SCENARIO ANALYSIS FOR A MULTI-PERIOD DIFFUSION MODEL OF RISK

BY

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ABSTRACT

This paper extends and develops the results of a previous paper Malinovskii (2007). Dealing with a simplistic diffusion multi-year model of insurance operations, this paper illustrates the adaptive control approach when the object of control is the balance of solvency and equity. Compared to the previous paper, a new element is the "scenario of nature", or the incomplete knowledge of future risk, which is quite often the case in insurance. It introduces a new and inevitable randomness in the model and leads to a qualitative difference in its behavior.

KEYWORDS AND PHRASES

Multi-period insurance process, Diffusion annual mechanisms, Volatile scenario, Solvency, Equity, Adaptive control strategies.

1. INTRODUCTION

In the papers Malinovskii (2007) – Malinovskii (2008b) the insurance process is viewed as a series of periods (years). Each period starts with a manager's adaptive, or sensitive to the previous years-'s financial results, control decision. Insurance operations are represented by a "probability mechanism". The manager's decision concerns tariffs, reserves and other operational characteristics of this mechanism. By the nature of insurance, that decision typically remains in force throughout the entire year until the next year-end financial report and appropriate changes are made.

The adaptive control approach in insurance modeling is inspired by many scholars including Karl Borch who claimed back in 1967 that "general formulation of the actuary's problem leads directly to the general theory of *optimal control processes*[1] or *adaptive control processes*[1]" and "the theory of control processes seems to be tailor-made for the problems which actuaries have struggled to formulate for more than a century" (Borch (1967), p. 451).

[1] Emphasis is original.

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The object of control set forth in Malinovskii (2007) – Malinovskii (2008b) was the balance of solvency and equity. Solvency means that a prescribed value of the probability of non-ruin always must be secured, whatever the previous years-'s financial results may be and regardless of outside factors. Equity requires premiums well-balanced with claims and loaded with an amount necessary to provide adequate security for the insureds rather than to benefit those who seek unearned profit. It means that the insureds ought to pay premiums which are sensibly concentrated around the long-term mean value of their losses. In that sense the customers will not be overcharged; but only in the long run (i.e., over a period of several years).

In some years the premiums may be above or below that mean value. Insurers spreading the cost of random losses among the policyholders, and over time, act as a buffer against claim fluctuations in consecutive years.

A related problem is the discrimination of the risk reserves, capital and special purpose provisions[2] (see, e.g., Dacorogna and Rüttener (2006)). Bearing in mind the principle of equity, risk reserves must be large enough to remain solvent, but at the expected level called "target" or "fair" capital value. Otherwise, one could argue that the reserves are being used to cover the unexpected. However, the probability that the risk reserve will end up at the expected level at the end of the year is small. It will most probably be above or below and occasionally much above or below. The more the manager's prediction disagrees with reality, the greater is the difference. Thus, developing the appropriate risk-based provisions to keep the risk reserve over many years at the expected level is an important problem for insurance management. This paper addresses that problem from the theoretical premises of annual dynamic solvency control.

The economics of the object of control considered in Malinovskii (2007) – Malinovskii (2008b) and in this paper is therefore a cautious and equitable asset-liability and solvency adaptive management. Sophistication of the model may lead to additional rational priorities and to more complicated objects of control but does not change the fundamental nature of the adaptive control concept of this paper.

It should be emphasized that adaptive, rather than optimal, control is the main theme of this paper. Optimal control usually directs single-purposed objectives, like maximization of the insurer's profit[3]. Even under some restrictions on ruin and some kind of equity determined by the market, it yields quite a different mathematical game which lies outside the scope of this paper. Optimal control is the traditional setup in actuarial mathematics (see Asmussen

[2] Many parties to the insurance business are very attentive to that problem by other reasons than equity: reserves belong legally to the policyholders while capital belongs to the shareholders; risk reserve should be invested at the risk free rate, while the capital can be invested in riskier and more rewarding assets; taxation of risk reserves and capital is different.

[3] Standing by the side of insurers, wise is to care for the insureds as good shepherd cares for his sheep. In that sense the position of those who wish to win clients' loyalty, or merely avoid they outflow, may agree with the object of control set forth in the paper. More technical discussion is deferred to Section 2.4.

and Taksar (1997), Taksar and Zhou (1998)) but the objectives of optimal control are "to find the policy which maximizes the expected total discounted dividend pay-outs until the time of bankruptcy" (Taksar and Zhou (1998), p. 105). These objectives were severely criticized as deficient in the insurance context (see quotation from C.-O. Segerdhal on p. 392 of Malinovskii (2007)).

It is clear that insurance deals with such uncertainties as random claim arrival and random claim severity. Even more uncertainty looms for the insurance business because of the randomness called scenario of nature. Quoting Norbert Wiener (see Wiener (1966), p. 90), it results in a resemblance of that economic game — insurer vs. nature — to the Queen's croquet game in "Alice in Wonderland". Wiener emphasized that such resemblance exists in all economic games where the rules are subject to important and, additionally, random revisions. In particular, the changes of climate around us impact and will increasingly impact many sectors of business and society. The most profound effects are likely to be associated with changes in rainfall and severe weather.

It is recognized (see e.g., Borch (1967), p. 451) that the insurance company, being incompletely informed, needs to devise

(i) an *information system* for observing the insurance process as it develops,

(ii) a *decision function*: a set of rules for translating the observations into action. The latter means that a manager's control needs to fine-tune tariffs, reserves and other operational characteristics of the probability mechanism of insurance over several years. This is called a strategy and is developed with a lack of information. It should be thoroughly analyzed by actuaries to insure that its impact on the insurer's business is clearly understood.

Two commonly accepted techniques used to evaluate the impact of the lack of information on the insurer's business are scenario analysis and stress testing. The former considers typical, favorable and unfavorable scenarios. The latter refers to the shifting of values of individual parameters in the model that affects critically the insurer's financial position. Largely, both use simulation.

This paper's purpose is to accentuate the risk theory-based, analytical approach. As to the general multi-period model of risk (the control-oriented reader may wish to start from formal definitions in Section 3 below), each trajectory may be diagrammed as

$$\underbrace{\mathbf{w}_0 \xrightarrow{\gamma_0} \mathbf{u}_0 \xrightarrow{\pi_1} \mathbf{w}_1}_{\text{1-st year}} \cdots \xrightarrow{\pi_{k-1}} \underbrace{\mathbf{w}_{k-1} \xrightarrow{\gamma_{k-1}} \mathbf{u}_{k-1} \xrightarrow{\pi_k} \mathbf{w}_k}_{k\text{-th year}} \cdots. \tag{1}$$

According to this diagram[4] (for $k = 1, 2, \ldots$), at the end of $(k-1)$-th year the state variable $\mathbf{w}_{k-1}$ is observed. It describes the insurer's position at that moment and may be more complex than just a real-valued surplus. At the beginning of

[4] In the case of our particular interest (see Section 3) the state variables $\mathbf{w}_k$, the control variables $\mathbf{u}_k$ and the other components of the scheme (1) are yielded explicitly.

the k-th year the nature selects, in a certain scenario, a value influencing the forthcoming annual risk. Then the insurer applies the control rule γ_{k-1} to choose the control variable $\mathbf{u}_{k-1}$. The structural assumption is that nature is acting first, before the insurer, at the beginning of the insurance year and the lag between these actions is negligible. It may be easily weakened. In what follows in this paper this is accepted for simplicity. Typically, making his control decision, the insurer remains ignorant about nature's choice though he may be aware of nature's scenario. If so, he acts bearing in mind the limitations induced by the scenario and applying the past-year data[5] $\mathbf{w}_{k-1}$ to the control rule γ_{k-1}. Thereupon the k-th year "probability mechanism" of insurance unfolds; the transition function of this mechanism is denoted by π_k. It defines the insurer's position at the end of the k-th year, and the process repeats anew.

Paramount in (1) is the annual probability mechanism of insurance[6]. In Malinovskii (2008a) and Malinovskii (2008b) it is generated by the Poisson-Exponential collective risk model. In Malinovskii (2007) and in this paper it is diffusion. Thus, the annual probability mechanism of insurance is produced (see Section 3) by the claim payout process $V_s(M) = Ms + \sigma(M)\mathsf{W}_s$, $0 \leqslant s \leqslant t$. The annual risk reserve process is

$$R_s(u, c, M) = u + cs - V_s(M), \ 0 \leqslant s \leqslant t. \tag{2}$$

Here u is the risk reserve at the beginning of the year, called either initial risk reserve or starting capital, c is the premium intensity, M is the random claim payout rate, $\sigma(\cdot)$ is a known function assuming positive values and $\sigma^2(M)$ is the random volatility; W_s, $0 \leqslant s \leqslant t$, is the standard Brownian motion. In (2), sensible control leverages are both the initial risk reserve and the premium intensity, so that the control variable is bivariate.

Having specified the annual mechanism of insurance, it is paramount to keep track of how the information is revealed in time. Going back to the diagram (1), introduce the sequence $\{\mathsf{W}_s^{[k]}, 0 \leqslant s \leqslant t\}$, $k = 1, 2, \ldots$, of independent Brownian motions and the sequence M_k, $k = 1, 2, \ldots$, of random claim intensities. Assume that these sequences are independent of each other. These two independent assumptions are sensible. The former guarantees mutual independence of the annual claim payout processes $V_s^{[k]}(M_k)$, $k = 1, 2, \ldots$, as the claim intensities are fixed. The latter reflects independence of nature's choice from the annual insurance process. To concatenate the annual probability mechanisms (see formalities in Section 3) introduce a simplistic scenario of nature.

Definition 1.1. By the volatile (homogeneous and with known generic risk) scenario of nature associated with the multi-period model (1) and with the

[5] Or, more generally, all the past history $\mathcal{Y}_{k-1} = (\mathbf{u}_0, \ldots, \mathbf{u}_{k-2}, \mathbf{w}_0, \ldots, \mathbf{w}_{k-1})$. The control based on $\mathcal{Y}_{k-1}$ is called non-Markov (see Malinovskii (2007)), the control $\mathbf{u}_{k-1} = \gamma_{k-1}(\mathbf{w}_{k-1})$ is called Markov.

[6] The scheme (1) is fit to model dissimilar dynamics of the insurance process by means of addressing different annual probability mechanisms of insurance.

annual mechanisms of insurance (2) we mean the sequence of independent and identically distributed claim intensities M_k, $k = 1, 2, \ldots$, with known generic distribution G.

Bearing in mind the independence of M and $\{\mathsf{W}_s, 0 \leqslant s \leqslant t\}$ under the volatile scenario of nature, one has

$$\mathsf{E}V_t(M) = \mathsf{E}\big(Mt + \sigma(M)\mathsf{W}_t\big) = \mathsf{E}M \cdot t, \tag{3}$$

and in the multi-period model (1)-(2) the "fair" premium rate $c = \mathsf{E}M$. Indeed, by the law of large numbers the annual premium on the right hand side of Eq. (3) equalizes in the long run the multi-year average value of the annual claims.

The goal of the adaptive control in (1)-(2) is to find the strategy which compensates for fluctuations of the claims payout process around the "target" or "fair" capital value which corresponds to the "fair" premium rate. This "fair" premium rate will be defined later on the base of solvency requirements. The origin of the fluctuations may be twofold: the pure randomness and the difference between the unknown, but actual, realization of the random variable M and the heuristic, but average, value $\mathsf{E}M$.

