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PACKING SUBORDINACY WITH APPLICATION TO SPECTRAL CONTINUITY

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Abstract

By using methods of subordinacy theory, we study packing continuity properties of spectral measures of discrete one-dimensional Schrödinger operators acting on the whole line. Then we apply these methods to Sturmian operators with rotation numbers of quasibounded density to show that they have purely α-packing continuous spectrum. A dimensional stability result is also mentioned.

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1. Introduction

We are interested in packing dimensional properties of spectral measures for discrete Schrödinger operators H, in $l^2(\mathbb{Z})$, of the form

$$(H\psi)(n) = \psi(n+1) + \psi(n-1) + V(n)\psi(n), \tag{1.1}$$

with (real) potentials $V = \{V(n)\}$. First, we extend some results from (the partial) packing subordinacy theory for one-dimensional operators on the half-line [3] to the whole-line case. This was initially proposed to provide information about packing dimensional properties of spectral measures and it was an adaptation of the (Hausdorff) power-law subordinacy introduced by Jitomirskaya and Last in [14, 15]. We refer to the latter as Hausdorff subordinacy theory.

The fractal (that is, Hausdorff and packing) subordinacy theories are generalizations of the subordinacy theory, introduced by Gilbert and Pearson in [10, 11] (see [17] for an adaptation to discrete operators). All of them exploit the relation between the asymptotic behavior of the solutions to the eigenvalue equation

$$(H\psi)(n) = E\psi(n) \tag{1.2}$$

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and the spectral nature of the operator H. The idea is to use the existence of subordinate or power-law subordinate solutions to (1.2) to investigate the standard decomposition of a spectral measure into its point part, absolutely singular continuous and absolutely continuous.

The general idea of this work is based on the fact that one can obtain results on packing dimension similarly to results on Hausdorff dimension using power-law subordinacy and interchanging liminfs and limsups. Some arguments are simple adaptations of existing ones, but others need additional care. Furthermore, by adapting them to the packing setting, we are able to say something about Sturmian operators with frequencies not covered in the Hausdorff case (see Theorem 1.5).

Fix $E \in \mathbb{R}$, $\varphi \in (-\pi/2, \pi/2]$, and denote by $u_{1,\varphi,E}$ and $u_{2,\varphi,E}$ the solutions to (1.2) which satisfy the initial conditions

$$\begin{cases} u_{1,\varphi,E}(0) = -\sin\varphi, & u_{2,\varphi,E}(0) = \cos\varphi, \\ u_{1,\varphi,E}(1) = \cos\varphi, & u_{2,\varphi,E}(1) = \sin\varphi. \end{cases} \tag{1.3}$$

A solution ψ to (1.2) is called subordinate at $+\infty$ if

$$\lim_{L\to\infty} \frac{\|\psi\|_L}{\|\Phi\|_L} = 0$$

holds for any linearly independent solution Φ to (1.2); here, $\|\cdot\|_L$ denotes the norm truncated at $L \in \mathbb{R}$ ($[L]$ is the integral part of L), that is,

$$\|\psi\|_L = \Bigg[\sum_{n=1}^{[L]} |\psi(n)|^2 + (L-[L])|\psi([L]+1)|^2\Bigg]^{1/2};$$

the subordinacy of a solution ψ at $-\infty$ is defined analogously.

Given $\alpha \in (0, 1]$, a solution ψ to (1.2) is called α-Hausdorff (packing) subordinate at $+\infty$ if

$$\liminf(\limsup)_{L\to\infty} \frac{\|\psi\|_L}{\|\Phi\|_L^{\alpha/(2-\alpha)}} = 0$$

holds for any other linearly independent solution.

In particular, the α-Hausdorff (packing) continuous part of the spectral measure is supported on the set of energies E for which (1.2) does not have α-Hausdorff (respectively, packing) subordinate solutions at $-\infty$ or at $+\infty$, and its α-Hausdorff singular part is supported on the set of energies E for which $u_{1,\varphi,E}$ is an α-Hausdorff subordinate solution at both $\pm\infty$ (note the absence of a characterization of the corresponding α-packing singular part; see below).

The possible existence of power-law bounds of the form [5, 6, 15]

$$C_1 L^{\gamma_1} \le \|u\|_L \le C_2 L^{\gamma_2}, \tag{1.4}$$

for positive constants $C_1(E), C_2(E), \gamma_1, \gamma_2$ and every solution u (with normalized initial conditions (NIC), that is, $|u(0)|^2 + |u(1)|^2 = 1$) to the generalized eigenvalue equation (1.2) implied the nonexistence of α-Hausdorff subordinate solutions at $+\infty$ (similarly at $-\infty$), with $\alpha = 2\gamma_1/(\gamma_1 + \gamma_2)$.

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In the following theorem we present a natural version of such a tool to prove the lack of α-packing subordinate solutions for some fixed energy E; its proof appears at the end of Section 2.

THEOREM 1.1. *Let $\sigma(H)$ be the spectrum of H and let μ_ϕ be the spectral measure of the pair (H, ϕ), with $\phi \in l^2(\mathbb{Z})$. Suppose that there are constants τ_1, τ_2 and a subsequence $L_j \to \infty$ such that, for each $E \in \sigma(H)$, every solution to (1.2) with NIC obeys the estimates*

$$C_1 L_j^{\tau_1} \le \|u\|_{L_j} \le C_2 L_j^{\tau_2}, \tag{1.5}$$

where $C_1 = C_1(E), C_2 = C_2(E)$ are suitable positive constants. Then H has purely α-packing continuous spectrum, with $\alpha = 2\tau_1/(\tau_1 + \tau_2)$, that is, for any $\phi \in l^2$, μ_ϕ is purely α-packing continuous.

REMARK 1.2. Similarly to [6, Remark 2], there is an analogous left half-line version of the previous result. If one is able to establish power-law bounds (1.5) on the restriction of the operator to the right half-line, then the resulting α-packing continuity is independent of the potential on the left half-line. In this sense, the most packing continuous half-line dominates and bounds the dimensionality of the whole-line problem from below.

We apply Theorem 1.1 to the family $\{H_{\lambda,\theta,\rho}\}$ of operators (1.1) with almost periodic Sturmian potentials

$$V(n) = V_{\lambda,\theta,\rho}(n) = \lambda\chi_{[1-\theta,1)}(n\theta + \rho \bmod 1), \quad n \in \mathbb{Z},$$

with coupling constant $0 \neq \lambda \in \mathbb{R}$, irrational rotation number $\theta \in [0, 1)$ and (initial) phase $\rho \in [0, 1)$.

Recall that any irrational $\theta \in [0, 1)$ has an infinite continued fraction expansion

$$\theta = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots}}} = [0; a_1, a_2, \ldots], \tag{1.6}$$

with uniquely determined $a_n \in \mathbb{N}$. The associated rational approximants p_n/q_n are obtained from

$$\begin{aligned} p_0 &= 0, \quad p_1 = 1, \quad p_n = a_n p_{n-1} + p_{n-2}, \\ q_0 &= 1, \quad q_1 = a_1, \quad q_n = a_n q_{n-1} + q_{n-2}. \end{aligned}$$

DEFINITION 1.3. Let $\theta \in [0, 1)$ be an irrational number and (1.6) its continued fraction expansion. Then θ is said to be a number of bounded (quasibounded) density if

$$\limsup(\liminf)_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} a_i < \infty.$$

REMARK 1.4. The set of quasibounded density numbers is uncountable, but has Lebesgue measure zero (see [18, page 93]). We have found that the (rather natural) proposal of the concept of 'quasibounded density' is the convenient one in the packing setting.

