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UNIMODULARITY UNIFIED

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Abstract. Unimodularity is localized to a complete stationary type, and its properties are analysed. Some variants of unimodularity for definable and type-definable sets are introduced, and the relationship between these different notions is studied. In particular, it is shown that all notions coincide for non-multidimensional theories where the dimensions are associated to strongly minimal types.

§1. Introduction. Unimodularity was defined by Hrushovski in [4] where he proved that a unimodular strongly minimal set is one-based, thus generalising Zilber's result that a locally finite strongly minimal set is 1-based. Recently, Hrushovski has re-visited unimodularity in the context of pseudofinite structures, aiming to develop an intersection theory for definable pseudofinite sets.

It was claimed in [4] that unimodularity was equivalent to an *a priori* weaker notion called *functional unimodularity* in [1] and [3]. This was then used by Elwes as part of a proof that measurable stable structures are 1-based [1, Lemma 6.4] and was repeated in [6] and the survey article [2]. In an attempt to clarify the situation, Pillay and Kestner [5] have distinguished two types of functional unimodularity: one for definable sets and one for type-definable sets. They also studied the relationships between various notions and definitions, mainly in the context of strongly minimal structures. In particular, they showed that for strongly minimal theories, unimodularity is equivalent to functional unimodularity for arbitrary types, and is also equivalent to the structures being measurable in the sense of [7]. They also presented an example intended to be a strongly minimal set which is functionally unimodular but not unimodular. However, the example actually turns out not to be functionally unimodular; in fact our Theorem 4.14 states that all variants of unimodularity coincide for non-multidimensional theories where the dimensions are associated to strongly minimal types.

This paper can be seen as yet another attempt to clarify the situation, and is organized as follows: In Section 2 we introduce the notion of a uniform correspondence, measurability of a (partial) type, and commensurability between (partial) types, and develop the basic properties. In Section 3 we introduce the concept of correspondence unimodularity and functional unimodularity for complete types, partial

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types, and definable sets, and give a correction to Proposition 3.2 in [5]. The main result in this section is Theorem 3.11, which states that unimodularity is equivalent to both correspondence unimodularity and to functional unimodularity for complete types, and Theorem 3.12, which says that in an ω-stable theory unimodularity is equivalent to both correspondence and functional unimodularity for partial types.

In Section 4 we localize unimodularity to complete stationary types and finally show that all concepts coincide for non-multidimensional theories where the dimensions are associated to strongly minimal types, and in particular for $\aleph_1$-categorical theories and groups of finite Morley rank.

We use standard model-theoretic notation and work in some big sufficiently saturated and ultrahomogeneous monster model of the theory. Lower case letters a, b, c, etc. will denote finite tuples. If a tuple a is algebraic over b, we use $m(a/b)$ for the (finite) number of realizations of $\mathrm{tp}(a/b)$. We shall not distinguish between singletons and tuples or between real and imaginary elements (i.e., we work in T^{eq}).

§2. Correspondences.

DEFINITION 2.1 (Correspondence). Let π and π' be two type-definable sets.

(1) A *correspondence* between π and π' is a nonempty type-definable set $C(x, y) \vdash \pi(x) \times \pi'(y)$ such that all fibres $C_x = \{y \models \pi' : C(x, y)\}$ and $C^y = \{x \models \pi : C(x, y)\}$ are finite. If $\pi' = \pi$ we call C a correspondence *on* π.
(2) A correspondence C is *complete* if it is a complete type.
(3) A correspondence C is *uniform* if the fibre sizes $k_C = |C_x|$ and $\ell_C = |C^y|$ are constant, independently of $x \models \pi$ and $y \models \pi'$.
(4) A (k, ℓ)-*correspondence* is a uniform correspondence with $k = k_C$ and $\ell = \ell_C$.
(5) For a uniform correspondence C, the *ratio* of C is $m_C = \frac{k_C}{\ell_C}$.
(6) A correspondence C is *balanced* if it is uniform and $k_C = \ell_C$ (equivalently, $m_C = 1$).

If π, π' and C are all type-definable over some parameters A, we say that C is *over* A.

Note that a uniform correspondence is actually relatively definable, by compactness. If $C(x, y)$ is a correspondence between $\pi(x)$ and $\pi'(y)$, then $C^{-1}(y, x) = C(x, y)$ is a correspondence between $\pi'(y)$ and $\pi(x)$. Clearly, $(C^{-1})_y = C^y$ and $(C^{-1})^x = C_x$. So C^{-1} is uniform/complete/balanced if and only if C is.

Correspondences between complete types are particularly well behaved.

LEMMA 2.2. *Let C be a correspondence between a complete type p and some partial type $\pi(y)$, all over the same parameters A. Then,*

(1) *$|C_x|$ does not depend on $x \models p$.*
(2) *C can be written as the disjoint union of finitely many complete correspondences $C = C_0 \dot\cup \cdots \dot\cup C_n$, with $n \leq |C_x|$.*

PROOF. (1) If $a, a' \models p$, then there is an automorphism σ fixing A with $\sigma(a) = a'$. Then $C_{a'} = \sigma(C_a)$, so $|C_{a'}| = |C_a|$.