Supplement the analysis of a model with a clear warning of its restricted applicability, as recommended in Daykin et al. (1996) [Chapter 1, Section 5.5, p. 154]: emphasize that the diffusion annual mechanism and the volatile scenario are simplistic indeed. That allows a transparent mathematics and yields a telling illustration of the adaptive control approach. Extensions on more general cases are straightforward. The computer-oriented person may apply a numerical solution of the basic equations of Section 2. A more realistic probability background may be achieved by applying the technique of Malinovskii (2008a) and Malinovskii (2008b) in the Poisson-Exponential framework. Overall, the simplistic models may hint on how to attack more realistic insurance risk models, where no explicit formulae may exist.

The rest of the paper is arranged as follows.

Section 2 describes the annual controls.

Section 3 deals with multi-period diffusion model under the volatile scenario of nature and addresses equity and solvency of several adaptive control strategies.

Section 4 formulates auxiliary results.

2. Synthesis of the annual adaptive controls

This section is devoted to the annual development of the insurance process. It precedes multi-period modeling and strategy design of Section 3. We denote by $\Phi(x)$ the standard normal distribution function and by $\phi(x)$ its density function. For $0 < \gamma < 1$, denote by $\kappa_\gamma = \Phi^{-1}(1 - \gamma)$ the $(1 - \gamma)$-quantile of $\Phi(x)$.

2.1. Annual solvency criteria

Formulate an assumption and two definitions.

ASSUMPTION 1. In the diffusion generic model (2) the random parameter M is non-degenerate, with c.d.f. G and support $\mathrm{M} \subset \mathsf{R}^+$.

In the framework of diffusion generic model (2), for $m \in \mathrm{M}$, set

$$\psi_t(u, c, m) = \mathsf{P}\{ \inf_{0 \leqslant s \leqslant t} R_s(u, c, M) < 0 \mid M = m\}, \quad t \geqslant 0. \tag{4}$$

The control variable is bivariate (u, c). In the sequel it will be a function of the previous year-end capital. Introduce two annual solvency criteria which modify the standard one.

DEFINITION 2.1. Assume that w is the previous year-end capital. The adaptive control $(u(w), c(w))$ satisfies the α-level $(0 < \alpha < 1)$ conservative, or uniform, solvency criterion if

$$\sup_{w > 0,\, m \in \mathrm{M}} \psi_t(u(w), c(w), m) \leqslant \alpha. \tag{5}$$

DEFINITION 2.2. Assume that w is the previous year-end capital. The adaptive control $(u(w), c(w))$ satisfies the α-level integral solvency criterion if

$$\sup_{w > 0} \mathsf{P}\Big\{ \inf_{0 \leqslant s \leqslant t} R_s(u(w), c(w), M) < 0 \Big\} = \sup_{w > 0} \int_M \psi_t(u(w), c(w), m)\, G(dm) \leqslant \alpha. \tag{6}$$

REMARK 2.1. In the particular case of the bounded support $\mathrm{M} = [\mu_{\min}, \mu_{\max}]$, $0 < \mu_{\min} < \mu_{\max} < \infty$, the claim intensity $\mu_{\max}$ is known to be the most unfavorable case for the insurer since

$$\sup_{w > 0,\, m \in \mathrm{M}} \psi_t(u(w), c(w), m) = \sup_{w > 0} \psi_t(u(w), c(w), \mu_{\max}).$$

The control $((u(w), c(w))$ satisfies the α-level conservative solvency criterion if

$$\sup_{w > 0} \psi_t(u(w), c(w), \mu_{\max}) \leqslant \alpha. \tag{7}$$

ASSUMPTION 2. The support $\mathrm{M} = [\mu_{\min}, \infty)$ is unbounded and known is the lower value $0 < \mu_{\min} < \infty$ of the claim intensity. The value $\mu_{\min}$ is the most favorable case for the insurer.

The conservative solvency criterion may be called "egalitarian" with respect to all realizations of M. It treats indifferently moderate and large values of M and looks exceedingly restrictive. More probabilistic is the integral solvency criterion which attributes (by means of c.d.f. G) proper weights to the different choices.

Recall that $\mu_\alpha (0 < \alpha < 1)$ such that $\mathsf{P}\{M > \mu_\alpha\} = \alpha$, or $G(\mu_\alpha) = 1 - \alpha$, is called $(1 - \alpha)$-quantile of c.d.f. G. Introduce for simplicity the following assumption which guarantees that μ_α exists and is unique. The cases of discrete c.d.f. G and bounded support M are similar.

ASSUMPTION 3. The distribution function G is absolutely continuous.

DEFINITION 2.3. Assume that w is the previous year-end capital, μ_{α_1} is the $(1 - \alpha_1)$-quantile of c.d.f. G and $\alpha_i \in (0, 1/2)$, $i = 1, 2$. The adaptive control $(u(w), c(w))$ satisfies the (α_1, α_2)-solvency criterion if

$$\sup_{w>0,\ m \leqslant \mu_{\alpha_1}} \psi_t(u(w), c(w), m) \leqslant \alpha_2. \tag{8}$$

The adaptive control $(u(w), c(w))$ satisfies the (α_1, α_2)-solvency criterion sharply if

$$\psi_t(u(w), c(w), \mu_{\alpha_1}) = \alpha_2$$

for all $w > 0$.

THEOREM 2.1. (Sufficient conditions of integral solvency). *If the adaptive control* $(u(w), c(w))$ *satisfies the* (α_1, α_2)*-solvency criterion, then it satisfies the* $(\alpha_1 + \alpha_2)$*-level integral solvency criterion.*

PROOF OF THEOREM 2.1. It is noteworthy that

$$\sup_{w>0,\ m \leqslant \mu_{\alpha_1}} \psi_t(u(w), c(w), m) = \sup_{w>0} \psi_t(u(w), c(w), \mu_{\alpha_1}).$$

Bearing in mind (6), the result is yielded by

$$\sup_{w>0} \mathsf{P}\Big\{\inf_{0 \leqslant s \leqslant t} R_s(u(w), c(w), M) < 0\Big\} \leqslant \sup_{w>0} \int_{m \leqslant \mu_{\alpha_1}} \psi_t(u(w), c(w), m)\, G(dm)$$
$$+ \int_{m > \mu_{\alpha_1}} G(dm) \leqslant \sup_{w>0,\ m \leqslant \mu_{\alpha_1}} \psi_t(u(w), c(w), m) + \mathsf{P}\{M > \mu_{\alpha_1}\}$$
$$= \sup_{w>0} \psi_t(u(w), c(w), \mu_{\alpha_1}) + \alpha_1 \leqslant \alpha_2 + \alpha_1.$$

□

We are dealing with controls which satisfy the (α_1, α_2)-solvency criterion and, consequently, the $(\alpha_1 + \alpha_2)$-level integral solvency criterion. It means that we can confine ourselves to $m \in [\mu_{\min}, \mu_{\alpha_1}]$, $\mu_{\min} > 0$, and disregard the other outcomes "of rare occurrence".

2.2. Level capital and premium intensity

Introduce two key components of the adaptive control rules. The existence and the structure of these components in the diffusion framework will be discussed in Theorems 4.4 and 4.5.

DEFINITION 2.4. For $\alpha_i \in (0, 1/2)$, $i = 1, 2$, and for the $(1 - \alpha_1)$-quantile μ_{α_1} of c.d.f. G the solution $u_{\alpha_2,t}(c, \mu_{\alpha_1})$ of the equation

$$\psi_t(u, c, \mu_{\alpha_1}) = \alpha_2 \tag{9}$$

with respect to u is called α_2-level initial capital corresponding to the claim intensity μ_{α_1} and to the premium intensity c. The solution $c_{\alpha_2,t}(u, \mu_{\alpha_1})$ of Eq. (9) with respect to c is called α_2-level premium intensity corresponding to the claim intensity μ_{α_1} and to the initial capital u.

REMARK 2.2. By definition, $c_{\alpha_2,t}(u_{\alpha_2,t}(c, \mu_{\alpha_1}), \mu_{\alpha_1}) = c$, $u_{\alpha_2,t}(c_{\alpha_2,t}(u, \mu_{\alpha_1}), \mu_{\alpha_1}) = u$.

2.3. Rigid (non-adaptive) controls

Solvent control may be safe, but unsatisfactory. Demonstrate it by means of two simple illustrations. For the previous year-end capital w consider $\alpha_i \in (0, 1/2)$, $i = 1, 2$, and $\mu \in [\mu_{\min}, \mu_{\alpha_1}]$, $\mu_{\min} > 0$, and set $c_{\min} = \mu_{\min}$ and $c_{\max} = \mu_{\alpha_1}$.

EXAMPLE 2.1. (Lowest premiums and highest starting capital). The control

$$\breve{u}(w) \equiv u_{\alpha_2,t}(c_{\min}, \mu_{\alpha_1}),\ \ \breve{c}(w) \equiv c_{\min} \tag{10}$$

with highest starting capital and lowest premiums[7] satisfies the (α_1, α_2)-solvency criterion sharply. Indeed,

$$\psi_t(\breve{u}(w), \breve{c}(w), \mu_{\alpha_1}) \equiv \psi_t(u_{\alpha_2,t}(c_{\min}, \mu_{\alpha_1}), c_{\min}, \mu_{\alpha_1}) = \alpha_2$$

by definition of $u_{\alpha_2,t}(c_{\min}, \mu_{\alpha_1})$. By Theorem 2.1, the control (10) satisfies the $(\alpha_1 + \alpha_2)$-level integral solvency criterion.

This control undercharges the insureds and contradicts the principle of equity even in its most primitive form: "no premium – no insurance". It implies borrowing and freezes the insurer's capital. Moreover, it is liable to a rightful argument that it is being used to cover the unexpected.

EXAMPLE 2.2. (Highest premiums and lowest starting capital). The opposite extreme case of hedging against insolvency is yielded by the control

$$\breve{u}(w) \equiv u_{\alpha_2,t}(c_{\max}, \mu_{\alpha_1}),\ \ \breve{c}(w) \equiv c_{\max} \tag{11}$$

7 Assume that a premium rate which is less than the least possible value of the claims intensity level $\mu_{\min}$ is considered a kind of self-inflicting behavior and is prohibited.

with lowest starting capital and highest premiums[8]. Again, it satisfies the (α_1, α_2)-solvency criterion sharply: by definition of $u_{\alpha_2,t}(c_{\max}, \mu_{\alpha_1})$, one has

$$\psi_t(\breve{u}(w), \breve{c}(w), \mu_{\alpha_1}) \equiv \psi_t(u_{\alpha_2,t}(c_{\max}, \mu_{\alpha_1}), c_{\max}, \mu_{\alpha_1}) = \alpha_2.$$

By Theorem 2.1, the control (11) satisfies the ($\alpha_1 + \alpha_2$)-level integral solvency criterion.

When this control is applied, the insurer's capital is not frozen, but the insureds are severely overcharged, which will not be appreciated by the customers or the regulatory authorities.

Both controls (10) and (11) are rigid (non-adaptive) in the sense that they are not sensitive to the previous year-'s financial results and extensively use the premium and the reserve capacities of the insurer.

2.4. "Fair" capital and ultimate equity

The controls (10) and (11) fail to comply with the principle of equity. That principle requires "fair" premiums, well-balanced with the claims, so the customers will not be overcharged in the long run. Since in our model

$$\mathsf{E}V_t(M) = \mathsf{E}M \cdot t, \tag{12}$$

"fair" long-time average premium rate is $\mathsf{E}M$.