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Theorem 1.5. *Let θ be an irrational number of quasibounded density. Then, for every $\lambda \neq 0$, there exists $\alpha = \alpha(\lambda, \theta) > 0$ so that, for every $\phi \in l^2(\mathbb{Z})$, the spectral measure of the pair $(H_{\lambda,\theta,\rho}, \phi)$ is purely α-packing continuous.*

In [16, Theorem 8], the authors proved that if

$$\beta(\theta) := \limsup_{n\to\infty} \frac{\log(q_{n+1})}{q_n} > 0$$

and ρ is θ-Diophantine (that is, there exist $\gamma > 0$, $\tau > 1$ such that, for each $m \in \mathbb{Z}$, $\|\rho + m\theta\|_{\mathbb{R}/\mathbb{Z}} \geq \gamma/(|m| + 1)^\tau$), then for every $\phi \in l^2(\mathbb{Z})$, the spectral measure of the pair $(H_{\lambda,\theta,\rho}, \phi)$ is purely 1-packing continuous. We note that the hypotheses used in the proofs of our Theorem 1.5 and [16, Theorem 8] are different. Namely, if $\beta(\theta) > 0$, then θ is not necessarily a number of quasibounded density and vice versa. Moreover, in Theorem 1.5, there is no restriction on the value of the real number ρ.

It is well known [5, 6, 15] that each operator $H_{\lambda,\theta,\rho}$, with coupling constant $0 \neq \lambda \in \mathbb{R}$, irrational rotation number of bounded density $\theta \in [0, 1)$, and phase $\rho \in [0, 1)$, has purely α_H-Hausdorff continuous spectrum (and that $\sigma(H_{\lambda,\theta,\rho})$ has zero Lebesgue measure) for some $\alpha_H \in (0, 1)$, with $\alpha_H = 2\gamma_1/(\gamma_1 + \gamma_2)$, where $\gamma_1 = \gamma_1(\theta, \lambda) \geq 0, \gamma_2 = \gamma_2(\theta, \lambda) > 0$ satisfy relation (1.4).

Since (1.4) is a particular instance of (1.5), a bounded density rotation number also implies α-packing continuity of the spectral measure of the operator $H_{\lambda,\theta,\rho}$, and here we will get the additional information $\alpha > \alpha_H$ (see Remark 3.7 in Section 3).

We are also interested in the extension, to this packing setting, of the spectral Hausdorff dimensional stability results presented in [1]. In this direction we have the following result.

Corollary 1.6. *Let θ be an irrational number of bounded density and γ_1, γ_2 as in (1.4). Then, for every $\rho \in [0, 1)$ and for large λ, the singular continuous component of each spectral measure of the operator*

$$(H^P_{\lambda,\theta,\rho}\psi)(n) := (H_{\lambda,\theta,\rho}\psi)(n) + P(n)\psi(n), \quad \psi \in l^2(\mathbb{Z}), \tag{1.7}$$

with the perturbation P satisfying $|P(n)| \leq C(1 + |n|)^{-p}$, for all $n \in \mathbb{Z}$, some $C > 0$ and $p > 3\gamma_2 - \gamma_1$, when it exists, is also purely α-packing continuous.

Remark 1.7. We note that a proof of Corollary 1.6 follows directly by [1, Theorem 1.1], since the packing dimension is larger than the Hausdorff dimension. However, we provide a specific proof below since the α obtained is now larger than that from [1, Theorem 1.1]. See details in Remark 3.7.

Remark 1.8. We emphasize that under certain perturbations no singular continuous component may be present, as is the case for rank-one perturbations of operators with singular continuous spectrum of zero Lebesgue measure (this follows directly from results due to Simon and Wolff [26]). However, if the perturbed operator (1.7) has some singular continuous component, then the property of α-packing continuity of the spectral measure of this operator is preserved.

REMARK 1.9. We also obtain the stability of spectral packing dimensional properties for some classes of sparse operators, that is, we obtain packing versions of [1, Theorem 1.2] (see Section 4 for details).

The organization of this paper is as follows. In Section 2, part of the subordinacy theory is recalled and the proof of Theorem 1.1 is presented. Section 3 is devoted to the proof of Theorem 1.5, after recalling some of the basics of Sturmian potentials. Some packing stability results of operators of the form (1.1), under suitable power decaying perturbations, are discussed in Section 4. For the reader's convenience, some definitions and concepts regarding Hausdorff and packing measures are recalled in Appendix A.

2. Subordinacy theory

Let us recall some important results of subordinacy theory and use them in order to obtain information about the spectral packing dimensional properties of (1.1). In what follows, we adopt the same strategy presented in [15].

The study of the spectral measure of an operator given by (1.1) is related to the study of the Weyl–Titchmarsh m-functions. For each such whole-line operator H, consider two operators, denoted by $H^{\pm}$, which correspond to the restrictions of (1.1) to $l^2(\mathbb{Z}^{\pm})$, respectively, where $\mathbb{Z}^{+} = \{1, 2, \ldots\}$ and $\mathbb{Z}^{-} = \{0, -1, -2, \ldots\}$. For each $z \in \mathbb{C}\backslash\mathbb{R}$, let $\psi^{\pm}(n; z)$ be the unique solutions to

$$H\psi^{\pm} = z\psi^{\pm}, \quad \text{satisfying } \psi^{\pm}(0; z) = 1 \text{ and } \sum_{n=0}^{\infty} |\psi^{\pm}(\pm n; z)|^2 < \infty.$$

With this notation, the m-functions are given, for every $z \in \mathbb{C}\backslash\mathbb{R}$, by

$$m^{+}(z) = \langle \delta_1 | (H^{+} - z)^{-1} \delta_1 \rangle = -\psi^{+}(1; z)/\psi^{+}(0; z),$$
$$m^{-}(z) = \langle \delta_0 | (H^{-} - z)^{-1} \delta_0 \rangle = \psi^{-}(1; z)/\psi^{-}(0; z),$$

where $\delta_j = (\delta_{ij})_{i \geq 1}$. We note that for the whole-line case, the m-function is a matrix-valued function $M(z)$ so that

$$\begin{bmatrix} a & b \end{bmatrix} M(z) \begin{bmatrix} a \\ b \end{bmatrix} = \langle (a\delta_0 + b\delta_1) | (H - z)^{-1} (a\delta_0 + b\delta_1) \rangle,$$

or, more explicitly (omitting the z dependence),

$$M = \frac{1}{-m^{+} - m^{-}} \begin{bmatrix} 1 & m^{+} \\ m^{+} & -m^{+}m^{-} \end{bmatrix}.$$

Let $m(z) = \mathrm{tr}(M(z))$, that is, the trace of M. These definitions relate the m-functions to resolvents, and hence to spectral measures. Explicitly, one has

$$m^{\pm}(z) = \int \frac{1}{t - z}\, d\mu^{\pm}(t),$$
$$m(z) = \int \frac{1}{t - z}\, d\mu(t),$$

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with μ^+ and μ^- respectively representing spectral measures of the pairs (H^+, δ_1), (H^-, δ_0), and with $\mu = \mu^+ + \mu^-$, that is, the sum of the spectral measures of the pairs (H^+, δ_1) and (H^-, δ_0). Note that the pair of vectors $\{\delta_0, \delta_1\}$ is cyclic for H.