(2) If $\mathrm{tp}(a_i, b_i/A)$ for $i \in I$ are the completions of C, then $a_i \models p$ and we may assume $a_i = a_0$ for all $i \in I$. But then $b_i \in C_{a_0}$; since the types $\mathrm{tp}(a_0, b_i/A)$ are all different, we have $b_i \neq b_j$ for $i \neq j$, and $|I| \leq |C_{a_0}|$. It follows that $C = \dot\bigcup_{i \in I} C_i$ with $C_i = \mathrm{tp}(a_i, b_i/A)$. ⊣

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Corollary 2.3. *A correspondence C between complete types is automatically uniform, and if all its completions have the same ratio m, then $m_C = m$.*

Proof. Suppose $C(x, y)$ is a correspondence between complete types $p(x)$ and $q(y)$. Then $|C_x| = k_C$ and $|C^y| = |(C^{-1})_y| = \ell_C$ are constant for $x \models p$ and $y \models q$ by Lemma 2.2, hence the correspondence is uniform. If $C_0, \dots, C_n$ are the completions of $C(x, y)$, then

$$k = |C_x| = \left| \dot{\bigcup_{i=0}^{n}} (C_i)_x \right| = \sum_{i=0}^{n} k_{C_i} \text{ and } \ell = |C^y| = \left| \dot{\bigcup_{i=0}^{n}} (C_i)^y \right| = \sum_{i=0}^{n} \ell_{C_i}. \qquad (1)$$

If all the completions C_i have the same ratio m, then $k_{C_i} = m\ell_{C_i}$ for all i, whence $k = m\ell$ and $m_C = m$. ⊣

Definition 2.4 (Measurable, Commensurable). Let π be a partial type over A. We say that π is *measurable over A* if every A-type-definable uniform correspondence C on π is balanced.

Two partial types π and π' over A are *commensurable over A* if there is a uniform correspondence C from π to π', and for any other uniform correspondence C' over A between π and π' one has $m_{C'} = m_C$. In this case we put $m_\pi^{\pi'} = m_C$. If π is measurable over any $B \supseteq A$, we say that π is measurable; if π and π' are commensurable over any $B \supseteq A$ we say that they are commensurable.

Thus π is measurable (over A) if and only if π and π are commensurable (over A). It follows from Corollary 2.3 that for complete types we may restrict ourselves to complete correspondences in Definition 2.4.

If $B \supseteq A$ and π and π' are commensurable over B, and if there is a correspondence between π and π' over A, then π and π' are commensurable over A. However, commensurability or measurability over A need not imply commensurability or measurability over B.

Lemma 2.5. *Two complete types p and q are commensurable over A if and only if there is a complete correspondence C over A between p and q, and all such complete correspondences take the same value $m_C = m_p^q$.*

Proof. The left to right direction follows directly from the definitions. Conversely, let $C_0, \dots, C_n$ be the completions of C. By (1),

$$k_C = \sum_{i=0}^{n} k_{C_i} = \sum_{i=0}^{n} m_p^q \cdot \ell_{C_i} = m_p^q \ell_C.$$

This yields the result. ⊣

We shall now study composition of correspondences.

Definition 2.6 (Composition). Let π, π', and π'' be partial types over A, and suppose C, C' are correspondences between π and π' and between π' and π'', respectively. The composition $C' \circ C$ is defined by

$$(a, c) \in C' \circ C \Leftrightarrow \exists b \, [(a, b) \in C \wedge (b, c) \in C'].$$

By compactness and saturation, $C' \circ C$ is type-definable; note that any witness b for the existential quantifier must automatically satisfy π'. It is clear that $(C' \circ C)_a$ and $(C' \circ C)^c$ are finite for every $a \models \pi$ and $c \models \pi''$, so $C' \circ C$ is a correspondence between π and π''.

If π and π'' are complete types over A, then $C' \circ C$ can be written as a finite union $D_0 \cup \cdots \cup D_n$ of complete correspondences between π and π'' by Lemma 2.2, each of which is uniform by Corollary 2.3. If moreover C and C' are both uniform (for instance if π' is also complete), given $(a, c) \in D_i$ define

$$r_i = |\{b \models \pi' : (a, b) \in C \text{ and } (b, c) \in C'\}|.$$

Since D_i is complete, this number only depends on D_i and not on the choice of $(a, c) \models D_i$. Then for $a \models \pi$

$$\begin{aligned} k_C \cdot k_{C'} &= |\{(b, c) : (a, b) \in C \wedge (b, c) \in C'\}| \\ &= |\bigcup_{i \leq n} \{(b, c) : (a, c) \in D_i \wedge (a, b) \in C \wedge (b, c) \in C'\}| = \sum_{i=0}^{n} r_i \cdot k_{D_i}. \end{aligned} \tag{2}$$

Similarly $\ell_C \cdot \ell_{C'} = \sum_{i=0}^{n} r_i \cdot \ell_{D_i}$.

Proposition 2.7. *Let p, q, and r be complete types, and suppose C is a correspondence between p and q and C' is a correspondence between q and r, all over A. If p and r are commensurable over A, then $m_{C' \circ C} = m_C \cdot m_{C'}$.*

Proof. By Lemma 2.2 the correspondences C, C', and $C' \circ C$ are all uniform; let $(D_i : i \leq n)$ be the finitely many completions of $C' \circ C$. Since p and r are commensurable over A, we have that $m_{D_i} = m_p^r$ for every $i \leq n$. By (2) we obtain

$$k_C \cdot k_{C'} = \sum_{i=1}^{n} r_i \cdot k_{D_i} = \sum_{i=1}^{n} r_i \cdot (m_p^r \cdot \ell_{D_i}) = m_p^r \sum_{i=1}^{n} r_i \cdot \ell_{D_i} = m_p^r \cdot (\ell_C \cdot \ell_{C'}),$$

whence

$$m_{C' \circ C} = m_p^r = \frac{k_C \cdot k_{C'}}{\ell_C \cdot \ell_{C'}} = m_C \cdot m_{C'}. \qquad \dashv$$

Corollary 2.8. *Let p and q be complete types over A.*

(1) *Suppose there is a correspondence C between p and q. If p is measurable over A, then so is q, and p and q are commensurable over A.*
(2) *If p and q are commensurable over A, then p and q are both measurable over A.*
(3) *For any three complete commensurable types p, q, and r over A we have $m_p^q m_q^r = m_p^r$.*

Proof. (1) If C' is any other correspondence between p and q over A, then $C'^{-1}(y, x) = C'(x, y)$ is a correspondence from q to p. Clearly $m_{C'^{-1}} = m_{C'}^{-1}$. By Lemma 2.7 we have

$$1 = m_p^p = m_C \cdot m_{C'^{-1}} = m_C / m_{C'},$$

so $m_{C'} = m_C = m_p^q$. Hence p and q are commensurable over A.

