Proceedings of the Royal Society of Edinburgh, **134A**, 1199–1217, *2004*

Existence and non-existence of global solutions of diffusion systems with nonlinear boundary conditions

Yuanwei Qi
Department of Mathematics, University of Central Florida,
Orlando, FL 32816, USA
(yqi@pegasus.cc.ucf.edu)

Mingxin Wang
Department of Applied Mathematics, Southeast University,
Nanjing 210018, People's Republic of China
(mxwang@seu.edu.cn)

Zejun Wang
Institute of Mathematics, Fudan University,
Shanghai 200433, People's Republic of China
(wangzejun@xinhuanet.com)

(MS received 5 December 2002; accepted 17 June 2004)

This paper deals with existence and non-existence of global solutions of certain fast–slow diffusion systems with nonlinear boundary conditions. Necessary and sufficient conditions for global existence of positive solutions are obtained in terms of various parameters which appear explicitly in the definition of the systems.

1. Introduction

In this paper we study the following fast–slow diffusion systems with nonlinear boundary conditions:

$$\begin{cases} (u_i^{m_i})_t = \Delta u_i, & x \in \Omega, \quad t > 0, \\ \dfrac{\partial u_i}{\partial \eta} = \displaystyle\prod_{j=1}^{n} u_j^{m_{ij}}, & x \in \partial\Omega, \quad t > 0, \\ u_i(x,0) = u_{i0}(x) > 0, & x \in \bar{\Omega}, \end{cases} \tag{1.1}$$

for $1 \leqslant i \leqslant n$, where $\Omega \subset R^N$ is a bounded domain with smooth boundary $\partial\Omega$, η is the unit outward normal vector on $\partial\Omega$, the exponents m_i are positive and indices m_{ij} are non-negative, $i,j = 1,\dots,n$. In addition, initial data $u_{i0}(x) \in C^1(\bar{\Omega})$, $(1 \leqslant i \leqslant n)$ are positive functions and satisfy the compatibility conditions.

When $n = 1$, we have the familiar equation $(u^m)_t = \Delta u$, or $v_t = \Delta v^{1/m}$. It is clear that $m > 1$ corresponds to the fast diffusion equation, whereas $m < 1$ the porous medium equation (PME) or slow diffusion equation. But, since the initial value is positive in Ω, we encounter no difficulty of the typical PME problem when one is forced to deal with the weak solution due to the finite speed propagation of initial disturbance whose support is strictly contained inside Ω.

Likewise, for the system under consideration, we say that (1.1) is a fast diffusion system if $m_i \geqslant 1$ for all i, a fast–slow diffusion system if there exist i and j such that $m_i > 1$ but $m_j < 1$, and a slow diffusion system if $m_i \leqslant 1$ for all i.

The system models diffusion of n different physical substances that are linked by the influx of energy input at the boundary. Our primary concern is to work out conditions on the exponents m_i and indices m_{ij} so that every solution with the given positive C^1 initial data exists globally or blows up in finite time.

We note that most previous works deal with special cases such as $n = 1$ (see [5]) or $n = 2$ (see [2,4,6,7]). For systems involving more than two equations, the special case of $m_i = 1$, $1 \leqslant i \leqslant n$, is discussed in [8], whereas the slow diffusion case is studied in [9].

In this paper, we focus on two remaining cases: (i) fast diffusion and (ii) fast–slow diffusion, which are our standing assumptions unless otherwise stated. We write $M = (m_{ij})_{n\times n}$, $b_i = \min\{m_i, \frac{1}{2}(m_i + 1)\}$, $b_{ij} = b_i\delta_{ij}$, $i, j = 1, \dots, n$, $B = (b_{ij})_{n\times n}$, $F = B - M$. The main results of this paper are as follows.

Theorem 1.1. *If all the principal minor determinants of F are non-negative, then every solution of (1.1) exists globally.*

Theorem 1.2. *If one of the following three conditions holds, then every solution of (1.1) blows up in finite time.*

(i) *There exists $i : 1 \leqslant i \leqslant n$ such that $b_i < m_{ii}$.*

(ii) *F has a negative principal minor determinant in which $b_i \leqslant 1$.*

(iii) *There exists a negative principal minor determinant of F and $\Omega = B(0, R)$ is the ball in R^N centred at the origin with radius R.*

Remark 1.3. For the special case $\Omega = B(0, R)$, these two theorems show that all solutions of (1.1) exist globally if and only if all the principal minor determinants of F are non-negative.

Remark 1.4. It is clear from the statement of our results that the global existence or blow-up depends entirely on the exponents m_i, $1 \leqslant i \leqslant n$, and m_{ij}, $1 \leqslant i, j \leqslant n$. It is a very different situation when one considers the same kind of systems in an unbounded domain, for example, in the whole space R^N. The spatial dimension plays a very important role in determining global existence or blow-up for the unbounded domain case. We refer the interested reader to the survey paper [1].

Remark 1.5. A very interesting feature of our results is that the fine structure of the matrix F, which is a nonlinear function of the exponents m_i and indices m_{ij}, $1 \leqslant i, j \leqslant n$, is crucially important. Therefore, the tools to study our problem are a combination of algebraic matrix theory and analytical partial-differential-equation theory.

We note that, using the method of [2], it can be proved that if there exists $T < \infty$ such that the solution $(u_1(\cdot, t), \dots, u_n(\cdot, t))$ of (1.1) exists on the interval $[0, T)$, but cannot be extended beyond time T, then

$$\lim_{t \to T^-} \sup \sum_{i=1}^{n} \|u_i(\cdot, t)\|_\infty = +\infty,$$

i.e. $(u_1, \dots, u_n)$ blows up in finite time. Moreover, the comparison principle for the positive upper and lower solutions holds (for details, see [2]). Consequently, $u_i(x, t) \geqslant \delta$, $1 \leqslant i \leqslant n$, where $\delta = \min_{1 \leqslant i \leqslant n} \min_{\bar{\Omega}} u_{i0}(x) > 0$.

2. Preliminaries

In this section, we prove some preliminary results which will be used in the proof of our main theorems.

NOTATION 2.1.

(i) $|F| = \det F$ is the determinant of matrix F.

(ii) F_k^l is the square sub-matrix of F with entries f_{ij}, $i, j = k, \dots, l$.

(iii) $\bar{F}_k^l$ is sub-matrix of F made of f_{ij}, $i \in \{k, \dots, l\}$, $j \in \{1, \dots, k-1, l+1, \dots, n\}$.

LEMMA 2.2 (cf. [3]). *Suppose that A is a non-negative matrix. If A is irreducible, then A has a positive eigenvalue λ which is the largest, i.e. $|\mu| \leqslant \lambda$ for any eigenvalue μ of A, and the corresponding eigenvector $\alpha = (\alpha_1, \dots, \alpha_n)^{\mathrm{T}}$ is positive, i.e. $\alpha_i > 0$ $(i = 1, \dots, n)$.*

LEMMA 2.3. *Assume that all the principal minor determinants of F are non-negative and $f_{nn} > 0$. Let $g_{ij} = f_{ij} - f_{in} f_{nj} / f_{nn}$ and $G = (g_{ij})_{(n-1)\times(n-1)}$. Then the following conclusions hold.*

(i) *Any kth-order $(1 \leqslant k \leqslant n-1)$ principal minor determinant of G has the same sign as some $(k+1)$th-order principal minor determinant of F, and vice versa.*

(ii) *$g_{ij} \leqslant 0$ for $i \neq j$.*

Proof. The second conclusion is obvious. To prove the first one, without loss of generality, we only prove that $|G_1^k|$ has the same sign as $|F_{(n)}^k|$, where

$$|F_{(n)}^k| = \begin{vmatrix} f_{11} & \cdots & f_{1k} & f_{1n} \\ \vdots & \ddots & \vdots & \vdots \\ f_{k1} & \cdots & f_{kk} & f_{kn} \\ f_{n1} & \cdots & f_{nk} & f_{nn} \end{vmatrix}.$$

Direct computation gives

$$|F_{(n)}^k| = f_{nn} |G_1^k|.$$

Since $f_{nn} > 0$, the conclusion holds. □

https://doi.org/10.1017/S030821050000370X Published online by Cambridge University Press

LEMMA 2.4 (cf. [8]). *Assume that all the principal minor determinants of F are non-negative. Then* $\det(F + xI) > 0$ *for any* $x > 0$.