Recall that the nature's choice is first, and assume that it selects the worst possible: the largest claims intensity μ_{α_1}. Assume that the insurer is unaware of and continues to apply the "long-time-average" premium rate $\mathsf{E}M$. In this case the least initial capital needed to keep the probability of non-ruin within time t equal to $1 - \alpha_2$ is equal to $u_{\alpha_2,t}(\mathsf{E}M, \mu_{\alpha_1})$. It follows from Eq. (9). Therefore, the capital $u_{\alpha_2,t}(\mathsf{E}M, \mu_{\alpha_1})$ may appear "fair" to those customers who prefer to pay premiums which are priced around the average for their *guaranteed* insurance protection.

We term *equitable* those controls $(u(w), c(w))$ which are keeping the risk reserve large enough to secure solvency, but at the expected level i.e., around the "fair" capital value $u_{\alpha_2,t}(\mathsf{E}M, \mu_{\alpha_1})$.

Definition 2.5. Assume that w is the previous year-end capital. The adaptive control $(u(w), c(w))$ is called ultimately equitable[9], if

$$\mathsf{E}R_t(u(w), c(w), \mu_{\alpha_1}) = u_{\alpha_2,t}(\mathsf{E}M, \mu_{\alpha_1})$$

uniformly in $w \in \mathsf{R}^+$.

[8] Assume that the highest premium rate $c_{\max}$ can not exceed the upper claims intensity μ_{α_1} for ethical reasons, or restrictions imposed by regulatory authorities.

[9] It may be also called balanced around the "fair" capital value $u_{\alpha_2,t}(\mathsf{E}M, \mu_{\alpha_1})$, or targeted at that "fair" capital value.

2.5. Adaptive control satisfying solvency criterion sharply

For $\alpha_i \in (0, 1/2)$, $i = 1, 2$, introduce

$$\hat{u}(w) = \begin{cases} u_{\alpha_2,t}\left(c_{\min}, \mu_{\alpha_1}\right), & w > u_{\alpha_2,t}\left(c_{\min}, \mu_{\alpha_1}\right), \\ w, & u_{\alpha_2,t}\left(c_{\max}, \mu_{\alpha_1}\right) \leqslant w \leqslant u_{\alpha_2,t}\left(c_{\min}, \mu_{\alpha_1}\right), \\ u_{\alpha_2,t}\left(c_{\max}, \mu_{\alpha_1}\right), & 0 < w < u_{\alpha_2,t}\left(c_{\max}, \mu_{\alpha_1}\right), \end{cases}$$
$$\hat{c}(w) = \begin{cases} c_{\min}, & w > u_{\alpha_2,t}\left(c_{\min}, \mu_{\alpha_1}\right), \\ c_{\alpha_2,t}\left(w, \mu_{\alpha_1}\right), & u_{\alpha_2,t}\left(c_{\max}, \mu_{\alpha_1}\right) \leqslant w \leqslant u_{\alpha_2,t}\left(c_{\min}, \mu_{\alpha_1}\right), \\ c_{\max}, & 0 < w < u_{\alpha_2,t}\left(c_{\max}, \mu_{\alpha_1}\right), \end{cases} \tag{13}$$

where $c_{\min} = \mu_{\min}$, $c_{\max} = \mu_{\alpha_1}$. The adaptive control $(\hat{u}(w), \hat{c}(w))$ is more sensitive to the previous year-end capital w than (10) and (11).

THEOREM 2.2. *The control* $(\hat{u}(w), \hat{c}(w))$ *satisfies the* (α_1, α_2)*-solvency criterion sharply. Moreover, it satisfies the* $(\alpha_1 + \alpha_2)$*-level integral solvency criterion.*

PROOF OF THEOREM 2.2. The proof is straightforward. By definition of $c_{\alpha_2,t}(w, \mu_{\alpha_1})$, for each $u_{\alpha_2,t}(c_{\max}, \mu_{\alpha_1}) \leqslant w \leqslant u_{\alpha_2,t}(c_{\min}, \mu_{\alpha_1})$

$$\psi_t(\hat{u}(w), \hat{c}(w), \mu_{\alpha_1}) = \psi_t(w, c_{\alpha_2,t}(w, \mu_{\alpha_1}), \mu_{\alpha_1}) \equiv \alpha_2,$$

for $w > u_{\alpha_2,t}(c_{\min}, \mu_{\alpha_1})$

$$\psi_t(\hat{u}(w), \hat{c}(w), \mu_{\alpha_1}) \equiv \psi_t(u_{\alpha_2,t}(c_{\min}, \mu_{\alpha_1}), c_{\min}, \mu_{\alpha_1}) = \alpha_2,$$

for $0 < w < u_{\alpha_2,t}(c_{\max}, \mu_{\alpha_1})$

$$\psi_t(\hat{u}(w), \hat{c}(w), \mu_{\alpha_1}) \equiv \psi_t(u_{\alpha_2,t}(c_{\max}, \mu_{\alpha_1}), c_{\max}, \mu_{\alpha_1}) = \alpha_2,$$

and the adaptive control $(\hat{u}(w), \hat{c}(w))$ satisfies the (α_1, α_2)-solvency criterion sharply. For the rest of the proof apply Theorem 2.1. □

REMARK 2.3. When $u_{\alpha_2,t}(c_{\max}, \mu_{\alpha_1}) \leqslant w \leqslant u_{\alpha_2,t}(c_{\min}, \mu_{\alpha_1})$ in (13), no need is to resort to borrowing. When the previous year-end capital w falls below $u_{\alpha_2,t}(c_{\max}, \mu_{\alpha_1})$, the capital deficiency must be covered by borrowing. In the opposite case, when w is above $u_{\alpha_2,t}(c_{\min}, \mu_{\alpha_1})$, the excess of capital must be adsorbed (e.g., distributed as dividends). It would be smarter to set provisions[10] to store in the

[10] That is suggested by practice. It is known that taxation of the risk reserves is less than taxation of capitals. It makes raising capital more expensive than, e.g., holding equalization reserves (see Dacorogna and Rüttener (2006)).

latter case, say, in the "years of plenty" in order to cover deficiencies in the former case, say, in the "years of famine".

2.6. Adaptive control with linearized premiums

A technical drawback of the control (13) is that $c_{\alpha_2,t}(w,\mu_{\alpha_1})$ has to be calculated for each w, i.e., the non-linear function has to be determined. Introduce

$$\bar{\tau}_{\alpha_2,t}(w) = -\frac{w - u_{\alpha_2,t}\left(\mathsf{E}M, \mu_{\alpha_1}\right)}{t}, \tag{14}$$

where $\mathsf{E}M$ is "fair" in the sense of Eq. (12), or ultimately equitable premium rate. Recall that $c_{\min} = \mu_{\min}$, $c_{\max} = \mu_{\alpha_1}$ and introduce the control with linearized adaptive premium rates

$$\begin{aligned} \bar{u}(w) &= \begin{cases} u_{\alpha_2,t}\left(c_{\min}, \mu_{\alpha_1}\right), & w > u_{\alpha_2,t}\left(c_{\min}, \mu_{\alpha_1}\right), \\ w, & u_{\alpha_2,t}\left(c_{\max}, \mu_{\alpha_1}\right) \leqslant w \leqslant u_{\alpha_2,t}\left(c_{\min}, \mu_{\alpha_1}\right), \\ u_{\alpha_2,t}\left(c_{\max}, \mu_{\alpha_1}\right), & 0 < w < u_{\alpha_2,t}\left(c_{\max}, \mu_{\alpha_1}\right), \end{cases} \\ \bar{c}(w) &= \begin{cases} \mathsf{E}M + \bar{\tau}_{\alpha_2,t}\left(u_{\alpha_2,t}\left(c_{\min}, \mu_{\alpha_1}\right)\right), & w > u_{\alpha_2,t}\left(c_{\min}, \mu_{\alpha_1}\right), \\ \mathsf{E}M + \bar{\tau}_{\alpha_2,t}(w), & u_{\alpha_2,t}\left(c_{\max}, \mu_{\alpha_1}\right) \leqslant w \leqslant u_{\alpha_2,t}\left(c_{\min}, \mu_{\alpha_1}\right), \\ \mathsf{E}M + \bar{\tau}_{\alpha_2,t}\left(u_{\alpha_2,t}\left(c_{\max}, \mu_{\alpha_1}\right)\right), & 0 < w < u_{\alpha_2,t}\left(c_{\max}, \mu_{\alpha_1}\right). \end{cases} \end{aligned} \tag{15}$$

On the one hand, the unique value $u_{\alpha_2,t}(\mathsf{E}M,\mu_{\alpha_1})$ is easier to calculate than a nonlinear function $c_{\alpha_2,t}(w,\mu_{\alpha_1})$. On the other hand, it casts more light on equity.

The rate $\mathsf{E}M + \bar{\tau}_{\alpha_2,t}(w)$, where $\mathsf{E}M$ is the average price and $\bar{\tau}_{\alpha_2,t}(w)$ is the adaptive loading, either positive or negative, depends on the deviation of the previous year-end risk reserve w from the "fair" capital value $u_{\alpha_2,t}(\mathsf{E}M, \mu_{\alpha_1})$ *linearly*. The case $\bar{\tau}_{\alpha_2,t}(w) > 0$ corresponds to the previous year-end deficit below $u_{\alpha_2,t}(\mathsf{E}M,\mu_{\alpha_1})$, whereas the case $\bar{\tau}_{\alpha_2,t}(w) < 0$ corresponds to the previous year-end surplus over $u_{\alpha_2,t}(\mathsf{E}M,\mu_{\alpha_1})$.

THEOREM 2.3. *The control* $(\bar{u}(w),\bar{c}(w))$ *is ultimately equitable.*

PROOF OF THEOREM 2.3. Set $z(w) = w - u_{\alpha_2,t}(\mathsf{E}M,\mu_{\alpha_1})$. When $u_{\alpha_2,t}(c_{\max},\mu_{\alpha_1}) \leqslant w \leqslant u_{\alpha_2,t}(c_{\min},\mu_{\alpha_1})$, one has

$$\begin{aligned} R_t(\bar{u}(w),\bar{c}(w),M) &= \bar{u}(w) + \bar{c}(w)t - V_t(M) \\ &= z(w) + u_{\alpha_2,t}(\mathsf{E}M,\mu_{\alpha_1}) + \left(\mathsf{E}M - \frac{z(w)}{t}\right)t - V_t(M) \\ &= u_{\alpha_2,t}(\mathsf{E}M,\mu_{\alpha_1}) + \mathsf{E}M\cdot t - V_t(M). \end{aligned}$$

