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Nilpotent centres via inverse integrating factors

ANTONIO ALGABA[1], CRISTÓBAL GARCÍA[1] and JAUME GINÉ[2]

[1]*Dept. Matemáticas, Facultad de Ciencias, University of Huelva, Huelva, Spain*
emails: algaba@uhu.es, cristoba@uhu.es
[2]*Departament de Matemàtica, Escola Politècnica Superior,*
Universitat de Lleida, Av. Jaume II, 69, 25001, Lleida, Catalonia, Spain
email: gine@matematica.udl.cat

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In this paper, we are interested in the nilpotent centre problem of planar analytic monodromic vector fields. It is known that the formal integrability is not enough to characterize such centres. More general objects are considered as the formal inverse integrating factors. However, the existence of a formal inverse integrating factor is not sufficient to describe all the nilpotent centres. For the family studied in this paper, it is enough.

Key words: non-linear differential systems, integrability problem, nilpotent centre problem

1 Introduction

We consider planar differential systems that can be written as

$$\dot{x} = P(x, y), \qquad \dot{y} = Q(x, y), \tag{1.1}$$

where P and Q are analytic functions that vanish at the origin. It is known that system (1.1) has a centre at a singular point only if it is monodromic and it has either linear part of centre type, i.e. with imaginary eigenvalues (non-degenerate point), or nilpotent linear part (nilpotent point) or null linear part (degenerate point). A non-degenerate singular point is a centre if and only if it has a formal (actually analytic) first integral around the singular point. In contrast, the formal integrability does not characterize the nilpotent or degenerate centres, see for instance [16,18] although some nilpotent or degenerate centres have analytic first integral, see for instance [6]. On the other hand, a non-degenerate singular point is a centre if, and only if, there is a non-zero analytic inverse integrating factor around the singular point, see [24] and [17].

In this work, we focus on the nilpotent centre problem which consists in characterizing when a monodromic nilpotent singular point is a centre. In fact this problem is solved and several methods can be used. One is blowing-up the singularity by means of generalized polar coordinates, see for instance [9]. Another one is applying normal form theory up to some order to transform the system into a suitable Liénard normal form, see [10]. Other method to find the centres of a family is to study the orbitally reversibility due to the result that any nilpotent system has a centre at the origin if and only if it is orbitally reversible, see [12]. This method has been used in [1] to study certain families of nilpotent systems. Finally, the last method is embedding the nilpotent singularity as the limit of a family of non-degenerate centres, see [16, 18] and the final corrected version in [14]. Moreover, in [4], it is proved that in order to determine certain nilpotent centres we can use the existence of a C^∞ Lyapunov function. However, no version is based in the construction of a formal power series which gives the obstructions to the existence of a nilpotent centre. This is because neither the first integral nor the inverse integrating factors are, in general, formal power series for nilpotent centres, see [2, 19]. In [19], it is proved that any (possibly degenerate) centre of an analytic planar system admits a C^∞ inverse integrating factor and a C^∞ Lie symmetry in a neighbourhood of the singularity.

The *monodromy problem* consists in characterizing when a singular point is either a focus or a centre. Andreev in [11] solved this problem for nilpotent singular points.

Theorem 1.1 (*Andreev*) *Consider the analytic system*

$$\dot{x} = y + P(x,y), \qquad \dot{y} = Q(x,y), \tag{1.2}$$

where P and Q are analytic functions around the origin without constant and linear terms and with an isolated singularity at the origin. Let $y = \phi(x)$ be the solution of $y + P(x,y) = 0$ passing through the origin. Consider the functions

$$\psi(x) = Q(x,\phi(x)) = ax^\alpha + \mathcal{O}(x^{\alpha+1})$$
$$\Delta(x) = \operatorname{div}(P,Q)(x,\phi(x)) = bx^{\tilde{\beta}} + \mathcal{O}(x^{\tilde{\beta}+1}),$$

with $a \neq 0$, $\alpha \geqslant 2$ and $b \neq 0$, $\tilde{\beta} \geqslant 1$ or $\Delta(x) \equiv 0$. Then, the origin of (1.2) is monodromic if and only if $a < 0$, $\alpha = 2\tilde{n} - 1$ is an odd number and one of the following conditions holds: (i) $\tilde{\beta} > \tilde{n} - 1$; (ii) $\tilde{\beta} = \tilde{n} - 1$ and $b^2 + 4\tilde{n}a < 0$; (iii) $\Delta(x) \equiv 0$.

The *Andreev number n* of a monodromic singular point at the origin of system (1.2) is the integer $\tilde{n} \geqslant 2$ given in Theorem 1.1. Recently in [13], using the Andreev number n is proved that if n is even and there exists an inverse integrating factor then the system (1.1) has a centre at the singular point. If $n = 2$, then the existence of an inverse integrating factor characterizes all the centres. Finally, if there is an inverse integrating factor with minimum "vanishing multiplicity" at the singularity then, generically, the system (1.1) has a centre at the singular point.

In this work, we are interested in deepening knowledge of the methods based on formal power series for the nilpotent centre problem. We prove that for certain nilpotent systems the existence of a centre is equivalent to the existence of a formal inverse integrating factor generalizing the results obtained in [20, 21]. For more general families, we do not know if there exists a formal power series to characterize if a nilpotent singular point is a centre or not.

2 Nilpotent centres and inverse integrating factors

A scalar function f is quasi-homogeneous of type $\mathbf{t} = (t_1, t_2) \in \mathbb{N}^2$ and degree k if

$$f(\varepsilon^{t_1}x, \varepsilon^{t_2}y) = \varepsilon^k f(x, y).$$

The vector space of quasi-homogeneous scalar function of type $\mathbf{t}$ and degree k is denoted by $\mathcal{P}_k^{\mathbf{t}}$. A vector field $\mathbf{F} = (P, Q)^T$ is quasi-homogeneous of type $\mathbf{t}$ and degree k if $P \in \mathcal{P}_{k+t_1}^{\mathbf{t}}$ and $Q \in \mathcal{P}_{k+t_2}^{\mathbf{t}}$. The vector space of quasi-homogeneous vector field of type $\mathbf{t}$ and degree k is denoted by $\mathcal{Q}_k^{\mathbf{t}}$. Given an analytic vector field $\mathbf{F}$, we can write it as a quasi-homogeneous expansion corresponding to a fixed type $\mathbf{t}$:

$$\mathbf{F}(\mathbf{x}) = \mathbf{F}_r(\mathbf{x}) + \mathbf{F}_{r+1}(\mathbf{x}) + \cdots,$$

where $\mathbf{x} \in \mathbb{R}^2$, $r \in \mathbb{Z}$ and $\mathbf{F}_k \in \mathcal{Q}_k^{\mathbf{t}}$ i.e. each term $\mathbf{F}_k$ is a quasi-homogeneous vector field of type $\mathbf{t}$ and degree k. Any $\mathbf{F}_k \in \mathcal{Q}_k^{\mathbf{t}}$ can be uniquely written as

$$\mathbf{F}_k = \mu_k \mathbf{D}_0 + \mathbf{X}_{h_k}, \tag{2.3}$$

where $\mu_k = \frac{1}{r+|\mathbf{t}|}\operatorname{div}(\mathbf{F}_k) \in \mathcal{P}_k^{\mathbf{t}}$, $h_k = \frac{1}{r+|\mathbf{t}|}\mathbf{D}_0 \wedge \mathbf{F}_k \in \mathcal{P}_{k+|\mathbf{t}|}^{\mathbf{t}}$, $\mathbf{D}_0 = (t_1x, t_2y)^T$, and $\mathbf{X}_{h_k} = (-\frac{\partial h_k}{\partial y}, \frac{\partial h_k}{\partial x})^T$ is the Hamiltonian vector field with Hamiltonian function h_k (see [3, Proposition 2.7]).