LEMMA 2.5. *Suppose that all the lower-order principal minor determinants of F are positive. If $|F| = 0$, then there does not exist $\alpha = (\alpha_1, \dots, \alpha_n)^{\mathrm{T}} > 0$ such that*

$$F\alpha \geqslant 0, \ \neq 0,$$

where a vector $\beta \geqslant 0, \neq 0$ means that β is non-negative and non-trivial.

Proof. We use induction to prove this lemma. If $n = 1$, the conclusion is obvious. Suppose that the conclusion holds for $n-1$. By the hypothesis, we have that $f_{ii} > 0$ for $1 \leqslant i \leqslant n$. Suppose there exists a positive vector α such that $F\alpha \geqslant 0, \ \neq 0$, i.e.

$$\sum_{j=1}^{n} f_{ij}\alpha_j \geqslant 0 \quad \text{for } 1 \leqslant i \leqslant n,$$

and at least one of them is strict. Without loss of generality, we assume that the first one is strict. From the last equation, we get $\alpha_n \geqslant (\sum_{j=1}^{n-1} m_{nj}\alpha_j)/f_{nn}$. Substituting it into the other equations, we find that

$$\sum_{j=1}^{n-1} g_{1j}\alpha_j > 0, \quad \sum_{j=1}^{n-1} g_{ij}\alpha_j \geqslant 0, \quad 2 \leqslant i \leqslant n-1.$$

By lemma 2.3 and the inductive assumption, we get a contradiction. □

LEMMA 2.6. *Assume that all the principal minor determinants of F are non-negative. If F has a lower-order principal minor determinant which equals zero, then F is reducible.*

Proof. Without loss of generality, we assume that $|F_1^l| = 0$ and l is the smallest, i.e. for any $k < l$, all kth-order principle minor determinants of F are positive. (This implies that all lower-order principle minor determinants of F_1^l are positive.) It is obvious that $l < n$. Since $F = B - M$ and M is non-negative, there exist a non-negative matrix D and a positive constant μ such that $F = \mu I - D$. If F is irreducible, so is D. From lemma 2.2, we know that D has the largest positive eigenvalue λ and the corresponding eigenvector $\alpha = (\alpha_1, \dots, \alpha_n)^{\mathrm{T}}$ is positive. Therefore,

$$F\alpha = \mu I\alpha - D\alpha = (\mu - \lambda)\alpha.$$

By lemma 2.4, it follows that $\mu - \lambda \geqslant 0$. Consequently, $F\alpha \geqslant 0$. We write

$$\alpha^{(l)} = (\alpha_1, \dots, \alpha_l)^{\mathrm{T}}, \qquad \bar{\alpha}^{(l)} = (\alpha_{l+1}, \dots, \alpha_n)^{\mathrm{T}}.$$

Then we have

$$F_1^l\alpha^{(l)} + \bar{F}_1^l\bar{\alpha}^{(l)} \geqslant 0.$$

Because $f_{ij} \leqslant 0$ for all $1 \leqslant i \leqslant l, l+1 \leqslant j \leqslant n$, F is irreducible and $\alpha_i > 0$ for all i, it follows that

$$F_1^l\alpha^{(l)} \geqslant 0, \ \neq 0.$$

But this is in clear contradiction with lemma 2.5. □

Similar to the proof of proposition 5 of [8], it can be shown that the following result holds.

LEMMA 2.7. *Suppose all the principal minor determinants of F are non-negative. If F is irreducible, then there exists $\alpha = (\alpha_1, \dots, \alpha_n)^{\mathrm{T}}$, with $\alpha_i > 0$ $(1 \leqslant i \leqslant n)$, such that $F\alpha \geqslant 0$, i.e. $b_i\alpha_i - \sum_{j=1}^{n} m_{ij}\alpha_j \geqslant 0$.*

LEMMA 2.8. *Suppose that all the lower-order principal minor determinants of F are non-negative and F is irreducible. For any s $(1 \leqslant s \leqslant n)$ and positive constant Q, there exist positive constant θ and large positive constants L_i $(1 \leqslant i \leqslant n)$ such that*

$$\left.\begin{aligned} \theta\prod_{j=1}^{n} L_j^{f_{ij}} &\geqslant Q, \quad 1 \leqslant i \leqslant s, \\ \prod_{j=1}^{n} L_j^{f_{ij}} &\geqslant Q, \quad s+1 \leqslant i \leqslant n. \end{aligned}\right\} \tag{2.1}$$

Proof. When $s = n$, we choose $\theta = Q$ and let $L_i = L^{\alpha_i}$ for some fixed constant $L > 1$. Then (2.1) is equivalent to $F\alpha \geqslant 0$, $\alpha = (\alpha_1, \dots, \alpha_n)^{\mathrm{T}}$. From lemma 2.7, we know that the conclusion holds. When $s \leqslant n-1$, consider the following inequalities:

$$\left.\begin{aligned} \theta\prod_{j=1}^{n} L_j^{f_{ij}} &\geqslant Q, \quad 1 \leqslant i \leqslant s, \\ \prod_{j=1}^{n} L_j^{f_{ij}} &\geqslant Q, \quad s+1 \leqslant i \leqslant n-1, \\ \prod_{j=1}^{n} L_j^{f_{nj}} &= Q. \end{aligned}\right\} \tag{2.2}$$

Using lemma 2.6, we know that all the lower-order principal minor determinants of F are positive. Consequently, $f_{nn} > 0$. By direct computations, equations (2.2) can be reduced to the following inequalities,

$$\left.\begin{aligned} \theta\prod_{j=1}^{n-1} L_j^{g_{ij}} &\geqslant Q', \quad 1 \leqslant i \leqslant s, \\ \prod_{j=1}^{n-1} L_j^{g_{ij}} &\geqslant Q', \quad s+1 \leqslant i \leqslant n-1, \\ \prod_{j=1}^{n} L_j^{f_{nj}} &= Q, \end{aligned}\right\} \tag{2.3}$$

for some positive constant Q'. Because the first $n-1$ inequalities of (2.3) do not depend on L_n and $f_{nn} > 0$, by induction, we can prove that there exist a suitable positive constant θ and large positive constants L_i $(1 \leqslant i \leqslant n)$ such that (2.3) holds. In consequence, so does (2.2). □

Lemma 2.9. *Suppose that all the lower-order principal minor determinants of F are non-negative and $|F| < 0$. Then F is irreducible and there exists*

$$\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)^{\mathrm{T}},$$

with $\alpha_i > 0$ $(1 \leqslant i \leqslant n)$ and $\varepsilon \in (0,1)$, such that

$$m_i\alpha_i - \sum_{j=1}^{n} m_{ij}\alpha_j < -1 \quad \text{for } m_i \leqslant 1,$$

$$\frac{m_i+\varepsilon}{1+\varepsilon}\alpha_i - \sum_{j=1}^{n} m_{ij}\alpha_j < -1 \quad \text{for } m_i > 1,$$

$$m_i\alpha_i - \sum_{j=1}^{n} m_{ij}\alpha_j < -1 \quad \text{for } m_i > 1.$$

Proof. Similar to the proof of proposition 6 of [8], it can be shown that there exist $\alpha_i > 0$ $(1 \leqslant i \leqslant n)$ such that $b_i\alpha_i - \sum_{j=1}^{n} m_{ij}\alpha_j < 0$. If we take α_i $(1 \leqslant i \leqslant n)$ to be large enough, then $b_i\alpha_i - \sum_{j=1}^{n} m_{ij}\alpha_j < -1$. When $m_i \leqslant 1$, the conclusion holds. When $m_i > 1$, since $b_i = \frac{1}{2}(1+m_i)$, it is obvious that $\exists \varepsilon \in (0,1)$ such that $((m_i+\varepsilon)/(1+\varepsilon))\alpha_i - \sum_{j=1}^{n} m_{ij}\alpha_j < -1$. □

3. Proof of theorem 1.1

First we note that if F is reducible, then the full system can be reduced to several sub-systems, independent of each other. Therefore, in the following, we assume that F is irreducible. In consequence, lemma 2.8 holds. In addition, we suppose that $m_1 \geqslant m_2 \geqslant \cdots \geqslant m_n$. We divide our proof into four different cases.

