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ON SPECIAL ATOMS

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Abstract

A characterization of all special atoms in the form of the upper radical generated by the class of all prime rings outside the smallest special class containing some prime ring is provided and prime rings for which the above mentioned upper radical coincides with the prime radical are investigated.

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All rings to be considered are associative and it will be assumed that any class of rings contains the one element ring 0 as well as all rings isomorphic to any member of the class. The fundamental definitions and properties of radicals of associative rings can be found in [1] and [3].

Atoms of the lattice of all special radicals will be called *special atoms*. For a ring R write $I \lhd R$ to mean I is an ideal of R and let $\overline{R}$ always denote some homomorphic image of R. For a class $\mathscr{C}$ of rings as usual let

$$\mathscr{U}(\mathscr{C}) = \left\{R : \text{ every } 0 \neq \overline{R} \notin \mathscr{C}\right\},$$
$$\mathscr{S}(\mathscr{C}) = \{R : \text{for every } 0 \neq I \lhd R \text{ we have } I \notin \mathscr{C}\},$$
$$\mathscr{C}_k = \{R : R \text{ is an essential extension of some } I \in \mathscr{C}\}.$$

The prime radical will be denoted by β and the smallest special radical containing a ring A will be denoted by $\widehat{l_A}$. The class of all prime rings will be denoted by π and the smallest special class containing a prime ring A will be denoted by π_A.

A semiprime ring A is called a *-ring if the factor ring $A/I \in \beta$ for every $0 \neq I \lhd A$. The class of all *-rings will be denoted by *.

It is known [4] that * is a hereditary class consisting of prime rings and for any non-zero *-ring R we have $0 \neq \overline{R} \in \pi$ implies $\overline{R} \simeq R$ so that $\overline{R} \in \pi_R$.

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A prime ring R such that $\overline{R} \in \pi_R$ for every $\overline{R} \in \pi$ is called a *ring with* PEI.

Using methods of [5], it was shown in [6] that for every non-zero ring A with PEI the radical $\mathscr{U}(\pi \backslash \pi_A)$ is a special atom and $\mathscr{U}(\pi \backslash \pi_A) = \widehat{l_A}$. We will now give a characterization of all special atoms of the form $\mathscr{U}(\pi \backslash \pi_A)$ where A is a non-zero prime ring. To do so we will need the following

LEMMA 1. *For any non-zero prime ring A and any special class $\mathscr{C}$, if $\pi \backslash \pi_A \subsetneqq \mathscr{C}$ then $\mathscr{C} = \pi$.*

PROOF. Suppose a non-zero prime ring $B \notin \mathscr{C}$. Then $B \in \pi \backslash \mathscr{C}$ and (as $\pi \backslash \mathscr{C}$ is a special class) we have $\pi_B \subseteq \pi \backslash \mathscr{C}$. Hence $\mathscr{C} \subseteq \pi \backslash \pi_B$. On the other hand, (as $B \notin \mathscr{C}$ and $\pi \backslash \pi_A \subsetneqq \mathscr{C}$) it follows that $B \notin \pi \backslash \pi_A$. Therefore (as $B \in \pi$) we have $B \in \pi_A$ which by [5, Proposition 2] implies that $\pi_B = \pi_A$. Consequently we have $\pi \backslash \pi_A \subsetneqq \mathscr{C} \subseteq \pi \backslash \pi_B = \pi \backslash \pi_A$, a contradiction. Thus $\pi \subseteq \mathscr{C}$ and (as $\mathscr{C} \subseteq \pi$) it follows that $\pi = \mathscr{C}$.

We say that a class $\mathscr{C} \subseteq \pi$ is *closed under subdirect sums* if any prime ring which is a subdirect sum of rings from $\mathscr{C}$ belongs to $\mathscr{C}$. It is clear that a special class $\mathscr{C}$ is closed under subdirect sums if and only if $\mathscr{C} = \mathscr{S}(\mathscr{U}(\mathscr{C})) \cap \pi$.

THEOREM 2. *For a non-zero prime ring A the following conditions are equivalent.*

(1) *$\mathscr{U}(\pi \backslash \pi_A)$ is a special atom;*
(2) *$\beta \subsetneqq \mathscr{U}(\pi \backslash \pi_A)$;*
(3) *$\pi \backslash \pi_A$ is closed under subdirect sums;*
(4) *A is an essential extension of a ring I with* PEI.

PROOF. (1) if and only if (2): Clearly if $\mathscr{U}(\pi \backslash \pi_A)$ is a special atom then $\beta \subsetneqq \mathscr{U}(\pi \backslash \pi_A)$.