In this work, we consider analytic nilpotent differential system of the form:

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}) := \overbrace{\begin{pmatrix} y \\ -x^3 \end{pmatrix} + d\,x\begin{pmatrix} x \\ 2y \end{pmatrix}}^{\mathbf{F}_1} + \sum_{j>1}\mathbf{F}_j, \tag{2.4}$$

with $d \in \mathbb{R}$, $\mathbf{F}_j \in \mathcal{Q}_j^{\mathbf{t}}$, $\mathbf{t} = (1, 2)$, and where the conservative-dissipative decomposition of the first quasi-homogeneous component of (2.4) is given by

$$\mathbf{F}_1 = \mathbf{X}_h + d\,x\,\mathbf{D}_0, \quad \text{with} \quad \mathbf{D}_0 = (x, 2y)^T, \quad \text{and} \quad h = -\tfrac{1}{4}\left(2y^2 + x^4\right).$$

Remark *Notice that any generic monodromic nilpotent vector field can be expressed in the form (2.4). More concretely, a normal form of system (1.2) is* $(\dot{x}, \dot{y}) = (y, \sum_{i\geqslant 2} a_i x^i + \sum_{i\geqslant 1} b_i x^i y)^T$. *If* $a_2 \neq 0$, *by Theorem 1.1 the origin of (1.2) is not monodromic. By Theorem 1.1, the origin of (1.2) is monodromic in the generic case* $a_3 \neq 0$, $b_1^2 + 8a_3 < 0$. *In this case, the change of variables* $u = x$, $v = \gamma y - \frac{\gamma}{4}b_1x^2$ *and the rescaling in the time* $dt = \gamma dT$, *with* $\gamma = \sqrt{8}/\sqrt{|b_2^2+8a_3|}$ *transforms the system (1.2) into (2.4) with* $d = \frac{b_1\gamma}{4}$.

A non-null $\mathscr{C}^1$ class function V is an inverse integrating factor of system $\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x})$ on U if satisfies the linear partial differential equation $\nabla V \cdot \mathbf{F} = \operatorname{div}(F)\,V$, being $\operatorname{div}(F) := \partial P/\partial x + \partial Q/\partial y$ the divergence of $\mathbf{F}$. We will say that V is a formal inverse integrating factor of system if $V \in \mathbb{C}[[x, y]]$, where $\mathbb{C}[[x, y]]$ is the algebra of the power series in x and y with coefficients in $\mathbb{C}$, convergent or not.

Lemma 2.2 *Let $V(\mathbf{x})$ be a formal function, $\mathbf{F}(\mathbf{x})$ a vector field, $\mathbf{x} = \Phi(\mathbf{y})$ a change of variables with $\mathbf{G} = \Phi_* \mathbf{F}$ (where $_*$ is the pull-back of Φ) and $\tilde{V} := V(\Phi(\mathbf{y}))$, then*

$$\nabla_y \tilde{V} \cdot \mathbf{G}(y) = \nabla_x V(\mathbf{x}) \cdot \mathbf{F}(\mathbf{x}).$$

Proof $\nabla \tilde{V} \cdot \mathbf{G} = \nabla V(\Phi(\mathbf{y})) \cdot \mathbf{G} = \nabla V(\Phi(\mathbf{y})) D\Phi(\mathbf{y}) \cdot [D\Phi(\mathbf{y})]^{-1} \mathbf{F}(\Phi(\mathbf{y})) = \nabla V(\mathbf{x}) \cdot \mathbf{F}(\mathbf{x})$. □

Lemma 2.3 *Let Φ be a diffeomorphism and μ a function on $U \subset \mathbb{R}^2$ such that $\det(D\Phi)$ has no zero on U and $\mu(0) \neq 0$, let $\mathbf{F}$ a vector field and $\mathbf{G} = \Phi_*(\mu \mathbf{F})$, let f, V functions. If*

$$\nabla V \cdot \mathbf{F} - \operatorname{div}(\mathbf{F})\, V = f,$$

then

$$\nabla W \cdot \mathbf{G} - \operatorname{div}(\mathbf{G})\, W = g,$$

where $W(\mathbf{y}) = \mu(\Phi(y)) V(\Phi(\mathbf{y})) \det(D\Phi(\mathbf{y}))^{-1}$ and $g(\mathbf{y}) = \mu(\Phi(\mathbf{y}))^2 \det(D\Phi(\mathbf{y}))^{-1} f(\Phi(\mathbf{y}))$.

Proof We define $\tilde{V}(\mathbf{y}) = V(\Phi(\mathbf{y}))$ and $J = \det(D\Phi(\mathbf{y}))$ then we can write $W(\mathbf{y}) = \mu(\Phi(\mathbf{y})) J^{-1} \tilde{V}(\mathbf{y})$. Applying Lemma 2.2 and the known result $\operatorname{div}(\mathbf{G}) = \operatorname{div}(\mathbf{F}) - \nabla J \cdot \mathbf{G}/J$ (see [22]), we have

$$\begin{aligned} g(\mathbf{y}) &= \nabla W \cdot \mathbf{G} - \operatorname{div}(\mathbf{G})\, W = \nabla(\tfrac{\mu}{J} \tilde{V}) \cdot \mathbf{G} - \operatorname{div}(\mathbf{G}) \tfrac{\mu}{J} \tilde{V} \\ &= \tfrac{\mu}{J} \nabla \tilde{V} \cdot \mathbf{G} + \tfrac{\tilde{V}}{J} \nabla \mu \cdot \mathbf{G} - \tfrac{\mu \tilde{V}}{J^2} \nabla J \cdot \mathbf{G} - \left(\operatorname{div}(\mu \mathbf{F}) - \tfrac{\nabla J \cdot \mathbf{G}}{J}\right) \tfrac{\mu}{J} \tilde{V} \\ &= \tfrac{\mu}{J} \nabla V \cdot (\mu \mathbf{F}) + \tfrac{\tilde{V}}{J} \nabla \mu \cdot (\mu \mathbf{F}) - \tfrac{\mu \tilde{V}}{J} \nabla \mu \cdot \mathbf{F} - \tfrac{\mu^2 V}{J} \operatorname{div}(\mathbf{F}) \\ &= \tfrac{\mu^2}{J} \nabla V \cdot \mathbf{F} + \tfrac{\mu \tilde{V}}{J} \nabla \mu \cdot \mathbf{F} - \tfrac{\mu \tilde{V}}{J} \nabla \mu \cdot \mathbf{F} - \tfrac{\mu^2 V}{J} \operatorname{div}(\mathbf{F}) \\ &= \tfrac{\mu^2}{J} (\nabla V \cdot \mathbf{F} - \operatorname{div}(\mathbf{F})\, V) = \tfrac{\mu^2}{J} f. \end{aligned}$$