Case 1. ($m_i > 1$, $1 \leqslant i \leqslant n$.) Let $\lambda_0 > 0$ and $\varphi(x)$ be the first eigenvalue and the corresponding eigenfunction of $-\Delta$ with homogeneous Dirichlet boundary conditions with $\max_{\bar{\Omega}} \varphi(x) = \frac{1}{2}$. Then $\varphi(x) > 0$ in Ω and $\partial\varphi/\partial\eta < 0$ on $\partial\Omega$. We write

$$\max_{\bar{\Omega}} |\nabla\varphi(x)| = c_1 \quad \text{and} \quad \min_{\partial\Omega} -\frac{\partial\varphi}{\partial\eta} = c_2 > 0.$$

Let

$$\bar{u}_i(x,t) = L_i A_i^{2/(1-m_i)}, \quad A_i(x,t) = \mathrm{e}^{-Mt} - \tfrac{1}{4}(1-\varphi(x))^{\theta p_i}, \quad 1 \leqslant i \leqslant n,$$

where

$$M = \max_{1 \leqslant i \leqslant n} \frac{\theta}{8m_i}\left[\lambda_0 + 2\theta c_1^2 + \frac{(m_i+1)c_1^2\theta}{2(m_i-1)}\right], \qquad p_i = {L_i}^{(m_i-1)/2} > 1,$$

and θ and $L_i (1 \leqslant i \leqslant n)$ will be determined later. We only assume $\theta p_i > 2$ for now. Take $T = M^{-1}\log 2$, $\mathrm{e}^{-Mt} \geqslant \frac{1}{2}$ for all $0 \leqslant t \leqslant T$. We do our analysis for $0 \leqslant t \leqslant T$.

By direct computations, we have

$$
\begin{aligned}
(\bar{u}_i^{m_i})_t &= \frac{2m_i M}{m_i-1} L_i^{m_i} A_i^{(3m_i-1)/(1-m_i)} \mathrm{e}^{-Mt},\\
\nabla \bar{u}_i &= \frac{\theta p_i}{2(1-m_i)} L_i A_i^{(1+m_i)/(1-m_i)} (1-\varphi(x))^{\theta p_i - 1} \nabla\varphi(x),\\
\Delta \bar{u}_i &= \frac{1}{m_i-1} L_i{}^{(m_i+1)/2}\\
&\qquad \times \Big[\tfrac{1}{2}\theta\lambda_0\varphi A_i^{(1+m_i)/(1-m_i)}(1-\varphi(x))^{\theta p_i-1}\\
&\qquad\qquad + \tfrac{1}{2}\theta(\theta p_i - 1)(1-\varphi(x))^{\theta p_i - 2}|\nabla\varphi(x)|^2 A_i^{(1+m_i)/(1-m_i)}\\
&\qquad\qquad\quad + \frac{\theta^2 p_i(m_1+1)}{8(m_i-1)}(1-\varphi(x))^{2(\theta p_i - 1)}|\nabla\varphi(x)|^2 A_i^{(2m_i)/(1-m_i)}\Big]\\
&= \frac{1}{m_i-1} L_i^{m_i} A_i^{(3m_i-1)/(1-m_i)}\\
&\qquad \times \Big[\tfrac{1}{2}\theta\lambda_0\varphi(1-\varphi(x))^{\theta p_i - 1} L_i^{(1-m_i)/2} A_i^2\\
&\qquad\qquad + \tfrac{1}{2}\theta(\theta p_i - 1)(1-\varphi(x))^{\theta p_i - 2}|\nabla\varphi(x)|^2 L_i^{(1-m_i)/2} A_i^2\\
&\qquad\qquad\qquad + \frac{\theta^2(m_i+1)}{8(m_i-1)}(1-\varphi(x))^{2(\theta p_i-1)} A_i |\nabla\varphi(x)|^2\Big].
\end{aligned}
$$

Since $A_i \leqslant \mathrm{e}^{-Mt} \leqslant 1$ and $(\theta p_i - 1)L_i^{(1-m_i)/2} = \theta - L_i^{(1-m_i)/2} < \theta$, we get

$$
\Delta \bar{u}_i \leqslant \frac{1}{m_i-1} L_i^{m_i} A_i^{(3m_i-1)/(1-m_i)} \mathrm{e}^{-Mt}\Big[\tfrac{1}{4}\lambda_0\theta + \tfrac{1}{2}\theta^2 c_1^2 + \frac{(m_i+1)c_1^2\theta^2}{8(m_i-1)}\Big].
$$

Thus we have, by the choice of M,

$$
(\bar{u}_i^{m_i})_t \geqslant \Delta \bar{u}_i. \tag{3.1}
$$

It is clear that $\frac{1}{4} \leqslant A_i \leqslant 1$ for all $(x,t) \in \bar{\Omega} \times [0,T]$. Consequently, for $x \in \partial\Omega$ and $1 \leqslant i \leqslant n$,

$$
\begin{aligned}
\frac{\partial \bar{u}_i}{\partial \eta} &= L_i^{(m_i+1)/2} \frac{\theta}{2(m_i-1)}(1-\varphi(x))^{\theta p_i - 1}\Big(-\frac{\partial\varphi}{\partial\eta}\Big) A_i^{(1+m_i)/(1-m_i)}\\
&\geqslant \frac{\theta c_2}{2(m_i-1)} L_i^{(m_i+1)/2}
\end{aligned}
$$

and

$$
\begin{aligned}
\prod_{j=1}^{n} \bar{u}_j^{m_{ij}} &= \prod_{j=1}^{n} L_j{}^{m_{ij}} A_j^{(2m_{ij})/(1-m_j)}\\
&\leqslant \prod_{j=1}^{n} L_j{}^{m_{ij}} (16)^{\sum_{j=1}^{n} m_{ij}/(m_j-1)}.
\end{aligned}
$$

https://doi.org/10.1017/S030821050000370X Published online by Cambridge University Press

Then we have

$$\frac{\partial \bar{u}_i}{\partial \eta} \geqslant \prod_{j=1}^{n} \bar{u}_j^{m_{ij}}, \quad x \in \partial\Omega, \quad 1 \leqslant i \leqslant n, \tag{3.2}$$

if

$$\frac{\theta c_2}{2(m_i-1)} L_i^{(m_i+1)/2} \prod_{j=1}^{n} L_j^{-m_{ij}} \geqslant (16)^{\sum_{j=1}^{n} m_{ij}/(m_j-1)}, \quad 1 \leqslant i \leqslant n. \tag{3.3}$$

From lemma 2.8, we know that inequality (3.3) holds for suitable choices of θ and L_i $(1 \leqslant i \leqslant n)$. Moreover, if we choose L_i to be large enough, then

$$\bar{u}_i(x,0) \geqslant L_i \geqslant u_{i0}(x), \quad x \in \bar{\Omega}, \quad 1 \leqslant i \leqslant n. \tag{3.4}$$

From (3.1), (3.2) and (3.4), we know that $(\bar{u}_1, \dots, \bar{u}_n)$ is an upper solution of (1.1) on $\Omega \times [0,T]$. By the comparison principle, it follows that $u_i(x,t) \leqslant \bar{u}_i(x,t)$ $(1 \leqslant i \leqslant n)$ on $\bar{\Omega} \times [0,T]$, and hence $(u_1, \dots, u_n)$ exists on $[0,T]$. Since T is a constant and does not depend on the initial data $u_{i0}(x)$, $1 \leqslant i \leqslant n$, we can prove that $(u_1, \dots, u_n)$ exists on $[T, 2T]$ by a similar argument to the above. Repeating this procedure, we get that $(u_1, \dots, u_n)$ exists globally by a standard *continuation-of-solutions method*.

In the following, we always take $T = M^{-1} \log 2$. From the above discussion, it is clear that we need only prove that solution of (1.1) exists on $[0,T]$.