It was shown in [14] that

$$(\overline{D}^\alpha \mu)(E) = \infty \quad \Leftrightarrow \quad \limsup_{\varepsilon \to 0} \varepsilon^{1-\alpha} |m(E + i\varepsilon)| = \infty,$$

whereas for the inferior derivative one may only conclude that

$$(\underline{D}^\alpha \mu)(E) = \infty \quad \Rightarrow \quad \liminf_{\varepsilon \to 0} \varepsilon^{1-\alpha} |m(E + i\varepsilon)| = \infty. \tag{2.1}$$

There is a mistake in the discussion in [3] (in Theorem 14 there) and currently one guarantees that only the implication in (2.1) holds true (there is no proof or counterexample to the converse statement). However, we emphasize that (2.1) is exactly what we need in this work.

These results, together with Remark A.7, show that the study of the dimensional spectral properties of Schrödinger operators (1.1) can sometimes be reduced to the study of the behavior of $m(E + i\varepsilon)$ as $\varepsilon \to 0$, which in turn reduces to the study of the behavior of $m^\pm(E + i\varepsilon)$ as $\varepsilon \to 0$.

Given an operator H of the form (1.1) and $E \in \mathbb{R}$, let $u^\pm_{1,\varphi,E}$ and $u^\pm_{2,\varphi,E}$ be the solutions to (1.2), defined in $\mathbb{Z}^\pm$, satisfying (1.3). Now, given $\varepsilon > 0$, define the lengths $L(\varepsilon)^\pm \in (0, \infty)$ by

$$\|u^\pm_{1,\varphi,E}\|_{L(\varepsilon)^\pm} \|u^\pm_{2,\varphi,E}\|_{L(\varepsilon)^\pm} = \frac{1}{2\varepsilon}. \tag{2.2}$$

By the constancy of the Wronskian (namely, $W[u^\pm_{1,\varphi,E},\ u^\pm_{2,\varphi,E}] = 1$), at most one of the solutions $u^\pm_{1,\varphi,E}$, $u^\pm_{2,\varphi,E}$ belongs to $l^2(\mathbb{Z}^\pm)$, the functions $L(\varepsilon)$ are well defined by (2.2), and $L(\varepsilon) \to \infty$ as $\varepsilon \to 0$ (see [15]).

As a consequence of the Jitomirskaya–Last inequality [14, Theorem 1.1], we have the following results that connect Hausdorff and packing continuity of the spectral measure of H to the scaling behavior of the (generalized) eigenfunctions of H.

Theorem 2.1 [14, Theorem 1.2] and part of [3, Theorem 14]. *Let H be defined by the action (1.1) in $l^2(\mathbb{Z}^+)$, and let μ denote the spectral measure of H associated with the cyclic vector δ_1. Let $E \in \mathbb{R}$ and $\alpha \in (0, 1)$. Then, for any $\varphi \in (-\pi/2, \pi/2]$, $(\overline{D}^\alpha \mu)(E) = \infty$ holds if and only if*

$$\liminf_{L \to \infty} \frac{\|u_{1,\varphi,E}\|_L}{\|u_{2,\varphi,E}\|_L^{\alpha/(2-\alpha)}} = 0,$$

and $(\underline{D}^\alpha \mu)(E) < \infty$ holds if

$$\limsup_{L \to \infty} \frac{\|u_{1,\varphi,E}\|_L}{\|u_{2,\varphi,E}\|_L^{\alpha/(2-\alpha)}} > 0.$$

Theorem 2.1 provides a tool for the analysis of the dimensional properties of some spectral measures of Schrödinger operators.

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Lemma 2.2. *Pick $E \in \sigma(H)$, and suppose that there exists a sequence $L_j \to \infty$ such that every solution to $(H - E)u = 0$ with NIC obeys the estimate*

$$C_1 L_j^{\tau_1} \leq \|u\|_{L_j} \leq C_2 L_j^{\tau_2},$$

where C_1, C_2, τ_1, τ_2 are positive constants. Then, there exist a positive constant C_3 and a sequence $\varepsilon_j \to 0$ such that, for $\alpha = 2\tau_1/(\tau_1 + \tau_2)$,

$$|m(E + i\varepsilon_j)| = \left| \frac{m^+(E + i\varepsilon_j)m^-(E + i\varepsilon_j) - 1}{m^+(E + i\varepsilon_j) + m^-(E + i\varepsilon_j)} \right| \leq C_3 \varepsilon_j^{\alpha - 1}.$$

Consequently, μ is α-packing continuous.

Proof. The proof of Lemma 2.2 traces the same steps of the proof of [6, Theorem 4 and Corollary 2.1], with simple adaptations. We conclude from Theorem 2.1 and (2.1) that μ is α-packing continuous. □

Proof of Theorem 1.1. It follows from hypothesis (1.5) and Lemma 2.2 that μ is α-packing continuous. The result is a consequence of the fact that $\mu_\phi \ll \mu$. □

3. Spectral packing continuity for Sturmian operators

In this section we present the proof of Theorem 1.5, but first we recall some basic properties of the Sturmian potentials. Let us fix a rotation number θ and let (a_n) be the sequence of coefficients in its continued fraction expansion (1.6). Define the words S_n over the alphabet $\mathcal{A} = \{0, \lambda\}$ (with $0 \neq \lambda \in \mathbb{R}$ fixed) by

$$S_{-1} = \lambda, \quad S_0 = 0, \quad S_1 = S_0^{a_1 - 1} S_{-1}, \quad S_n = S_{n-1}^{a_n} S_{n-2}, \quad n \geq 2. \tag{3.1}$$

In particular, the word S_n has length q_n for each $n \geq 0$. For each potential of the form $V_{\lambda,\theta,\rho}(n) = \lambda\chi_{[1-\theta,1)}(n\theta + \rho \bmod 1)$, with $0 \neq \lambda \in \mathbb{R}$, $\theta \in [0, 1)$ an irrational number and $\rho \in [0, 1)$, it is possible to select a sequence (a_n) so that the potential is recovered through (3.1) (see [2, 6, 7] for more details).

Fix $E \in \mathbb{R}$; then, for each $w = w_1 \dots w_n \in \mathcal{A}^n$, the transfer matrix $M(E, w)$ is defined as

$$M(E, w) = \begin{pmatrix} E - w_n & -1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} E - w_1 & -1 \\ 1 & 0 \end{pmatrix}.$$

If u is a solution to (1.2), one has, for every $n \in \mathbb{N}$,

$$U(n + 1) = M(E, V_{\lambda,\theta,\rho}(1) \dots V_{\lambda,\theta,\rho}(n))U(1), \tag{3.2}$$

where

$$U(n) = \begin{pmatrix} u(n) \\ u(n - 1) \end{pmatrix}.$$

Observe that the behavior of $\|u\|_L$, for L large, can be investigated through

$$\|U\|_L = \left(\sum_{n=1}^{[L]} \|U(n)\|^2 + (L - [L])\|U([L] + 1)\|^2 \right)^{1/2},$$

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with $\|U(n)\|^2 = |u(n)|^2 + |u(n-1)|^2$. A simple calculation leads to

$$\tfrac{1}{2}\|U\|_L^2 \le \|u\|_L^2 \le \|U\|_L^2.$$

Since the spectrum of $H_{\lambda,\theta,\rho}$ is independent of ρ [2], we just denote it by $\sigma(H_{\lambda,\theta})$. Put $x_n = \mathrm{tr}(M(E, S_{n-1}))$, $y_n = \mathrm{tr}(M(E, S_n))$ and $z_n = \mathrm{tr}(M(E, S_n S_{n-1}))$, with the explicit dependence on λ and E suppressed. According to results in [2, 6], for every $0 \neq \lambda \in \mathbb{R}$ there exists a $C_\lambda > 1$ such that

$$\max_n \{|x_n|, |y_n|, |z_n|\} \le C_\lambda,$$

uniformly in $E \in \sigma(H_{\lambda,\theta})$ and every irrational θ. We emphasize that this property is important for obtaining lower bounds for the solutions u to (1.2).

