

Prescribed mean curvature equation on the unit ball in the presence of reflection or rotation symmetry

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Using the flow method, we prove some existence results for the problem of prescribing the mean curvature on the unit ball. More precisely, we prove that there exists a conformal metric on the unit ball such that its mean curvature is f , when f possesses certain reflection or rotation symmetry.

Keywords: mean curvature; unit ball; Nirenberg's problem

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1. Introduction

The problem of prescribing scalar curvature on a closed manifold has been studied extensively for last few decades. More precisely, let (M, g) be an n -dimensional compact smooth Riemannian manifold without boundary. Given a smooth function f on M , can we find a metric conformal to g such that its scalar curvature is f ? This has been studied in [3, 5, 14, 27, 32, 33, 37, 42]. When (M, g) is the n -dimensional standard sphere S^n , it is called the Nirenberg's problem and has been studied extensively. See [4, 7–10, 13, 24, 26, 38, 43, 44] and the references therein. In particular, Chang–Yang obtained in [9] a perturbation theorem which asserts that there exists a conformal metric whose scalar curvature is equal to f , provided that the degree condition holds for f which is a positive Morse function and is sufficiently closed to $n(n-1)$ in C^0 norm. See also [18, 19, 28–31, 36] for related results of prescribing the Webster scalar curvature on CR manifolds.

A geometric flow has been introduced to study the Nirenberg's problem by Struwe in [41] for $n = 2$, and has been generalized to $n \geq 3$ by Chen–Xu in [15]. More precisely, the scalar curvature flow is defined as

$$\frac{\partial}{\partial t} u = \frac{n-2}{4} (\alpha f - R_{\bar{g}}) u,$$

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where $R_{\tilde{g}}$ is the scalar curvature of $\tilde{g} = u^{4/n-2}g$ and α is a constant chosen to preserve the volume along the flow. Using the scalar curvature flow, Chen–Xu [15] was able to prove Chang–Yang’s result with the quantitative bound on $\|f - n(n - 1)\|_{C^0(S^n)}$.

Again using the scalar curvature flow, Leung–Zhou [34] proved an existence result for prescribing scalar curvature with symmetry. To describe their result, we have the following:

ASSUMPTION 1.1. f is symmetric under a mirror reflection upon a hyperplane $\mathcal{H} \subset \mathbb{R}^{n+1}$ passing through the origin.

Under assumption 1.1, without loss of generality, we may assume that \mathcal{H} is the hyperplane perpendicular to the x_1 -axis. Then the symmetry can be expressed as

$$f(\gamma_m(x)) = f(x)$$

where $\gamma_m : S^n \rightarrow S^n$ is given by

$$\gamma_m(x_1, x_2, \dots, x_{n+1}) = (-x_1, x_2, \dots, x_{n+1}) \quad \text{for } x = (x_1, x_2, \dots, x_{n+1}) \in S^n.$$

Then

$$\mathcal{F} = \mathcal{H} \cap S^n = \{(0, x_2, \dots, x_{n+1}) \in S^n\}$$

is the fixed point set.

ASSUMPTION 1.2. f is invariant under a rotation γ_θ of angle $\theta = \pi/k$ with the rotation axis being a straight line in \mathbb{R}^{n+1} passing through the origin. Here $k > 1$ is an integer.

Under assumption 1.2, without loss of generality, we may assume that the rotation axis is the x_{n+1} -axis. In this case, $\mathcal{F} = \{N, S\}$ is the fixed point set, where $N = (0, \dots, 0, 1)$ is the north pole and $S = (0, \dots, 0, -1)$ is the south pole.

With these assumptions, we can state the result of Leung–Zhou in [34].

THEOREM 1.1 (Leung–Zhou [34]). *Suppose that f is a positive smooth function on S^n satisfying assumptions 1.1 or 1.2. Assume that*

$$x_m \in \mathcal{F} \text{ with } f(x_m) = \max_{\mathcal{F}} f \Rightarrow \Delta_{S^n} f(x_m) > 0$$

where Δ_{S^n} is the Laplacian of the standard metric of S^n , and

$$\max_{S^n} f < 2^{2/(n-2)} \left(\max_{\mathcal{F}} f \right).$$

Then f can be realized as the scalar curvature of some metric conformal to the standard metric of S^n .

Note that existence results for prescribing scalar curvature with symmetry were obtained earlier by Moser [35] and by Escobar–Schoen [23].