Case 2. ($m_i > 1$ for $1 \leqslant i \leqslant s$; $m_i = 1$ for $i = s+1, \dots, n$.) Let

$$\bar{u}_i(x,t) = L_i A_i^{2/(1-m_i)} \quad \text{for } 1 \leqslant i \leqslant s, \tag{3.5}$$

$$\bar{u}_i(x,t) = L_i h(x) \mathrm{e}^{\sigma t} \qquad \text{for } s+1 \leqslant i \leqslant n, \tag{3.6}$$

where

$$A_i = \mathrm{e}^{-Mt} - \tfrac{1}{4}(1-\varphi(x))^{\theta p_i}, \qquad M = \max_{1 \leqslant i \leqslant s} \frac{\theta}{8m_i}\left[\tfrac{1}{4}\lambda_0 + 2\theta c_1^2 + \frac{(m_i+1)c_1^2\theta}{2(m_i-1)}\right],$$

θ and L_i $(1 \leqslant i \leqslant n)$ are determined as in lemma 2.8, $\sigma = |\partial\Omega|/c|\Omega|$, $p_i = L_i^{(m_i-1)/2}$, $1 \leqslant i \leqslant s$, and $h(x)$ is a positive solution of the linear elliptic equation

$$\Delta h = \frac{|\partial\Omega|}{|\Omega|}, \quad x \in \Omega,$$

$$\frac{\partial h}{\partial \eta} = 1, \qquad x \in \partial\Omega,$$

with $c \leqslant h(x) \leqslant d$ for some positive constants c and d. It is easy to prove that

$$(\bar{u}_i^{m_i})_t \geqslant \Delta \bar{u}_i, \quad 1 \leqslant i \leqslant n. \tag{3.7}$$

Now consider the boundary condition. For $1 \leqslant i \leqslant s$ and $x \in \partial\Omega$, we have

$$\frac{\partial \bar{u}_i}{\partial \eta} \geqslant \frac{\theta c_2}{2(m_i-1)} L_i^{(m_i+1)/2}.$$

For $s+1 \leqslant i \leqslant n$ and $x \in \partial\Omega$, we have

$$\frac{\partial \bar{u}_i}{\partial \eta} \geqslant L_i \mathrm{e}^{\sigma t} \frac{\partial h}{\partial \eta} \geqslant L_i.$$

For $1 \leqslant i \leqslant n$ and $x \in \partial\Omega$, we have

$$\prod_{j=1}^{n} \bar{u}_j^{m_{ij}} \leqslant \prod_{j=i}^{s} L_j^{m_{ij}} (16)^{\sum_{j=1}^{s} m_{ij}/(m_j-1)} \prod_{j=s+1}^{n} (L_j h(x) \mathrm{e}^{\sigma t})^{m_{ij}} \leqslant \prod_{j=i}^{n} L_j^{m_{ij}} Q,$$

where

$$Q = (16)^{\sum_{j=1}^{s} m_{ij}/(m_j-1)} d^{\sum_{j=s+1}^{n} m_{ij}} \exp\biggl\{\frac{\sigma}{M} \log 2 \sum_{j=s+1}^{n} m_{ij}\biggr\}.$$

Hence we get

$$\frac{\partial \bar{u}_i}{\partial \eta} \geqslant \prod_{j=1}^{n} \bar{u}_j^{m_{ij}}, \quad x \in \partial\Omega, \quad 1 \leqslant i \leqslant n, \tag{3.8}$$

if

$$\frac{\theta c_2}{2(m_i-1)} \prod_{j=1}^{n} L_j^{f_{ij}} \geqslant Q, \quad 1 \leqslant i \leqslant s, \tag{3.9}$$

and

$$\prod_{j=1}^{n} L_j^{f_{ij}} \geqslant Q, \quad s+1 \leqslant i \leqslant n. \tag{3.10}$$

From lemma 2.8, we know that both (3.9) and (3.10) hold if we choose suitable L_i $(1 \leqslant i \leqslant n)$ and θ. Moreover, if we choose L_i to be large enough, then

$$\bar{u}_i(x,0) \geqslant u_{i0}(x), \quad x \in \bar{\Omega}, \quad 1 \leqslant i \leqslant n. \tag{3.11}$$

Inequalities (3.7), (3.8) and (3.11) show that $(\bar{u}_1, \bar{u}_1, \dots, \bar{u}_n)$ is an upper solution of (1.1) in $\Omega \times [0,T]$. Therefore, $(u_1, \dots, u_n)$ exists on $[0,T]$.

Case 3. ($m_i > 1$ for $1 \leqslant i \leqslant s$; $m_i < 1$ for $s+1 \leqslant i \leqslant n$.) For $1 \leqslant i \leqslant s$, let $\bar{u}_i(x,t)$ be as in (3.5). For $s+1 \leqslant i \leqslant n$, let

$$\bar{u}_i(x,t) = L_i^{q_i} [B_i^{1/q_i} - (1-m_{ii})\varphi(x)]^{1/(1-m_{ii})}, \tag{3.12}$$

where

$$B_i = (2-m_{ii})^{q_i} + \biggl(\frac{m_i - m_{ii}}{m_i}\biggr) L_i^{q_i(1-m_i)} (m_{ii} c_1^2 + \tfrac{1}{2}\lambda_0) t, \qquad q_i = \frac{m_i - m_{ii}}{1 - m_{ii}}$$

and θ_i and L_i $(1 \leqslant i \leqslant n)$ are to be determined. From lemma 2.6, we know that $q_i > 0$, $s+1 \leqslant i \leqslant n$. Now we prove that $(\bar{u}_1, \dots, \bar{u}_n)$ is an upper solution of (1.1) in $\Omega \times [0,T]$. As in the proof of case 2, we find that

$$(\bar{u}_i^{m_i})_t \geqslant \Delta \bar{u}_i, \quad 1 \leqslant i \leqslant s. \tag{3.13}$$

https://doi.org/10.1017/S030821050000370X Published online by Cambridge University Press

For $s+1 \leqslant i \leqslant n$, we have

$$
\begin{aligned}
(\bar{u}_i^{m_i})_t &= L_i^{q_i}(m_{ii}c_1^2 + \tfrac{1}{2}\lambda_0)[B_i^{1/q_i} - (1-m_{ii})\varphi(x)]^{(m_i-1+m_{ii})/(1-m_{ii})} B_i^{1/q_i - 1}, \\
\nabla \bar{u}_i &= -L_i^{q_i}[B_i^{1/q_i} - (1-m_{ii})\varphi(x)]^{m_{ii}/(1-m_{ii})}\nabla\varphi(x), \\
\Delta \bar{u}_i &= m_{ii}L_i^{q_i}[B_i^{1/q_i} - (1-m_{ii})\varphi(x)]^{(2m_{ii}-1)/(1-m_{ii})}|\nabla\varphi(x)|^2 \\
&\qquad + \lambda_0 L_i^{q_i}[B_i^{1/q_i} - (1-m_{ii})\varphi(x)]^{m_{ii}/(1-m_{ii})}\varphi(x) \\
&\leqslant m_{ii}c_1^2[B_i^{1/q_i} - (1-m_{ii})\varphi(x)]^{(2m_{ii}-1)/(1-m_{ii})} \\
&\qquad + \tfrac{1}{2}\lambda_0 L_i^{q_i}[B_i^{1/q_i} - (1-m_{ii})\varphi(x)]^{m_{ii}/(1-m_{ii})}.
\end{aligned}
$$

Since $B_i \geqslant (2-m_{ii})^{q_i}$, it follows that

$$B_i^{1/q_i} - (1-m_{ii})\varphi(x) \geqslant 2 - m_{ii} - (1-m_{ii}) = 1.$$

Using $(2m_{ii}-1)/(1-m_{ii}) \leqslant m_{ii}/(1-m_{ii})$, we have

$$
\begin{aligned}
\Delta \bar{u}_i &\leqslant L_i^{q_i}(m_{ii}c_1^2 + \tfrac{1}{2}\lambda_0)[B_i^{1/q_i} - (1-m_{ii})\varphi(x)]^{m_{ii}/(1-m_{ii})} \\
&\leqslant L_i^{q_i}(m_{ii}c_1^2 + \tfrac{1}{2}\lambda_0)[B_i^{1/q_i} - (1-m_{ii})\varphi(x)]^{(m_{ii}-1+m_i)/(1-m_{ii})} \\
&\qquad \times [B_i^{1/q_i} - (1-m_{ii})\varphi(x)]^{(1-m_i)/(1-m_{ii})} \\
&\leqslant L_i^{q_i}(m_{ii}c_1^2 + \tfrac{1}{2}\lambda_0)[B_i^{1/q_i} - (1-m_{ii})\varphi(x)]^{(m_{ii}-1+m_i)/(1-m_{ii})} B_i^{1/q_i-1}.
\end{aligned}
$$

Thus

$$(\bar{u}_i^{m_i})_t \geqslant \Delta \bar{u}_i, \quad s+1 \leqslant i \leqslant n. \tag{3.14}$$