In two other cases, when $w > u_{\alpha_2,t}(c_{\min}, \mu_{\alpha_1})$ and $0 < w < u_{\alpha_2,t}(c_{\max}, \mu_{\alpha_1})$, the similar expression for $R_t(\bar{u}(w), \bar{c}(w), M)$ is evident. Bearing in mind Eq. (12), one has

$$\mathsf{E}R_t(\bar{u}(w), \bar{c}(w), \mu_{\alpha_1}) = u_{\alpha_2,t}(\mathsf{E}M, \mu_{\alpha_1})$$

uniformly in $w \in \mathsf{R}^+$. □

REMARK 2.4. It is noteworthy that the lower premium intensity $\mathsf{E}M + \bar{\tau}_{\alpha_2,t}(u_{\alpha_2,t}(c_{\min}, \mu_{\alpha_1}))$ in Eq. (15) is close to $c_{\min}$ because of the following. While the insurance years in the model (1) are numbered, the time within each separate insurance year is set *operational* rather than calendar. The operational time is known to be proportional to the size of the insurance portfolio. Therefore, provided the insurance portfolio is large, time t in the generic model (2) may be assumed large. It makes the asymptotical analysis eligible, as $t \to \infty$. By Theorem 4.4,

$$\begin{aligned} \mathsf{E}M + \bar{\tau}_{\alpha_2,t}(u_{\alpha_2,t}(c_{\min}, \mu_{\alpha_1})) &= \mathsf{E}M - \frac{u_{\alpha_2,t}\left(c_{\min}, \mu_{\alpha_1}\right) - u_{\alpha_2,t}\left(\mathsf{E}M, \mu_{\alpha_1}\right)}{t} \\ &= c_{\min} + \sigma(\mu_{\alpha_1}) \frac{z_{\alpha_2}\left(\left(\mu_{\alpha_1} - c_{\min}\right)\sqrt{t}/\sigma\left(\mu_{\alpha_1}\right)\right) + z_{\alpha_2}\left(\left(\mu_{\alpha_1} - \mathsf{E}M\right)\sqrt{t}/\sigma\left(\mu_{\alpha_1}\right)\right)}{\sqrt{t}}, \end{aligned} \tag{16}$$

where $z_{\alpha_2}(\cdot)$ is the function introduced in Theorem 4.4, $0 < \kappa_{\alpha_2} \leqslant z_{\alpha_2}((\mu_{\alpha_1} - \mathsf{E}M)\sqrt{t}/\sigma(\mu_{\alpha_1})) \leqslant \kappa_{\alpha_2/2}$, $0 < \kappa_{\alpha_2} \leqslant z_{\alpha_2}((\mu_{\alpha_1} - c_{\min})\sqrt{t}/\sigma(\mu_{\alpha_1})) \leqslant \kappa_{\alpha_2/2}$. As $t \to \infty$, the second summand in the right hand side of Eq. (16) is tending to zero, being the term of order $\underline{O}(t^{-1/2})$. The lower premium intensity in Eq. (15) remains close to $c_{\min}$. By the similar arguments, the upper premium intensity in Eq. (15) is close to $c_{\max}$.

Unfortunately, the control $(\bar{u}(w), \bar{c}(w))$ with linearized adaptive premium rates satisfies no more the (α_1, α_2)-solvency criterion: the upper bound for the annual probabilities of ruin

$$\sup_{m \leqslant \mu_{\alpha_1}} \psi_t(\bar{u}(w), \bar{c}(w), m) = \psi_t(\bar{u}(w), \bar{c}(w), \mu_{\alpha_1})$$

may exceed α_2 for some $w \in \mathsf{R}^+$.

THEOREM 2.4. *One has*

$$c_{\alpha_2,t}\left(w, \mu_{\alpha_1}\right) \begin{cases} < \mathsf{E}M + \bar{\tau}_{\alpha_2,t}(w), & w > u_{\alpha_2,t}\left(\mathsf{E}M, \mu_{\alpha_1}\right), \\ = \mathsf{E}M + \bar{\tau}_{\alpha_2,t}(w), & w = u_{\alpha_2,t}\left(\mathsf{E}M, \mu_{\alpha_1}\right), \\ > \mathsf{E}M + \bar{\tau}_{\alpha_2,t}(w), & 0 < w < u_{\alpha_2,t}\left(\mathsf{E}M, \mu_{\alpha_1}\right). \end{cases} \tag{17}$$

PROOF OF THEOREM 2.4. Introduce

$$L(w) = c_{\alpha_2,t}(w,\mu_{\alpha_1}) - (\mathsf{E}M + \bar{\tau}_{\alpha_2,t}(w)), \quad w \geqslant 0, \tag{18}$$

and note that $L(u_{\alpha_2,t}(\mathsf{E}M,\mu_{\alpha_1})) = 0$. It is straightforward from $c_{\alpha_2,t}(u_{\alpha_2,t}(\mathsf{E}M,\mu_{\alpha_1}),\mu_{\alpha_1}) = \mathsf{E}M$ and $\bar{\tau}_{\alpha_2,t}(u_{\alpha_2,t}(\mathsf{E}M,\mu_{\alpha_1})) = 0$ (see Remark 2.2 and Eq. (14)). Theorem 4.4. and Theorem 4.5 yield

$$L(w) = \frac{w}{t} - \frac{\sigma(\mu_{\alpha_1})}{\sqrt{t}} v_{\alpha_2}\left(\frac{w}{\sigma(\mu_{\alpha_1})\sqrt{t}}\right) - \frac{\sigma(\mu_{\alpha_1})}{\sqrt{t}} z_{\alpha_2}\left(\frac{(\mu_{\alpha_1} - \mathsf{E}M)\sqrt{t}}{\sigma(\mu_{\alpha_1})}\right), \quad w \geqslant 0, \tag{19}$$

where $z_{\alpha_2}(\cdot)$ and $v_{\alpha_2}(\cdot)$ are the functions introduced in Theorems 4.4 and 4.5. Continuous function $L(w)$ is monotone decreasing since $v'_{\alpha_2}(z) > 1$ for $z \geqslant 0$ by Theorem 4.5, and

$$L'(w) = \frac{1}{t}\left[1 - v'_{\alpha_2}\left(\frac{w}{\sigma(\mu_{\alpha_1})\sqrt{t}}\right)\right] < 0, \quad w \geqslant 0.$$

It completes the proof. □

Theorem 2.4 claims that the linearized control overcharges the insureds when the past year capital w exceeds the target value $u_{\alpha_2,t}(\mathsf{E}M,\mu_{\alpha_1})$ and undercharges them otherwise. In that sense linearization deteriorates the control (13). Our next goal is to cure that deficiency.

2.7. Zone-adaptive control with linearized premiums

Improve the control with linear adaptive loading (15) seeking for controllable solvency. For the level β such that $0 < \alpha_2 \leqslant \beta < 1/2$, introduce the strip zone with the lower bound $\underline{u}_{\beta,t} = u_{\alpha_2,t}(\mathsf{E}M,\mu_{\alpha_1}) + z_{\beta,t}$, where $z_{\beta,t} < 0$ is a solution of the equation

$$\psi_t\left(z + u_{\alpha_2,t}(\mathsf{E}M,\mu_{\alpha_1}), \mathsf{E}M - \frac{z}{t}, \mu_{\alpha_1}\right) = \beta \tag{20}$$

with respect to z, and with an upper bound $\bar{u}_{\beta,t}$ such that

$$u_{\alpha_2,t}(c_{\max},\mu_{\alpha_1}) \leqslant \underline{u}_{\beta,t} \leqslant u_{\alpha_2,t}(\mathsf{E}M,\mu_{\alpha_1}) \leqslant \bar{u}_{\beta,t} \leqslant u_{\alpha_2,t}(c_{\min},\mu_{\alpha_1}).$$

There are different ways to select the upper bound $\bar{u}_{\beta,t}$. For example (recall that $c_{\min} = \mu_{\min}$, $c_{\max} = \mu_{\alpha_1}$), one may take $\bar{u}_{\beta,t} = u_{\alpha_2,t}(c_{\min},\mu_{\alpha_1})$, or[11] $\bar{u}_{\beta,t} = u_{\alpha_2,t}(\mathsf{E}M,\mu_{\alpha_1})$.

[11] That selection is sensible because the premiums will not be larger than $\mathsf{E}M$ (i.e., $\bar{\mu}_{\beta,t} = \mathsf{E}M$ in (21)) and no capital exceeding one least necessary to guarantee the non-ruin with probability α_2 is "frozen" as solvency reserve. For $\bar{u}_{\beta,t}$ selected in that way, $|z_{\beta,t}|$ is the width of the strip zone. However, these reasons may look unconvincing for a decision maker with other preferences.

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Introduce the control

$$\hat{u}(w) = \begin{cases} \overline{u}_{\beta,t}, & w > \overline{u}_{\beta,t}, \\ w, & \underline{u}_{\beta,t} \leqslant w \leqslant \overline{u}_{\beta,t}, \\ \underline{u}_{\beta,t}, & 0 < w < \underline{u}_{\beta,t}, \end{cases}$$
$$\hat{c}(w) = \begin{cases} \overline{\mu}_{\beta,t}, & w > \overline{u}_{\beta,t}, \\ \mathsf{E}M + \overline{\tau}_{\alpha_2,t}(w) & \underline{u}_{\beta,t} \leqslant w \leqslant \overline{u}_{\beta,t}, \\ \underline{\mu}_{\beta,t}, & 0 < w < \underline{u}_{\beta,t} \end{cases} \tag{21}$$

called zone-adaptive annual control with linearized premiums, where

$$\overline{\mu}_{\beta,t} = \mathsf{E}M - \frac{\overline{u}_{\beta,t} - u_{\alpha_2,t}\left(\mathsf{E}M, \mu_{\alpha_1}\right)}{t},$$
$$\underline{\mu}_{\beta,t} = \mathsf{E}M - \frac{\underline{u}_{\beta,t} - u_{\alpha_2,t}\left(\mathsf{E}M, \mu_{\alpha_1}\right)}{t} = \mathsf{E}M - \frac{z_{\beta,t}}{t}.$$

THEOREM 2.5. *For* $0 < \alpha_1 < 1/2$, $0 < \alpha_2 \leqslant \beta < 1/2$, *the control* $(\hat{u}(w), \hat{c}(w))$ *is ultimately equitable and satisfies the* (α_1, β)*-solvency criterion sharply.*

PROOF OF THEOREM 2.5. The proof of the first assertion is straightforward. It consists in verification, similarly to the proof of Theorem 2.3, that the equation

$$\mathsf{E}R_t(\hat{u}(w), \hat{c}(w), M) = u_{\alpha_2,t}(\mathsf{E}M, \mu_{\alpha_1})$$

holds true uniformly in $w \in \mathsf{R}^+$. The second assertion needs no proof since

$$\sup_{m \leqslant \mu_{\alpha_1}} \psi_t(\hat{u}(w), \hat{c}(w), m) = \psi_t(\hat{u}(w), \hat{c}(w), \mu_{\alpha_1}) \equiv \beta \tag{22}$$

uniformly in $w \in \mathsf{R}^+$, by Eq. (20).

THEOREM 2.6. *For* $z \in [a,b]$*, where* $-u_{\alpha_2,t}(\mathsf{E}M, \mu_{\alpha_1}) < a < 0 < b < \mathsf{E}M \cdot t$*, the probability in the left hand side of Eq.* (20) *regarded as a function of z, is monotone decreasing, as z increases.*

PROOF OF THEOREM 2.6. Bearing in mind Eq. (4), the proof is straightforward from

$$\psi_t\left(z + u_{\alpha_2,t}(\mathsf{E}M, \mu_{\alpha_1}), \mathsf{E}M - \frac{z}{t}, \mu_{\alpha_1}\right)$$
$$= \mathsf{P}\left\{\inf_{0 < s \leqslant t}\left[\left(1 - \frac{s}{t}\right) z + (\mathsf{E}M - \mu_{\alpha_1}) s - \sigma(\mu_{\alpha_1}) \mathsf{W}_s\right] < -u_{\alpha_2,t}(\mathsf{E}M, \mu_{\alpha_1})\right\},$$

since $1 - \frac{s}{t} \geqslant 0$ under the infimum sign. □

2.8. Strip width

Address existence, uniqueness and analytical structure of $z_{\beta,t} < 0$, which is a solution of Eq. (20).