3.1. Lower bound for the solutions. In order to prove Theorem 1.5, we need to obtain lower bounds for all solutions to $(H_{\lambda,\theta,\rho} - E)u = 0$ with NIC and corresponding to energies $E \in \sigma(H_{\lambda,\theta})$. Thus, we will prove the following proposition.

PROPOSITION 3.1. *Suppose that θ is an irrational number of quasibounded density. Then, for every $\lambda > 0$, there exist positive constants τ_1, C_1 and a sequence $(n_j)_{j\in\mathbb{N}}$ such that, for every solution u to (1.2) with NIC and corresponding to energies $E \in \sigma(H_{\lambda,\theta})$, one has*

$$\|u\|_{q_{n_j}} \ge C_1 q_{n_j}^{\tau_1}.$$

In the proof of Proposition 3.1, we will use the ideas of [5, 6] with obvious adaptations for a sequence.

LEMMA 3.2 [6, Lemma 4.1]. *Let λ, θ, ρ be arbitrary, $E \in \sigma(H_{\lambda,\theta})$ and u a solution to (1.2) with NIC. Then, for every $n \ge 8$, the inequality*

$$\|U\|_{q_n} \ge D_\lambda \|U\|_{q_{n-8}}$$

holds true, where $D_\lambda = (1 + 1/4C_\lambda^2)^{1/2}$.

LEMMA 3.3. *Suppose that θ is an irrational number of quasibounded density. Then there exist a constant C_θ and a sequence $(n_j)_{j\in\mathbb{N}}$ such that $q_{n_j} \le C_\theta^{n_j}$.*

PROOF. The proof of Lemma 3.3 traces the same steps of the proof of [5, Lemma 2.3], with the obvious adaptations for a subsequence. More precisely, we consider the sequence $(r_n)_{n\in\mathbb{N}}$ defined recursively by

$$r_{n+1} = 2a_{n+1} r_n, \quad n \in \mathbb{N},$$

with initial condition $r_1 = 2a_1$.

Note that $q_n \le r_n$ and $r_n = \prod_{i=1}^n 2a_i$, for all $n \in \mathbb{N}$. Now, since θ is an irrational number of quasibounded density, there exist a constant B_θ and a sequence $(n_j)_{j\in\mathbb{N}}$ such that

$$\frac{1}{n_j}\sum_{i=1}^{n_j} 2a_i \le B_\theta. \tag{3.3}$$

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Thus,

$$\ln(q_{n_j})^{1/n_j} \le \ln(r_{n_j})^{1/n_j} = \frac{1}{n_j}\sum_{i=1}^{n_j}\ln(2a_i) \le B_\theta,$$

and consequently, $q_{n_j} \le C_\theta^{n_j}$, with $C_\theta = e^{B_\theta}$. □

PROOF OF PROPOSITION 3.1. We have by Lemma 3.2 that, for all $n_j \ge 8$,

$$\begin{aligned}\|U\|_{q_{n_j}} &\ge D_\lambda \|U\|_{q_{n_j-8}} \ge \cdots \ge D_\lambda^{[n_j/8]}\|U\|_{q_{n_j-8[n_j/8]}}\\ &\ge D_\lambda^{[n_j/8]}\|U\|_{q_0} \ge D_\lambda^{(n_j/8)-1},\end{aligned}$$

where $D_\lambda > 1$ and $[n_j/8]$ is the integral part of $n_j/8$.

Thus, the existence of a sequence $(n_j)_{j\in\mathbb{N}}$ such that $q_{n_j} \le C_\theta^{n_j}$ follows from Lemma 3.3. Then choose $\tau_1 > 0$, satisfying $C_\theta^{8\tau_1} \le D_\lambda$,

$$\frac{\|U\|_{q_{n_j}}}{q_{n_j}^{\tau_1}} \ge \frac{D_\lambda^{(n_j/8)-1}}{C_\theta^{n_j\tau_1}} = \frac{1}{D_\lambda}\Big(\frac{D_\lambda^{1/8}}{C_\theta^{\tau_1}}\Big)^{n_j} \ge \frac{1}{D_\lambda},$$

which implies that $\|U\|_{q_{n_j}} \ge D_\lambda^{-1} q_{n_j}^{\tau_1}$. Therefore,

$$\|u\|_{q_{n_j}} \ge C_1 q_{n_j}^{\tau_1},$$

with $C_1 = 1/D_\lambda\sqrt{2}$. □

3.2. Upper bound for the solutions.

We now obtain power-law upper bounds for the solutions to (1.2). We prove the following proposition.

PROPOSITION 3.4. *Suppose that θ is an irrational number of quasibounded density. Then, for every $\lambda > 0$, there exist positive constants τ_2, C_2 and a sequence $(n_j)_{j\in\mathbb{N}}$ (which is the same as in Proposition 3.1) such that, for every solution u to (1.2), with NIC and corresponding to energies $E \in \sigma(H_{\lambda,\theta})$, one has*

$$\|u\|_{q_{n_j}} \le C_2 q_{n_j}^{\tau_2}.$$

In the proof of Proposition 3.4, we will use ideas employed in [12, 13] which produce estimates for the norm of the transfer matrices associated with the operator $H_{\lambda,\theta,\rho}$. In order to avoid cumbersome notation, we set

$$M(m) := M(E, V_{\lambda,\theta,0}(1)\ldots V_{\lambda,\theta,0}(m)).$$

LEMMA 3.5 [12, Theorem 9]. *For any integer m, written as $m = \sum_{i=0}^{n}\epsilon_i q_i$, with all ϵ_i integers, one has*

$$\|M(m)\| \le J_1^{\sum_{i=1}^{n+1} a_i} J_2^{\sum_{i=0}^{n}\epsilon_i},$$

where J_1 and J_2 are positive constants such that $J_1 \ge J_2$.

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LEMMA 3.6. *Let $(q_{n_j})_{j\in\mathbb{N}}$ be the sequence obtained in Lemma 3.3, and let m be a positive integer with $m < q_{n_j}$, for some n_j. Then one has the expansion*

$$m = \sum_{i=0}^{n_j-1} \epsilon_i q_i, \tag{3.4}$$

with $0 \le \epsilon_i \le a_{i+1}$, $i = 0, 1, \dots, n_j - 1$.

PROOF. For the sequence $(q_{n_j})_{j\in\mathbb{N}}$, we can suppose, without loss of generality, that $q_{n_0} = q_0 = 1$ and $q_{n_1} = q_1 = a_1$. We prove the result by induction on $j \in \mathbb{N}$. If $q_{n_0} = 1 \le m < q_{n_1} = a_1$, then $m = \epsilon_0 q_0 = \epsilon_0$, with $1 \le \epsilon_0 \le a_1$. Suppose that the result is valid for $m < q_{n_{j-1}}$.

Now suppose that $q_{n_{j-1}} \le m < q_{n_j}$. We will analyze all possible values that m may assume in the interval $[q_{n_{j-1}}, q_{n_j})$.