Suppose that $\beta \subsetneqq \mathscr{U}(\pi \backslash \pi_A)$ and let α be a special radical such that $\beta \subsetneqq \alpha \subseteqq \mathscr{U}(\pi \backslash \pi_A)$. Then $\pi \backslash \pi_A \subseteqq \mathscr{S}(\alpha) \cap \pi \subsetneqq \pi$ and by Lemma 1 it follows that $\pi \backslash \pi_A = \mathscr{S}(\alpha) \cap \pi$. Hence (as α is a special radical) we have $\alpha = \mathscr{U}(\mathscr{S}(\alpha) \cap \pi) = \mathscr{U}(\pi \backslash \pi_A)$. Thus $\mathscr{U}(\pi \backslash \pi_A)$ is a special atom.

(2) if and only if (3): Suppose $\beta \subsetneqq \mathscr{U}(\pi \backslash \pi_A)$. Since $\pi \backslash \pi_A \subseteq \mathscr{S}(\mathscr{U}(\pi \backslash \pi_A)) \cap \pi \subseteq \pi$, it follows from Lemma 1 that either $\mathscr{S}(\mathscr{U}(\pi \backslash \pi_A)) \cap \pi = \pi$ or $\pi \backslash \pi_A = \mathscr{S}(\mathscr{U}(\pi \backslash \pi_A)) \cap \pi$. The first case is impossible since $\mathscr{U}(\pi \backslash \pi_A)$ is a special radical strictly containing β. Therefore $\pi \backslash \pi_A = \mathscr{S}(\mathscr{U}(\pi \backslash \pi_A)) \cap \pi$ which means that $\pi \backslash \pi_A$ is closed under subdirect sums.

Conversely, let $\pi \backslash \pi_A$ be closed under subdirect sums and suppose that $\beta = \mathscr{U}(\pi \backslash \pi_A)$. Then we would have $\pi \backslash \pi_A = \mathscr{S}(\mathscr{U}(\pi \backslash \pi_A)) \cap \pi = \mathscr{S}(\beta) \cap \pi = \pi$ which is impossible since $0 \neq A \in \pi_A = \pi \backslash (\pi \backslash \pi_A)$.

(1) if and only if (4): Suppose $\mathscr{U}(\pi\backslash\pi_A)$ is a special atom. If $A \in \mathscr{S}(\mathscr{U}(\pi\backslash\pi_A))$ then A is a subdirect sum of rings from $\pi\backslash\pi_A$ and (as A is prime) it follows from part 3 that $A \in \pi\backslash\pi_A$. This however is impossible since $0 \neq A \in \pi_A$. Thus A contains a non-zero ideal $I \in \mathscr{U}(\pi\backslash\pi_A)$ and (as $A \in \pi$) A is an essential extension of I. We will show that I is a ring with PEI. Since $0 \neq I \lhd A$ and $A \in \pi$ it follows from [5, Proposition 2] that $\pi_I = \pi_A$. Then $\mathscr{U}(\pi\backslash\pi_A) = \mathscr{U}(\pi\backslash\pi_I)$ and (as $I \in \mathscr{U}(\pi\backslash\pi_A)$) we have $I \in \mathscr{U}(\pi\backslash\pi_I)$ which implies that every prime homomorphic image of I belongs to π_I. Thus I is with PEI.

Conversely, let A be an essential extension of some ring I with PEI. Then (as $A \in \pi$) we have $0 \neq I \lhd A$ and $I \in \mathscr{U}(\pi\backslash\pi_I) \cap \pi$. But by [5, Proposition 2] it then follows that $\pi_A = \pi_I$ which implies $\mathscr{U}(\pi\backslash\pi_A) = \mathscr{U}(\pi\backslash\pi_I)$. But (as $0 \neq I \in \mathscr{U}(\pi\backslash\pi_I) \cap \pi$) it follows that $\beta \subsetneqq \mathscr{U}(\pi\backslash\pi_I)$. Hence $\beta \subsetneqq \mathscr{U}(\pi\backslash\pi_A)$ which by part 2 implies that $\mathscr{U}(\pi\backslash\pi_A)$ is a special atom.

COROLLARY 3. *A special atom α is of the form $\mathscr{U}(\pi\backslash\pi_A)$ for some $0 \neq A \in \pi$ if and only if $\alpha = \widehat{l_I}$ for some non-zero ring I with* PEI.

PROOF. Clearly, if $\alpha = \widehat{l_I}$ for some non-zero ring I with PEI then by [6, Theorem 4] $\alpha = \mathscr{U}(\pi\backslash\pi_I)$ and we are done.