(2) Suppose that p and q are commensurable over A. If C is a complete correspondence on p over A, then $m_C m_p^q = m_p^q$ by Proposition 2.7, and $m_C = 1$. Thus p is measurable over A; measurability of q over A follows by symmetry.

(3) This follows immediately from Proposition 2.7. $\dashv$

Theorem 2.9. *Let π be a partial type over A and suppose $MR(\pi) < \infty$. If all completions of π over A of maximal Morley rank are measurable over A, so is π.*

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PROOF. Suppose C is a (k_C, ℓ_C)-correspondence on π over A. Let $(p_i : i \in I)$ be the finitely many completions of π over A of maximal Morley rank. Then for all $i, j \in I$, if $C_{ij} = C \cap (p_i \times p_j)$ is nonempty, it is a correspondence between p_i and p_j, so the two types are commensurable by Corollary 2.8. If $C_{ij} = \emptyset$ put $k_{C_{ij}} = \ell_{C_{ij}} = 0$. Put

$$I_0 = \{i \in I : p_1 \text{ and } p_i \text{ are commensurable over } A\}.$$

If $(a, b) \in C$ with $a \models p_i$ for some $i \in I_0$, then by interalgebraicity

$$RM(b/A) = RM(ab/A) = RM(a/A),$$

so $b \models p_j$ for some $j \in I_0$. It follows that for each $i \in I_0$

$$\sum_{j \in I_0} k_{C_{ij}} = k_C \qquad \text{and} \qquad \sum_{j \in I_0} \ell_{C_{ji}} = \ell_C.$$

For $i \in I_0$ put $m_i = m_{p_1}^{p_i}$. If $C_{ij} \neq \emptyset$ we have $m_j = m_i \cdot m_{C_{ij}}$ by Corollary 2.8, that is

$$m_j \cdot \ell_{C_{ij}} = m_i \cdot k_{C_{ij}}.$$

Note that the latter equation trivially holds if $C_{ij} = \emptyset$.

Put $\mu = \sum_{i \in I_0} m_i$. Then $\mu \neq 0$ and

$$\begin{aligned} \mu \cdot k_C &= \sum_{i \in I_0} (m_i \cdot k_C) = \sum_{i \in I_0} \Big(m_i \sum_{j \in I_0} k_{C_{ij}}\Big) = \sum_{i \in I_0} \sum_{j \in I_0} (m_i \cdot k_{C_{ij}}) \\ &= \sum_{i \in I_0} \sum_{j \in I_0} (m_j \cdot \ell_{C_{ij}}) = \sum_{j \in I_0} \sum_{i \in I_0} (m_j \cdot \ell_{C_{ij}}) = \sum_{j \in I_0} \Big(m_j \sum_{i \in I_0} \ell_{C_{ij}}\Big) \\ &= \sum_{j \in I_0} (m_j \cdot \ell_C) = \mu \cdot \ell_C. \end{aligned}$$

It follows that $k_C = \ell_C$. ⊣

EXAMPLE 2.10. Let $M = \mathbb{Z} \times 2^\omega$ in the language $\{f, E_n : n \in \omega\}$, where the E_n are equivalence relations with 2^n classes given by

$$(z, \eta)\, E_n\, (z', \eta') \Leftrightarrow z \equiv z' \mod 2^n$$

and

$$f(z, \eta) = (z + 1, \eta \circ S),$$

where S is the successor function on ω. Then E_{n+1} cuts each E_n-class in half, and $f : M \to M$ is a surjective function with fibres of size two. Moreover, $x\, E_n\, y \Leftrightarrow f(x)\, E_n\, f(y)$, and for any $m \in M$ the 2^n elements $m, f(m), f^2(m), \dots, f^{2^n-1}(m)$ are in different E_n-classes. This theory is complete of Lascar rank one, but not ω-stable. Every stationary complete type is measurable, but the model itself (equivalently, the partial type $x = x$) is not. So ω-stability is necessary in Theorem 2.9.

§3. Unimodularity and its variations. We shall now study the relationship between unimodularity introduced in [4], functional unimodularity and its variants formally introduced in [5], and *correspondence unimodularity* for definable sets, complete types or types. We start with some definitions.

DEFINITION 3.1 (Unimodularity). A complete theory is *unimodular* if for any two tuples a, b, and parameters A in the monster model, if $a \equiv_A b$ and a and b are interalgebraic over A, then $m(a/Ab) = m(b/Aa)$.

LEMMA 3.2. *A theory is unimodular if and only if every complete type is measurable over its domain.*

PROOF. Let $p(x)$ be a complete type. Note that two realizations $a, b \models p$ are A-interalgebraic if and only if $C = \text{tp}(a, b/A)$ is a complete correspondence on p over A. Then $m(b/Aa) = k_C$ and $m(a/Ab) = \ell_C$. So $m(a/Ab) = m(b/Aa)$ if and only if C is balanced. By Corollary 2.3, any correspondence on p is balanced if and only if all complete correspondences on p are balanced. Thus shows the equivalence. ⊣

DEFINITION 3.3 (Functional unimodularity). Let T be a complete theory. Then T is

(1) *functionally unimodular* (FU) if for any two definable sets X and Y we have the following:
 (*) If two definable functions $f, g : X \to Y$ have constant fibre sizes k and ℓ, respectively, then $k = \ell$;
(2) *functionally unimodular for types* (FU-t) if property (*) holds for any type-definable sets X, Y;
(3) *functionally unimodular for complete types* (FU-ct) if property (*) holds for any complete types X, Y.

Kestner and Pillay [5] proved that if T is strongly minimal, then unimodularity is equivalent to functional unimodularity for types, and in this case it is also equivalent to *MS-measurability*. We shall now show that functional unimodularity allows finitely many exceptional finite fibres.