For $q_{n_{j-1}} \le m < q_{n_{j-1}+1}$, write $\epsilon_{n_{j-1}} = [m/q_{n_{j-1}}]$. Then $m - \epsilon_{n_{j-1}} q_{n_{j-1}} < q_{n_{j-1}}$ and

$$m - \epsilon_{n_{j-1}} q_{n_{j-1}} = \sum_{i=0}^{n_{j-1}-1} \epsilon_i q_i,$$

for $\epsilon_i \le a_{i+1}$, $i = 0, \dots, n_{j-1} - 1$. We also have

$$\epsilon_{n_{j-1}} < \left[\frac{q_{n_{j-1}+1}}{q_{n_{j-1}}}\right] = \left[\frac{a_{n_{j-1}+1} q_{n_{j-1}} + q_{n_{j-1}-1}}{q_{n_{j-1}}}\right] = a_{n_{j-1}+1},$$

and $\epsilon_i = 0$ for $i = n_{j-1} + 1, \dots, n_j - 1$.

The next step is to consider the case $q_{n_{j-1}+1} \le m < q_{n_{j-1}+2}$, which follows from the same considerations as the previous case. Thus, proceeding inductively with this analysis on the values of m, we obtain (3.4) for $q_{n_j-1} \le m < q_{n_j}$, that is,

$$m - \epsilon_{n_j-1} q_{n_j-1} = \sum_{i=0}^{n_j-2} \epsilon_i q_i,$$

where $\epsilon_i \le a_{i+1}$ for $i = 0, \dots, n_j - 2$ and $\epsilon_{n_j-1} = [m/q_{n_j-1}] < a_{n_j}$. □

PROOF OF PROPOSITION 3.4. We will check here in detail the proposition for $\rho = 0$, which represents one of the main differences between our article and [6, 8]. However, as noted in [6], this proposition is valid for every $\rho \in [0, 1)$. We discuss the generalization of this result for $\rho \in [0, 1)$ in Appendix B, which proof follows the same steps presented in [8].

If u is a solution to (1.2) with NIC, one has, by (3.2), that $|u(m)| \le \|M(m)\|$. Then

$$\begin{aligned}\|u\|^2_{q_{n_j}} &= \sum_{m=1}^{q_{n_j}} |u(m)|^2 \le \sum_{m=1}^{q_{n_j}} \|M(m)\|^2 \\ &\le q_{n_j}(J_1)^{4\sum_{i=1}^{n_j} a_i} \le q_{n_j}(J_1)^{2B_\theta n_j} \\ &\le q_{n_j}^{2\tau_2},\end{aligned}$$

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with $\tau_2 \geq 1/2 + \ln(J_1)$, where we have used Lemmas 3.5 and 3.6 in the second inequality, (3.3) in the third, and Lemma 3.3 in the last. □

Proof of Theorem 1.5. Let θ be an irrational number of quasibounded density. Thus, by Propositions 3.1 and 3.4, there exists a sequence $(q_{n_j})_{j\in\mathbb{N}}$ such that for every $\lambda > 0$, there exist τ_1, τ_2 and constants C_1, C_2 such that, for any solution to (1.2) with NIC, one has

$$C_1 q_{n_j}^{\tau_1} \leq \|u\|_{q_{n_j}} \leq C_2 q_{n_j}^{\tau_2}.$$

Therefore, by Theorem 1.1, the spectrum of $H_{\lambda,\theta,\rho}$ is purely α-packing continuous, with $\alpha = 2\tau_1/\tau_1 + \tau_2$. □

Remark 3.7. We have, as a particular case of Theorem 1.5, that if θ is an irrational number of bounded density, then the spectral measure of the operator $H_{\lambda,\theta,\rho}$ is purely α_P-packing continuous with $\alpha_P = 2\tau_1/(\tau_1 + \tau_2)$, where $\tau_1, \tau_2 > 0$ are of the form

$$C_1 q_n^{\tau_1} \leq \|u\|_{q_n} \leq C_2 q_n^{\tau_2},$$

for any solution to (1.2) with NIC. However, it is well known [5, 6, 12, 15] that if θ is an irrational number of bounded density, then $H_{\lambda,\theta,\rho}$ has purely α_H-Hausdorff continuous spectrum for $\alpha_H = 2\gamma_1/(\gamma_1 + \gamma_2)$, where $\gamma_1, \gamma_2 > 0$ satisfy relation (1.4).

Due to the way that τ_1 and τ_2 were obtained in Propositions 3.1 and 3.4, we note that these estimates have important relations with γ_1, γ_2. More specifically, we observe that $\tau_1 > \gamma_1$ and $\tau_2 < \gamma_2$; consequently, it follows that $\alpha_P > \alpha_H$ from such estimates.

This observation is verified by rewriting the proof of [5, Proposition 2.1], which has $\gamma_1 \equiv \tau_1 - \varepsilon$, with $\varepsilon \in (((\ln C_{\theta,1} - \ln C_{\theta,2})/\ln C_{\theta,1})\tau_1, \tau_1)$ and $C_{\theta,2}^n \leq q_n \leq C_{\theta,1}^n$.

The estimate for γ_2 was obtained by [12, Corollary 10], where one has $\gamma_2 \geq 1/2 + (4/\ln 2)\ln J_1$, with J_1 as in Lemma 3.5 above. Therefore,

$$\gamma_2 \geq \frac{1}{2} + \frac{4}{\ln 2}\ln J_1 > \frac{1}{2} + \ln J_1 = \tau_2.$$

Remark 3.8. We present an example of an irrational number $\theta = [0; a_1, a_2, \ldots]$ of quasibounded density that is not of bounded density. Let

$$A_n = \frac{1}{n}\sum_{i=1}^{n} a_i;$$

it is enough to build (a_n) and a subsequence of indices $(n_j)_{j\in\mathbb{N}}$ such that $A_{n_j} \leq 2$ and $A_{n_j+1} \geq j$, for all j.

For this purpose write $n_1 = 2$ and take $a_1 = a_2 = 1$, $a_3 = 3$; choose $n_2 = 5$ with $a_4 = a_5 = 1$, $a_6 = 2 * 6$; now consider $n_3 = 6A_6 = 19$ and take $a_7 = \ldots = a_{19} = 1$, $a_{20} = 3 * 20$. Proceeding this way, we obtain the subsequence $(n_j)_{j\in\mathbb{N}}$ defined recursively by

$$n_{j+1} = A_{n_j+1}(n_j + 1), \quad \forall j > 2,$$

with the terms (a_n) given by

$$a_n = \begin{cases} j(n_j + 1), & n \in J, \\ 1, & n \notin J, \end{cases}$$

where $J = \{n_j + 1 : j \in \mathbb{N}\}$.

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4. Stability of spectral packing dimension

We present in this section stability results of spectral packing dimensional properties for some discrete Schrödinger operators of the form (1.1) under suitable (real) polynomially decaying perturbations $P = \{P(n)\}$, that is, when V is replaced by $V + P$. The results obtained here are analogous to the results presented in [1] for the Hausdorff dimensional setting. As in [1], we are interested in energies in the set

$$S(H) := \{E \mid \exists\, \varphi \text{ s.t. } u_{1,\varphi,E} \text{ is a subordinate solution to (1.2) and } u_{1,\varphi,E} \notin l^2(\mathbb{Z}^+)\}.$$

It is known [19] that, for any $\varphi \in [-\pi/2, \pi/2)$, the singular continuous part of the spectral measure of H_φ is supported in $S(H)$. In the case of whole-line operators, $S(H)$ should be defined as [10]

$$\{E \mid \exists \text{ a solution to (1.2) which is subordinate at both ends } \pm\infty \text{ and which is not in } l^2(\mathbb{Z})\},$$

and the singular continuous parts of the spectral measures are supported in this set; note that if no solution to (1.2) satisfies such condition at one end, then the corresponding energy E does not belong to the singular continuous component.