□

Definition 2.4 *Let us consider two vector fields $\mathbf{F}$, $\mathbf{G}$. We say that $\mathbf{F}$ is orbital equivalent to $\mathbf{G}$ if there exist a diffeomorphism Φ and a function μ with $\mu(\mathbf{0}) = 1$ such that $\mathbf{G} = \Phi_* (\mu \mathbf{F})$.*

Proposition 2.5 *System (2.4) is orbitally equivalent to*

- **(a)** $(\dot{x}, \dot{y}) = (y, -x^3)^T + h f_0(h) \mathbf{D}_0 + x h f_1(h) \mathbf{D}_0 + x^2 f_2(h) \mathbf{D}_0$ *for* $d = 0$ *where* $f_i(h) = \sum_{j \geqslant 0} a_j^{(i)} h^j$, $i = 0, 1, 2$.
- **(b)** $(\dot{x}, \dot{y}) = (y + dx^2, -x^3 + 2dxy)^T + h f_0(h) \mathbf{D}_0 + x^2 f_2(h) \mathbf{D}_0$ *for* $d \neq 0$ *where* $f_i(h) = \sum_{i \geqslant 0} a_j^{(i)} h^j$, $i = 0, 2$.

Proof The proof of statement **(a)** is given in [7, Theorem 16]. Following the same ideas it is possible to prove **(b)**.

Remark *In Proposition 2.5, it is not necessary neither the convergence of the transformation nor the convergence of the transformed vector field. In fact, we only need the formal part*

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of the transformed vector field since when system (2.4) is monodromic the Poincaré map is analytic, see [23].

More specifically, in this case, we consider a vector field $\mathbf{F} \in \mathbb{C}^\omega$ *with a monodromic nilpotent singular point at the origin. The transformed system is given by* $\tilde{\mathbf{F}} + \tau(x, y)$*, where* $\tilde{\mathbf{F}}$ *is a formal vector field and where* τ *is a flat vector field at the origin. By Moussu, see [23], we know that the Poincaré map for* $\mathbf{F}$ *is given by* $\Pi(x) = x + \sum_{n\geqslant 2} a_n x^n \in \mathbb{C}^\omega$ *and the Poincaré map of* $\tilde{\mathbf{F}}$ *is the formal map* $\tilde{\Pi}(x) = x + \sum_{n\geqslant 2} \tilde{a}_n x^n$*.*

Moreover, if $a_2 = \ldots = a_{n-1} = 0$ *and* $a_n \neq 0$*, that is* $\mathbf{F}$ *has a focus at the origin this implies that* $\tilde{a}_2 = \ldots = \tilde{a}_{n-1} = 0$ *and* $\tilde{a}_n \neq 0$*. Otherwise if* $\tilde{a}_n = 0$ *for* $n \geqslant 2$*, then* $\tilde{\Pi}(x) = x$ *that implies* $a_n = 0$ *for* $n \geqslant 2$ *and* $\Pi(x) = x$*, that is, the origin is a centre for* $\mathbf{F}$ *and hence we do not need the convergence of the transformation.*

The following proposition is proved in [7, Theorem 6].

Proposition 2.6 *System (2.4) with* $d = 0$ *has a formal inverse integrating factor if and only if* $\mathbf{F}$ *is orbitally equivalent to* $(\dot{x}, \dot{y}) = (y, -x^3)^T + xhf_1(h)\mathbf{D}_0$*, where* $f_1(h) = \sum_{j\geqslant 0} a_j^{(1)} h^j$*.*

The next proposition characterizes when system (2.4) has a centre at the origin.

Proposition 2.7 *System (2.4) has a centre at the origin if and only if* $\mathbf{F}$ *is orbitally equivalent to*

(a) $(\dot{x}, \dot{y}) = (y, -x^3)^T + xhf_1(h)\mathbf{D}_0$ *for* $d = 0$*, where* $f_1(h) = \sum_{j\geqslant 0} a_j^{(1)} h^j$*.*

(b) $(\dot{x}, \dot{y}) = (y + dx^2, -x^3 + 2dxy)^T$ *for* $d \neq 0$*.*

Proof We give the proof for each statement.

Case $d = 0$ Applying Proposition 2.5 statement **(a),** the origin of $\mathbf{F}$ is a centre if and only if the origin of $(\dot{x}, \dot{y}) = \mathbf{G} := (y, -x^3)^T + hf_0(h)\mathbf{D}_0 + xhf_1(h)\mathbf{D}_0 + x^2 f_2(h)\mathbf{D}_0$ is a centres.

The sufficient condition is trivial because, if $f_0 \equiv f_2 \equiv 0$, then $\mathbf{G}$ is monodromic and is invariant by the symmetry $(x, y, t) \mapsto (-x, y, -t)$ and therefore has a centre at the origin.

To prove the necessity of the condition, we assume that $f_0^2 + f_2^2 \not\equiv 0$ and we define $\hat{\mathbf{G}} = (y, -x^3)^T + xhf_1(h)\mathbf{D}_0$ whose origin is a centre. The wedge product $\mathbf{G} \wedge \hat{\mathbf{G}}$ measures the direction of the orbits of $\dot{\mathbf{x}} = \mathbf{G}(\mathbf{x})$ crossing the ovals of the centre $\dot{\mathbf{x}} = \hat{\mathbf{G}}(x)$.

$$\begin{aligned}\mathbf{G} \wedge \hat{\mathbf{G}} &= \left(\hat{\mathbf{G}} + (hf_0(h) + x^2 f_2(h))\mathbf{D}_0\right) \wedge \hat{\mathbf{G}} = (hf_0(h) + x^2 f_2(h))\mathbf{D}_0 \wedge \hat{\mathbf{G}} \\ &= (hf_0(h) + x^2 f_2(h))\mathbf{D}_0 \wedge \mathbf{X}_h = 4h(hf_0(h) + x^2 f_2(h)).\end{aligned}$$

We claim that this wedge product is semi-definite. To prove our claim, we consider $k = \min\{i \in \mathbb{N} \cup \{0\} : a_i^{(2)} \neq 0 \text{ or } a_i^{(0)} \neq 0\}$. If the minimum is $a_k^{(0)}$, then $\mathbf{G} \wedge \hat{\mathbf{G}} = 4h^{k+2}(a_k^{(0)} + \cdots)$ which is definite. If the minimum is $a_k^{(2)}$, then $\mathbf{G} \wedge \hat{\mathbf{G}} = 4x^2 h^{k+1}(a_k^{(2)} + \cdots)$ which is semi-definite. In both cases, the orbits of $\dot{\mathbf{x}} = \mathbf{G}(\mathbf{x})$ cross the ovals of the centre of $\dot{\mathbf{x}} = \hat{\mathbf{G}}(\mathbf{x})$ in the same direction which implies that the origin of system (2.4) cannot be a centre.