PROPOSITION 3.4. *Let X and Y be two infinite definable sets, and $f, g : X \to Y$ two definable functions with finite fibres, such that $|f^{-1}(y)| = k$ and $|g^{-1}(y)| = \ell$ for all but finitely many $y \in Y$. If $k \neq \ell$, there are definable sets X' and Y', as well as definable functions $f', g' : X' \to Y'$ such that the fibres of f' and g' have constant sizes k and ℓ, respectively.*

PROOF. Put

$$Y_0 = \{y \in Y : |f^{-1}(y)| \neq k \text{ or } |g^{-1}(y)| \neq \ell\}.$$

Let $F = f^{-1}(Y_0)$ and $G = g^{-1}(Y_0)$. Without loss of generality we may assume that $|F| \leq |G|$; modifying f definably on finitely many points we may further assume $F \subseteq G$. Put

$$X'' = X \setminus F, \quad Y'' = Y \setminus Y_0, \quad G' = G \setminus F, \quad \text{and} \quad f'' = f\restriction_{X''} : X'' \to Y''.$$

Then f'' has constant fibre size k, and

$$g\restriction_{X'' \setminus G'} : X'' \setminus G' \to Y'$$

has constant fibre size ℓ. Put $n = |G'|$.

CASE 1. $k < \ell$. Let $n' = \ell - k$. Let P be a set of cardinality kn and Q a set of cardinality n. Put

$$X' = (X \times n') \cup P, \quad Y' = (Y'' \times n') \cup Q,$$

and define $f' : X' \to Y'$ via $f'((y,i)) = (f''(y), i)$ for $(y,i) \in X'' \times n'$, and $f' : P \to Q$ arbitrarily with fibres of constant size k. Finally, define $g' : X' \to Y'$ via $g'((y,i)) = (g(y), i)$ for $(y,i) \in (X'' \setminus G') \times n'$, and $g' : (G' \times n') \cup P \to Q$ arbitrarily with fibres of constant size ℓ, which is possible since

$$|(G' \times n') \cup P| = nn' + kn = n(\ell - k + k) = \ell n = \ell\,|Q|.$$

Case 2. $\ell < k$. Let $n' = k - \ell - 1$. Let $Q \subset Y''$ have cardinality n, and put $P = f''^{-1}(Q) \subset X''$, of cardinality kn. We choose Q such that $P \cap G' = \emptyset$. Put

$$X' = (X'' \times n') \cup ((X'' \setminus P) \times \{n'\}), \quad Y' = (Y'' \times n') \cup ((Y'' \setminus Q) \times \{n'\}),$$

and define $f' : X' \to Y'$ via $f'((y,i)) = (f''(y), i)$, with fibres of constant size k. Note that the map

$$g'' : (X'' \setminus G') \times (n' + 1) \to Y'' \times (n' + 1)$$

defined by $g''((y,i)) = (g(y), i)$ has constant fibre size ℓ. Now X' has

$$|P| - |G' \times (n' + 1)| = kn - n(n' + 1) = (k - (k - \ell))n = \ell n$$

points less than $(X'' \setminus G') \times (n'+1)$, and Y' has $|Q| = n$ points less than $Y'' \times (n'+1)$. Modifying g'' on finitely many points, we can thus define a map $g' : X' \to Y'$ with constant fibre size ℓ. ⊣

Corollary 3.5. *Let T be functionally unimodular. If X and Y are two definable sets, and $f, g : X \to Y$ are two definable maps of constant fibre sizes k and ℓ, respectively, except for finitely many exceptional fibres which are still finite, then $k = \ell$.*

Proof. This follows immediately from Proposition 3.4. ⊣

Example 3.6. Consider the structure $M = \langle 2^{<\omega}, S \rangle$ where S is interpreted as the successor relation, that is, $D \models S(a,b)$ if and only if $a\hat{\ }0 = b$ or $a\hat{\ }1 = b$. This structure is strongly minimal and was proposed in [5] as an example of a strongly minimal structure which is functionally unimodular but not unimodular. The nonunimodularity follows from the fact that if $S(a,b)$ holds, then a and b are interalgebraic but $m(a/b) = 1 \neq 2 = m(b/a)$.

Contrary to [5, Proposition 3.2], in fact this structure is *not* functionally unimodular: The identity function id_M is clearly 1-to-1, while the predecessor function f defined by the formula

$$\varphi(x,y) = S(y,x) \vee (\forall z(\neg S(z,x)) \wedge x = y)$$

is 2-to-1 almost everywhere, with an exceptional fibre of size 3 at $\emptyset$. So M is not functionally unimodular by Corollary 3.5. This can also be seen directly: Add an additional point ∞ to the structure and define $f'(x) = f(x)$ for $x \neq \emptyset$, and $f'(\emptyset) = f'(\infty) = \infty$. Then f' is surjective and 2-to-1 on $M \cup \{\infty\}$, contradicting functional unimodularity.

Definition 3.7 (Correspondence unimodularity). A complete theory T is *correspondence unimodular* (CU) if for any two definable sets X and Y we have the following:

(**) If C_1 and C_2 are uniform correspondences between X and Y, then $m_{C_1} = m_{C_2}$.

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We say that T is correspondence unimodular for (complete) types (CU-t and CU-ct, respectively), if (**) holds whenever X and Y are (complete) types.

Lemma 3.8. *A theory T is correspondence unimodular* (*resp. for types or complete types*) *if and only if all definable sets* (*resp. types or complete types*) *are measurable.*

Proof. ($\Rightarrow$) Suppose C is a uniform correspondence on π. Then C^{-1} is again a uniform correspondence on π. By correspondence unimodularity, $m_C = m_{C^{-1}} = 1/m_C$, whence $m_C = 1$ and C is balanced.