Conversely, suppose a special atom $\alpha = \mathscr{U}(\pi\backslash\pi_A)$ for some $0 \neq A \in \pi$. Then by Theorem 2 it follows that there exists a non-zero ring I with PEI such that $0 \neq I \lhd A$. Then by [5, Proposition 2] we have $\pi_A = \pi_I$. Thus $\mathscr{U}(\pi\backslash\pi_A) = \mathscr{U}(\pi\backslash\pi_I)$. But (as I is with PEI) we have $I \in \mathscr{U}(\pi\backslash\pi_I)$. Thus $\beta \subsetneqq \widehat{l_I} \subseteqq \mathscr{U}(\pi\backslash\pi_I) = \mathscr{U}(\pi\backslash\pi_A)$ which implies that $\widehat{l_I} = \mathscr{U}(\pi\backslash\pi_A)$ since $\mathscr{U}(\pi\backslash\pi_A)$ is a special atom and $\widehat{l_I}$ is a special radical.

It is now natural to ask the following question:

Does there exist a prime ring A such that $\beta = \mathscr{U}(\pi\backslash\pi_A)$ and yet $\widehat{l_A}$ is a special atom?

A negative answer to this question would mean that special atoms are precisely the special radicals generated by individual non-zero rings with PEI.

Although we are unable to construct a prime ring satisfying both conditions specified in the above given question, we provide a plentiful supply of non-zero rings A such that $\beta = \mathscr{U}(\pi\backslash\pi_A)$.

THEOREM 4. *For a prime ring A, $\beta = \mathscr{U}(\pi\backslash\pi_A)$ if and only if A is a subdirect sum of rings from $\pi\backslash\pi_A$.*

PROOF. If $\beta = \mathscr{U}(\pi\backslash\pi_A)$ for some $A \in \pi$ then A is a subdirect sum of rings from $\pi\backslash\pi_A$ because $A \in \pi \subseteqq \mathscr{S}(\beta) = \mathscr{S}(\mathscr{U}(\pi\backslash\pi_A))$ and $\mathscr{U}(\pi\backslash\pi_A)$ (as a special radical) has the intersection property with respect to the special class $\pi\backslash\pi_A$.

Conversely, let A be a prime ring such that A is a subdirect sum of rings from $\pi\backslash\pi_A$. Then (as $\pi\backslash\pi_A \subseteqq \mathscr{S}(\mathscr{U}(\pi\backslash\pi_A))$) it follows that $A \in \mathscr{S}(\mathscr{U}(\pi\backslash\pi_A)) \cap \pi$. Since $\mathscr{S}(\mathscr{U}(\pi\backslash\pi_A)) \cap \pi$ is a special class we thus have $\pi_A \subseteqq \mathscr{S}(\mathscr{U}(\pi\backslash\pi_A)) \cap \pi$. On the other hand, $\pi\backslash\pi_A \subseteqq \mathscr{S}(\mathscr{U}(\pi\backslash\pi_A)) \cap \pi$. Thus $\pi = \pi_A \cup (\pi\backslash\pi_A) \subseteqq \mathscr{S}(\mathscr{U}(\pi\backslash\pi_A)) \cap \pi$ which implies that $\mathscr{S}(\mathscr{U}(\pi\backslash\pi_A)) \cap \pi = \pi$. Consequently $\mathscr{U}(\pi\backslash\pi_A) = \mathscr{U}(\mathscr{S}(\mathscr{U}(\pi\backslash\pi_A)) \cap \pi) = \mathscr{U}(\pi) = \beta$.

COROLLARY 5. *If Z is the ring of all integers then $\beta = \mathscr{U}(\pi\backslash\pi_Z)$.*

PROOF. It is clear that Z is a subdirect sum of rings isomorphic to Z_p, p prime and each Z_p (as a field) is a simple prime ring with identity. If $Z_p \in \pi_Z$ for some prime p, then by [1, Proposition 5, p. 239] Z_p would contain a non-zero ideal I which is isomorphic to a non-zero accessible subring J of Z. But (as Z_p is simple) it follows that $I = Z_p$. Thus J is isomorphic to Z_p which implies that J is a finite ideal of Z, a contradiction. Thus $Z_p \in \pi\backslash\pi_z$ for every prime p and by Theorem 4 it follows that $\beta = \mathscr{U}(\pi\backslash\pi_Z)$.

Beidar [2] constructed a non-zero prime ring A such that $\pi_A \cap \pi_{\overline{A}} = 0$ and $\mathscr{U}(\pi_A) = \mathscr{U}(\pi_{\overline{A}})$ for some $0 \neq \overline{A} \in \pi$.

