continuous and dotted lines show the final solutions. V_x is the x component of velocity as measured in the frame of the emerged intermediate shock.

impossible to set up a shock whose speed is exactly equal to a characteristic speed. However, if we compute a Riemann problem that corresponds to a compound wave of any of the types discussed above, the wave that is expected – or, strictly speaking, a solution close to such a wave – always emerges. This is hardly surprising, because all of them have neighbouring solutions containing shocks with a steady dissipative structure.

As shown in Fig. 3, evolutionary shocks always survive interactions with small-amplitude waves and persist if the Riemann problem is perturbed. Figure 4 shows how a small variation of the initial data forces an overdetermined $1 \rightarrow 4$ shock to split into two evolutionary shocks. Depending on the form of the perturbation,

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the shock splits either into a $1 \to 2$ shock followed by a $2 \to 4$ shock or into a $1 \to 3$ shock followed by a $3 \to 4$ shock. This is to be expected, because, as one can see from Fig. 1, a $1 \to 4$ shock is exactly equivalent to one or other of these shock pairs propagating with the same speed. In fact, this result is in complete agreement with the analysis of the Riemann problem for planar MHD in Myong and Roe (1997b). $1 \to 4$ shocks (*O shocks* in their notation), are only required on the boundary between the two domains of parameter space in which their solution involves a combination of fast and slow planar shocks (S_2 and S_1).

The results for compound waves involving non-evolutionary shocks are similar. Figure 1 shows that the non-evolutionary $1 \to s$ limit shock can be understood as a *double-layer* shock composed of two evolutionary shocks: a $1 \to 2$ and a $2 \to s$. Indeed, if the Riemann problem corresponding to a compound wave containing such a shock is perturbed, then in some cases the outcome is a $1 \to 2$ shock and a slow compound wave, while in other cases it is a $1 \to 3$ shock and a detached slow rarefaction.

All of this can be summed up by saying that, for planar MHD, the behaviour of shocks in our numerical simulations is entirely consistent with the classical evolutionary theory of shocks and the theory of dissipative shock structures as described in Sec. 2 and 3.

5.2. Full MHD

Since both fast ($1 \to 2$) and slow ($3 \to 4$) shocks satisfy the strong evolutionary condition in full MHD, they are expected to have unique dissipative structure and be stable with respect to small perturbations of any kind. This is precisely what we find from our simulations.

$1 \to 3$ and $2 \to 4$ shocks are overdetermined in full MHD, and it is therefore possible that they might have a non-unique steady dissipative structure – indeed, it turns out that they do. These shocks, as well as $1 \to 4$ shocks, can now have a non-vanishing z component of magnetic field inside the shock layer even if $B_z = 0$ outside. For, given the dissipative coefficients, their stucture can be parametrized by the value of the following integral:

$$I_z = \int_{-\infty}^{\infty} B_z \, dx.$$

We can gradually increase or decrease the value of I_z by sending from the downstream side of the shock an Alfvén wave that first rotates the magnetic field by a small angle and then restores the original state. This wave is absorbed by the shock, which develops a new steady structure (see the left-hand panels of Fig. 5). However, like Kennel et al. (1990), we found that there is a maximum value of $|I_z|$ that the shock can manage. If this limit is exceeded, then the shock disintegrates (see the right-hand panels of Fig. 5). This does not occur in the case of fast and slow shocks, because the Alfvén waves do not get trapped inside the shocks, but instead pass straight through.

$2 \to 3$ shocks have the right number of incoming characteristics, and may therefore have a unique dissipative structure in full MHD. Since such a structure does not exist in planar MHD, we can only expect to find them in our simulations by allowing a non-zero B_z. In order to do this, we modified the initial data by inserting a layer in which the transverse field rotates smoothly from that in the original left state to that in the original right state. We found that the solution never relaxed

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to a smooth steady $2 \to 3$ transition, and were about to conclude that no steady structure exists until we realized that the solution shown in the right-hand panels of Fig. 5 actually contains a $2 \to 3$ shock, which was produced by the disintegration of the $1 \to 3$ shock. We therefore studied the reaction of a $1 \to 3$ shock to an increase in I_z. After absorbing another Alfvén wave, the shock splits, and one of the emerging waves is again a $2 \to 3$ shock but of smaller amplitude (Fig. 6). This behaviour is consistent with the existence of a unique dissipative structure for $2 \to 3$ shocks. In fact, what happens is that, as I_z increases, the shock tends to an Alfvén shock that rotates the transverse field by π. This situation has been considered by Wu and Kennel (1992), who showed that if I_z increases linearly with time, then the width of the shock increases like $t^{1/2}$ and its strength decreases like $t^{-1/2}$.