($\Leftarrow$) Suppose C_1, C_2 are uniform correspondences between π_1 and π_2. Define C on $\pi_1 \times \pi_2$ by

$$(a_1, b_1)\ C\ (a_2, b_2) \Leftrightarrow a_1\ C_1\ b_2 \wedge a_2\ C_2\ b_1.$$

It is easy to see that C is a uniform correspondence on $\pi_1 \times \pi_2$, with

$$k_C = k_{C_1} \cdot \ell_{C_2} \qquad \text{and} \qquad \ell_C = k_{C_2} \cdot \ell_{C_1}.$$

By assumption $k_C = \ell_C$, whence $m_{C_1} = m_{C_2}$. So T is correspondence unimodular. ⊣

Example 3.9. It is easy to show that all pseudofinite structures are correspondence unimodular (for definable sets): If $M = \prod_{\mathcal{U}} M_i$ is an ultraproduct of finite structures and C is a uniform correspondence on a definable set $X \subseteq M$, then in the finite structures M_i we have that

$$|C_i| = \Big|\bigcup_{x \in X_i} \{(a,b) \in C_i : a = x\}\Big| = \sum_{x \in X_i} |(C_i)_x| = |X_i| \cdot k_C$$

for $\mathcal{U}$-almost all indices i. Similarly, $|C_i| = |X_i| \cdot \ell_C$, whence $k_C = \ell_C$ and C is balanced. Therefore all definable sets are measurable; by Lemma 3.8 we have correspondence unimodularity.

We shall now identify various implications between the different notions of unimodularity. It is clear that functional unimodularity for types implies both functional unimodularity for complete types and for definable sets, and similarly for correspondence unimodularity. We shall show the implications given by the dotted arrows in the diagram below, sometimes under additional model-theoretic hypotheses.

$$\begin{tikzcd}
 & \text{CU-ct} \arrow[r, dotted, shift left, "(1)"] \arrow[dl, dotted, leftrightarrow] \arrow[dd, dotted, leftrightarrow] & \text{CU-t} \arrow[l, shift left] \arrow[r, shift right] \arrow[dd, dotted, leftrightarrow] & \text{CU} \arrow[l, dotted, shift right, "(2)"'] \arrow[dd, dotted, leftrightarrow] \\
\text{Unimodularity} \arrow[dr, dotted, leftrightarrow] & & & \\
 & \text{FU-ct} \arrow[r, dotted, shift left, "(1)"] & \text{FU-t} \arrow[l, shift left] \arrow[r, shift right] & \text{FU} \arrow[l, dotted, shift right, "(2)"']
\end{tikzcd}$$

(1) T ω-stable.
(2) T non-multidimensional, with strongly minimal dimensions.

We first note that the functional and correspondence versions of unimodularity are equivalent.

Proposition 3.10. *A theory is functionally unimodular* (*resp. FU-t or FU-ct*) *if and only it is correspondence unimodular* (*resp. CU-t or CU-ct*).

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PROOF. $(\Rightarrow)$: Let $C(x, y)$ be a uniform correspondence on a definable set X (resp. type-definable set or complete type). Note that if X is a complete type, by Corollary 2.3 we may assume that C is complete. Consider the two functions $f, g : C \to X$, where f is the projection to the first and g the projection to the second coordinate. Then f is k_C-to-1 and g is ℓ_C-to-1. By functional unimodularity (resp. FU-t or FU-ct) we have $k_C = \ell_C$, and C is balanced. By Lemma 3.8 we are done.

$(\Leftarrow)$: Suppose X and Y are type-definable sets, and $f, g : X \to Y$ are relatively definable surjective functions that are, respectively, k-to-1 and ℓ-to-1. Consider the correspondence C on X defined by

$$(a, a') \in C \Leftrightarrow f(a) = g(a').$$

Then C is a (ℓ, k)-correspondence on X, and $k = \ell$ by correspondence unimodularity. ⊣

As a corollary, we obtain in general the equivalence between unimodularity and functional unimodularity for complete types, originally shown by Kestner and Pillay for strongly minimal theories.

THEOREM 3.11. *Let T be a complete theory. The following are equivalent*:

(1) *T is unimodular.*
(2) *T is correspondence unimodular for complete types.*
(3) *T is functionally unimodular for complete types.*

PROOF. This follows from Lemmas 3.2 and 3.8 and Proposition 3.10. ⊣

Example 3.6 shows that our next theorem does need ω-stability.

THEOREM 3.12. *Let T be ω-stable unimodular. Then T is correspondence unimodular for types.*

PROOF. This follows from Lemmas 3.2 and 3.8 and Theorem 2.9. ⊣

The following is an example of a functionally unimodular structure which is not unimodular. We shall show in Theorem 4.14 that for a non-multidimensional theory with strongly minimal dimensions, functional unimodularity does imply unimodularity.

EXAMPLE 3.13. For each $n < \omega$, let $M_n = 2^{<n}$. We consider M_n as a finite structure in the language $\mathcal{L} = \{R_i : i < \omega\} \cup \{f\}$ by interpreting the predicates as $R_i^{M_n} = \{\eta \in M_n : \text{length}(\eta) = n - i\}$ for $i \leq n$, and $R_i^{M_n} = \emptyset$ for $i > n$. To interpret the function f we put:

$$f(\eta)) = \begin{cases} \eta \restriction_{\text{length}(\eta)\text{-}1} & \text{if length}(\eta) > 1, \\ \emptyset & \text{if } \eta = \emptyset. \end{cases}$$

Let $M = \prod_{\mathcal{U}} M_n$, where $\mathcal{U}$ is a nonprincipal ultrafilter over ω. Note that in the ultraproduct, $f : M \to M$ is a definable function such that $f \restriction_{R_i} : R_i \twoheadrightarrow R_{i+1}$ is a 2-to-1 function.

Since M is pseudofinite, it is correspondence unimodular (Example 3.9). It is easy to check that M is ω-stable, even non-multidimensional of Morley rank 2. However, M is not correspondence unimodular for complete types: Consider the complete type given by

$$q(x) = \{\neg R_i(x) : i < \omega\} \cup \{f^i(x) \neq x : i < \omega\}.$$

Then $f(q) = q$, and $f \restriction_q$ is 2-to1, so q is not measurable.

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§4. Unimodularity for types. Throughout this section we shall work in a stable theory with elimination of imaginaries. We first introduce some notions from geometric stability theory. For further reading, the reader can consult [9] or [11].