Case $d \neq 0$ Applying Proposition 2.5 statement **(b),** the origin of $\mathbf{F}$ is a centre if and only if the origin of $(\dot{x}, \dot{y}) = \mathbf{G} := (y + dx^2, -x^3 + 2dxy)^T + hf_0(h)\mathbf{D}_0 + x^2 f_2(h)\mathbf{D}_0$ is a centre.

The sufficient condition, as before, is trivial because if $f_0 \equiv f_2 \equiv 0$ then $\mathbf{G}$ is monodromic and is invariant by the symmetry $(x, y, t) \mapsto (-x, y, -t)$ and therefore has a centre at the origin.

To prove the necessity of the condition, we assume that $f_0^2 + f_2^2 \not\equiv 0$ and we define $\hat{\mathbf{G}} = (y + dx^2, -x^3 + 2dxy)^T$ whose origin is a centre. We compute the wedge product $\mathbf{G} \wedge \hat{\mathbf{G}}$ that is given by

$$\mathbf{G} \wedge \hat{\mathbf{G}} = \left(\hat{\mathbf{G}} + (hf_0(h) + x^2 f_2(h))\mathbf{D}_0\right) \wedge \hat{\mathbf{G}} = (hf_0(h) + x^2 f_2(h))\mathbf{D}_0 \wedge \hat{\mathbf{G}}$$
$$= (hf_0(h) + x^2 f_2(h))\mathbf{D}_0 \wedge \mathbf{X}_h = 4h(hf_0(h) + x^2 f_2(h)).$$

We are going to see that this expression is also semi-definite. We define, as before, $k = \min\{i \in \mathbb{N} \cup \{0\} : a_i^{(2)} \neq 0\text{or } a_i^{(0)} \neq 0\}$. If the minimum is $a_k^{(0)}$, then $\mathbf{G} \wedge \hat{\mathbf{G}} = 4h^{k+2}(a_k^{(0)} + \cdots)$ which is definite. If the minimum is $a_k^{(2)}$, then $\mathbf{G} \wedge \hat{\mathbf{G}} = 4x^2 h^{k+1}(a_k^{(2)} + \cdots)$ which is semi-definite. In both cases, the orbits of system $\dot{\mathbf{x}} = \mathbf{G}(\mathbf{x})$ cross the ovals of the centre of $\dot{\mathbf{x}} = \hat{\mathbf{G}}(\mathbf{x})$ in the same direction which implies that the origin of system (2.4) cannot be a centre.

The next theorems are the main results of this work and they give a new characterization of the centres of the nilpotent differential system (2.4). This characterization provides a scalar procedure to determine the centres of system (2.4).

Theorem 2.8 *The origin of system (2.4) with $d = 0$ is a centre if and only if* $\mathbf{F}$ *has a formal inverse integrating factor.*

Proof Applying Proposition 2.7 statement **(a),** we have that the origin of system (2.4) is a centre if and only if $\mathbf{F}$ is orbitally equivalent to $\mathbf{G} := (y, -x^3) + xf_1(h)\mathbf{D}_0$ and by Proposition 2.6, this condition is equivalent to have a formal inverse integrating factor.

Theorem 2.8 is not satisfied for others nilpotent vector fields different from system (2.4). For instance, we provide an example where the necessary condition of Theorem 2.8 is not satisfied, that is, we give a vector field which has a centre at the origin but it does not have a formal inverse integrating factor.

Proposition 2.9 *The origin of system* $(\dot{x}, \dot{y})^T = \mathbf{F} := (y, -x^5) + d\,x^3(x, 3y)^T$ *is a centre but it does not have any formal inverse integrating factor.*

Proof The origin of the system is a centre because the system is monodromic and is invariant by the symmetry $(x, y, t) \mapsto (-x, y, -t)$. However, the system has not a formal inverse integrating factor. If we define $h = -\frac{1}{6}\left(3y^2 + x^6\right)$ and $\mathbf{D}_0 = (x, 3y)^T$, then $\mathbf{F} = \mathbf{X}_h + dx^3\mathbf{D}_0$ and $V = h^{7/6}$ is a non-formal inverse integrating factor. In [7] is proved that this inverse integrating factor is unique up to a constant multiplicative because any

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invariant curve has the form $h^\alpha u$, where $u \in \mathbb{R}[[x,y]]$ is a unity element. Moreover, in order to be $h^\alpha u$ an inverse integrating factor it is enough to take $\alpha = 7/6$ and $u = 1$.

Now we give an example where the sufficient condition of Theorem 2.8 is not satisfied for nilpotent systems different from system (2.4), that is, we find a vector field that has a formal inverse integrating factor. Nevertheless, this vector field has not a centre at the origin.

Proposition 2.10 *The origin of system* $(\dot{x},\dot{y})^T = \mathbf{F} := (y, -x^5) + d\,x^2 h(x,3y)^T$, *with* $h = -\frac{1}{6}(3y^2 + x^6)$, *is not a centre but this system has a formal inverse integrating factor.*

Proof The system has a formal inverse integrating factor $V = h^2$. However, the origin of the system is not a centre because the wedge product

$$\mathbf{F} \wedge \mathbf{X}_h = dx^2 h \mathbf{D}_0 \wedge \mathbf{X}_h = dx^2 h \nabla h \cdot \mathbf{D}_0 = 6dx^2h^2$$

is definite and therefore the origin is a focus. □

For these last systems, we do not have a characterization of the centres using an scalar algorithm and these systems require further studies.

3 Characterization of the centres of system (2.4)

The following results give the characterization of the nilpotent centres for system (2.4).

Lemma 3.11 *Let* $\mathbf{F}$ *be the associated vector field to system (2.4). There exist a formal function* $V = h + \cdots$ *and some numbers* $\alpha \in \mathbb{R}$, $\alpha \neq 0$ $m \in \mathbb{N}$ *and* $A \in \mathbb{N} \cup \{0\} \cup \{+\infty\}$ *such that one of the following conditions is satisfied:*

(a) $\nabla V \cdot \mathbf{F} - (1+A)^{-1}\mathrm{div}\,(\mathbf{F})\, V = 0$,

(b) $\nabla V \cdot \mathbf{F} - (1+A)^{-1}\mathrm{div}\,(\mathbf{F})\, V = \alpha x^2 h^m + \mathcal{O}(|x,y|^{4m+2})$,

(c) $\nabla V \cdot \mathbf{F} - (1+A)^{-1}\mathrm{div}\,(\mathbf{F})\, V = \alpha h^m + \mathcal{O}(|x,y|^{4m})$.