Finally, we have also verified that all intermediate shocks and compound waves disintegrate when exposed to perturbations that render the left and right states non-coplanar. For example, Fig. 7 shows how $1 \to 3$ and $2 \to 4$ shocks split into evolutionary waves after interaction with a small-amplitude Alfvén wave. After the Alfvén wave has been absorbed, the transverse fields on either side of the shock are no longer parallel or antiparallel, as required by the shock equations. The shock can only become coplanar by emitting Alfvén waves, which, for an intermediate shock, can only be done in the downstream direction. However, since there is no downstream-travelling Alfvén wave that can restore the original post-shock state, the shock must split. This argument is not new – in fact it was used by Kantrowitz and Petschek (1966) to prove that intermediate shocks are unphysical. The wave designated as AW in Fig. 7 can be called a dissipative Alfvén wave, but it could also be described as an evolving $2 \to 3$ shock with a gradually increasing value of I_z.

There appears to be little danger that any of these results are artefacts of our numerical method, since it seems that all numerical calculations of which we are aware give similar results. In particular, our results are enirely compatible with those that Wu (1998a) obtained for the interaction of an Alfvén wave with a $2 \to 4$ shock using an entirely different numerical method.

We therefore conclude that, for full MHD, the behaviour of shocks in our numerical simulations is also entirely consistent with the classical evolutionary theory of shocks and the theory of dissipative shock structures as described in Secs 2 and 3.

6. Discussion

The results described in the previous sections have clarified many aspects of shock theory in general and MHD shocks in particular, and provide a basis upon which we can discuss other important, related, issues.

The first question is the sense in which the ideal theory is a useful approximation to the real world – a subject about which there appears to be some confusion. Since the dissipation coefficients are the coefficients of the highest-order derivatives in the dissipative equations, it is clear that we have a singular perturbation problem when the dissipation is small. The classical inviscid theory gives the the large-scale, or outer, solution, which is valid outside shock structures, and this has to be matched to an inner, steady solution for the shock structure. Uniqueness of the inviscid solutions requires that all shocks be evolutionary, and the dissipation must also be such that shocks possess a steady shock structure. We have pointed out that these conditions are compatible in the sense that if an evolutionary shock has a steady