DEFINITION 4.1. Let π be a partial type over A, and Σ an A-invariant family of partial types. Then π is

- (*almost*) Σ-*internal* if for every realization a of π there is $B \mathop{\smile\!\!\!\!\!\mid}_A a$ and a tuple $\bar{b}$ of realizations of types in Σ based on B, such that $a \in \operatorname{dcl}(B\bar{b})$ (or $a \in \operatorname{acl}(B\bar{b})$, respectively),
- Σ-*analysable* if for any realization a of π there are $(a_i : i < \alpha) \in \operatorname{dcl}(Aa)$ such that $\operatorname{tp}(a_i/A, a_j : j < i)$ is Σ-internal for all $i < \alpha$, and $a \in \operatorname{acl}(A, a_i : i < \alpha)$. We call α the *length* of the analysis.

We shall say that a is (almost) Σ-internal or Σ-analysable over b if $\operatorname{tp}(a/b)$ is.

DEFINITION 4.2. Two types $p \in S(A)$ and $q \in S(B)$ are *orthogonal* if for all $C \supseteq AB$, $a \models p$, and $b \models q$ with $a \mathop{\smile\!\!\!\!\!\mid}_A C$ and $b \mathop{\smile\!\!\!\!\!\mid}_B C$ we have $a \mathop{\smile\!\!\!\!\!\mid}_C b$.
A type p is *regular* if it is orthogonal to all its forking extensions.
A theory is *non-multidimensional* if every type is nonorthogonal to a type over $\emptyset$.

Equivalently, a theory is non-multidimensional if there are only boundedly many pairwise orthogonal types.

DEFINITION 4.3 (Unimodularity). A complete stationary type p is *unimodular* if over any set A of parameters containing $\operatorname{dom}(p)$, whenever a and b are A-interalgebraic realizations of the nonforking extension of p to A, then $m(a/Ab) = m(b/Aa)$.

REMARK 4.4. Equivalently, p is unimodular if all its nonforking extensions are measurable over their domain.

LEMMA 4.5. *Let p and p' be unimodular stationary types of finite Lascar rank over A. Let aa' and bb' be A-interalgebraic realizations of the free product $p \otimes p'$. Suppose $a \mathop{\smile\!\!\!\!\!\mid}_A b'$ and $a' \mathop{\smile\!\!\!\!\!\mid}_A b$. Then $m(aa'/Abb') = m(bb'/Aaa')$.*

PROOF. By stationarity and independence, a and b both realize $p|Aa'$. Moreover, $b \in \operatorname{acl}(Aaa')$. By the Lascar equalities in finite rank,

$$U(a/Aa'b) = U(aa'b/A) - U(a'b/A) = U(aa'b/A) - U(aa'/A) = U(b/Aaa') = 0.$$

So a and b are Aa'-interalgebraic, whence $m(a/Aa'b) = m(b/Aaa')$ by unimodularity of p. Thus

$$m(bb'/Aaa') = m(b'/Aaa'b)\cdot m(b/Aaa') = m(b'/Aaa'b)\cdot m(a/Aba') = m(ab'/Aba').$$

Similarly, b' and a' are Ab-interalgebraic realizations of $p'|Ab$. So $m(b'/Aba') = m(a'/Abb')$ by unimodularity of p', and

$$m(ab'/Aba') = m(a/Aba'b') \cdot m(b'/Aba') = m(a/Aba'b') \cdot m(a'/Abb') = m(aa'/Abb'). \quad \dashv$$

COROLLARY 4.6. *If p and q are orthogonal unimodular stationary types of finite Lascar rank, then their free product $p \otimes q$ is unimodular.*

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PROOF. This follows immediately from the definitions and Lemma 4.5. ⊣

COROLLARY 4.7. *If p is a unimodular regular stationary type of finite Lascar rank, then the free power $p^{(n)}$ is unimodular for all $n \geq 1$.*

PROOF. We can assume $p \in S(\emptyset)$. If $(a_i : i < n)$ and $(b_i : i < n)$ are two interalgebraic realizations of $p^{(n)}$, put $\bar{a} = (a_i : i > 0)$. Let $\tilde{b} = (b_i : b_i \not\perp \bar{a})$. Since $a_0 \perp \bar{a}$ we have $a_0 \perp \tilde{b}$. Let $\bar{b} \supseteq \tilde{b}$ be maximal with $a_0 \perp \bar{b}$. Then $\bar{b}$ has length $n-1$, and there is a unique $b_j \notin \bar{b}$. Note that $b_j \perp \bar{a}$. As $\bar{a}$ and $\bar{b}$ satisfy $p^{(n-1)} =: p'$, and a_0 and b_j satisfy p, the hypotheses of Lemma 4.5 are satisfied, and we conclude. ⊣

LEMMA 4.8. *Let π and π' be partial types over A, and $A \subseteq B$. Put*

$$\bar{\pi}(x) := \pi(x) \wedge x \underset{A}{\perp} B \qquad \textit{and} \qquad \bar{\pi}'(y) := \pi'(y) \wedge y \underset{A}{\perp} B.$$

If C is a uniform correspondence between π and π' over A, then $C' = C \cap (\bar{\pi} \times \bar{\pi}')$ is a uniform correspondence between $\bar{\pi}$ and $\bar{\pi}'$ with $m_{C'} = m_C$.