For $s+1 \leqslant i \leqslant n$, if we choose L_i large enough, it yields $(2-m_{ii})^{q_i} L_i^{q_i(m_i-1)} \leqslant 1$. Consequently, for $x \in \partial\Omega$,

$$
\begin{aligned}
\bar{u}_i(x,t) &\leqslant L_i^{q_i}\left[(2-m_{ii})^{q_i} + \left(\frac{m_i - m_{ii}}{m_i}\right)L_i^{q_i(1-m_i)}(m_{ii}c_1^2 + \tfrac{1}{2}\lambda_0)t\right]^{1/(m_i-m_{ii})} \\
&= L_i\left[(2-m_{ii})^{q_i}L_i^{q_i(m_i-1)} + \left(\frac{m_i - m_{ii}}{m_i}\right)(m_{ii}c_1^2 + \tfrac{1}{2}\lambda_0)t\right]^{1/(m_i-m_{ii})} \\
&\leqslant L_i\left[1 + \left(\frac{m_i - m_{ii}}{m_i}\right)(m_{ii}c_1^2 + \tfrac{1}{2}\lambda_0)M^{-1}\log 2\right]^{1/(m_i-m_{ii})} \\
&\triangleq L_i D_i, \quad s+1 \leqslant i \leqslant n.
\end{aligned}
$$

For $1 \leqslant i \leqslant s$ and $x \in \partial\Omega$, we get

$$\frac{\partial \bar{u}_i}{\partial \eta} \geqslant \frac{\theta c_2}{2(m_i-1)} L_i^{(m_i+1)/2}$$

and

$$
\begin{aligned}
\prod_{j=1}^{n} \bar{u}_j^{m_{ij}} &\leqslant \prod_{j=1}^{s} L_j^{m_{ij}}(16)^{\sum_{j=1}^{s} m_{ij}/(m_j-1)} \prod_{j=s+1}^{n} L_j^{m_{ij}} D_j^{m_{ij}} \\
&\leqslant \prod_{j=1}^{n} L_j^{m_{ij}}(16)^{\sum_{j=1}^{s} m_{ij}/(m_j-1)} \prod_{j=s+1}^{n} D_j^{m_{ij}}.
\end{aligned}
$$

For $s+1 \leqslant i \leqslant n$ and $x \in \partial\Omega$, we have

$$\begin{aligned}\frac{\partial \bar{u}_i}{\partial \eta} &= L_i^{q_i} B_i^{m_{ii}/(m_i-m_{ii})}\left(-\frac{\partial \varphi}{\partial \eta}\right) \\ &\geqslant c_2 L_i^{q_i} B_i^{m_{ii}/(m_i-m_{ii})} \bar{u}_i^{m_{ii}} L_i^{-m_{ii}q_i} B_i^{-m_{ii}/(m_i-m_{ii})} \\ &= c_2 L_i^{(m_i-m_{ii})} \bar{u}_i^{m_{ii}}\end{aligned}$$

and

$$\begin{aligned}\prod_{j=1}^{n} \bar{u}_j^{m_{ij}} &\leqslant \bar{u}_i^{m_{ii}} \prod_{j=1}^{s} L_j^{m_{ij}} (16)^{\sum_{j=1}^{s} m_{ij}/(m_j-1)} \prod_{j=s+1,\, j\neq i}^{n} L_j^{m_{ij}} D_j^{m_{ij}} \\ &= \bar{u}_i^{m_{ii}} \prod_{j=1,\, j\neq i}^{n} L_j^{m_{ij}} (16)^{\sum_{j=1}^{s} m_{ij}/(m_j-1)} \prod_{j=s+1,\, j\neq i}^{n} D_j^{m_{ij}}.\end{aligned}$$

Therefore, when $1 \leqslant i \leqslant n$ and $x \in \partial\Omega$,

$$\frac{\partial \bar{u}_i}{\partial \eta} \geqslant \prod_{j=1}^{n} \bar{u}_j^{m_{ij}} \tag{3.15}$$

holds if

$$\begin{aligned}&\frac{\theta c_2}{2(m_i-1)} L_i^{(m+1)/2} \prod_{j=1}^{n} L_j^{-m_{ij}} \\ &\qquad \geqslant (16)^{\sum_{j=1}^{s} m_{ij}/(m_j-1)} \prod_{j=s+1}^{n} D_j^{m_{ij}}, \qquad i=1,\dots,s,\end{aligned} \tag{3.16}$$

and

$$c_2 L_i^{m_i} \prod_{j=1}^{n} L_j^{-m_{ij}} \geqslant (16)^{\sum_{j=1}^{s} m_{ij}/(m_j-1)} \prod_{j=s+1,\, j\neq i}^{n} D_j^{m_{ij}}, \quad i=s+1,\dots,n. \tag{3.17}$$

By an argument similar to that of case 2, we know that (3.16) and (3.17) hold. At the same time, for $x \in \bar{\Omega}$,

$$\begin{aligned}&\bar{u}_i(x,0) \geqslant L_i, && 1 \leqslant i \leqslant s, \\ &\bar{u}_i(x,0) \geqslant L_i^{q_i}(2-m_{ii})^{1/(1-m_{ii})}, && s+1 \leqslant i \leqslant n.\end{aligned}$$

If we choose L_i to be large enough, then

$$\bar{u}_i(x,0) \geqslant u_{i0}(x), \quad x \in \partial\Omega, \quad 1 \leqslant i \leqslant n. \tag{3.18}$$

Inequalities (3.13), (3.14), (3.15) and (3.18) show that $(\bar{u}_1,\dots,\bar{u}_n)$ is an upper solution of (1.1) in $\Omega \times [0,T]$, so $(u_1,\dots,u_n)$ exists on $[0,T]$.

Case 4. ($m_i > 1$ for $1 \leqslant i \leqslant s_1$; $m_i = 1$ for $s_1+1 \leqslant i \leqslant s_2$; $m_i < 1$ for $s_2+1 \leqslant i \leqslant n$.) Let $\bar{u}_i(x,t)$ be as in (3.5), (3.6), (3.12) for $1 \leqslant i \leqslant s_1$, $s_1+1 \leqslant i \leqslant s_2$ and $s_2+1 \leqslant i \leqslant n$, respectively. Using the same methods as in the above, we can prove that $(\bar{u}_1,\dots,\bar{u}_n)$ is an upper solution of (1.1) on $\Omega \times [0,T]$, so $(u_1,\dots,u_n)$ exists on $[0,T]$.

The proof of theorem 1.1 is complete.

4. Proof of theorem 1.2

All three cases in theorem 1.2 imply that there exists a negative principal minor determinant. Hence we assume that one lth-order ($1 \leqslant l \leqslant n$) principal minor determinant of F is negative, and, without loss of generality, $|F_1^l| < 0$, and all the pth-order ($1 \leqslant p < l$) principal minor determinants of F are non-negative. We consider the following problem,

$$\left.\begin{aligned} (u_i^{m_i})_t &= \Delta u_i, & x \in \Omega, \quad t > 0,\\ \frac{\partial u_i}{\partial \eta} &= K_i \prod_{j=1}^{l} u_j^{m_{ij}}, & x \in \partial\Omega, \quad t > 0,\\ u_i(x,0) &= u_{i0}(x) > 0, & x \in \bar{\Omega},\\ & i = 1,\dots,l, \end{aligned}\right\} \tag{4.1}$$

where

$$K_i = \prod_{j=l+1}^{n} \delta_j^{m_{ij}} \quad \text{and} \quad \delta_i = \min_{\bar{\Omega}} u_{i0}(x).$$

If we can prove that the solution $(u_1^*, \dots, u_l^*)$ of (4.1) blows up in finite time, then $(u_1^*, \dots, u_l^*, \delta_{l+1}, \dots, \delta_n)$ is a lower solution of (1.1) that blows up in finite time. Therefore, the solution of (1.1) blows up in finite time. In the following, we focus on (4.1). Denote $\delta = \min_{1\leqslant i\leqslant n} \delta_i$.

(i) If there exists i, $1 \leqslant i \leqslant n$, such that $b_i < m_{ii}$, then $l = 1$. It follows from [5] that the solution of (4.1) blows up in finite time.

(ii) If $b_i \leqslant 1$, then $m_{ii} \leqslant 1$, $i = 1, \dots, l$. The case follows from the result in [9].

(iii) If $\Omega = B(0,R)$, without loss of generality, we assume that $K_i = 1$ ($1 \leqslant i \leqslant l$) and $l = n$. We establish the result for different cases.