We have the following theorem.

Theorem 4.1. *Let $E \in S(H)$ and $u_{1,\varphi,E}$, $u_{2,\varphi,E}$ be solutions to (1.2) satisfying (1.3). Suppose that there exist positive constants $C_1, C_2, \gamma_1, \gamma_2$ such that every solution to (1.2) with NIC obeys the estimates (1.4) for $L > 0$ sufficiently large. Suppose also that, for every $n \in \mathbb{N}$ and $p > 3\gamma_2 - \gamma_1$, there exists a positive constant C_3 such that*

$$|P(n)| \le C_3(1+n)^{-p}.$$

Then $E \in S(H + P)$ and, for all $\kappa \in [0, 1]$,

$$\limsup_{L\to\infty} \frac{\|u_{1,\varphi,E}\|_L}{\|u_{2,\varphi,E}\|_L^\kappa} = \limsup_{L\to\infty} \frac{\|v_{1,\tilde\varphi,E}\|_L}{\|v_{2,\tilde\varphi,E}\|_L^\kappa},$$

where $v_{1,\tilde\varphi,E}$ is the solution to (1.2) for $H + P$, which satisfies the initial condition (1.3) with some phase $\tilde\varphi$, and $v_{2,\tilde\varphi,E}$ satisfying the corresponding orthogonal condition (always for the operator $H + P$).

Proof. The proof of this theorem is analogous to the proof of [1, Theorem 1.3], with simple modification to the upper limit. □

We emphasize that condition (1.4) is essential in Theorem 4.1 (that is, bounds as in (1.5) are not enough for the result). Therefore, we have considered, in Corollary 1.6, Schrödinger operators with Sturmian potentials whose rotation number is of bounded density.

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Proof of Corollary 1.6. The proof of Corollary 1.6 is analogous to the proof of [1, Theorem 1.1]; however the constant α_P-packing continuous obtained in Remark 3.7 is preserved. More specifically, we have, by [5, 6, 12], that for Schrödinger operators with Sturmian potentials whose rotation numbers are of bounded density, there exist power-law bounds of the form (1.4) for every solution u to (1.2) (with NIC). We have, by Theorem 1.5, that if θ is of bounded density (Remark 3.7) then, for every $\lambda \neq 0$, there exists $\alpha_P = \alpha(\lambda, \theta) > 0$ such that, for every $\rho \in [0, 1)$, the spectral measure $H_{\lambda,\theta,\rho}$ is purely α-packing continuous, with $\alpha_P = 2\tau_1/(\tau_1 + \tau_2)$.

We note again [5, 6] that if one is able to establish uniform power-law bounds on the restriction of the operator to the right half-line, then the resulting α-continuity is independent of the potential on the left half-line.

Suppose that the spectrum $\sigma(H^P_{\lambda,\theta,\rho})$ has some singular continuous component (that is, $\sigma_{\rm sc}(H^P_{\lambda,\theta,\rho}) \neq \emptyset$); now, since the perturbation decays as $|P(n)| \leq C(1 + |n|)^{-p}$, with $p > 3\gamma_2 - \gamma_1$, it is a compact perturbation and the essential spectrum is preserved; thus $\sigma_{\rm sc}(H^P_{\lambda,\theta,\rho})$ is contained in the spectrum $\sigma(H_{\lambda,\theta,\rho})$.

Note that $S(H_{\lambda,\theta,\rho})$ cannot be a proper subset of $\sigma(H_{\lambda,\theta,\rho})$, because by relation (1.4) the operator $H_{\lambda,\theta,\rho}$ has no solution u to (1.2) (with NIC) in $l^2(\mathbb{Z})$. And it has been shown in [21, 22, 24] that, for large λ, the spectrum of $H_{\lambda,\theta,\rho}$ (as a set) has Hausdorff dimension strictly less than 1; this implies that for such λs the spectrum is also β-Hausdorff singular for some $\beta < 1$, so $H_{\lambda,\theta,\rho}$ has a β-Hausdorff subordinate solution; in particular, this solution is subordinate.

Therefore, this singular continuous component is supported in $S(H_{\lambda,\theta,\rho})$ and, by Theorem 4.1, we obtain that the asymptotic behavior of generalized eigenfunctions of the operators $H^P_{\lambda,\theta,\rho}$ (that is, the solutions to (1.2) in (1.7)) is the same as the eigenfunctions of the unperturbed operators $H_{\lambda,\theta,\rho}$; and again, by the α-subordinacy theory (Theorem 2.1), such component is still α_P-packing continuous for these perturbed operators, with $\alpha_P = 2\tau_1/(\tau_1 + \tau_2)$. □

According to results in [1], we can also apply Theorem 4.1 to operators with sparse potentials. We reconsider here the class of operators H^α_φ [3, 14, 27] defined by the action (1.1) in $l^2(\mathbb{Z}^+)$, along with a phase boundary condition

$$\psi(0)\cos\varphi + \psi(1)\sin\varphi = 0, \quad \varphi \in (-\pi/2, \pi/2],$$

and, for each $\alpha \in (0, 1)$, sparse potentials

$$V(n) = \begin{cases} x_j^{(1-\alpha)/2\alpha}, & n = x_j \in \mathcal{B}, \\ 0, & n \notin \mathcal{B}, \end{cases}$$

where $\mathcal{B} = (x_j)_j = (2^{j^j})_j$. It is known that the restriction of its spectral measure to the interval $(-2, 2)$ is 1-packing dimensional [3, 4] for every boundary phase φ.

Theorem 4.2. *Fix $\alpha \in (0, 1)$. Let H^α_φ be as above and*

$$(H^{P,\alpha}_\varphi \psi)(n) := (H^\alpha_\varphi \psi)(n) + P(n)\psi(n), \quad \psi \in l^2(\mathbb{Z}^+),$$

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with $|P(n)| \le C(1+n)^{-p}$ *for all* n *and some* $C > 0$, $p > \min\{3/(2\alpha), (2-\alpha)/\alpha\}$. *Then, the restriction of the spectral measure of the operator* $H_\varphi^{P,\alpha}$ *to* $(-2, 2)$ *is also* 1*-packing dimensional, for all boundary phase* $\varphi \in (-\pi/2, \pi/2]$.

Proof. The proof of Theorem 4.2 follows the same steps as the proof of [1, Theorem 1.2], following the same lines as in Corollary 1.6. □

A. Hausdorff and packing dimensions

In this appendix we recall some definitions and concepts regarding Hausdorff and packing measures, and fix notation. Most of the material presented here is based on [3, 9, 14, 20, 23, 25].

Definition A.1. Given a set $S \subset \mathbb{R}$ and $\alpha \in [0, 1]$, consider the number

$$Q_{\alpha,\delta}(S) = \inf\Big\{\sum_{k=1}^{\infty} |I_k|^\alpha \mid |I_k| < \delta, \forall k; S \subset \bigcup_{k=1}^{\infty} I_k\Big\},$$

with the infimum taken over all covers of S by intervals I_k of size at most δ. The limit

$$h^\alpha(S) = \lim_{\delta \to 0} Q_{\alpha,\delta}(S)$$

is called the α-dimensional Hausdorff measure of S.