PROOF. For $a \models \bar{\pi}$ we have $a \perp_A B$. If $(a, b) \in C$, then $b \models \pi'$ and $b \in \mathrm{acl}(Aa)$, whence $b \perp_A B$ and $b \models \bar{\pi}'$. Thus $(a, b) \in C'$, and $|(C')_a| = k_C$. Similarly $|(C')^b| = \ell_C$ for all $b \models \bar{\pi}'$. Therefore C' is uniform, with $k_{C'} = k_C$ and $\ell_{C'} = \ell_C$, whence $m_{C'} = m_C$. ⊣

COROLLARY 4.9. *Suppose q is a nonforking extension of a stationary type p. Then p is unimodular if and only if q is unimodular.*

PROOF. ($\Rightarrow$) follows from the definition. For the converse, consider a nonforking extension p' of p, and the common nonforking extension q' of $p' \cup q$. Take $\pi = \pi' = p'$ and $\bar{\pi} = \bar{\pi}' = q'$ in Lemma 4.8. As $m_{C'} = 1$ by measurability of q, we get $m_C = 1$ and p' is measurable. Hence p is unimodular. ⊣

COROLLARY 4.10. *Let p and q be stationary types over A whose realizations are A-interalgebraic. Suppose p is unimodular.*

(1) *Then q is unimodular, and p and q are commensurable.*
(2) *If p' and q' are nonforking extension of p and q to the same domain, then p' and q' are again commensurable, and $m_p^q = m_{p'}^{q'}$.*

PROOF. As p is measurable, p and q are commensurable by Corollary 2.8. Moreover, p' and q' are also commensurable by Lemma 4.8, and $m_p^q = m_{p'}^{q'}$. Hence all nonforking extensions of q are measurable, and q is unimodular. ⊣

COROLLARY 4.11. *Let P be an $\emptyset$-invariant family of unimodular weakly minimal stationary types. If q is almost P-internal, then q is unimodular.*

PROOF. Since q is almost P-internal, there is a realization $a \models q$, some set A of parameters independent of a, and realisations $\bar{b}$ of types in P over A with $a \in \mathrm{acl}(A\bar{b})$. As P consists of weakly minimal types, we may assume that $\bar{b}$ is independent over A. Let $\bar{b} = \bar{b}'\bar{b}''$, where $\bar{b}'$ is a maximal subtuple of $\bar{b}$ independent of a over A. Then $\mathrm{tp}(a/A\bar{b}')$ is a nonforking extension of q, and a and $\bar{b}''$ are interalgebraic over $A\bar{b}'$ by weak minimality of the types in P. Moreover, $\bar{b}''$ is independent over $A\bar{b}'$. The result now follows from Corollaries 4.6, 4.7, 4.9, and 4.10. ⊣

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We now turn to analysability. Let us first consider an example which shows that non-multidimensionality is necessary in Theorem 4.14.

EXAMPLE 4.12. Let E be an equivalence relation with infinitely many infinite classes, and f a unary surjective function with fibres of size two, such that $x\,E\,x' \Leftrightarrow f(x)\,E\,f(x')$ and that neither f nor the induced relation f_E on E-classes have any nonoriented cycles (and in particular $\neg x\,E\,f(x)$). It is easy to see that this theory is multidimensional of Morley rank 2; one dimension is carried by the type $\mathrm{tp}(a_E)$ of the E-classes, and the other dimensions by $\mathrm{tp}(a/a_E)$, for any a. Each dimension has Morley rank 1 and is unimodular. Nevertheless, $\mathrm{tp}(a)$ is clearly not unimodular, as $a \equiv f(a)$, $m(f(a)/a) = 1$ but $m(a/f(a)) = 2$.

THEOREM 4.13. *Let P be a set of unimodular strongly minimal types over $\emptyset$. Then any P-analysable stationary type is unimodular.*

PROOF. By Corollary 4.9 we may add parameters to the language and suppose that the types in P are over $\emptyset$. Note that as the types in P are strongly minimal, any P-analysable stationary type q is contained in a definable set φ which is P-analysable of finite length. Then φ is non-multidimensional, and its dimensions are strongly minimal. So φ is ω-stable by [10, Corollaire 2.14].

We shall use induction on the length of a P-analysis of q. If it is 1, then q is almost P-internal, and we are done by Corollary 4.11.

So suppose q has a P-analysis of length $n+1$. For $b \models q$ put

$$B = \{e \in \mathrm{acl}(b) : \mathrm{tp}(e) \text{ has a } P\text{-analysis of length at most } n\},$$

the n-th P-level $\ell_n^P(b)$ (see [8, Definition 3.1]). Put $A = B \cap \mathrm{dcl}(b)$. If $e \in B$ and $e' \equiv_b e$, then $e' \in B$, and there are only finitely many such e'. Let $\bar{e}$ be any imaginary element coding this finite set. Then $\bar{e} \in \mathrm{dcl}(b)$, and $\bar{e} \in \mathrm{dcl}\{e' : e' \equiv_b e\} \subseteq B$, so $\bar{e} \in A$. Hence $B = \mathrm{acl}(A)$. Moreover, the type $\mathrm{tp}(b/A)$ is stationary, as $\mathrm{tp}(b/B)$ is stationary, $b \mathop{\smile\!\!\!\!\!\!\!\;|}_A B$, and for every A-definable finite equivalence relation E the class b_E of b modulo E is in

$$\mathrm{dcl}(Ab) \cap \mathrm{acl}(B) = \mathrm{dcl}(b) \cap B = A.$$

By ω-stability of φ we can choose $a \in A$ such that $b \mathop{\smile\!\!\!\!\!\!\!\;|}_a A$ and $\mathrm{tp}(b/a)$ is stationary; note that then $A = \mathrm{dcl}(a)$. Since $\mathrm{tp}(b)$ has a P-analysis of length $n+1$, the type $\mathrm{tp}(b/B)$ and thus also $\mathrm{tp}(b/a)$ is almost P-internal, whence unimodular. Finally, $\mathrm{tp}(a)$ is stationary since $\mathrm{tp}(b)$ is, and unimodular by inductive hypothesis.

If $b' \models q$ and b and b' are interalgebraic, choose a' with $a'b' \equiv ab$. Note that $\mathrm{Cb}(a'/b)$ is definable over a Morley sequence in $\mathrm{tp}(a/b')$, and thus has a P-analysis of length at most n. It follows that $\mathrm{Cb}(a'/b) \in B$ and $a' \mathop{\smile\!\!\!\!\!\!\!\;|}_B b$, whence $a' \mathop{\smile\!\!\!\!\!\!\!\;|}_a b$. Similarly, $a \mathop{\smile\!\!\!\!\!\!\!\;|}_{a'} b'$. But

$$a \in \mathrm{dcl}(b) \subseteq \mathrm{acl}(b) = \mathrm{acl}(b') \qquad \text{and} \qquad a' \in \mathrm{dcl}(b') \subseteq \mathrm{acl}(b') = \mathrm{acl}(b),$$

so the independences above imply that a and a' are interalgebraic.