Case 1. ($m_i > 1$ for $1 \leqslant i \leqslant n$.) Let

$$\underline{u}_i(x,t) = \delta A_i^{(1+\varepsilon)/(1-m_i)} \quad \text{and} \quad A_i = (1-\sigma t)^{k_i} + \theta(R-r), \tag{4.2}$$

where $r = |x|$,

$$\theta = \min_{1\leqslant i\leqslant n} \frac{m_i - 1}{1+\varepsilon} \delta^{\sum_{j=1}^{n} m_{ij} - 1}, \quad \sigma = \min_{1\leqslant i\leqslant n} \frac{\theta^2(m_i+\varepsilon)}{m_i(m_i-1)k_i} \delta^{1-m_i}, \quad k_i = \frac{m_i - 1}{1+\varepsilon}\alpha_i$$

and the α_i are as given in lemma 2.9 and satisfy

$$\alpha_i > \frac{1+\varepsilon}{(1-\varepsilon)(m_i-1)}.$$

In particular, $k_i > 1/(1-\varepsilon)$.

It is clear that functions $\underline{u}_i(x,t)$ are defined on $[0,1/\sigma)$ and $\max_{\bar{\Omega}}\underline{u}_i(x,t) = \underline{u}_i(R,t) \to \infty$ as $t \to 1/\sigma$, $1 \leqslant i \leqslant n$, i.e. $(\underline{u}_1,\dots,\underline{u}_n)$ blows up in finite time. For $0 \leqslant t \leqslant 1/\sigma$, by direct computations, we have

$$\begin{aligned}
(\underline{u}_i^{m_i})_t &= \frac{m_ik_i\sigma}{m_i-1}(1+\varepsilon)\delta^{m_i}(1-\sigma t)^{k_i-1}A_i^{((2+\varepsilon)m_i-1)/(1-m_i)},\\
(\underline{u}_i)_r &= \delta\theta\frac{1+\varepsilon}{m_i-1}A_i^{(m_i+\varepsilon)/(1-m_i)},\\
(\underline{u}_i)_{rr} &= \delta\theta^2\frac{(1+\varepsilon)(m_i+\varepsilon)}{(m_i-1)^2}A_i^{(2m_i+\varepsilon-1)/(1-m_i)},\\
\Delta\underline{u}_i &= (\underline{u}_i)_{rr} + \frac{N-1}{r}(\underline{u}_i)_r\\
&\geqslant (\underline{u}_i)_{rr}\\
&= \delta\theta^2\frac{(1+\varepsilon)(m_i+\varepsilon)}{(m_i-1)^2}A_i^{((2+\varepsilon)m_i-1)/(1-m_i)}(1-\sigma t)^{k_i-1}A_i^{\varepsilon}(1-\sigma t)^{1-k_i}\\
&\geqslant \delta\theta^2\frac{(1+\varepsilon)(m_i+\varepsilon)}{(m_i-1)^2}A_i^{((2+\varepsilon)m_i-1)/(1-m_i)}(1-\sigma t)^{k_i-1}(1-\sigma t)^{\varepsilon k_i-k_i+1}\\
&\geqslant \delta\theta^2\frac{(1+\varepsilon)(m_i+\varepsilon)}{(m_i-1)^2}A_i^{((2+\varepsilon)m_i-1)/(1-m_i)}(1-\sigma t)^{k_i-1}\\
&\geqslant \frac{m_ik_i\sigma}{m_i-1}(1+\varepsilon)\delta^{m_i}(1-\sigma t)^{k_i-1}A_i^{((2+\varepsilon)m_i-1)/(1-m_i)},
\end{aligned}$$

since $k_i > 1/(1-\varepsilon)$. Hence

$$(\underline{u}_i^{m_i})_t \leqslant \Delta\underline{u}_i, \quad 1 \leqslant i \leqslant n. \tag{4.3}$$

For $x \in \partial\Omega$, i.e. $r = R$, we find that

$$\begin{aligned}
\frac{\partial\underline{u}_i}{\partial\eta} = (\underline{u}_i)_r &= \delta\theta\frac{1+\varepsilon}{m_i-1}A_i^{(m_i+\varepsilon)/(1-m_i)}\\
&= \delta\theta\frac{1+\varepsilon}{m_i-1}(1-\sigma t)^{((m_i+\varepsilon)/(1-m_i))k_i}\\
&= \delta\theta\frac{1+\varepsilon}{m_i-1}(1-\sigma t)^{-((m_i+\varepsilon)/(1+\varepsilon))\alpha_i}
\end{aligned}$$

and

$$\begin{aligned}
\prod_{j=1}^n \underline{u}_j^{m_{ij}} &= \prod_{j=1}^n \delta^{m_{ij}}A_j^{((1+\varepsilon)/(1-m_j))m_{ij}}\\
&= \delta^{\sum_{j=1}^n m_{ij}}(1-\sigma t)^{-\sum_{j=1}^n m_{ij}\alpha_j}.
\end{aligned}$$

Thus

$$\frac{\partial\underline{u}_i}{\partial\eta} \leqslant \prod_{i=1}^n \underline{u}_j^{m_{ij}}, \quad 1 \leqslant i \leqslant n, \tag{4.4}$$

if

$$\theta\delta^{1-\sum_{j=1}^n m_{ij}}\frac{m_i+\varepsilon}{1+\varepsilon}(1-\sigma t)^{-(((m_i+\varepsilon)/(1+\varepsilon))\alpha_i-\sum_{j=1}^n m_{ij}\alpha_j)} \leqslant 1, \quad 1 \leqslant i \leqslant n. \tag{4.5}$$

https://doi.org/10.1017/S030821050000370X Published online by Cambridge University Press

Since

$$-\left(\frac{m_i+\varepsilon}{1+\varepsilon}\alpha_i-\sum_{j=1}^{n}m_{ij}\alpha_j\right)>0,\quad 1\leqslant i\leqslant n,$$

it yields that

$$(1-\sigma t)^{-(((m_i+\varepsilon)/(1+\varepsilon))\alpha_i-\sum_{j=1}^{n}m_{ij}\alpha_j)}\leqslant 1,\quad 1\leqslant i\leqslant n.$$

Using

$$\theta\leqslant\frac{m_i-1}{1+\varepsilon}\delta^{\sum_{j=i}^{n}m_{ij}-1},$$

it follows that inequality (4.5) holds.

It is obvious that

$$\underline{u}_i(x,0)\leqslant\delta\leqslant u_{i0}(x),\quad x\in\bar{\Omega},\quad 1\leqslant i\leqslant n. \tag{4.6}$$

Inequalities (4.3), (4.4) and (4.6) show that $(\underline{u}_1,\dots,\underline{u}_n)$ is a lower solution of problem (4.1). Therefore, $\underline{u}_i\leqslant u_i$, $1\leqslant i\leqslant n$. Consequently, $(u_1,\dots,u_n)$ blows up in finite time.

CASE 2. ($m_i>1$ for $1\leqslant i\leqslant s$; $m_i=1$ for $s+1\leqslant i\leqslant n$.) Let $\underline{u}_i(x,t)$ be as given in (4.2), $1\leqslant i\leqslant s$,

$$\underline{u}_i(x,t)=\delta\left[\left(1-\frac{\theta}{2N}(R^2-r^2)\right)^{-1}-\sigma t\right]^{-\alpha_i}\triangleq\delta B_i^{-\alpha_i},\quad s+1\leqslant i\leqslant n, \tag{4.7}$$

where

$$\theta=\min\left\{\frac{N}{R^2},\min_{1\leqslant i\leqslant s}\frac{m_i-1}{1+\varepsilon}\delta^{\sum_{j=1}^{n}m_{ij}-1},\min_{s+1\leqslant i\leqslant n}\frac{N}{R}\delta^{\sum_{j=1}^{n}m_{ij}-1}\right\},$$

$$\sigma=\min\left\{\frac{\theta}{N},\min_{1\leqslant i\leqslant s}\frac{\theta^2(m_i+\varepsilon)}{m_i(m_i-1)k_i}\delta^{1-m_i}\right\},\quad k_i=\frac{m_i-1}{1+\varepsilon}\alpha_i\quad(1\leqslant i\leqslant s)$$

and the α_i are as given in lemma 2.9 and satisfy $\alpha_i\geqslant(1+\varepsilon)/((1-\varepsilon)(m_i-1))$ for $1\leqslant i\leqslant s$, $\alpha_i\geqslant 1$ for $s+1\leqslant i\leqslant n$. By direct computation, we have

$$(\underline{u}_i^{m_i})_t\leqslant\Delta\underline{u}_i,\quad x\in\Omega,\quad 1\leqslant i\leqslant s. \tag{4.8}$$