THEOREM 2.7. *For* $0 < \alpha_1 < 1/2$, $0 < \alpha_2 \leqslant \beta < 1/2$, *set*[12] $z_{\alpha_2,t} = z_{\alpha_2}((\mu_{\alpha_1} - \mathsf{E}M)\sqrt{t}/\sigma(\mu_{\alpha_1}))$, *where* $z_{\alpha_2}(\cdot)$ *is the function introduced in Theorem 4.4. The unique solution of Eq.* (20) *may be written as*

$$z_{\beta,t} = -[(\mu_{\alpha_1} - \mathsf{E}M)t + \sigma(\mu_{\alpha_1})\sqrt{t}\,x_{\beta,t}],$$

where $x_{\beta,t} > 0$ *is the unique root of the equation*

$$1 - \Phi(z_{\alpha_2,t}) + \exp\{-2x(z_{\alpha_2,t} - x)\}\,\Phi(2x - z_{\alpha_2,t}) = \beta. \tag{23}$$

REMARK 2.5. For any $0 < \alpha_2 \leqslant \beta \leqslant 1/2$ and $t \geqslant 0$ the solution of Eq. (23) is bounded from above by a constant, $0 < x_{\beta,t} \leqslant z_{\alpha_2,t} \leqslant \kappa_{\alpha_2/2}$.

PROOF OF THEOREM 2.7. Bearing in mind Theorem 4.4, Theorem 2.6 and Eq. (32), it requires just some direct algebra. □

2.9. Asymptotic analysis and rules of thumb

Summarize the results of the previous sections as recommendations. It yields certain "rules of thumb", as t is large (see Remark 2.4).

The magnitude of the target capital value is paramount: it yields a benchmark for the size of the "long-run mean value", or "appropriate risk-based provisions". That magnitude appears dramatically larger for the volatile scenario than for the complete knowledge case[13] since

$$\begin{aligned} u_{\alpha_2,t}(\mathsf{E}M, \mu_{\alpha_1}) &= (\mu_{\alpha_1} - \mathsf{E}M)t + \sigma(\mu_{\alpha_1})\sqrt{t}\,z_{\alpha_2,t} \\ &= (\mu_{\alpha_1} - \mathsf{E}M)t + \underline{O}(\sqrt{t}), \quad \text{as } t \to \infty. \end{aligned}$$

In particular, unless μ_{α_1} equals $\mathsf{E}M$, the "fair" or "target" capital value is the term of order t rather than of order $\sqrt{t}$. The latter was the case for the completely known risk (see Theorem 2.1 in Malinovskii (2007)).

[12] Recall that $\kappa_{\alpha_2/2} = z_{\alpha_2}(0) \geqslant z_{\alpha_2}(v) \geqslant z_{\alpha_2}(+\infty) = \kappa_{\alpha_2} \geqslant 0$ for $v > 0$.
[13] Recall that $0 < \kappa_{\alpha_2} \leqslant z_{\alpha_2,t} = z_{\alpha_2}((\mu_{\alpha_1} - \mathsf{E}M)\sqrt{t}/\sigma(\mu_{\alpha_1})) \leqslant \kappa_{\alpha_2/2}$.

Besides the above asymptotics for $u_{\alpha_2,t}(\mathsf{E}M, \mu_{\alpha_1})$, which is the upper bound of the strip zone defined in (21), for the lower bound one has

$$\begin{aligned}\underline{u}_{\beta,t} &= u_{\alpha_2,t}(\mathsf{E}M, \mu_{\alpha_1}) + z_{\beta,t}\\ &= \sigma(\mu_{\alpha_1})\sqrt{t}\,[z_{\alpha_2,t} - x_{\beta,t}] = \underline{O}(\sqrt{t}), \quad \text{as } t \to \infty.\end{aligned}$$

For the width of the strip zone one has therefore

$$|z_{\beta,t}| = (\mu_{\alpha_1} - \mathsf{E}M)t + \sigma(\mu_{\alpha_1})\sqrt{t}\,x_{\beta,t}.$$

Bearing in mind Remark 2.5, one has $0 < x_{\beta,t} \leqslant z_{\alpha_2,t} = z_{\alpha_2}((\mu_{\alpha_1} - \mathsf{E}M)\sqrt{t}\,/\,\sigma(\mu_{\alpha_1})) \leqslant \kappa_{\alpha_2/2}$.

Extend Remark 2.4 as follows. For $0 \leqslant \alpha_2 \leqslant 1/2$ and for the capital w_t such that $w_t - \sigma(\mu_{\alpha_1})\sqrt{t} \to +\infty$, as $t \to \infty$, the linearized premium rate $\mathsf{E}M + \bar{\tau}_{\alpha_2,t}(w_t)$ differs from the original premium rate $c_{\alpha_2,t}(w_t, \mu_{\alpha_1})$ by the terms of order $\sqrt{t}$. Deterioration of the original premium rate is therefore rather small in magnitude. By Lemma 4.1, one has $v_{\alpha_2}(z) = z - \kappa_{\alpha_2} + \bar{o}(1)$, as $z \to +\infty$, for the function $v_{\alpha_2}(\cdot)$ introduced in Theorem 4.5. Eq. (19) yields

$$L(w_t) = c_{\alpha_2,t}(w_t, \mu_{\alpha_1}) - \left(\mathsf{E}M + \bar{\tau}_{\alpha_2,t}(w_t)\right) = -\frac{\sigma\left(\mu_{\alpha_1}\right)}{\sqrt{t}}(z_{\alpha_2,t} - \kappa_{\alpha_2}) + \bar{o}(t^{-1/2}), \quad \text{as } t \to \infty.$$

Bearing in mind that $0 < \kappa_{\alpha_2} \leqslant z_{\alpha_2,t} = z_{\alpha_2}((\mu_{\alpha_1} - \mathsf{E}M)\sqrt{t}\,/\,\sigma(\mu_{\alpha_1})) \leqslant \kappa_{\alpha_2/2}$, the right hand side of this equation is $\underline{O}(t^{-1/2})$, as $t \to \infty$. It is also noteworthy that for $0 < \alpha_2 \leqslant \beta \leqslant 1/2$

$$0 < c_{\alpha_2,t}(w_t, \mu_{\alpha_1}) - c_{\beta,t}(w_t, \mu_{\alpha_1}) = \frac{\sigma\left(\mu_{\alpha_1}\right)}{\sqrt{t}}(\kappa_{\alpha_2} - \kappa_{\beta}) + \bar{o}(t^{-1/2}), \text{ as } t \to \infty.$$

3. Multi-period model of risk under volatile scenario

A rigorous definition of the multi-period controlled risk model, with realizations matching diagram (1), over the elementary state space $(\Omega, \mathcal{F})$ is equivalent to the definition of a controlled random sequence (see Malinovskii (2007) – Malinovskii (2008b)). In our particular case of the

(a) annual mechanisms of insurance introduced by Eq. (2),

(b) volatile scenario of nature introduced in Definition 1.1,

(c) adaptive controls introduced in Section 2

the state space W and the control space U are $\mathsf{R} \times \{0,1\} \times \mathrm{M}$ and $\mathsf{R}^+ \times \mathsf{R}^+$ respectively.

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It is noteworthy that assuming[14] all probability mechanisms of insurance π_k, $k = 1, 2, \ldots$, complying with the same generic model (2), we deal with the *homogeneous* multi-period model. It matches well the homogeneous volatile scenario of nature set forth in Definition 1.1.

REMARK 3.1. Bearing in mind e.g., inflation, the "long run" multi-period model becomes more realistic when the probability mechanisms of insurance π_k are endowed with discount factors. It will be done elsewhere, since our concern in this paper is the homogeneous case. Bearing in mind Remark 2.4, emphasize it that "long run" refers here to a large number of insurance years rather than to the length of each separate insurance year.

The first component of the state vector $\mathbf{w}_k = (\mathbf{w}_k^{\langle 1\rangle}, \mathbf{w}_k^{\langle 2\rangle}, \mathbf{w}_k^{\langle 3\rangle}) \in \mathrm{W}$ is the k-th year-end capital of the company. The second component indicates whether ruin has occurred, or not, in the k-th year. The third component is the outcome of the next-year claims intensity which is the choice of the nature. The two components of the control vector $\mathbf{u}_{k-1} = (\mathbf{u}_{k-1}^{\langle 1\rangle}, \mathbf{u}_{k-1}^{\langle 2\rangle}) \in \mathrm{U}$ are the starting capital and the premium intensity in Eq. (2), respectively.

It is known (see, e.g., §1 of Chapter 1 in Gihman and Skorokhod (1979)) that under certain mild regularity conditions the couple $\boldsymbol{\pi} = \{\pi_k, k = 1, 2, \ldots\}$ and $\boldsymbol{\gamma} = \{\gamma_k, k = 0, 1, \ldots\}$ is a sufficient background for a rigorous definition of the controlled random sequence (W_k, M_{k+1}, U_k), $k = 0, 1, \ldots$, on the probability space $(\Omega, \mathcal{F}, \mathbf{P}^{\boldsymbol{\pi},\boldsymbol{\gamma}})$. This random sequence will assume values in the product space $(\mathrm{W} \times \mathrm{U}, \mathcal{W} \otimes \mathcal{U})$.

In this paper we deal with Markov (see, e.g., Section 3 of Malinovskii (2007) for definitions and particulars) annual probability mechanisms of insurance π_k and pure Markov strategies[15] $\boldsymbol{\gamma} = \{\gamma_k, k = 0, 1, \ldots\}$. Therefore, the controlled random sequence (W_k, M_{k+1}, U_k), $k = 0, 1, \ldots$, may be reduced to a homogeneous Markov chain on the state space W with the transition probability

$$P(\mathbf{w}_{k-1}; d\mathbf{w}_k) = P_{\mathbf{w}_{k-1}^{\langle 3\rangle}}(\mathbf{w}_{k-1}^{\langle 1\rangle}; d\mathbf{w}_k^{\langle 1\rangle} \times d\mathbf{w}_k^{\langle 2\rangle})\, G(d\mathbf{w}_k^{\langle 3\rangle}),$$

where

$$\begin{aligned} P_m(\mathbf{w}_{k-1}^{\langle 1\rangle}; d\mathbf{w}_k^{\langle 1\rangle} \times \{0\}) &= \mathsf{P}\{R_t(\gamma_{k-1}(\mathbf{w}_{k-1})) \\ &\qquad \in d\mathbf{w}_k^{\langle 1\rangle}, \inf_{0 \leqslant s \leqslant t} R_s(\gamma_{k-1}(\mathbf{w}_{k-1})) > 0 \mid M_k = m\}, \\ P_m(\mathbf{w}_{k-1}^{\langle 1\rangle}; d\mathbf{w}_k^{\langle 1\rangle} \times \{1\}) &= \mathsf{P}\{R_t(\gamma_{k-1}(\mathbf{w}_{k-1})) \\ &\qquad \in d\mathbf{w}_k^{\langle 1\rangle}, \inf_{0 \leqslant s \leqslant t} R_s(\gamma_{k-1}(\mathbf{w}_{k-1})) < 0 \mid M_k = m\} \end{aligned} \tag{24}$$

[14] For simplicity's sake. A straightforward and useful generalization is different annual mechanisms of insurance and different annual controls. The unique technical difficulty is that the non-homogeneous Markov chains will appear in Section 3. An other self-suggesting generalization is the non-homogeneous volatile scenario of nature, i.e., M_k, $k = 1, 2, \ldots$, independent, but not identically distributed.

[15] Or $\gamma_n = \{\gamma_k, k = 0, 1, \ldots, n-1\}$, to introduce a more specific notation for the n-years horizon strategy.

and c.d.f. G is the common distribution of the independent random variables M_k, $k = 1, 2, \ldots$.