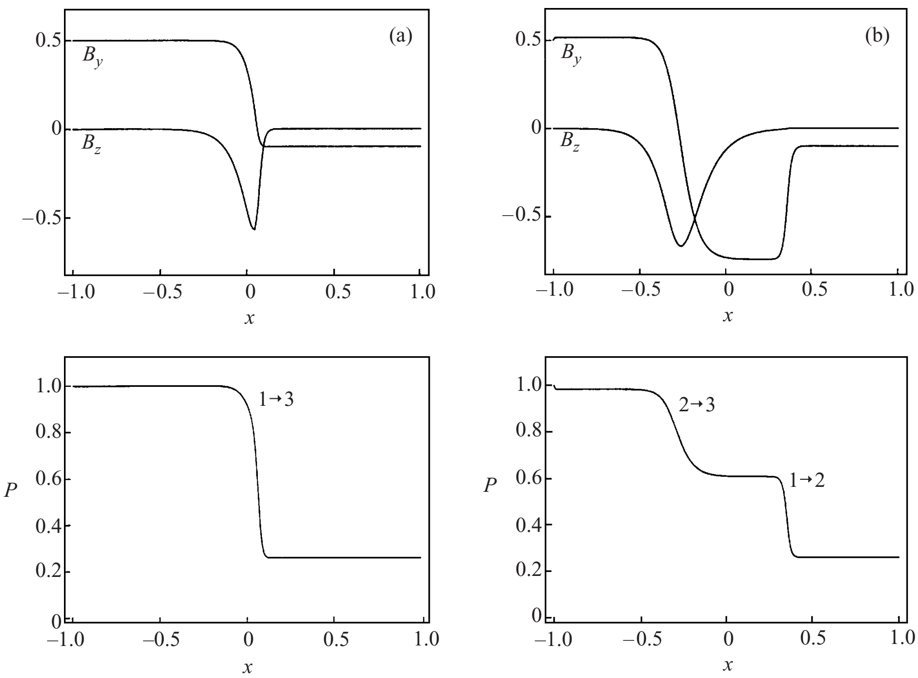

Figure 5. Dissipative structure of a $1 \rightarrow 3$ shock in full MHD for different values of I_z. This shock has a non-unique steady dissipative structure that depends upon I_z. For relatively small values of I_z, this structure is steady (left panel, $I_z = -0.085$) but for larger values it splits into $1 \rightarrow 2$ and $1 \rightarrow 2$ shocks (right panel, $I_z = -0.20$).

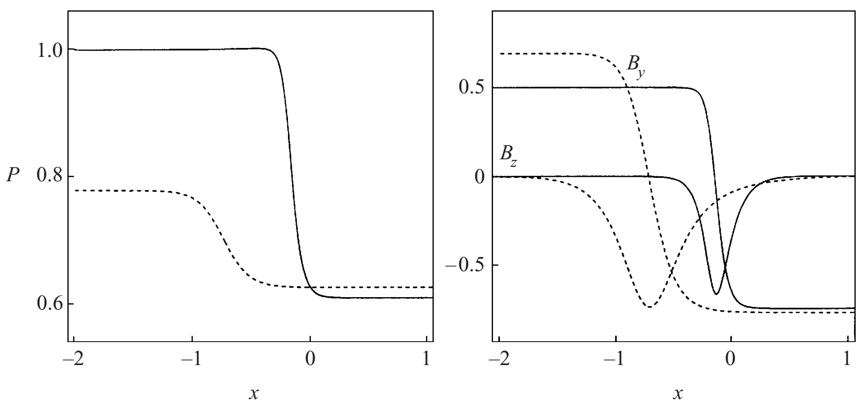

Figure 6. Dissipative structure of a $2 \rightarrow 3$ shock in full MHD. This shock has a unique steady dissipative structure, and therefore reacts to a change in I_z by emitting some waves and turning into a different $2 \rightarrow 3$ shock. The continuous lines show the solution for $I_z = -0.20$ and the dashed lines that for $I_z = -0.52$.

shock structure, then it is unique, whereas this is not true for non-evolutionary shocks. Furthermore, the fact that in all physical systems in which they are known to occur, MHD shocks are indeed very thin compared with the scale of the flow means that the small-dissipation limit is the appropriate one.