When $s+1\leqslant i\leqslant n$,

$$\begin{aligned}
(\underline{u}_i)_t&=\delta\alpha_i\sigma B_i^{-\alpha_i-1},\\
(\underline{u}_i)_r&=\delta\alpha_i\theta rB_i^{-\alpha_i-1}\frac{[1-(\theta/2N)(R^2-r^2)]^{-2}}{N},\\
(\underline{u}_i)_{rr}&=\delta\alpha_i\theta B_i^{-\alpha_i-1}\frac{[1-(\theta/2N)(R^2-r^2)]^{-2}}{N}\\
&\quad+\delta\alpha_i\theta^2r^2(\alpha_i+1)B_i^{-\alpha_i-2}\frac{[1-(\theta/2N)(R^2-r^2)]^{-4}}{N^2}\\
&\quad-2\delta\alpha_i\theta^2r^2B_i^{-\alpha_i-1}\frac{[1-(\theta/2N)(R^2-r^2)]^{-3}}{N^2}.
\end{aligned}$$

https://doi.org/10.1017/S030821050000370X Published online by Cambridge University Press

Since $B_i \leqslant [1-(\theta/2N)(R^2-r^2)]^{-1}$ and $\alpha_i \geqslant 1$, we have

$$\frac{\delta\alpha_i\theta^2 r^2}{N^2}(\alpha_i+1)B_i^{-\alpha_i-2}\left[1-\frac{\theta}{2N}(R^2-r^2)\right]^{-4} \geqslant \frac{2\delta\alpha_i\theta^2 r^2}{N^2}B_i^{-\alpha_i-1}\left[1-\frac{\theta}{2N}(R^2-r^2)\right]^{-3}.$$

Consequently,

$$(\underline{u}_i)_{rr} \geqslant \frac{\delta\alpha_i\theta}{N}B_i^{-\alpha_i-1}\left[1-\frac{\theta}{2N}(R^2-r^2)\right]^{-2} \geqslant \frac{\delta\alpha_i\theta}{N}B_i^{-\alpha_i-1}.$$

Since $\sigma \leqslant \theta/N$ and $(\underline{u}_i)_r \geqslant 0$, we have

$$(\underline{u}_i)_t \leqslant \Delta\underline{u}_i, \quad s+1 \leqslant i \leqslant n, \quad x \in \Omega. \tag{4.9}$$

When $r = R$, we have

$$\frac{\partial\underline{u}_i}{\partial\eta} = (\underline{u}_i)_r = \delta\theta\frac{1+\varepsilon}{m_i-1}(1-\sigma t)^{-(m_i+\varepsilon)/(1+\varepsilon)\alpha_i}, \quad 1 \leqslant i \leqslant s,$$
$$\frac{\partial\underline{u}_i}{\partial\eta} = (u_i)_r = \delta\alpha_i\frac{\theta R}{N}(1-\sigma t)^{-\alpha_i-1}, \quad s+1 \leqslant i \leqslant n,$$

and

$$\begin{aligned}\prod_{j=1}^n \underline{u}_j^{m_{ij}} &= \delta^{\sum_{j=1}^s m_{ij}}(1-\sigma t)^{-\sum_{j=1}^s m_{ij}\alpha_j}\delta^{\sum_{j=s+1}^n m_{ij}}(1-\sigma t)^{-\sum_{j=s+1}^n m_{ij}\alpha_j} \\ &= \delta^{\sum_{j=1}^n m_{ij}}(1-\sigma t)^{\sum_{j=1}^n -m_{ij}\alpha_j}, \quad 1 \leqslant i \leqslant n.\end{aligned}$$

Therefore, when $r = R$,

$$\frac{\partial\underline{u}_i}{\partial\eta} \leqslant \prod_{j=1}^n \underline{u}_j^{m_{ij}}, \quad 1 \leqslant i \leqslant n, \tag{4.10}$$

hold if

$$\theta\delta^{1-\sum_{j=1}^n m_{ij}}\frac{1+\varepsilon}{m_i-1}(1-\sigma t)^{-(((m_i+\varepsilon)/(1+\varepsilon))\alpha_i-\sum_{j=1}^n m_{ij}\alpha_j)} \leqslant 1, \quad 1 \leqslant i \leqslant s, \tag{4.11}$$

and

$$\frac{\delta^{1-\sum_{j=1}^n m_{ij}}\alpha_i\theta R}{N}(1-\sigma t)^{-(\alpha_i-\sum_{j=1}^n m_{ij}\alpha_j+1)} \leqslant 1, \quad s+1 \leqslant i \leqslant n. \tag{4.12}$$

Since

$$-\left(\frac{m_i+\varepsilon}{1+\varepsilon}\alpha_i - \sum_{j=1}^n m_{ij}\alpha_j\right) > 0 \quad \text{for } 1 \leqslant i \leqslant s,$$

https://doi.org/10.1017/S030821050000370X Published online by Cambridge University Press

and

$$-\left(\alpha_i - \sum_{j=1}^{n} m_{ij}\alpha_j + 1\right) > 0 \quad \text{for } s+1 \leqslant i \leqslant n,$$

it follows that

$$(1-\sigma t)^{-(((m_i+\varepsilon)/(1+\varepsilon))\alpha_i - \sum_{j=1}^{n} m_{ij}\alpha_j)} < 1 \quad \text{for } 1 \leqslant i \leqslant s.$$

From these two inequalities and the choice of θ, we know that (4.11) and (4.12) hold, and, in consequence, so does (4.10).

It is obvious that

$$\underline{u}_i(x,0) \leqslant \delta \leqslant u_{i0}(x), \quad x \in \bar{\Omega}, \quad 1 \leqslant i \leqslant n. \tag{4.13}$$

Inequalities (4.8), (4.9) and (4.10) show that $(\underline{u}_1, \dots, \underline{u}_n)$ is a lower solution of (4.1). Since $(\underline{u}_1, \dots, \underline{u}_n)$ blows up in finite time, it follows that the solution of (4.1) blows up in finite time.

Case 3. ($m_i > 1$ for $1 \leqslant i \leqslant s$; $m_i < 1$ for $s+1 \leqslant i \leqslant n$.) Let $\underline{u}_i(x,t)$, $1 \leqslant i \leqslant s$, be as given in case 2. Then

$$\underline{u}_i(x,t) = \delta_0\left[(1-\sigma t)^{-l_i} + \frac{\theta}{2R}r^2\right]^{\alpha_i/l_i} \triangleq \delta_0 B_i^{\alpha_i/l_i}, \quad s+1 \leqslant i \leqslant n, \tag{4.14}$$

where

$$\begin{aligned}
\delta_0 &= \min_{s+1\leqslant i\leqslant n} 2^{-\alpha_i/l_i}\delta,\\
k_i &= \frac{m_i-1}{1+\varepsilon}\alpha_i, \quad 1 \leqslant i \leqslant s,\\
l_i &= \alpha_i(1-m_i) - 1, \quad s+1 \leqslant i \leqslant n,\\
\theta &= \min\left(\frac{2}{R}, \min_{1\leqslant i\leqslant s} \frac{m_i-1}{1+\varepsilon}\delta_0^{\sum_{j=1}^{n} m_{ij}-1}, \min_{s+1\leqslant i\leqslant n} \frac{l_i}{\alpha_i}2^{1-\alpha_i/l_i}\delta_0^{\sum_{j=1}^{n} m_{ij}-1}\right),\\
\sigma &= \min\left\{\frac{\theta}{N}, \min_{1\leqslant i\leqslant s} \frac{\theta^2(m_i+\varepsilon)}{m_i(m_i-1)k_i}\delta^{1-m_i}, \min_{s+1\leqslant i\leqslant n} \frac{\theta}{Rl_i m_i}\delta_0^{1-m_i}\right\}
\end{aligned}$$

and the α_i are as given in lemma 2.9 and satisfy

$$\alpha_i > \begin{cases} \dfrac{1+\varepsilon}{(1-\varepsilon)(m_i-1)} & \text{for } 1 \leqslant i \leqslant s,\\[2ex] \dfrac{1}{1-m_i} & \text{for } s+1 \leqslant i \leqslant n. \end{cases}$$

Hence $l_i > 0$, $s+1 \leqslant i \leqslant n$.