Moreover, if $d \neq 0$, *then* $A = 0$.

Proof We divide the proof in two cases.

(i) If $d \neq 0$, by Proposition 2.5 statement **(b)**, $\mathbf{F}$ is orbitally equivalent to $\mathbf{G} := (y + dx^2, -x^3 + 2dxy)^T + x^2 f_2(h)\mathbf{D}_0 + hf_4(h)\mathbf{D}_0$ with $f_i(h) = \sum_{j=0}^{\infty} a_j^{(i)} h^j$ for $i = 2,4$, where $a_i^{(j)} \in \mathbb{R}$. Taking $A = 0$, we have

$$\begin{aligned}\nabla h \cdot \mathbf{G} - \mathrm{div}\,(\mathbf{G})\, h &= 4h\left(dx + x^2 f_2(h) + hf_4(h)\right) \\ &\quad - h\left(4dx + 5x^2 f_2(h) + 4x^2 f_2'(h)h + 7hf_4(h) + 4h^2 f_4'(h)\right), \\ &= x^2 g_2(h) + hg_4(h),\end{aligned}$$

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where $g_2(h) = -hf_2(h) - 4h^2 f_2'(h)$, $g_4(h) = -3hf_4(h) - 4h^2 f_4'(h)$, that is, $g_i(h) = \sum_{j>0} c_j^{(i)} h^j$ for $i = 2, 4$.

(i.1) If $g_2(h) \equiv g_4(h) \equiv 0$, then we are in the same situation **(a)** for $A = 0$.

(i.2) If $g_2(h)^2 + g_4(h)^2 \not\equiv 0$, we consider

$$m_0 = \min\left\{m \in \mathbb{N} \cup \{0\} . \left(c_m^{(2)}\right)^2 + \left(c_m^{(4)}\right)^2 \neq 0\right\},$$
$$i_0 = \min\left\{i \in \{2,4\} : c_{m_0}^{(i)} \neq 0\right\}.$$

(i.2.a) If $i_0 = 2$, we have

$$\begin{aligned}\nabla h \cdot \mathbf{G} - \operatorname{div}(\mathbf{G})\, h &= 4c_{m_0}^{(2)} x^2 h^{m_0+1} - (5 + 4m_0) c_{m_0}^{(2)} x^2 h^{m_0} h + \cdots, \\ &= -(4m_0 + 1) c_{m_0}^{(2)} x^2 h^{m_0+1} + \cdots\end{aligned}$$

defining $\alpha := -(4m_0 + 1)c_{m_0}^{(2)} \neq 0$ and $m := m_0 + 1$. If $\mathbf{G} = \Phi(\mu\mathbf{F})$, applying Proposition 2.3 and considering $V = \mu(\Phi(\mathbf{x}))\,(D\Phi)^{-1}\, h\,(\Phi(\mathbf{x})) = h + \cdots$ the statement **(b)** is satisfied for $A = 0$.

(i.2.b) If $i_0 = 4$, we have

$$\begin{aligned}\nabla h \cdot \mathbf{G} - \operatorname{div}(\mathbf{G})\, h &= 4c_{m_0}^{(4)} h^{m_0+1} - (7 + 4m_0) c_{m_0}^{(2)} h^{m_0} h + \cdots, \\ &= -(4m_0 + 3) c_{m_0}^{(4)} h^{m_0+1} + \cdots\end{aligned}$$

defining $\alpha := -(4m_0 + 3)c_{m_0}^{(4)} \neq 0$ and $m := m_0 + 1$. If $\mathbf{G} = \Phi(\mu\mathbf{F})$, applying Proposition 2.3 and considering $V = \mu(\Phi(\mathbf{x}))\,(D\Phi)^{-1}\, h\,(\Phi(\mathbf{x})) = h + \cdots$ statement **(c)** is satisfied.

(ii) If $d = 0$, by Proposition 2.5 statement **(a)**, **F** is orbitally equivalent to $\mathbf{G} := (y, -x^3)^T + x^2 f_2(h)\mathbf{D}_0 + hf_4(h)\mathbf{D}_0 + xhf_5(h)\mathbf{D}_0$ with $f_i(h) = \sum_{j=0}^{\infty} a_j^{(i)} h^j$ for $i = 2, 4, 5$, where $a_i^{(j)} \in \mathbb{R}$. If $f_2(h) \equiv f_4(h) \equiv f_5(h) \equiv 0$, then h is a first integral of **G** and **F** is also integrable with a first integral of the form $V = h + \cdots$ and the statement **(a)** with $A = +\infty$ is satisfied.

Assume now that $f_2^2(h) + f_4^2(h) + f_5^2(h) \not\equiv 0$ in this case, we consider

$$m_0 = \min\left\{m \in \mathbb{N} \cup \{0\} . \left(a_m^{(2)}\right)^2 + \left(a_m^{(4)}\right)^2 + \left(a_m^{(5)}\right)^2 \neq 0\right\},$$
$$i_0 = \min\left\{i \in \{2,4,5\} : a_{m_0}^{(i)} \neq 0\right\}.$$

- If $i_0 = 2$ for any value of A, we have

$$\begin{aligned}\nabla h \cdot \mathbf{G} - \frac{1}{1+A}\operatorname{div}(\mathbf{G})\, h &= 4a_{m_0}^{(2)} x^2 h^{m_0+1} - \frac{1}{1+A}(5 + 4m_0) a_{m_0}^{(2)} x^2 h^{m_0} h + \cdots, \\ &= \frac{4(A - m_0) - 1}{1+A} a_{m_0}^{(2)} x^2 h^{m_0+1} + \cdots\end{aligned}$$

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defining $\alpha := \frac{4(A-m_0)-1}{1+A} a_{m_0}^{(2)} \neq 0$ and $m := m_0 + 1$. If $\mathbf{G} = \Phi(\mu\mathbf{F})$, applying Proposition 2.3 and considering $V = \mu(\Phi(\mathbf{x}))\,(D\Phi)^{-1}\,h\,(\Phi(\mathbf{x})) = h + \cdots$ statement **(b)** is satisfied.

- If $i_0 = 4$ for any value of A, we have

$$\nabla h \cdot \mathbf{G} - \frac{1}{1+A}\mathrm{div}\,(\mathbf{G})\,h = 4a_{m_0}^{(4)} h^{m_0+2} - \frac{1}{1+A}(3 + 4(m_0+1))a_{m_0}^{(4)} h^{m_0+1} h + \cdots ,$$
$$= \frac{4(A-m_0)-3}{1+A} a_{m_0}^{(4)} h^{m_0+2} + \cdots$$

defining $\alpha := \frac{4(A-m_0)-3}{1+A} a_{m_0}^{(4)} \neq 0$ and $m := m_0 + 2$. If $\mathbf{G} = \Phi(\mu\mathbf{F})$, applying Proposition 2.3 and considering $V = \mu(\Phi(\mathbf{x}))\,(D\Phi)^{-1}\,h\,(\Phi(\mathbf{x})) = h + \cdots$ statement **(c)** is satisfied.