The α-dimensional Hausdorff measure, h^α, is an outer measure on subsets of $\mathbb{R}$ [25]. It is known that, for every set S, there is a unique α_S such that $h^\alpha(S) = 0$ if $\alpha > \alpha_S$ and $h^\alpha(S) = \infty$ if $\alpha_S < \alpha$. The number α_S is called the Hausdorff dimension of the set S, usually denoted by $\dim_H(S)$. Particular examples of h^α are the counting measure for $\alpha = 0$ and the Lebesgue measure for $\alpha = 1$.

Now the definition of packing measure. A δ-packing of an arbitrary set $S \subset \mathbb{R}$ is a countable disjoint collection $(B(x_k, r_k))_{k\in\mathbb{N}}$ of closed intervals centered at $x_k \in S$ with radius $r_k \le \delta/2$. The (α, δ)-premeasure $P_\delta^\alpha(S)$ is defined by

$$P_\delta^\alpha(S) = \sup\Big\{\sum_{k=1}^{\infty} (2r_k)^\alpha : (B(x_k, r_k))_{k\in\mathbb{N}} \text{ is a } \delta\text{-packing of } S\Big\},$$

the supremum taken over all δ-packings of S.

Definition A.2. The α-packing measure $P^\alpha(S)$ of S is constructed by a procedure in two steps: first, take the decreasing limit

$$\underline{P}^\alpha(S) = \lim_{\delta \to 0} P_\delta^\alpha(S),$$

and then

$$P^\alpha(S) = \inf\Big\{\sum_{k=1}^{\infty} \underline{P}^\alpha(S_k) : S \subset \bigcup_{k=1}^{\infty} S_k, S_k \text{ disjoint Borel sets}\Big\}.$$

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It follows, by Definition A.2, that $P^\alpha(S)$ is an outer measure on $\mathbb{R}$. The so-called packing dimension of the set S, denoted by $\dim_P(S)$, is defined as the infimum of all α such that $P^\alpha(S) = 0$, which coincides with the supremum of all α with $P^\alpha(S) = \infty$. It is possible to show (see [9]) that the Hausdorff and packing dimensions are related by the inequality $\dim_H(S) \leq \dim_P(S)$.

Definition A.3. Let μ be a Borel measure in $\mathbb{R}$ and $\alpha \in [0, 1]$.

(i) μ is called α-Hausdorff (α-packing) continuous if $\mu(S) = 0$ for every Borel set S with $h^\alpha(S) = 0$ (respectively, $P^\alpha(S) = 0$).
(ii) μ is called α-Hausdorff (α-packing) singular if μ is supported on some Borel set S, that is, $\mu(\mathbb{R}\backslash S) = 0$ with $h^\alpha(S) = 0$ (respectively, $P^\alpha(S) = 0$).

Definition A.4. A Borel measure μ in $\mathbb{R}$ is said to have exact Hausdorff (packing) dimension α, for some $\alpha \in (0, 1)$, and denoted by $\dim_H(\mu)$ (respectively, $\dim_P(\mu)$), if two requirements hold:

(i) for every set S with $\dim_H(S) < \alpha$ (respectively, $\dim_P(S) < \alpha$), one has $\mu(S) = 0$;
(ii) there is a Borel set, S_0, of Hausdorff (respectively, packing) dimension α, which supports μ.

A Borel measure μ in $\mathbb{R}$ is said to be 0-Hausdorff (0-packing) dimensional if it is supported on a set with $\dim_H(S) = 0$ (respectively, $\dim_P(S) = 0$) and 1-Hausdorff (1-packing) dimensional if $\mu(S) = 0$ for any set S with $\dim_H(S) < 1$ (respectively, $\dim_P(S) < 1$).

Remark A.5. According to Definitions A.3 and A.4, a Borel measure μ in $\mathbb{R}$ is of exact Hausdorff (packing) dimension α if, for every $\varepsilon > 0$, it is simultaneously $(\alpha - \varepsilon)$-Hausdorff (respectively, packing) continuous and $(\alpha + \varepsilon)$- Hausdorff (respectively, packing) singular.

Given a finite Borel measure μ and $\alpha \in [0, 1]$, write

$$(\overline{D}^\alpha \mu)(E) := \limsup_{\varepsilon \to 0} \frac{\mu((E - \varepsilon, E + \varepsilon))}{(2\varepsilon)^\alpha} \quad \text{and} \quad (\underline{D}^\alpha \mu)(E) := \liminf_{\varepsilon \to 0} \frac{\mu((E - \varepsilon, E + \varepsilon))}{(2\varepsilon)^\alpha}.$$

Theorem A.6. *Let $\alpha \in [0, 1]$ and μ a Borel measure on $\mathbb{R}$, and denote*

$$T^\alpha_\infty = \{E \in \mathbb{R} : (\overline{D}^\alpha \mu)(E) = \infty\}, \quad U^\alpha_\infty = \{E \in \mathbb{R} : (\underline{D}^\alpha \mu)(E) = \infty\}.$$

Then T^α_∞ and U^α_∞ are Borel sets, and

(1) $h^\alpha(T^\alpha_\infty) = 0$;
(2) $P^\alpha(U^\alpha_\infty) = 0$;
(3) $\mu(S \cap (\mathbb{R}\backslash T^\alpha_\infty)) = 0$, *for any S with* $h^\alpha(S) = 0$;
(4) $\mu(S \cap (\mathbb{R}\backslash U^\alpha_\infty)) = 0$, *for any S with* $P^\alpha(S) = 0$.

Proof. Items (1) and (3) are well known and proved in [25, Ch. 3], and the proofs of items (2) and (4) are in [3]. □

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REMARK A.7. The restriction $\mu_{\alpha Hs} := \mu(T_\infty^\alpha \cap \cdot)$ is α-Hausdorff singular, $\mu_{\alpha Ps} := \mu(U_\infty^\alpha \cap \cdot)$ is α-packing singular; and $\mu_{\alpha Hc} := \mu((\mathbb{R}\backslash T_\infty^\alpha) \cap \cdot)$ is α-Hausdorff continuous, $\mu_{\alpha Pc} := \mu((\mathbb{R}\backslash T_\infty^\alpha) \cap \cdot)$ is α-packing continuous. Thus, each measure decomposes uniquely into an α-Hausdorff (packing) continuous part and an α-Hausdorff (packing) singular part: $\mu = \mu_{\alpha Hs} + \mu_{\alpha Hc}$ (respectively, $\mu = \mu_{\alpha Ps} + \mu_{\alpha Pc}$).

Moreover, an α-Hausdorff (packing) singular measure is such that $(\overline{D}^\alpha\mu)(E) = \infty$ (respectively, $(\underline{D}^\alpha\mu)(E) = \infty$) almost everywhere (with respect to it), while an α-Hausdorff (packing) continuous measure is such that $(\overline{D}^\alpha\mu)(E) < \infty$ (respectively, $(\underline{D}^\alpha\mu)(E) < \infty$) almost everywhere (see [25, Ch. 3] and [3]).

B. Upper bounds of solutions: $\rho \in [0, 1)$

In this appendix we present a proof of Proposition 3.4 for every $\rho \in [0, 1)$. We consider a potential of the form

$$V_{\lambda,\theta,\rho}(n) = \lambda\chi_{[1-\theta,1)}(n\theta + \rho \bmod 1), \tag{B.1}$$

with $0 \neq \lambda \in \mathbb{R}$, $\theta \in [0, 1)$ an irrational number and $\rho \in [0, 1)$; then we set

$$\mathcal{W}(\theta) \equiv \bigcup_{\rho\in[0,1)} \mathrm{Sub}(V_{\lambda,\theta,\rho}),$$

where $V_{\lambda,\theta,\rho}$ are words over $\{0, \lambda\}$ and $\mathrm{Sub}(w)$ denotes the set of all finite nonempty subwords of w. It is well known that $\mathrm{Sub}(V_{\lambda,\theta,\rho})$ does not depend on ρ (see Proposition B.2).