REMARK 3.2. In the premises of the diffusion model (2), one can easily write the explicit expression for

$$P_m(\mathbf{w}_{k-1}^{\langle 1\rangle}; d\mathbf{w}_k^{\langle 1\rangle} \times \{0,1\}) = \mathsf{P}\{R_t(\gamma_{k-1}(\mathbf{w}_{k-1})) \in d\mathbf{w}_k^{\langle 1\rangle} \mid M_k = m\}$$
$$= \mathsf{P}\{\gamma_{k-1}^{\langle 1\rangle}(\mathbf{w}_{k-1}) + \gamma_{k-1}^{\langle 2\rangle}(\mathbf{w}_{k-1})\, t - (mt + \sigma(m)\mathsf{W}_t) \in d\mathbf{w}_k^{\langle 1\rangle}\}.$$

Theorem 4.3 provides the explicit expressions for the right hand sides of Eq. (24) i.e., for the transition probability $P(\mathbf{w}_{k-1}; d\mathbf{w}_k)$.

We use notation $\mathbf{P}^{\pi,\gamma}\{\cdot\}$ for the Markov chain with transition probability P. We denote by $\mathbf{E}^{\pi,\gamma}$ the mean with respect to that measure. Further, we denote by $\mathbf{P}_{\boldsymbol{m}}^{\pi,\gamma}\{\cdot\}$ the conditional distribution $\mathbf{P}^{\pi,\gamma}\{\cdot \mid \boldsymbol{M} = \boldsymbol{m}\}$, where $\boldsymbol{M} = \{M_k, k = 1, 2, \ldots\} \in \mathbf{M} = \mathrm{M}^\infty$ is the sequence of independent and identically distributed random variables and $\boldsymbol{m}$ is its realization. In words, $\mathbf{P}_{\boldsymbol{m}}^{\pi,\gamma}\{\cdot\}$ corresponds to the case when the trajectory $\boldsymbol{m}$ of the scenario of nature is fixed. Write $\mathbf{E}_{\boldsymbol{m}}^{\pi,\gamma}$ for the respective conditional expectation.

3.1. Solvency

The following results are fundamental.

THEOREM 3.1. *Assume that the starting capital in the homogeneous multi-period model is $w \in \mathsf{R}^+$. For the homogeneous pure Markov strategy γ generated by the annual control* (13)*,*

$$\sup_{w \in \mathsf{R}^+} \mathbf{P}^{\pi,\gamma} \left\{ \begin{array}{c} \text{first ruin in year } k, \\ \text{as starting capital is } w \end{array} \right\} \leqslant \alpha_1 + \alpha_2, \quad k = 1, 2, \ldots. \tag{25}$$

For the homogeneous pure Markov strategy γ generated by the zone-adaptive annual control with linearized premiums (21)*,*

$$\sup_{w \in \mathsf{R}^+} \mathbf{P}^{\pi,\gamma} \left\{ \begin{array}{c} \text{first ruin in year } k, \\ \text{as starting capital is } w \end{array} \right\} \leqslant \alpha_1 + \beta, \quad k = 1, 2, \ldots. \tag{26}$$

PROOF OF THEOREM 3.1. The proof of (25) is immediate from

$$\mathbf{P}^{\pi,\gamma} \left\{ \begin{array}{c} \text{first ruin in year } k, \\ \text{as starting capital is } w \end{array} \right\} = \int_{\mathsf{R}\times\mathrm{M}} G(dm_1)\, P_{m_1}(w; d\mathbf{w}_1^{\langle 1\rangle} \times \{0\}) \ldots$$
$$\ldots \int_{\mathsf{R}\times\mathrm{M}} G(dm_{k-1})\, P_{m_{k-1}}(\mathbf{w}_{k-2}^{\langle 1\rangle}; d\mathbf{w}_{k-1}^{\langle 1\rangle} \times \{0\}) \int_{\mathsf{R}\times\mathrm{M}} G(dm_k)\, P_{m_k}(\mathbf{w}_{k-1}^{\langle 1\rangle}; \mathsf{R} \times \{1\}),$$

$$\sup_{\mathbf{w}_{k-1}^{\langle 1\rangle}\in \mathsf{R}} \int_{\mathrm{M}} G(dm_k)\, P_{m_k}(\mathbf{w}_{k-1}^{\langle 1\rangle}; \mathsf{R}\times\{1\})$$
$$\leqslant \sup_{\mathbf{w}_{k-1}^{\langle 1\rangle}\in \mathsf{R}} \psi_t(\hat{u}(\mathbf{w}_{k-1}^{\langle 1\rangle}), \hat{c}(\mathbf{w}_{k-1}^{\langle 1\rangle}), \mu_{\alpha_1}) + \mathsf{P}\{M_k > \mu_{\alpha_1}\}$$

and Theorem 2.2. The proof of the bound (26) is quite analogous and applies Theorem 2.5. □

COROLLARY 3.1. Assume that the starting capital in the homogeneous multi-period model is $w \in \mathsf{R}^+$. For the homogeneous pure Markov strategy γ generated by the annual control (13),

$$\sup_{w\in\mathsf{R}^+} \mathbf{P}^{\pi,\gamma}\left\{\begin{matrix}\text{ruin within } n \text{ years,}\\ \text{as starting capital is } w\end{matrix}\right\} \leqslant \sum_{k=1}^{n} \sup_{w\in\mathsf{R}^+} \mathbf{P}^{\pi,\gamma}\left\{\begin{matrix}\text{first ruin in year } k,\\ \text{as starting capital is } w\end{matrix}\right\} \leqslant n(\alpha_1+\alpha_2)$$

for $n = 1, 2, \ldots$. For the homogeneous pure Markov strategy γ generated by the zone-adaptive annual control with linearized premiums (21) the above relations hold true with $n(\alpha_1+\beta)$ instead of $n(\alpha_1+\alpha_2)$.

3.2. Equity

In the homogeneous multi-period model with the starting capital $w \in \mathsf{R}^+$ the homogeneous pure Markov strategy γ generated by the annual control (15) with linearized premiums is equitable by Theorem 2.3. It means that uniformly in $w \in \mathsf{R}^+$ and for $k = 1, 2, \ldots$,

$$\mathsf{E}\left[\mathbf{E}_{\boldsymbol{m}}^{\pi,\gamma}\left(\begin{matrix}\text{capital at the end of year } k,\\ \text{as starting capital is } w\end{matrix}\right)\right] = u_{\alpha_2,t}(\mathsf{E}M, \mu_{\alpha_1}).$$

That strategy directs the risk reserve at the "target" value $u_{\alpha_2,t}(\mathsf{E}M, \mu_{\alpha_1})$ which makes the risk reserver process balanced around that value in a long-time perspective.

The similar property holds true for the strategy γ generated by Eq. (21).

THEOREM 3.2. *For the homogeneous pure Markov strategy γ generated by the zone-adaptive annual control with linearized premiums* (21), *Eq.* (27) *holds true.*

PROOF OF THEOREM 3.2. Note first that for $\boldsymbol{m} = (m_1, m_2, \ldots)$

$$\mathbf{E}_{\boldsymbol{m}}^{\pi,\gamma}\left(\begin{matrix}\text{capital at the end of year } k,\\ \text{as starting capital is } w\end{matrix}\right) = \int_{\mathsf{R}} P_{m_1}(w; d\mathbf{w}_1^{\langle 1\rangle}\times\{0,1\})$$
$$\ldots \int_{\mathsf{R}} P_{m_{k-1}}(\mathbf{w}_{k-2}^{\langle 1\rangle}; d\mathbf{w}_{k-1}^{\langle 1\rangle}\times\{0,1\}) \int_{\mathsf{R}} \mathbf{w}_k^{\langle 1\rangle} P_{m_k}(\mathbf{w}_{k-1}^{\langle 1\rangle}; d\mathbf{w}_k^{\langle 1\rangle}\times\{0,1\}). \quad (28)$$

Bearing in mind Remark 3.2 and Eq. (21), one has

$$\int_{\mathsf{R}} \mathbf{w}_k^{\langle 1\rangle} P_{m_k}(\mathbf{w}_{k-1}^{\langle 1\rangle}; d\mathbf{w}_k^{\langle 1\rangle} \times \{0,1\})$$

$$= \int_{\mathsf{R}} \mathbf{w}_k^{\langle 1\rangle} \mathsf{P}\{\hat{u}(\mathbf{w}_{k-1}^{\langle 1\rangle}) + \hat{c}(\mathbf{w}_{k-1}^{\langle 1\rangle})t - (m_k t + \sigma(m_k)\mathsf{W}_t) \in d\mathbf{w}_k^{\langle 1\rangle}\}$$

$$= \begin{cases} (\mathsf{E}M - m_k)t + u_{\alpha_2,t}(\mathsf{E}M, \mu_{\alpha_1}), & \mathbf{w}_{k-1}^{\langle 1\rangle} > \overline{u}_{\beta,t}, \\ (\mathsf{E}M - m_k)t + u_{\alpha_2,t}(\mathsf{E}M, \mu_{\alpha_1}), & \underline{u}_{\beta,t} \leqslant \mathbf{w}_{k-1}^{\langle 1\rangle} \leqslant \overline{u}_{\beta,t}, \\ (\mathsf{E}M - m_k)t + u_{\alpha_2,t}(\mathsf{E}M, \mu_{\alpha_1}), & 0 < \mathbf{w}_{k-1}^{\langle 1\rangle} < \underline{u}_{\beta,t}. \end{cases}$$

It is noteworthy that the right hand side is independent on $\mathbf{w}_{k-1}^{\langle 1\rangle}$. Put it in (28). The proof completes by taking expectation over the outcomes $\boldsymbol{m}$ of the scenario of nature. □

REMARK 3.3. The homogeneous pure Markov strategy γ generated by the zone-adaptive annual control with linearized premiums (21) is both solvent and equitable.

3.3. Dynamic solvency provisions

Provisions similar to equalization reserves are set to face a large deficit at the end of the insurance year. Commonly, these provisions are invested, but in this paper we disregard the investment aspects; one may see that the price which we do not wish to pay for it is just more cumbersome transition probabilities.

For zone-adaptive control with $\overline{u}_{\beta,t} = u_{\alpha_2,t}(\mathsf{E}M, \mu_{\alpha_1})$ and with linearized premiums (21) introduce the variable

$$\Delta_t(w) = \begin{cases} 0, & \underline{u}_{\beta,t} \leqslant w \leqslant \overline{u}_{\beta,t}, \\ w - \overline{u}_{\beta,t}, & w > \overline{u}_{\beta,t}, \\ -\left(\underline{u}_{\beta,t} - w\right), & 0 < w < \underline{u}_{\beta,t} \end{cases}$$

called annual excess (of either sign) of capital. The mean aggregate excess (of either sign) of capital within n years for the strategy γ, or the mean aggregate dynamic solvency provisions, is

$$\mathsf{E}\left[\mathbf{E}_{\boldsymbol{m}}^{\pi,\gamma} \sum_{k=1}^{n} \Delta_t\left(W_k^{\langle 1\rangle}\right)\right] = \sum_{k=1}^{n} \mathsf{E}\left[\mathbf{E}_{\boldsymbol{m}}^{\pi,\gamma} \Delta_t\left(W_k^{\langle 1\rangle}\right)\right].$$

The following theorem demonstrates that application of the homogeneous pure Markov strategy γ generated by the zone-adaptive annual control with linearized premiums (21) increases the mean aggregate dynamic solvency provisions.

THEOREM 3.3. *For the homogeneous pure Markov strategy γ generated by the zone-adaptive annual control with linearized premiums* (21) *and for* $k = 1, 2, \ldots,$ *one has*

$$\mathsf{E}\left[\mathbf{E}_{\boldsymbol{m}}^{\pi,\gamma}\Delta_t\left(W_k^{\langle 1\rangle}\right)\right] > 0$$

uniformly in $w \in \mathsf{R}^+$.