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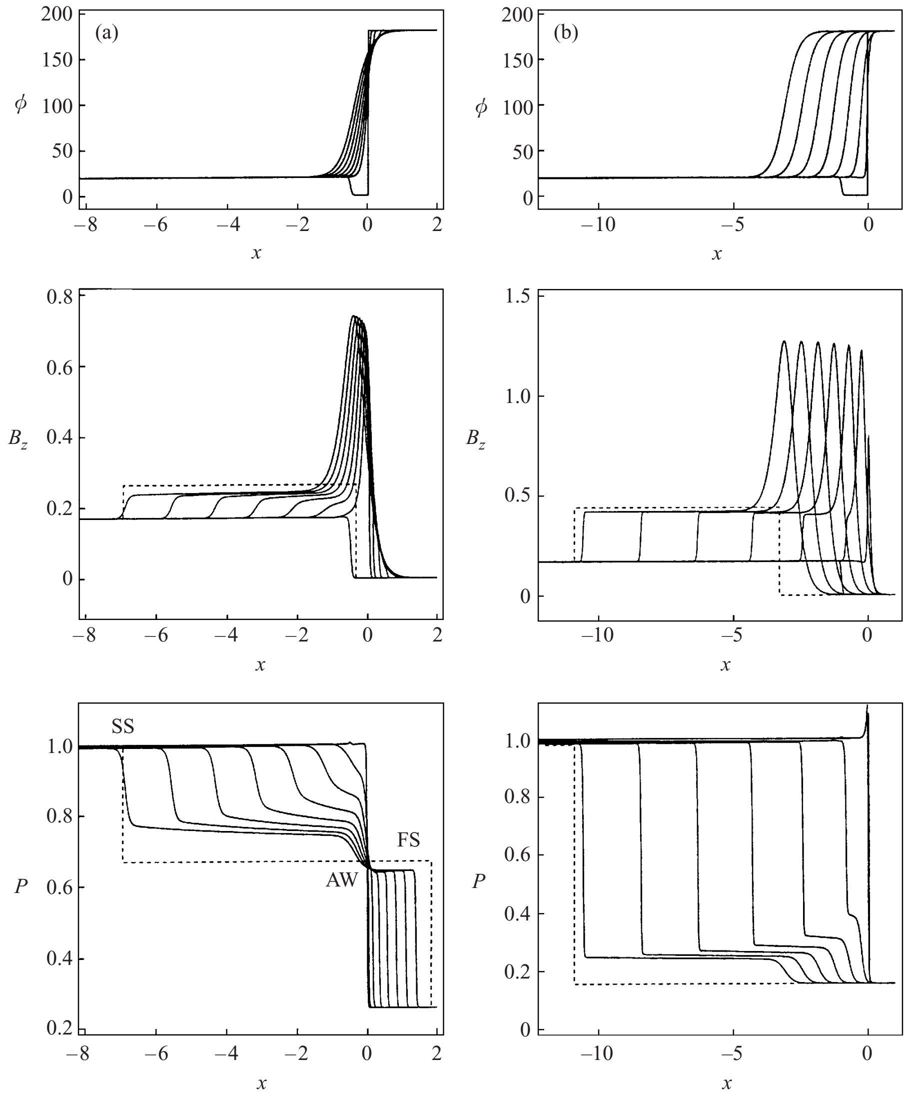

Figure 7. MHD shocks interacting with small-amplitude Alfvén waves. The evolutionary shocks survive, but the non-evolutionary ones split. Left panels: A $1 \rightarrow 3$ shock splits into a fast shock (FS), an Alfvén wave (AW), and a slow shock (SS). Right panels: A $2 \rightarrow 4$ shock splits into an Alfvén wave and a slow shock. Other small-amplitude waves are also emitted. The dashed line shows the exact ideal solution of the Riemann problem for the initial state formed by the collision of the intermediate and Alfvén shocks.

6.1. Riemann problems and evolutionary conditions

One of the arguments in favour of non-evolutionary shocks used in the current literature is that some Riemann problems do not have a solution unless non-evolutionary shocks are admitted (see e.g. Glimm 1988; Myong and Roe 1997a,b). This is presumably based on the belief that any Riemann problem must have a physically admissible solution. Although this is certainly true for gas dynamics, there is surely no reason why it has to be so for any system. As Markovskii (1998a)

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has pointed out, it all comes down to the notion of structural stability. One has to ask the following question: Is it, or is it not, possible to carry out the relevant experiment in a laboratory? If the qualitative result of the experiment does not change when the initial conditions are slightly changed, then the problem is structurally stable and the experiment is possible – at least in principle. However, if this is not true, then the problem is structurally unstable and no appropriate experiment is possible. It therefore follows that the set of structurally unstable Riemann problems are confined to regions of parameter space whose total volume is zero. Now suppose that there is an MHD Riemann problem that has no other solutions than those containing non-evolutionary shocks. Since there are arbitrary small perturbations of the parameters that cause these shocks to split into evolutionary shocks, this Riemann problem must be structurally unstable. In full MHD, the only known case for which a non-evolutionary shock, a $1 \to 4$ shock, is required is a piston problem in which the piston velocity is parallel to the magnetic field (Jeffrey and Taniuti 1964). If this condition is not exactly satisfied, then the non-evolutionary shock does not arise. Close inspection of the solution of the Riemann problem for planar MHD presented by Myong and Roe (1997b) shows that non-evolutionary shocks are required only on the boundaries between domains in parameter space that contain only evolutionary shocks.

It is evident from the above that intermediate shocks are not structurally stable, but the linear stability analysis discussed in Sec. 2 also shows that they are unstable to linear perturbations whose wavelength is either large or small compared with the shock width. This would in itself be sufficient to preclude their existence even if they were structurally stable.