- If $i_0 = 5$, we will see that it is possible to choose a value of A and a function $b(h) = 1 + \sum_{i=1}^{\infty} b_i h^i$, $b_i \in \mathbb{R}$ such that

$$\nabla V \cdot \mathbf{G} - \tfrac{1}{1+A}\,\mathrm{div}\,(\mathbf{G})\,V = x^2 g_2(h) + h g_4(h),$$

where $V := hb(h)$ with $g_2(h) = \sum_{i>m_0} c_i^{(2)} h^i$ and $g_4(h) = \sum_{i>m_0} c_i^{(4)} h^i$. We must eliminate the odd terms in h.

$$\begin{aligned}\nabla hb(h) \cdot \mathbf{G} - \tfrac{1}{1+A}\mathrm{div}\,(\mathbf{G})\,hb(h) &= \left[b(h) + hb'(h)\right] \nabla h \cdot \mathbf{G} - \tfrac{1}{1+A}\mathrm{div}\,(\mathbf{G})\,hb(h)\\ &= 4h\left[b(h) + hb'(h)\right]\left(x^2 f_2(h) + hf_4(h) + xhf_5(h)\right)\\ &\quad -\tfrac{1}{1+A}\left[5x^2 f_2(h) + 4x^2 f_2'(h)h + 7hf_4(h) + 4h^2 f_4'(h)\right.\\ &\quad \left.+8xhf_5(h) + 4xh^2 f_5'(h)\right] hb(h).\end{aligned}$$

The odd terms in h of the right-hand side are

$$\tfrac{4xh^2 b(h) f_5(h)}{1+A}\left[(1+A)\left(1 + \tfrac{hb'(h)}{b(h)}\right) - 2 - \tfrac{hf_5'(h)}{f_5(h)}\right]. \tag{3.5}$$

Taking into account that $\frac{hf_5'(h)}{f_5(h)} = \frac{m_0 a_{m_0}^{(5)} h^{m_0} + \cdots}{a_{m_0}^{(5)} h^{m_0} + \cdots} = m_0 + \cdots := m_0 + \tilde{f}_5(h)$ with $\tilde{f}_5(0) = 0$ and imposing $1 + A - 2 - m_0 = 0$, that is, $A = m_0 + 1$, the odd terms in h are

$$4xh^3 b(h) f_5(h)\left[(\log(b(h)))' - \tfrac{\tilde{f}_5(h)}{(m_0+2)h}\right].$$

In order to vanish these terms, we just take $b(h) = \exp\left(\frac{\tilde{f}_5(h)}{(m_0+2)h}\right)$.

Now we come back to the functions $g_2(h)$ and $g_4(h)$. If $g_2(h) \equiv g_4(h) \equiv 0$, we have that statement **(a)** is satisfied. If $(g_2(h))^2 + (g_4(h))^2 \not\equiv 0$ considering the non-zero terms of lower degree, we have that statement **(b)** or **(c)** would be satisfied.

Theorem 3.12 *The origin of system (2.4) is a centre if and only if there exist a formal function $V = h + \cdots$ and a unique value $A \in \mathbb{N} \cup \{0\} \cup \{+\infty\}$ such that*

$$\nabla V \cdot \mathbf{F} - \tfrac{1}{1+A}\mathrm{div}\,(\mathbf{F})\,V = 0.$$

Moreover, if $A = +\infty$, then the centre is integrable and if $d \neq 0$, then $A = 0$.

Proof Applying Theorem 2.8, the origin of system (2.4) is a centre if and only if its vector field associated $\mathbf{F}$, has a formal inverse integrating factor. The factors of the inverse integrating factor are invariant curves. Furthermore, the first quasi-homogeneous component of the formal invariant curve must be invariant curve of the first quasi-homogeneous component of $\mathbf{F}$, that is, of $\mathbf{F}_1$. Notice that the unique irreducible invariant curve of $\mathbf{F}_1$ are the factors of h. Therefore, the inverse integrating factor is of the form $W = h^{1+A} + \cdots := V^{1+A}$ with $A \in \mathbb{N} \cup \{0\}$ and $V = h + \cdots$. Consequently,

$$\begin{aligned} 0 &= \nabla W \cdot \mathbf{F} - \operatorname{div}(\mathbf{F})\, W = (1+A)V^A \nabla V \cdot \mathbf{F} - \operatorname{div}(\mathbf{F})\, V^{1+A} \\ &= (1+A)V^A \left(\nabla V \cdot \mathbf{F} - \tfrac{1}{1+A} \operatorname{div}(\mathbf{F})\, V \right), \end{aligned}$$

and from here the statement of the theorem follows. We can include the case $A = +\infty$ and in this case $\nabla V \cdot \mathbf{F} = 0$, that is, $\mathbf{F}$ would be integrable. Applying Lemma 3.11, we have the condition $A = 0$ if and only if $d \neq 0$.

We note that Lemma 3.11 and Theorem 3.12 give us a method to compute the necessary conditions to have a nilpotent centre for system (2.4).

Theorem 3.13 *Consider system (2.4) with $d \neq 0$ and $\mathbf{F}$ its associated vector field. The following conditions are equivalent:*

- **(a)** *The origin of system (2.4) is a centre.*
- **(b)** $\mathbf{F}$ *is orbitally reversible.*
- **(c)** *There exists a formal inverse integrating factor $V = h + \cdots$ of* $\mathbf{F}$.
- **(d)** *There exist $\mathbf{G} = \mathbf{D}_0 + \cdots$ and a scalar function μ with $\mu(\mathbf{0}) = 1$ such that $[\mathbf{F}, \mathbf{G}] = \mu \mathbf{F}$.*

Proof To see that statement **(a)** is equivalent to statement **(b)**, it is sufficient to apply Proposition 2.7 statement **(b)**. The equivalence between statement **(a)** and **(c)** follows from Theorem 3.12. Finally, the equivalence between statements **(a)** and **(d)** follows from Proposition 2.7 statement **(b)** and Theorem 1.3 of [5].

4 Applications

In this section, we apply the scalar algorithm obtained in order to find nilpotent centres.

Example 1 *We consider the planar differential system*

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} y \\ -x^3 \end{pmatrix} + \begin{pmatrix} a_{11}xy + a_{30}x^3 \\ b_{02}y^2 + b_{21}x^2y + b_{40}x^4 \end{pmatrix}. \tag{4.6}$$

The centre conditions of this family are studied, using a generalization of the Cherkas' method, in [15, Proposition 33]. We find the same centre conditions using the algorithm derived in the previous section.

Theorem 4.14 *System (4.6) has a centre at the origin if and only if one of the following conditions holds:*

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(i) $a_{30} = b_{21} = 0$.

(ii) $3a_{30} + b_{21} = 2b_{02} + a_{11} = 0$, $a_{30} \neq 0$.