First, by direct computation, we get

$$(\underline{u}_i^{m_i})_t \leqslant \Delta \underline{u}_i, \quad x \in \Omega, \quad 0 \leqslant t < \frac{1}{\sigma}, \quad 1 \leqslant i \leqslant s. \tag{4.15}$$

https://doi.org/10.1017/S030821050000370X Published online by Cambridge University Press

When $s+1 \leqslant i \leqslant n$, since $B_i \geqslant (1-\sigma t)^{-l_i}$, we obtain

$$\begin{aligned}(\underline{u}_i^{m_i})_t &= \alpha_i m_i \sigma \delta_0^{m_i} B_i^{\alpha_i m_i/l_i - 1}(1-\sigma t)^{-l_i-1} \\ &= \alpha_i m_i \sigma \delta_0^{m_i} B_i^{(\alpha_i m_i - l_i)/l_i}(1-\sigma t)^{-\alpha_i(1-m_i)} \\ &\leqslant \alpha_i m_i \sigma \delta_0^{m_i} B_i^{(\alpha_i - l_i)/l_i}, \\ (\underline{u}_i)_r &= \frac{\delta_0 \alpha_i \theta r B_i^{(\alpha_i - l_i)/l_i}}{l_i R} \geqslant 0\end{aligned}$$

and

$$\begin{aligned}\Delta \underline{u}_i &= \frac{N\delta_0 \alpha_i \theta}{l_i R} B_i^{(\alpha_i - l_i)/l_i} + \frac{\delta_0 \alpha_i \theta^2 r^2}{l_i R^2}\left(\frac{\alpha_i}{l_i} - 1\right) B_i^{(\alpha_i - 2l_i)/l_i} \\ &\geqslant \frac{N\delta_0 \alpha_i \theta}{l_i R} B_i^{(\alpha_i - l_i)/l_i},\end{aligned}$$

since $\alpha_i > l_i$. Therefore,

$$(\underline{u}_i^{m_i})_t \leqslant \Delta \underline{u}_i, \quad i = s+1, \dots, n. \tag{4.16}$$

When $1 \leqslant i \leqslant s$ and $r = R$,

$$\frac{\partial \underline{u}_i}{\partial \eta} = \delta_0 \theta \frac{1+\varepsilon}{m_i - 1}(1-\sigma t)^{-((m_i+\varepsilon)/(1+\varepsilon))\alpha_i}.$$

When $s+1 \leqslant i \leqslant n$ and $r = R$, since $\frac{1}{2}\theta R \leqslant 1 \leqslant (1-\sigma t)^{-l_i}$, we have

$$\begin{aligned}\frac{\partial \underline{u}_i}{\partial \eta} &= \frac{1}{l_i}\theta \delta_0 \alpha_i((1-\sigma t)^{-l_i} + \tfrac{1}{2}\theta R)^{(\alpha_i - l_i)/l_i} \\ &\leqslant \frac{1}{l_i}\theta \delta_0 \alpha_i 2^{(\alpha_i - l_i)/l_i}(1-\sigma t)^{-m_i \alpha_i - 1}.\end{aligned}$$

When $1 \leqslant i \leqslant n$ and $r = R$,

$$\begin{aligned}\prod_{j=1}^n \underline{u}_j^{m_{ij}} &= \delta_0^{\sum_{j=1}^s m_{ij}}(1-\sigma t)^{-\sum_{j=1}^s m_{ij}\alpha_j} \delta_0^{\sum_{j=s+1}^n m_{ij}} \\ &\qquad \times \prod_{j=s+1}^n [(1-\sigma t)^{-l_j} + \tfrac{1}{2}\theta R]^{m_{ij}\alpha_j/l_j} \\ &\geqslant \delta_0^{\sum_{j=1}^n m_{ij}}(1-\sigma t)^{-\sum_{j=1}^n m_{ij}\alpha_j}.\end{aligned}$$

Consequently, for $r = R$,

$$\frac{\partial \underline{u}_i}{\partial \eta} \leqslant \prod_{j=1}^n \underline{u}_j^{m_{ij}}, \quad i = 1, \dots, n, \tag{4.17}$$

hold if

$$\theta \delta_0^{1-\sum_{j=1}^n m_{ij}} \frac{1+\varepsilon}{m_i - 1}(1-\sigma t)^{-(((m_i+\varepsilon)/(1+\varepsilon))\alpha_i - \sum_{j=1}^n m_{ij}\alpha_j)} \leqslant 1, \quad 1 \leqslant i \leqslant s, \tag{4.18}$$

https://doi.org/10.1017/S030821050000370X Published online by Cambridge University Press

and

$$\frac{1}{l_i}\alpha_i\theta\delta_0^{1-\sum_{j=1}^n m_{ij}}2^{(\alpha_i-l_i)/l_i}(1-\sigma t)^{-(m_i\alpha_i-\sum_{j=1}^n m_{ij}\alpha_j+1)} \leqslant 1, \quad s+1 \leqslant i \leqslant n. \tag{4.19}$$

Since

$$-\left(\frac{m_i+\varepsilon}{1+\varepsilon}\alpha_i - \sum_{j=1}^n m_{ij}\alpha_j\right) > 0, \quad 1 \leqslant i \leqslant s,$$
$$-\left(m_i\alpha_i - \sum_{j=1}^n m_{ij}\alpha_j + 1\right) > 0, \quad s+1 \leqslant i \leqslant n,$$

and

$$\theta\delta_0^{1-\sum_{j=1}^n m_{ij}}\frac{1+\varepsilon}{m_i-1} \leqslant 1, \quad 1 \leqslant i \leqslant s,$$
$$\frac{\alpha_i}{l_i}\theta\delta_0^{1-\sum_{j=1}^n m_{ij}}2^{(\alpha_i-l_i)/l_i} \leqslant 1, \quad s+1 \leqslant i \leqslant n,$$

we know that (4.18) and (4.19) hold.

It is obvious that

$$\underline{u}_i(x,0) \leqslant \delta_0 \leqslant \delta \leqslant u_{i0}(x,0), \qquad 1 \leqslant i \leqslant s, \tag{4.20}$$
$$\underline{u}_i(x,0) \leqslant \delta_0 2^{\alpha_i/l_i} \leqslant \delta \leqslant u_{i0}(x,0), \quad s+1 \leqslant i \leqslant n. \tag{4.21}$$

Inequalities (4.15)–(4.17), (4.20) and (4.21) show that $(\underline{u}_1,\dots,\underline{u}_n)$ is a lower solution of (4.1). Since $(\underline{u}_1,\dots,\underline{u}_n)$ blows up in finite time, so does $(u_1,u_2,\dots,u_n)$.

Case 4. ($m_i > 1$ for $1 \leqslant i \leqslant s_1$; $m_i = 1$ for $s_1+1 \leqslant i \leqslant s_2$; $m_i < 1$ for $s_2+1 \leqslant i \leqslant n$.) Let $\underline{u}_i(x,t)$ be as in (4.2), (4.7), (4.16) for $1 \leqslant i \leqslant s_1$, $s_1+1 \leqslant i \leqslant s_2$ and for $s_2+1 \leqslant i \leqslant n$, respectively. Using the same method as in the above, it can be proved that $(\underline{u}_1,\dots,\underline{u}_n)$ is a lower solution of (4.1). Therefore, $(u_1,\dots,u_n)$ blows up in finite time.

The proof of theorem 1.2 is now complete.

Acknowledgments

Y.Q. and M.W. were supported in part by HK RGC grant HKUST630/95p. M.W. and Z.W. were partly supported by PRC grants NSFC 19771015, 19831060.

References

1 V. Galaktionov and H. Levine. On critical Fujita exponents for heat equations with nonlinear flux boundary conditions on the boundary. *Israel J. Math.* **94** (1996), 125–146.
2 C. V. Pao. *Nonlinear parabolic and elliptic equations* (New York: Plenum, 1992).
3 N. J. Pullman. *Matrix theory and its applications* (New York: Dekker, 1976).
4 M. X. Wang. Parabolic systems with nonlinear boundary conditions. *Chin. Sci. Bull.* **40** (1995), 1412–1414.
5 M. X. Wang. Long time behaviors of solutions of quasilinear parabolic equations with nonlinear boundary conditions. *Acta Math.* **39** (1996), 118–124. (In Chinese.)

6 M. X. Wang. Fast–slow diffusion systems with nonlinear boundary conditions. *Nonlin. Analysis* **46** (2001), 893–908.

7 M. X. Wang and S. Wang. Quasilinear reaction–diffusion systems with nonlinear boundary conditions. *J. Math. Analysis Applic.* **231** (1999), 21–33.

8 M. X. Wang and Y. M. Wang. Reaction diffusion systems with nonlinear boundary conditions. *Sci. China* A **27** (1996), 834–840.

9 S. Wang, M. X. Wang and C. H. Xie. Quasilinear parabolic systems with nonlinear boundary conditions. *J. Diff. Eqns* **166** (2000), 251–265.

(*Issued 7 December 2004*)