By stationarity of $\mathrm{tp}(b/a)$, the independence $b \mathop{\smile\!\!\!\!\!\!\!\;|}_a a'$ and unimodularity of $\mathrm{tp}(a)$ we have

$$m(a'/ab) = m(a'/a) = m(a/a') = m(a/a'b').$$

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Since $\mathrm{tp}(b)$ is almost P-internal, there is $D \mathop{\smile\!\!\!\!\!\mid}_a b$ containing a and some tuple d of realizations of types in P over D such that b and d are D-interalgebraic. As $\mathrm{tp}(d)$ is P-internal, it is unimodular by Corollary 4.11, as is $\mathrm{tp}(b/a)$. Put $p = \mathrm{tp}(b/a)$, $q = \mathrm{tp}(b'/a')$, and $r = \mathrm{tp}(d)$. Let p^* and q^* be the nonforking extensions of p and q to aa'. As b and b' are aa'-interalgebraic and p is unimodular, p^* and q^* are commensurable and

$$m_{p^*}^{q^*} = \frac{m(b'/aa'b)}{m(b/aa'b')}.$$

Let σ be a strong $\emptyset$-automorphism mapping a to a', and put $D' = \sigma(D)$. Let p' and r' be the nonforking extensions of p and r to D, and q' and r^* the nonforking extensions of q and r to D'. As p is unimodular, p' and q' are commensurable, as are $\sigma(p') = q'$ and $\sigma(r') = r^*$. Clearly $m_{p'}^{r'} = m_{q'}^{r^*}$.

Finally, let p'', q'', and r'' be the nonforking extensions of p, q, and r to DD'. Then p'', q'', and r'' are commensurable by Corollary 4.10, and by Lemma 2.7 we get

$$m_{p''}^{r''} = m_{p''}^{q''} m_{q''}^{r''}.$$

But now by Corollary 4.10 again,

$$\frac{m(b'/aa'b)}{m(b/aa'b')} = m_{p^*}^{q^*} = m_{p''}^{q''} = \frac{m_{p''}^{r''}}{m_{q''}^{r''}} = \frac{m_{p'}^{r'}}{m_{q'}^{r^*}} = 1.$$

Hence $m(b'/aa'b) = m(b/aa'b')$. As $a \in \mathrm{dcl}(b)$ and $a' \in \mathrm{dcl}(b')$, we finally obtain

$$\begin{aligned} m(b/b') &= m(ab/a'b') = m(b/aa'b')m(a/a'b') \\ &= m(b'/aa'b)m(a'/ab) = m(a'b'/ab) = m(b'/b). \end{aligned}$$

It follows that q is unimodular. ⊣

Theorem 4.14. *Let T be a non-multidimensional theory whose dimensions are associated to strongly minimal types. The following are equivalent*:

(1) *T is unimodular.*
(2) *T is functionally unimodular.*
(3) *All strongly minimal types are unimodular.*

Proof. (1) $\Rightarrow$ (2) : By [10, Corollaire 2.14] the theory T is ω-stable, so unimodularity implies functional unimodularity for partial types by Theorem 3.12. Functional unimodularity (for sets) follows.

(2) $\Rightarrow$ (3) : Let p be a strongly minimal type which is not unimodular. We may assume p is over $\emptyset$. So there are interalgebraic realizations $a, b \models p$ with $m(a/b) \neq m(b/a)$. Then $\mathrm{tp}(a, b)$ has Morley rank 1. Choose definable sets $X \in \mathrm{tp}(a, b)$ and $Y \in p$ of Morley rank 1, such that Y has Morley degree 1 and $X \subset Y \times Y$. Consider the functions $f, g : X \to Y$, where f is the projection to the first coordinate, and g is the projection to the second coordinate. Restricting Y we may assume that f has fibres of size at most $m(b/Aa)$, and g has fibres of size at most $m(a/Ab)$. As Y is strongly minimal and the fibre sizes are bounded, there are only a finite number of exceptional fibres, of size less than $m(b/a)$ for f and of size less than $m(a/b)$ for g. By Proposition 3.4 there are definable sets X' and Y' and definable functions $f', g' : X' \to Y'$ whose fibres all have size $m(b/a)$ and $m(a/b)$, respectively. As $m(b/a) \neq m(a/b)$, this contradicts functional unimodularity.

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$(3) \Rightarrow (1)$: Let P be a set of strongly minimal types containing a representative for each dimension. Then every type is P-analysable, and hence unimodular by Theorem 4.13. $\dashv$

Examples of non-multidimensional theories whose dimensions are associated to strongly minimal types are almost strongly minimal theories, uncountably categorical theories, and groups of finite Morley rank.

§5. Further remarks.

Although we have defined unimodularity for arbitrary stationary types, we could only show that it is well behaved for types of finite rank. The problem obviously comes from the fact that in infinite rank, say close to a regular type p, we should work with p-closure rather than algebraic closure, which is unbounded. Thus multiplicity is not the correct measure.

A possibility might be to define *Lascar unimodularity*: Let us say that a stationary type p over A is *Lascar unimodular* if for any realizations $a, b \models p$ we have $U(a/Ab) = U(b/Aa)$. Theories of finite Lascar rank are clearly Lascar unimodular. This notion may be particularly pertinent if p is a regular type, as then a and b are dependent if and only if either one is in the p-closure of the other. However, we have not studied the properties of Lascar unimodularity, nor have we looked for interesting examples.

Another question concerns unimodularity for nonstationary types. Section 2 of our paper does not assume stationarity, so one might be tempted to develop unimodularity, at least for Lascar strong types, in a simple theory in analogy with Section 4.

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