(iii) $3a_{30} + b_{21} = b_{02} + b_{40} - a_{11} = 2b_{40}^2 + b_{40}b_{02} + 6a_{30}^2 = 0$, $a_{30}(2b_{02} + a_{11}) \neq 0$.

(iv) $3a_{30} + b_{21} = b_{02} + b_{40} - a_{11} = 3b_{40} + 2b_{02} = 0$, $a_{30}(2b_{02} + a_{11})(2b_{40}^2 + b_{40}b_{02} + 6a_{30}^2) \neq 0$.

Proof First, we will see the necessity. We impose the condition $\nabla V \cdot \mathbf{F} - \frac{1}{1+A}\mathrm{div}\,(\mathbf{F})\,V = 0$ of Theorem 3.12 degree by degree of quasi-homogeneity. We consider $V = h + \sum_{j\geqslant 5} V_j$, where $V_j \in \mathcal{P}_j^{(1,2)}$ and we chose V_5 in order to cancel the maximum number of terms and we obtain

$$\nabla V \cdot \mathbf{F} - \tfrac{1}{1+A}\mathrm{div}\,(\mathbf{F})\,V = \tfrac{2(4A-1)(3a_{30}+b_{21})}{15(1+A)} x^2 h + \cdots .$$

Taking into account that $4A - 1 \neq 0$, the first condition is $3a_{30} + b_{21} = 0$.

Now we chose $V_6 \in \mathcal{P}_6^{(1,2)}$. In this case, we can choose V_6 such that all the terms of degree 7 cancel in $\nabla V \cdot \mathbf{F} - \frac{1}{1+A}\mathrm{div}\,(\mathbf{F})\,V$.

Next, we consider $V_7 \in \mathcal{P}_7^{(1,2)}$. After a good choice of V_7, we have that

$$\nabla V \cdot \mathbf{F} - \tfrac{1}{1+A}\mathrm{div}\,(\mathbf{F})\,V = \tfrac{4(4A-3)a_{30}(2b_{02}+a_{11})(b_{02}-a_{11}+b_{40})}{189(1+A)} h^2 + \cdots .$$

Taking into account that $4A - 3 \neq 0$, we have three possibilities either $a_{30} = 0$, or $2b_{02} + a_{11} = 0$, or $b_{02} - a_{11} + b_{40} = 0$.

The case $a_{30} = 0$ corresponds to statement **(i)**.

The case $2b_{02} + a_{11} = 0$, $a_{30} \neq 0$ corresponds to statement **(ii)**.

The remaining case is $a_{11} = b_{02} + b_{40}$, $a_{30}(2b_{02} + a_{11}) \neq 0$. After a good choice of $V_8 \in \mathcal{P}_8^{(1,2)}$, we obtain

$$\nabla V \cdot \mathbf{F} - \tfrac{1}{1+A}\mathrm{div}\,(\mathbf{F})\,V = \tfrac{a_{30}(b_{40}+3b_{02})(2b_{40}^2+b_{40}b_{02}+6a_{30}^2)(A-1)}{18(1+A)} x h^2 + \cdots .$$

Taking into account that $a_{30} \neq 0$, we have three possibilities **(c1)** $b_{40} + 3b_{02} = 0$, **(c2)** $b_{40} + 3b_{02} \neq 0$ and $2b_{40}^2 + b_{40}b_{02} + 6a_{30}^2 = 0$, and **(c3)** $(b_{40} + 3b_{02})(2b_{40}^2 + b_{40}b_{02} + 6a_{30}^2) \neq 0$ with $A = 1$. **(c1)** If $b_{40} + 3b_{02} = 0$, this implies $2b_{02} + a_{11} = 0$ which corresponds to statement **(ii)**. **(c2)** If $b_{40} + 3b_{02} \neq 0$ and $2b_{40}^2 + b_{40}b_{02} + 6a_{30}^2 = 0$, we have the case of statement **(iii)**. **(c3)** If $(b_{40} + 3b_{02})(2b_{40}^2 + b_{40}b_{02} + 6a_{30}^2) \neq 0$ with $A = 1$. After a good choice of $V_9 \in \mathcal{P}_9^{(1,2)}$, we obtain

$$\nabla V \cdot \mathbf{F} - \tfrac{1}{1+A}\mathrm{div}\,(\mathbf{F})\,V = \tfrac{4a_{30}(b_{40}+3b_{02})(3b_{40}+2b_{02})(2b_{40}^2+b_{40}b_{02}+6a_{30}^2)}{2025} x^2 h^2 + \cdots .$$

Taking into account that $a_{30}(b_{40} + 3b_{02})(2b_{40}^2 + b_{40}b_{02} + 6a_{30}^2) \neq 0$, the unique possibility is $3b_{40} + 2b_{02} = 0$ which corresponds with statement **(iv)**.

Now we will see the sufficiency.

In the case **(i)**, that is, $a_{30} = b_{21} = 0$, system (4.6) is time-reversible because is invariant by the the symmetry $(x, y, t) \mapsto (x, -y, -t)$ and, as the origin is monodromic, the origin is a centre.

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In the case **(ii)**, that is, $3a_{30}+b_{21} = 2b_{02}+a_{11} = 0$, we have that the divergence div $(\mathbf{F}) = 0$ and consequently $\mathbf{F}$ is a Hamiltonian vector field and, as the origin is monodromic, the origin is a centre.

In the case **(iii)**, that is, $3a_{30} + b_{21} = b_{02} + b_{40} - a_{11} = 2b_{40}^2 + b_{40}b_{02} + 6a_{30}^2 = 0$, with $a_{30}(2b_{02} + a_{11}) \neq 0$, the vector field $\mathbf{F}$ associated to system (4.6) has the inverse integrating factor $V = (1 + a_{30}b_{02}y + b_{02}x - \frac{b_{40}b_{02}}{2}x^2)^\alpha$, where $\alpha = (b_{40} + 3b_{02})/b_{02}$. We note that $b_{02} \neq 0$ because the case $b_{02} = 0$ implies $0 = b_{40}^2 + 3a_{30}^2$ which corresponds to a particular case of case **(i)**. Moreover, we also consider the case $b_{40} + 3b_{02} \neq 0$ because the case $b_{40} + 3b_{02} = 0$ implies $b_{02}^2 + 2a_{30}^2 = 0$ which also corresponds to a particular case of case **(i)**. Therefore, $\mathbf{F}$ is integrable and the origin of system (4.6) is a centre.