COROLLARY 6. *Let A be a prime ring such that $\pi_A \cap \pi_{\overline{A}} = 0$ and $\mathscr{U}(\pi_A) = \mathscr{U}(\pi_{\overline{A}})$ for some $0 \neq \overline{A} \in \pi$. Then $\beta = \mathscr{U}(\pi\backslash\pi_A)$.*

PROOF. Since $A \in \pi_A \subseteqq \mathscr{S}(\mathscr{U}(\pi_A)) \cap \pi \subseteqq \mathscr{S}(\mathscr{U}(\pi_{\overline{A}}))$ and $\mathscr{U}(\pi_{\overline{A}})$ (as a special radical) has the intersection property with respect to the special class $\pi_{\overline{A}}$, it follows that A is a subdirect sum of rings from $\pi_{\overline{A}}$. Moreover, since $\pi_A \cap \pi_{\overline{A}} = 0$ we must have $\pi_{\overline{A}} \subseteqq \pi\backslash\pi_A$. Thus A is a subdirect sum of rings from $\pi\backslash\pi_A$ which, in view of Theorem 4, means that $\beta = \mathscr{U}(\pi\backslash\pi_A)$.

We will now show how to construct prime rings A such that $\beta = \mathscr{U}(\pi\backslash\pi_A)$.

COROLLARY 7. *Let R be a prime ring with identity $1 \neq 0$ and with an infinite centre. Let X be an infinite set of cardinality $|X| > 2^{|R|}$ and let $R[X]$ be the ring of all polynomials in commutative indeterminates X over R. Then $\beta = \mathscr{U}\left(\pi\backslash\pi_{R[X]}\right)$.*

PROOF. Since R is a prime ring with identity $1 \neq 0$ and with an infinite centre it follows from [1, Lemma 5, p. 243] that $R[X] \in \pi$ and $R[X]$ is a subdirect sum of rings isomorphic to R. If $R \notin \pi\backslash\pi_{R[X]}$ then (as $R \in \pi$) we would have $R \in \pi_{R[X]}$.This in view of [1, Proposition 5, p. 239] implies that R has a non-zero ideal I which is isomorphic to an accessible subring of $R[X]$. Then it follows from [1, Lemma 3, p. 240] that $|R[X]| \leqslant 2^{|I|}$. But clearly $|I| \leqslant |R|$. Thus $|R[X]| \leqslant 2^{|R|}$. On the other hand, we have $|R[X]| \geqslant |X| > 2^{|R|}$, a contradiction. Thus $R \in \pi \backslash \pi_{R[X]}$ which, in view of Theorem 4, implies that $\beta = \mathcal{U}\left(\pi \backslash \pi_{R[X]}\right)$.

In [4, Theorem 6] rings of which some ideal can be homomorphically mapped onto a non-zero *-ring were discussed. For example, a prime ring with a non-zero centre is such a ring.

THEOREM 8. *If A is a prime ring of which some ideal I can be homomorphically mapped onto a non-zero *-ring $\overline{I}$ and such that $\beta = \mathcal{U}(\pi \backslash \pi_A)$ then $\widehat{l_A}$ is not a special atom.*

PROOF. Clearly, I is a non-zero ideal of A. So $A \neq 0$. Since $\widehat{l_A}$ (as a special radical) is hereditary and since $I \lhd A \in \widehat{l_A}$ it follows that $I \in \widehat{l_A}$ and consequently $\overline{I} \in \widehat{l_A}$. Thus $\widehat{l_{\overline{I}}} \leqslant \widehat{l_A}$. But since $\overline{I}$ (as a *-ring) is a ring with PEI it follows that $\widehat{l_{\overline{I}}} = \mathcal{U}(\pi \backslash \pi_{\overline{I}})$ is a special atom and so $\beta \subsetneqq \widehat{l_{\overline{I}}}$. Now, if $A \in \pi_{\overline{I}}$ then by [5, Proposition 2] it follows that $\pi_A = \pi_{\overline{I}}$ which implies that $\mathcal{U}(\pi \backslash \pi_{\overline{I}}) = \mathcal{U}(\pi \backslash \pi_A)$. This however, is impossible because $\beta = \mathcal{U}(\pi \backslash \pi_A)$ and $\beta \subsetneqq \mathcal{U}(\pi \backslash \pi_{\overline{I}})$. Therefore $A \in (\pi \backslash \pi_{\overline{I}}) \cap \widehat{l_A} \subseteqq \mathcal{S}\left(\widehat{l_{\overline{I}}}\right) \cap \widehat{l_A}$. Hence $\beta \subsetneqq \widehat{l_{\overline{I}}} \subsetneqq \widehat{l_A}$ which (as $\widehat{l_{\overline{I}}}$ is a special radical) implies that $\widehat{l_A}$ is not a special atom.

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