The set $\mathcal{W}(\theta)$ is, by its very definition, particularly appropriate to the study of uniform local properties of this family of operators. We also note that in order to prove Proposition 3.4 for each $\rho \in [0, 1)$, we need upper bounds for the solutions to (1.2) associated with the operator $H_{\lambda,\theta,\rho}$; this is equivalent to obtaining upper bounds for the norms of the transfer matrices

$$M(E, w) = \begin{pmatrix} E - w_n & -1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} E - w_1 & -1 \\ 1 & 0 \end{pmatrix},$$

for $w = w_1 \ldots w_n \in \mathcal{W}(\theta)$. If we denote, for $n \in \mathbb{N}$, the words

$$V^n_{\lambda,\theta,\rho} \equiv V_{\lambda,\theta,\rho}(1) \ldots V_{\lambda,\theta,\rho}(n),$$

then the matrices $M(E, V^n_{\lambda,\theta,\rho})$ are the usual transfer matrices associated with the solution to (1.2), as in (3.2).

In order to avoid cumbersome notation, we set $w = V_{\lambda,\theta,\rho}$; recall that $(n_j)_{j\in\mathbb{N}}$ is the same as in Proposition 3.1. Therefore, our main aim in this appendix is to prove the following result.

THEOREM B.1. *Suppose that θ is an irrational number of quasibounded density. Then there exist positive constants τ, C and a sequence $(n_j)_{j\in\mathbb{N}}$ such that, for each $E \in \sigma(H_{\lambda,\theta})$, one has*

$$\|M(E, w)\| \leq C q_{n_j}^\tau$$

for each word $w \in \mathcal{W}(\theta)$ such that $|w| \leq q_{n_j}$.

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We recall that for each potential of the form (B.1), the words S_n over the alphabet $\mathcal{A} = \{0, \lambda\}$, with $0 \neq \lambda \in \mathbb{R}$ fixed (as defined by (3.1)), have length q_n (that is, $|S_n| = q_n$).

By definition, for $n \geq 2$, S_{n-1} is a prefix of S_n. Therefore, the following limit exists:

$$c_\theta \equiv \lim_{n\to\infty} S_n.$$

The following proposition is an adapted version of [8, Propositions 2.1–2.3]; its proof follows the same lines as the proofs of these results.

Proposition B.2. *The following assertions hold.*

(i) *There exist palindromes π_n, $n \geq 2$, such that, for each $k \in \mathbb{N}$,*

$$S_{2k} = \pi_{2k}\lambda 0 \quad \textit{and} \quad S_{2k+1} = \pi_{2k}0\lambda.$$

(ii) *$V_{\lambda,\theta,0}$ restricted to $\{1, 2, 3, \ldots\}$ coincides with c_θ.*

(iii) *$\mathcal{W}(\theta) = \mathrm{Sub}(V_{\lambda,\theta,\rho}) = \mathrm{Sub}(V_{\lambda,\theta,0})$, for all $\theta \in [0, 1)$.*

We also need additional results from [8].

Lemma B.3 [8, Lemma 3.2]. *Let $w \in \mathcal{W}(\theta)$ be given. Then there exist $t \in \mathbb{N}$, a suffix x of S_t or S_{t-1}, and a prefix y of S_{t+1}, such that $w = xy$.*

Lemma B.4 [8, Lemma 5.1]. *For $w = w_1 \ldots w_n$, $w_i \in \mathcal{A}$, define $w^R = w_n \ldots w_1$. Then, for each $E \in \mathbb{C}$,*

$$\|M(E, w)\| = \|M(E, w^R)\|.$$

Lemma B.5. *Suppose that θ is an irrational number of quasibounded density. Then there exist constants $\tau > 0, C \geq 1$, and a sequence $(n_j)_{j\in\mathbb{N}}$ such that, for each $E \in \sigma(H_{\lambda,\theta})$, one has that*

$$\|M(E, w)\| \leq C q_{n_j}^{\tau}$$

holds for each prefix w of c_θ, with $|w| \leq q_{n_j}$.

Proof. We have, by Proposition B.2, that for every prefix w of c_θ, with $|w| \leq q_{n_j}$,

$$M(E, w) = M(E, V_{\lambda,\theta,0}(1) \ldots V_{\lambda,\theta,0}(m)) =: M(m),$$

with $1 \leq m \leq q_{n_j}$, where $(n_j)_{j\in\mathbb{N}}$ is the sequence obtained in Lemma 3.3, satisfying relation (3.3).

We have, by Lemmas 3.5 and 3.6, that there exists a positive constant J_1 such that

$$\|M(m)\| \leq J_1^{2\sum_{i=1}^{n_j} a_i}.$$

Therefore, by Lemma 3.3, there are positive constants $\tau \geq \ln J_1$ and $C \geq 1$ satisfying

$$\|M(m)\| \leq C q_{n_j}^{\tau}. \qquad \square$$

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Proof of Theorem B.1. Fix $w \in \mathcal{W}(\theta)$ with $|w| \le q_{n_j}$. By Lemma B.3, there exist $t \in \mathbb{N}$, a suffix x of S_t or S_{t-1}, and a prefix y of S_{t+1} such that $w = xy$.

We will consider the case $|y|, |x| \ge 1$; the submultiplicativity of the norm $\|\cdot\|$ implies

$$\|M(E, w)\| \le \|M(E, x)\|\|M(E, y)\|.$$

Next, we present estimates of $\|M(E, x)\|$ and $\|M(E, y)\|$. As y is a prefix of s_{t+1} and so, *a fortiori*, a prefix of c_θ, we can use Lemma B.5 to obtain

$$\|M(E, y)\| \le \hat{C} q_{n_j}^{\hat{\tau}}, \tag{B.2}$$

since $|y| \le q_{n_j}$.

Now let x be such that $|x| \in \{1, 2\}$; then

$$\|M(E, x)\| \le F^2 \le F^2 q_{n_j}^{\hat{\tau}}, \tag{B.3}$$

where, for $a, b \in \{0, \lambda\}$,

$$F \equiv \max\{1, \sup\{M(E, a), E \in \sigma(H_{\lambda,\theta})\}, \sup\{M(E, b), E \in \sigma(H_{\lambda,\theta})\}\}.$$

Otherwise, since x is a suffix of S_t, it follows by Proposition B.2 that

$$x^R = bav, \tag{B.4}$$

where v is a prefix of c_θ. We have, by Lemma B.4, the definition of F, and (B.4),

$$\|M(E, x)\| = \|M(E, x^R)\| \le F^2 \hat{C} |v|^{\hat{\tau}} \le F^2 \hat{C} q_{n_j}^{\hat{\tau}} \tag{B.5}$$

for some $\hat{C} \ge 1$. So, by combining (B.2), (B.3), and (B.5),

$$\|M(E, w)\| \le F^2 \hat{C}^2 q_{n_j}^{2\hat{\tau}}.$$

The result now follows by setting $C = F^2 \hat{C}^2$ and $\tau = 2\hat{\tau}$ in the previous estimate. □

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