PROOF OF THEOREM 3.3. With the starting capital $w \in \mathsf{R}^+$ and with $\boldsymbol{m} = (m_1, m_2, \ldots)$,

$$\begin{aligned}\mathbf{E}_{\boldsymbol{m}}^{\pi,\gamma}\Delta_t\left(W_k^{\langle 1\rangle}\right) &= \int_{\mathsf{R}} P_{m_1}\left(w, d\mathbf{w}_1^{\langle 1\rangle} \times \{0,1\}\right) \ldots \int_{\mathsf{R}} P_{m_{k-1}}\left(\mathbf{w}_{k-2}^{\langle 1\rangle}; d\mathbf{w}_{k-1}^{\langle 1\rangle} \times \{0,1\}\right) \\ &\times \left\{\int_{\left\{\mathbf{w}_k^{\langle 1\rangle} > \bar{u}_{\beta,t}\right\}} \left(\mathbf{w}_k^{\langle 1\rangle} - \bar{u}_{\beta,t}\right) P_{m_k}\left(\mathbf{w}_{k-1}^{\langle 1\rangle}; d\mathbf{w}_k^{\langle 1\rangle} \times \{0,1\}\right)\right. \\ &\left. - \int_{\left\{\mathbf{w}_k^{\langle 1\rangle} < \underline{u}_{\beta,t}\right\}} \left(\underline{u}_{\beta,t} - \mathbf{w}_k^{\langle 1\rangle}\right) P_{m_k}\left(\mathbf{w}_{k-1}^{\langle 1\rangle}; d\mathbf{w}_k^{\langle 1\rangle} \times \{0,1\}\right)\right\}.\end{aligned} \tag{29}$$

Bearing in mind Remark 3.2, apply the explicit expression to the integrand

$$P_{m_k}(\mathbf{w}_{k-1}^{\langle 1\rangle}; d\mathbf{w}_k^{\langle 1\rangle} \times \{0,1\}) = \mathsf{P}\{\hat{u}(\mathbf{w}_{k-1}^{\langle 1\rangle}) + \hat{c}(\mathbf{w}_{k-1}^{\langle 1\rangle})t - (m_k t + \sigma(m_k)\mathsf{W}_t) \in d\mathbf{w}_k^{\langle 1\rangle}\}$$

in (29). Direct integration completes the proof. □

4. AUXILIARY RESULTS

4.1. Mill's ratio and Brownian motion

The most well-known results (see, e.g., Patel and Read (1982)) for Mill's ratio

$$\mathcal{M}(x) = \frac{1 - \Phi(x)}{\phi(x)} = e^{x^2/2}\int_x^\infty e^{-t^2/2}dt, \; x \in \mathsf{R}, \tag{30}$$

are

$$\mathcal{M}(x) > 0, \frac{d}{dx}\mathcal{M}(x) = x\mathcal{M}(x) - 1 < 0, \; \frac{d^2}{dx^2}\mathcal{M}(x) = \mathcal{M}(x)(1 + x^2) - x > 0, \; x \in \mathsf{R}, \tag{31}$$

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so that $\mathcal{M}(x)$ is concave and decreasing from ∞ to 0, as x increases from $-\infty$ to ∞. Since

$$\frac{d}{dx}\big(x\mathcal{M}(x)\big) = \frac{d^2}{dx^2}\mathcal{M}(x) > 0, \quad x \in \mathsf{R},$$

the function $x\mathcal{M}(x)$ is increasing from $-\infty$ to 1, as x increases from $-\infty$ to $+\infty$.

For reader's convenience, we collect some formulae for real-valued Brownian motion with linear drift, $\theta t + \sigma \mathsf{W}_t$, $t \geqslant 0$, where $\theta \in \mathsf{R}$, $\sigma > 0$.

THEOREM 4.1. *For* $x \geqslant 0$, *one has* $\mathsf{P}\{\sup_{0\leqslant s\leqslant t} \mathsf{W}_s > x\} = 2\mathsf{P}\{\mathsf{W}_t > x\}$.

THEOREM 4.2. *For* $x \geqslant 0$ *and* $\theta \in \mathsf{R}$, $\sigma > 0$, *one has*

$$\mathsf{P}\left\{\sup_{0\leqslant s\leqslant t}(\theta s + \sigma W_s) \leqslant x\right\} = \Phi\left(\frac{x-\theta t}{\sigma\sqrt{t}}\right) - \exp\left\{2\theta x/\sigma^2\right\}\Phi\left(\frac{-x-\theta t}{\sigma\sqrt{t}}\right).$$

THEOREM 4.3. *For* $x \geqslant 0$ *and* $\theta \in \mathsf{R}$, $\sigma > 0$, *one has*

$$\mathsf{P}\{\theta t + \sigma\mathsf{W}_t \in dy, \sup_{0\leqslant s\leqslant t}(\theta s + \sigma\mathsf{W}_s) \leqslant x\} = \mathsf{P}\{\theta t + \sigma\mathsf{W}_t \in dy\}$$
$$- \mathsf{P}\{\theta t + \sigma\mathsf{W}_t \in dy, \sup_{0\leqslant s\leqslant t}(\theta s + \sigma\mathsf{W}_s) \geqslant x\},$$

where $\mathsf{P}\{\theta t + \sigma\mathsf{W}_t \in dy\} = \dfrac{1}{\sigma\sqrt{2\pi t}}\exp\{-(y-\theta t)^2/2\sigma^2 t\}\,dy$ *and*

$$\mathsf{P}\{\theta t + \sigma\mathsf{W}_t \in dy, \sup_{0\leqslant s\leqslant t}(\theta s + \sigma\mathsf{W}_s) \geqslant x\}$$
$$= \frac{1}{\sigma\sqrt{2\pi t}}\exp\{(2\theta y t - \theta^2 t^2 - (|y-x|+x)^2)/2\sigma^2 t\}dy.$$

These three results are well known. Theorem 4.1 is formula 1.1.4 in Part II, Chapter 1 of Borodin and Salminen (1996). For Theorem 4.2 see formula 1.1.4 in Part II, Chapter 2 of Borodin and Salminen (1996). For Theorem 4.3 see formulae 1.0.6 and 1.1.8 in Part II, Chapter 2 of Borodin and Salminen (1996).

4.2. Level values

Bearing in mind the level values introduced in Definition 2.4, address the solution $u_{\alpha,t}(c,m)$ of the equation

$$\boldsymbol{\psi}_t(u, c, m) = \alpha$$

with respect to u and the solution $c_{\alpha,t}(u, m)$ of that same equation with respect to c. In the diffusion framework this problem may be solved analytically in a

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comprehensive way. The generalization of the results of this paper would require mostly the alternative methods of that analysis, e.g. numerical evaluation or more complicated analytical technique.

THEOREM 4.4. *For* $0 \leqslant \alpha \leqslant 1/2$, *the* α*-level initial capital corresponding to the claim intensity* m *is*

$$u_{\alpha,t}(c,m) = \begin{cases} \sigma(m)\sqrt{t}\left[\dfrac{(m-c)\sqrt{t}}{\sigma(m)} + z_\alpha\left(\dfrac{(m-c)\sqrt{t}}{\sigma(m)}\right)\right], & m \geqslant c, \\ \sigma(m)\sqrt{t}\, z_\alpha\left(\dfrac{(m-c)\sqrt{t}}{\sigma(m)}\right), & m \leqslant c, \end{cases}$$

where $z_\alpha(v)$ *is continuous and monotone increasing, as* v *increases from* $-\infty$ *to* 0, *with*

$$0 = z_\alpha(-\infty) \leqslant z_\alpha(v) \leqslant z_\alpha(0) = \kappa_{\alpha/2},$$

and monotone decreasing, as v *increases from* 0 *to* $+\infty$, *with*

$$\kappa_{\alpha/2} = z_\alpha(0) \geqslant z_\alpha(v) \geqslant z_\alpha(-\infty) = \kappa_\alpha \geqslant 0.$$

REMARK 4.1. One can supplement Theorem 4.4 by the observation that $u_{\alpha,t}(c,m)$, which depends on m and c only through the difference $m-c$, is a monotone function of this difference. To be more specific, if $m-c$ increases from $-\infty$ to 0, the capital $u_{\alpha,t}(c,m) = u_{\alpha,t}(m-c)$ is monotone increasing from 0 to $\sigma(m)\sqrt{t}\,\kappa_{\alpha/2}$. If $m-c$ increases from 0 to $+\infty$, the capital $u_{\alpha,t}(m-c)$ is monotone increasing from $\sigma(m)\sqrt{t}\,\kappa_{\alpha/2}$ to $+\infty$.

Theorem 4.4 is illustrated by the following numerical calculations.

TABLE 4.1.

VALUES OF $u_{\alpha,t}(m-c)$ FOR $t=100$, $\sigma(m)=1$.

	$m-c=0$	$m-c=0.01$	$m-c=0.02$	$m-c=0.03$	$m-c=0.04$
$\alpha = 0.3$	10.3643	10.6166	10.8823	11.1841	11.5440
$\alpha = 0.1$	16.4485	16.8379	17.2536	17.7159	18.2422
$\alpha = 0.05$	19.5996	20.0331	20.4988	21.0161	21.6000
	$m-c=0$	$m-c=-0.01$	$m-c=-0.02$	$m-c=-0.03$	$m-c=-0.04$
$\alpha = 0.3$	10.3643	9.7120	9.0895	8.4983	7.9396
$\alpha = 0.1$	16.4485	15.6601	14.8907	14.1422	13.4161
$\alpha = 0.05$	19.5996	18.7682	17.9517	17.1515	16.3691

THEOREM 4.5. *For $u \geqslant 0$, the α-level premium intensity corresponding to the claim intensity m is*

$$c_{\alpha,t}(u,m) = m - \frac{\sigma(m)}{\sqrt{t}}\; v_\alpha\left(\frac{u}{\sigma(m)\sqrt{t}}\right),$$

where $v_\alpha(z)$, $z \geqslant 0$, is continuous, convex and monotone increasing from $-\infty$ to 0, as z increases from 0 to $\kappa_{\alpha/2}$, zero at $z = \kappa_{\alpha/2}$ and monotone increasing from 0 to ∞, as z increases from $\kappa_{\alpha/2}$ to ∞. Furthermore, $v'_\alpha(z) > 1$ for $z \geqslant 0$.