6.2. Steepening of continuous waves

Another argument that appears to justify the existence of intermediate shocks is based on the results of numerical simulations by Wu (1987), which suggest that intermediate shocks can be formed by nonlinear steepening of simple magnetosonic waves. Since the transverse component of the magnetic field changes sign across an intermediate shock, the simple wave must have the same property, which means that the transverse component of the magnetic field must vanish somewhere within the wave. However, at this point, the magnetosonic speed is equal to the Alfvén speed, and it is impossible to assign a unique eigenvector to the simple wave. As a result, the direction of the tangential component of the field can rotate by an arbitrary angle at this point, so that a simple wave really consists of two distinct parts, which are disconnected as far as the direction of the magnetic field is concerned. This can be put in a slightly different way. Alfvén waves propagating in the same direction as such a simple wave cannot pass throught the Alfvén point. During the steepening, they will accumulate near this point, giving rise to a net field rotation, so that the discontinuity that forms has non-coplanar left and right states and therefore cannot be a single shock. Instead, it must split into evolutionary shocks, one of which must be an Alfvén shock. Incidentally, this seems to be the only way of generating Alfvén shocks.

However, in planar MHD, the transition through the Alfvén point is unique, and, as we have seen, some of the intermediate shocks are in fact evolutionary. This is the explanation for the outcome of the planar simulations performed by Wu (1987). He also found that the results were not very different if the initial data was perturbed so that it was no longer exactly coplanar. However, because of the periodic boundary conditions used in this simulation, there was no net rotation in the perturbed problem, which makes it rather artificial. The reason why this perturbation did not destroy the intermediate shock is that these boundary conditions, together with the initial data, only allowed a small value of I_z per shock. It is therefore hardly surprising that an intermediate shock appeared, since, as we have shown, these shocks can survive if I_z is small enough.

6.3. Time scale for disintegration

Let us suppose that an intermediate shock has somehow been formed, and then interacts with an Alfvén wave that rotates the magnetic field by a small angle $\delta\phi$. It is clearly of some importance to know how long it takes for the shock to split. Our simulations show that it splits when the value of I_z associated with the shock structure becomes comparable to lB_y, where l is the shock thickness. If the incident Alfvén wave has a small amplitude, $\delta\phi$, then this gives us the following estimate for the disintegration time t_s:

$$t_s \approx \frac{l}{c_a \delta\phi}, \tag{6.1}$$

where we have used the Alfvén speed as a characteristic fluid velocity in the shock frame. This also tells us that the shock will only propagate for a distance $\approx l/\delta\phi$ before it falls apart. We conclude from this that, in all cases for which the dissipative scale is much smaller than the characteristic length scale of the flow, intermediate shocks can only appear as very short lived time-dependent phenomena.

It is instructive to apply (6.1) to the interplanetary intermediate shock for which Chao et al. (1993) claim to have found evidence in the Voyager 1 data. In this case $c_a = 40$ km s^{-1} and $l = 5 \times 10^4$ km, which gives $t_s = 1.2 \times 10^3 \delta\phi^{-1}$ s. The flow time for the solar wind at this distance ($\approx$ 9 AU) is $\approx 3 \times 10^7$ s. It is therefore clear that $\delta\phi$ would have to be ridiculously small for the shock to survive for a significant fraction of a flow time. This is most unlikely, since the flow of the solar wind is sufficiently complex to contain plenty of Alfvén waves for which $\delta\phi \approx 1$, and indeed Chao et al. find plenty of evidence for strong Alfvén waves in the data. Actually, the evidence for an intermediate shock is not really very convincing. The uncertainties are such that it could just as well be a slow shock.

Exactly the same arguments can be applied to MHD shocks in the interstellar medium. Not only does the theory of collisionless shocks (see e.g. Tidman and Krall 1971) predict that, under these conditions, such shocks are extremely thin compared with the scale of the flow, but also there are numerous observations that confirm that this is indeed true (see e.g. Draine and McKee 1993).

6.4. Convexity of MHD

It is quite clear from the above discussion that a hyperbolic system is genuinely non-convex if it allows structurally stable compound waves that only contain evolutionary shocks. Planar MHD is therefore genuinely non-convex, whereas full MHD is convex.