In the case **(iv),** system (4.6) takes the form

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} y \\ -x^3 \end{pmatrix} + \begin{pmatrix} a_{30}x^3 - \frac{b_{40}}{2}xy \\ b_{40}x^4 - 3a_{30}x^2y - \frac{3b_{40}}{2}y^2 \end{pmatrix}.$$

Applying the change of variables $u = (1 - \frac{b_{40}}{2}x)^{-1}$, $v = \left[y\left(1 - \frac{b_{40}}{2}x\right) + a_{30}x^3\right]\left(1 - \frac{b_{40}}{2}x\right)^{-4}$, $dT = \left(1 + \frac{b_{40}}{2}u\right)dt$, and denoting by $' = d/dT$, system (4.6) becomes

$$\begin{pmatrix} u' \\ v' \end{pmatrix} = \begin{pmatrix} v \\ -u^3 \end{pmatrix} + \begin{pmatrix} 0 \\ \left(3a_{30}^2 + \frac{b_{40}^2}{4}\right)u^5 - \frac{7}{2}a_{30}b_{40}u^3v \end{pmatrix} - \begin{pmatrix} 0 \\ \frac{3}{4}a_{30}^2b_{40}^2u^7 \end{pmatrix},$$

which is monodromic and time-reversible because is invariant by the symmetry $(u, v, t) \mapsto (-u, v, -t)$ and consequently the origin is a centre.

Example 2 *We now consider the differential system*

$$\begin{aligned} \dot{x} &= y + axy + by^2, \\ \dot{y} &= -x^3 + kxy + cy^3. \end{aligned} \tag{4.7}$$

Applying the change of variables $x = u$, $y = v + \frac{k}{4}u^2$, system (4.7) can be written as

$$\begin{aligned} \begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = \overbrace{\begin{pmatrix} v \\ (\frac{k^2}{8} - 1)u^3 \end{pmatrix} + \frac{k}{4}u\begin{pmatrix} u \\ 2v \end{pmatrix}}^{\mathbf{F}_1} + \begin{pmatrix} \frac{ak}{4}u^3 + auv \\ -\frac{k^2a}{8}u^4 - \frac{ka}{2}u^2v \end{pmatrix} \\ + \begin{pmatrix} \frac{k^2b}{16}u^4 + \frac{kb}{2}u^2v + bv^2 \\ -\frac{k^3b}{32}u^5 - \frac{k^2b}{4}u^3v - \frac{kb}{2}uv^2 \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{k^3c}{64}u^6 + \frac{3k^2c}{16}u^4v + \frac{3kc}{4}u^2v^2 + cv^3 \end{pmatrix}. \end{aligned} \tag{4.8}$$

Lemma 4.15 *The origin of system (4.7) is monodromic if, and only if, $k^2 < 8$.*

Proof The Hamiltonian function of system (4.8) is

$$h(x, y) = \frac{1}{4}\left(\frac{k^2}{8} - 1\right)u^4 - \frac{1}{2}v^2.$$

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If $k^2-8>0$, then h has two simple factor and consequently the origin of system (4.7) is not monodromic, see [8, Proposition 6].

If $k^2-8=0$, then $h=-\frac{1}{2}v^2$ has a double factor v which is not a factor of the divergence and applying [8, Proposition 6], we have that the origin of system (4.7) is not monodromic.

Finally, if $k^2-8<0$, then h is negative defined, does not vanish in a neighbourhood of the origin and hence, the origin of system (4.7) is monodromic, see [8, Proposition 5].

Theorem 4.16 *System (4.7) has a centre at the origin if and only if $k^2<8$ and any of the following conditions holds:*

(i) $a=c=0,\ k\neq 0$.

(ii) $k=a=c=0$.

(iii) $k=b=c=0$.

Proof Necessity: We recall that to have a monodromic singular point at the origin by Lemma 4.15, we have that $k^2<8$. From Theorem 3.12, we impose the condition $\nabla V\cdot\mathbf{F}-\frac{1}{1+A}\mathrm{div}\,(\mathbf{F})\,V=0$, where $V=h+\sum_{j\geqslant 5}V_j$, $V_j\in\mathcal{P}_j^{(1,2)}$ degree by degree of homogeneity in the equivalent system (4.8). We divide the study in the two case $k\neq 0$ and $k=0$.

In the case $k\neq 0$, we have $A=0$. We chose V_5 in order to cancel the maximum number of terms and we obtain

$$\nabla V\cdot\mathbf{F}-\mathrm{div}\,(\mathbf{F})\,V=-\tfrac{2ka}{3k^2-25}x^2h+\cdots.$$

Taking into account that $k\neq 0$, we have that the first condition is $a=0$. Now we chose $V_6\in\mathcal{P}_6^{(1,2)}$. In this case, we can choose V_6 in such way that all the terms of degree 7 in $\nabla V\cdot\mathbf{F}-\mathrm{div}\,(\mathbf{F})\,V$ vanish. Now we consider $V_7\in\mathcal{P}_7^{(1,2)}$. After a good choice of V_7, we have

$$\nabla V\cdot\mathbf{F}-\mathrm{div}\,(\mathbf{F})\,V=\tfrac{(2k^2-21)c}{5k^2-49}h^2+\cdots.$$

Taking into account that $2k^2-21\neq 0$, we get $c=0$ which corresponds to the statement **(i)**.

In the case $k=0$, we have $A>0$ and we can chose V_5 and V_6 in such way that all the terms of degree 6 and 7 vanish in $\nabla V\cdot\mathbf{F}-\frac{1}{1+A}\mathrm{div}\,(\mathbf{F})\,V$. Now we chose V_7 in order to cancel the maximum number of terms and we obtain

$$\nabla V\cdot\mathbf{F}-\tfrac{1}{1+A}\mathrm{div}\,(\mathbf{F})\,V=\tfrac{(4A-3)(ab-3c)}{21(1+A)}h^2+\cdots.$$

Taking into account that $4A-3\neq 0$, we get that $ab-3c=0$. Next, we chose V_8 in order to cancel the maximum number of terms and we have

$$\nabla V\cdot\mathbf{F}-\tfrac{1}{1+A}\mathrm{div}\,(\mathbf{F})\,V=-\tfrac{(A-1)a^2b}{8(1+A)}xh^2+\cdots.$$

In this case, we have three possibilities either $a=0$ or $b=0$ with $a\neq 0$ or $A=1$ with $ab\neq 0$.

The case $a = 0$ corresponds to statement **(ii)**. The case $b = 0$ with $a \neq 0$ corresponds to statement **(iii)**. In the case $A = 1$ with $ab \neq 0$, we chose V_9 in order to cancel the maximum number of terms and we get

$$\nabla V \cdot \mathbf{F} - \tfrac{1}{1+A}\mathrm{div}\,(\mathbf{F})\, V = -\tfrac{7a^3b}{450}x^2h^2 + \cdots .$$

Taking into account that $a^3b \neq 0$, we have that the origin cannot be a centre in this case.

Sufficiency:

In the case $a = c = 0$ with $k \neq 0$, system (4.7) is time-reversible because it is invariant by the symmetry $(x, y, t) \mapsto (-x, y, -t)$ and as the origin is monodromic then it is a centre. In the case $a = c = k = 0$, system (4.7) is also time-reversible because it has the same symmetry. In the case $k = b = c = 0$, system (4.7) is time-reversible because it is invariant by the symmetry $(x, y, t) \mapsto (x, -y, -t)$ and as the origin is monodromic then it is a centre.

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