PROOF OF THEOREM 4.4. Applying Theorem 4.2, one has

$$\begin{aligned} \psi_t(u,c,m) &= \mathsf{P}\{\inf_{0 \leqslant s \leqslant t}[u + (c-m)s - \sigma(m)\mathsf{W}_s] < 0\} \\ &= \mathsf{P}\{\sup_{0 \leqslant s \leqslant t}[(m-c)s + \sigma(m)\mathsf{W}_s] > u\} \\ &= 1 - \Phi\left(\frac{u}{\sigma(m)\sqrt{t}} - \frac{(m-c)\sqrt{t}}{\sigma(m)}\right) \\ &\quad + \exp\left\{2\frac{u}{\sigma(m)\sqrt{t}}\,\frac{(m-c)\sqrt{t}}{\sigma(m)}\right\}\left(1 - \Phi\left(\frac{u}{\sigma(m)\sqrt{t}} + \frac{(m-c)\sqrt{t}}{\sigma(m)}\right)\right). \end{aligned} \tag{32}$$

Consider the cases $m \geqslant c$ and $m \leqslant c$ separately. In the former case, bearing in mind (32), rewrite Eq. (9) with respect to u as

$$F_1\left(\frac{u}{\sigma(m)\sqrt{t}} - \frac{(m-c)\sqrt{t}}{\sigma(m)}, \frac{(m-c)\sqrt{t}}{\sigma(m)}\right) = \alpha, \tag{33}$$

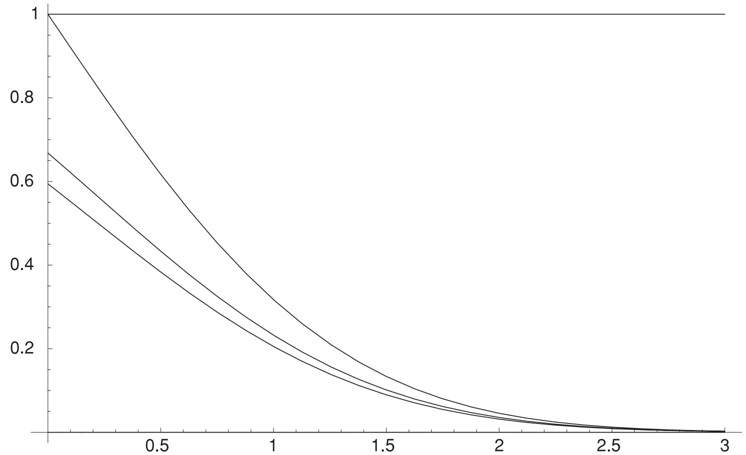

FIGURE 1: Three graphs: $F_1(z,0) \geqslant F_1(z,1) \geqslant F_1(z,2)$ with $z \geqslant 0$.
It is noteworthy that $F_1(0, +\infty) = 1/2$.

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where $F_1(z, v) = 1 - \Phi(z) + \exp\{2v[v + z]\}(1 - \Phi(2v + z))$. The solution $z = z_\alpha(v)$ of the equation

$$F_1(z, v) = \alpha,\ \ v \geqslant 0,\ \ 0 \leqslant \alpha \leqslant 1/2,$$

with respect to z exists, is unique, for α fixed is monotone decreasing, as v increases from 0 to $+\infty$, and is bounded, $\kappa_{\alpha/2} = z_\alpha(0) \geqslant z_\alpha(v) \geqslant z_\alpha(+\infty) = \kappa_\alpha \geqslant 0$. For α fixed, $v + z_\alpha(v)$ is monotone increasing, as v increases from 0 to $+\infty$. Moreover, for v fixed, $z_\alpha(v)$ is monotone decreasing, as α increases, and $\infty = z_0(v) \geqslant z_\alpha(v) \geqslant z_1(v) = 0$.

To prove monotonicity of $z_\alpha(v)$, apply the implicit function derivative theorem and note that[16]

$$\frac{d}{dv} z_\alpha(v) = -\left(\frac{d}{dv} F_1(z, v)\right)\left(\frac{d}{dz} F_1(z, v)\right)^{-1}\Bigg|_{z = z_\alpha(v)} = -\frac{2v\mathcal{M}(2v + z) - 1}{v\mathcal{M}(2v + z) - 1}\Bigg|_{z = z_\alpha(v)} \leqslant 0$$

since for $z, v \geqslant 0$

$$\begin{aligned}
\frac{d}{dv} F_1(z,v) &= \exp\{2v[v+z]\}\, 4v(1 - \Phi(2v+z)) - 2\exp\{2v[v+z]\}\,\phi(2v+z) \\
&= 2\exp\{2v[v+z]\}\,\phi(2v+z)\,[2v\mathcal{M}(2v+z) - 1] \leqslant 0, \\
\frac{d}{dz} F_1(z,v) &= -\phi(z) + \exp\{2v[v+z]\}\, 2v(1 - \Phi(2v+z)) - \exp\{2v[v+z]\}\,\phi(2v+z) \\
&= 2\exp\{2v[v+z]\}\,\phi(2v+z)\,[v\mathcal{M}(2v+z) - 1] \leqslant 0.
\end{aligned}$$

The inequalities $2v\mathcal{M}(2v + z) - 1 < 2v/(2v + z) - 1 \leqslant 0$ and $v\mathcal{M}(2v + z) - 1 < v/(2v + z) - 1 \leqslant -1/2$ follow from Eq. (31). Furthermore,

$$\begin{aligned}
\frac{d}{dv}(v + z_\alpha(v)) = 1 + \frac{d}{dv} z_\alpha(v) &= 1 - \frac{2v\mathcal{M}(2v + z) - 1}{v\mathcal{M}(2v + z) - 1}\Bigg|_{z = z_\alpha(v)} \\
&= -\frac{v\mathcal{M}(2v + z)}{v\mathcal{M}(2v + z) - 1}\Bigg|_{z = z_\alpha(v)} \geqslant 0,
\end{aligned}$$

which yields monotonicity of $v + z_\alpha(v)$. Bearing in mind that $F_1(z, \infty) = 1 - \Phi(z)$ and $F_1(z, 0) = 2(1 - \Phi(z))$, the analysis in the case $m \geqslant c$ is completed.

Address the case $m \leqslant c$. Bearing in mind (32), rewrite Eq. (9) with respect to u as

$$F_2\left(\frac{u}{\sigma(m)\sqrt{t}}, \frac{(m - c)\sqrt{t}}{\sigma(m)}\right) = \alpha, \tag{34}$$

[16] Here $\mathcal{M}(x)$, $x \in \mathsf{R}$, is Mill's ratio, see Eq. (30).

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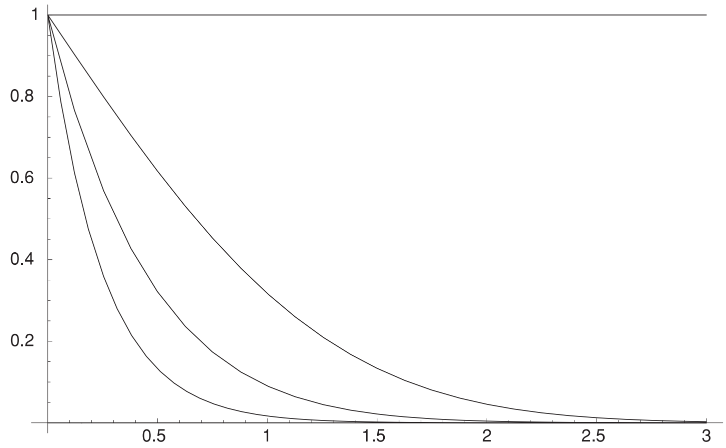

FIGURE 2: Three graphs: $F_2(z,0) \geqslant F_2(z,-1) \geqslant F_2(z,-2)$ with $z \geqslant 0$.

where[17] $F_2(z,v) = 1 - \Phi(z-v) + \exp\{2zv\}(1-\Phi(z+v))$. The solution $z = z_\alpha(v)$ of the equation

$$F_2(z,v) = \alpha, \;\; v \leqslant 0,\, 0 \leqslant \alpha \leqslant 1/2,$$

with respect to z exists, is unique, for α fixed is monotone increasing, as v increases from $-\infty$ to 0, and is bounded, $0 = z_\alpha(-\infty) \leqslant z_\alpha(v) \leqslant z_\alpha(0) = \kappa_{\alpha/2}$. For v fixed, $z_\alpha(v)$ is monotone decreasing, as α increases, and $\infty = z_0(v) \geqslant z_\alpha(v) \geqslant z_1(v) = 0$.

To prove monotonicity of $z_\alpha(v)$, apply the implicit function derivative theorem and note that

$$\frac{d}{dv} z_\alpha(v) = -\left(\frac{d}{dv} F_2(z,v)\right)\left(\frac{d}{dz} F_2(z,v)\right)^{-1}\Bigg|_{z=z_\alpha(v)} = -\frac{z\mathcal{M}(z+v)}{v\mathcal{M}(z+v)-1}\Bigg|_{z=z_\alpha(v)} \geqslant 0$$

since for $z \geqslant 0$ and $v \leqslant 0$

$$\begin{aligned}\frac{d}{dv}F_2(z,v) &= \phi(z-v) + \exp\{2zv\}\,2z(1-\Phi(z+v)) - \exp\{2zv\}\phi(z+v)\\ &= \exp\{2zv\}2z(1-\Phi(z+v)) \geqslant 0,\\ \frac{d}{dz}F_2(z,v) &= -\phi(z-v) + \exp\{2zv\}\,2v(1-\Phi(z+v)) - \exp\{2zv\}\phi(z+v)\\ &= \exp\{2zv\}2v(1-\Phi(z+v)) - 2\exp\{2zv\}\phi(z+v)\\ &= 2\exp\{2zv\}\phi(z+v)\,[v\mathcal{M}(z+v)-1] < 0.\end{aligned}$$

[17] One may put $F_2(z,v) = F_1(z-v,v)$ for $v \in \mathsf{R}$, though our concern is $F_1(z,v)$ for $v \geqslant 0$ and $F_2(z,v)$ for $v \leqslant 0$.

The inequality $v\mathcal{M}(z+v)-1 < v\mathcal{M}(v)-1 < 0$ is evident from Eq. (31). Bear also in mind that $F_2(z,0) = 2(1-\Phi(z))$. □

PROOF OF THEOREM 4.5. Bearing in mind (32), rewrite Eq. (9) with respect to c as

$$F_2\left(\frac{u}{\sigma(m)\sqrt{t}}, \frac{(m-c)\sqrt{t}}{\sigma(m)}\right) = \alpha. \tag{35}$$

The solution $v = v_\alpha(z)$ of the equation

$$F_2(z,v) = \alpha, \;\; z \geqslant 0, \, 0 \leqslant \alpha \leqslant 1/2, \tag{36}$$

with respect to v exists, is unique and has the following properties. For α fixed, $v_\alpha(z)$, $z \geqslant 0$, is continuous, convex and monotone increasing from $-\infty$ to 0, as z increases from 0 to $\kappa_{\alpha/2}$, zero at $z = \kappa_{\alpha/2}$ and monotone increasing from 0 to ∞, as z increases from $\kappa_{\alpha/2}$ to ∞. Furthermore, $v'_\alpha(z) > 1$ for $z \geqslant 0$.

To prove monotonicity of $v_\alpha(z)$, apply the implicit function derivative theorem and note that

$$\frac{d}{dz}v_\alpha(z) = -\left(\frac{d}{dz}F_2(z,v)\right)\left(\frac{d}{dv}F_2(z,v)\right)^{-1}\Bigg|_{v=z_\alpha(z)} = -\frac{v\mathcal{M}(z+v)-1}{z\mathcal{M}(z+v)}\Bigg|_{v=v_\alpha(z)} > 1$$

since (see Eq. (31)) for $z \geqslant 0$ and $v \in \mathsf{R}$

$$-\frac{v\mathcal{M}(z+v)-1}{z\mathcal{M}(z+v)} - 1 = \frac{1-(z+v)\mathcal{M}(z+v)}{z\mathcal{M}(z+v)} > 0.$$

The proof is completed. □

LEMMA 4.1. *The solution of Eq.* (36) *with respect to v is such that*

$$v_\alpha(z) = z - \kappa_\alpha + \bar{o}(1), \quad \text{as } z \to +\infty.$$

PROOF OF LEMMA 4.1. Note that

$$\exp\{2zv\}(1-\Phi(z+v)) = \exp\{2zv\}\,\phi(z+v)\,\mathcal{M}(z+v) = \phi(z-v)\,\mathcal{M}(z+v)$$

and rewrite Eq. (36) as

$$1-\Phi(z-v) = \alpha - \phi(z-v)\,\mathcal{M}(z+v).$$

By Theorem 4.5, $v_\alpha(z) \to +\infty$, as $z \to +\infty$. Since $\mathcal{M}(x) \to 0$, as $x \to +\infty$, and $\phi(x)$ is bounded, the proof is completed. □

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