6.5. Non-evolutionary shocks in numerical simulations

The appearance of non-evolutionary shocks in numerical calculations is not something that is unique to MHD, since it is well known that, even in gas dynamics, some numerical schemes can generate expansion shocks in certain circumstances. However, this phenomenon is both more subtle and more interesting in the case of

MHD. The essential point is that, unlike gas dynamics, planar MHD is very different from full MHD in the sense that there are shocks that are non-evolutionary in full MHD but evolutionary in planar MHD, and vice versa. Unfortunately, this property means that the results of planar MHD simulations can be very misleading, because, although most upwind schemes seem to give perfectly good solutions for planar MHD, these are of no relevance to the real universe with its three spatial dimensions. This is not at all unusual – indeed, it may very well be the rule rather than the exception. For example, the properties of fluid turbulence are very different in two and three dimensions, as are those of MHD dynamos.

The other properties of non-evolutionary MHD shocks that are not shared by gas-dynamical expansion shocks are that all of them satisfy the second law of thermodynamics and most of them also possess a steady dissipative structure. This, together with the fact that the ratio of the thickness of numerical shock structures to the overall scale of the flow is almost always many orders of magnitude greater than in the corresponding physical system, means that they can persist for a significant time, even in non-planar problems. For example, if the piston problem discussed by Jeffrey and Taniuti (1964, pp. 256–258) is slightly modified so that it has a small transverse component of the field, then the evolutionary solution contains fast, slow, and Alfvén shocks, all propagating with very similar speeds. In a numerical simulation, this complex would remain unresolved for some time, during which it would be classified as a $1 \rightarrow 4$ shock.

The only truly satisfactory solution to this difficulty is to devise schemes that only allow evolutionary shocks. Figure 8 shows that there are schemes that will do this. Here we have a numerical solution to the Brio and Wu problem obtained with our MHD version of Glimm's scheme (Glimm 1965). This method requires a nonlinear Riemann solver, and we employ the one described in Falle et al. (1998), which specifically excludes intermediate shocks. In fact, we do not use Glimm's scheme everywhere, but only to track the Alfvén shock. One can see that, in this way, we can avoid the appearance of intermediate shocks even in planar problems. Unfortunately, it is not a simple matter to generalize this to more than one dimension.

The only viable option that we can think of is to subject all numerical calculations to a careful analysis using the theory described in this paper. As an example of this, it is instructive look at some recent calculations of steady MHD flow past a cylinder.

6.6. *2D bow-shock simulations*

Recently, De Sterck et al. (1998) have carried out numerical MHD calculations of the flow past an infinite, perfectly conducting cylinder. These are planar simulations, and must therefore be interpreted in the light of the theory of planar MHD. The parameters are chosen in such way that the usual convex bow shock is impossible. Instead, the analysis given in Steinolfson and Hundhausen (1990) suggests that the shock has a dimple. They assumed that there is only a single shock, in which case a consistent solution requires the shock type to change from $1 \rightarrow 2$ to $1 \rightarrow 3$ and then to $1 \rightarrow 4$ as the distance from the symmetry axis decreases. Although the $1 \rightarrow 4$ shock is non-evolutionary even in planar MHD, it seems in this case that such a shock must occur on the symmetry axis for the same reason that it occurs when a piston moves parallel to the magnetic field. However, one would expect it to split into $1 \rightarrow 2$ and $2 \rightarrow 4$ or $1 \rightarrow 3$ and $3 \rightarrow 4$ shocks further away from the the axis. Indeed, De Sterck et al. (1998) find that, not far from the axis, the $1 \rightarrow 4$ shock

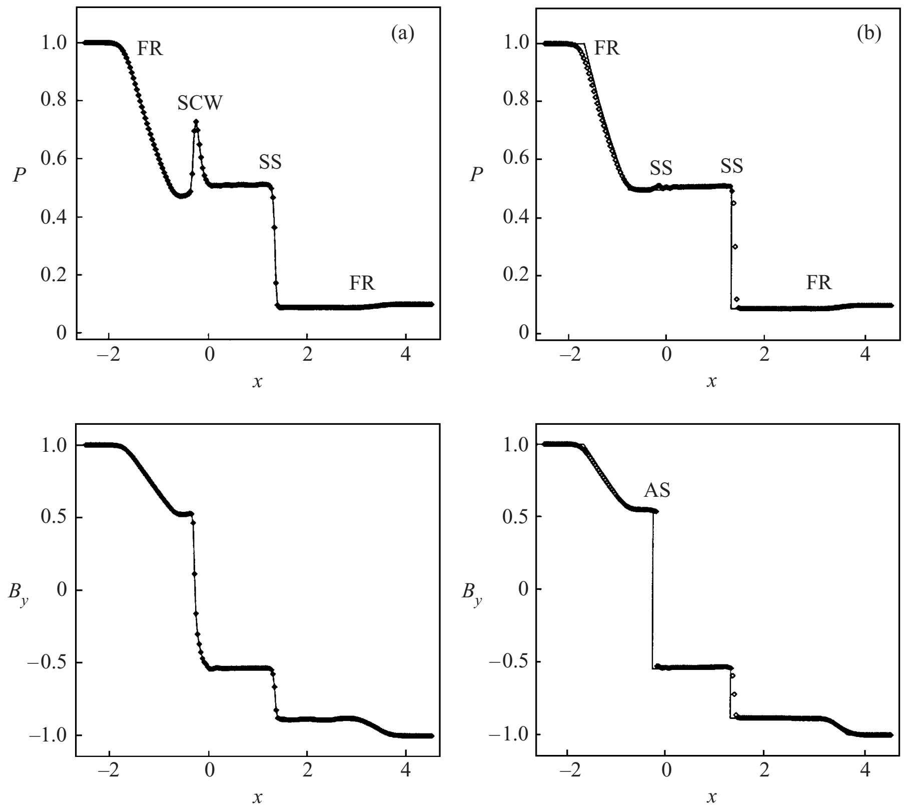

Figure 8. Brio and Wu problem (Brio and Wu 1988). Left panels: Numerical solution found using a Godunov-type scheme. This is a proper solution of the reduced system of planar MHD but is inadmissible in full MHD. Right panels: Numerical solution found using Glimm's scheme to track Alfvén discontinuities (markers) and the exact solution involving only evolutionary shocks (lines). This is a proper solution for full MHD, and is the only physically admissible solution for this problem.

splits and the leading shock (ED in their notation) is a $1 \to 2$. At some distance from this branching point, the other shock (EG) is identified by them as $f \to s$, but this is unlikely to be true everywhere for such an inhomogeneous flow. One would also expect another branching at the point where Steinolfson and Hundhausen (1990) predict a transition from $1 \to 3$ to $1 \to 4$. The results of De Sterck et al. (1998) do, indeed, show this branching (DE and DG), with the trailing shock being clearly identifiable as a $2 \to 4$ shock.

7. Conclusions

Both our analysis and numerical results show that the evolutionary conditions for existence and uniqueness of discontinuous solutions of the equations of ideal MHD are not only compatible with the conditions for existence and uniqueness of steady dissipative shock structures – they are actually complementary to them. The general theory suggests that this will be true for all nonlinear hyperbolic systems that can arise in nature. Non-evolutionary shocks can have a non-unique dissipative structure, and may, perhaps, appear under some exceptional circumstances as transient phenomena. However, they are not persistent, and are bound to split when

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subjected to small perturbations. In the case of MHD, Alfvén waves are the most effective killers, since not only our calculations but also those described by Wu (1988a) show that intermediate MHD shocks are destroyed by interactions with Alfvén waves. It is true that it takes a finite time for this interaction to take place, but, in any physical system of which we know, this time is so short that it is most unlikely that such shocks can be detected.

The occurrence of intermediate MHD shocks in planar numerical simulations is consistent with the mathematical properties of planar MHD, in which $1 \to 3$ and $2 \to 4$ shocks become evolutionary but the Alfvén shock becomes non-evolutionary. However, the planar limit is a singular limit of full MHD, and we suggest that planar numerical simulations should be avoided, especially since they are hardly any cheaper than those for full MHD.

Intermediate shocks may even pollute full MHD simulations, because numerical shock structures are usually not very thin compared with the length scale of the flow. It is therefore essential that the results of such simulations be subjected to a careful analysis in order to make sure that they do not contain any intermediate shocks. If they do, then additional work is required to determine the extent to which they are corrupted. The ideal solution would be to devise an algorithm that does not generate intermediate shocks, but, although we have shown that a variant of Glimm's scheme can do this in one dimension, there is no obvious way to extend it to multidimensional cases.

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