The problem of prescribing the scalar curvature or the mean curvature has been studied on manifolds with boundary. See [12, 16, 20, 21, 25, 39, 46] for example. In

this paper, we consider the following problem of prescribing the mean curvature on the unit ball, which is a natural analogy of the prescribing scalar curvature problem: Let B^{n+1} be the $(n + 1)$ -dimensional unit ball equipped with the flat metric g_0 , that is,

$$B^{n+1} = \left\{ (x_1, \dots, x_n, x_{n+1}) \in \mathbb{R}^{n+1} : \sum_{i=1}^{n+1} x_i^2 \leq 1 \right\}.$$

The boundary of B^{n+1} is the n -dimensional unit sphere S^n , that is,

$$\partial B^{n+1} = S^n = \left\{ (x_1, \dots, x_n, x_{n+1}) \in \mathbb{R}^{n+1} : \sum_{i=1}^{n+1} x_i^2 = 1 \right\}.$$

The mean curvature H_{g_0} of S^n with respect to g_0 is equal to 1, that is, $H_{g_0} = 1$. Conversely, the metric induced by g_0 on S^n is the standard metric g_{S^n} of S^n . We study the following problem: given a smooth function f on S^n , find a metric conformal to g_0 such that it is flat in the interior of B^{n+1} and its mean curvature is equal to f on $\partial B^{n+1} = S^n$. The problem is equivalent to finding a positive harmonic function u in the ball with nonlinear boundary condition:

$$\begin{aligned} \Delta_{g_0} u &= 0 && \text{in } B^{n+1}, \\ \frac{2}{n-1} \frac{\partial u}{\partial \nu_{g_0}} + u &= f u^{((n+1)/(n-1))} && \text{on } S^n. \end{aligned} \tag{1.1}$$

Here, Δ_{g_0} is the Laplacian of g_0 and $((\partial)/(\partial \nu_{g_0}))$ is the outward normal derivative of g_0 . See [1, 2, 6, 17, 20–22, 39, 40] and references therein for the results related to this problem. In particular, Chang–Xu–Yang proved in [11] that (1.1) has a solution when f is a positive Morse function satisfying the degree condition and being sufficiently closed to 1 in C^0 norm. By the method of geometric flow, Xu–Zhang [45] proved Chang–Xu–Yang’s result with the quantitative bound on $\|f - 1\|_{C^0(S^n)}$.

Inspired by the result of Leung–Zhou [34], we study the problem of prescribing the mean curvature on the unit ball with symmetry. We prove the following:

THEOREM 1.2. *Suppose that f is a positive smooth function on S^n satisfying assumption 1.1 or 1.2. Assume that*

$$x_m \in \mathcal{F} \quad \text{with } f(x_m) = \max_{\mathcal{F}} f \Rightarrow \Delta_{S^n} f(x_m) > 0 \tag{1.2}$$

where Δ_{S^n} is the Laplacian of the standard metric of S^n , and

$$\max_{S^n} f < 2^{1/(n-1)} \left(\max_{\mathcal{F}} f \right). \tag{1.3}$$

Then f can be realized as the mean curvature of some conformal metric on the unit ball B^{n+1} .

We remark that Escobar has also studied in [21] the existence of a positive solution to (1.1) when f has symmetry. The proof of Theorem 1.2 follows the

arguments of Leung–Zhou in [34]. We also remark that we have used the arguments of Leung–Zhou to study the problem of prescribing Webster scalar curvature on the CR sphere with symmetry. See [31].

2. The flow and its properties

Let B^{n+1} be the $(n + 1)$ -dimensional unit ball equipped with the flat metric g_0 . Then the n -dimensional sphere $S^n = \partial B^{n+1}$ is the boundary of B^{n+1} . Let f be a positive smooth function defined on S^n . Given a smooth function u_0 in B^{n+1} , we consider the flow

$$u_t = \frac{n - 1}{4}(\alpha f - H_g)u \quad \text{on } S^n \tag{2.1}$$

for $t \geq 0$ with the initial condition

$$u|_{t=0} = u_0,$$

where

$$H_g = u^{-((n+1)/(n-1))} \left(\frac{2}{n - 1} \frac{\partial u}{\partial \nu_{g_0}} + u \right) \quad \text{on } S^n \tag{2.2}$$

is the mean curvature of the conformal metric $g = u^{4/n-1}g_0$, and

$$\Delta_{g_0} u = 0 \quad \text{in } B^{n+1}. \tag{2.3}$$

Here, Δ_{g_0} is the Laplacian of g_0 and $((\partial)/(\partial \nu_{g_0}))$ is the outward normal derivative of g_0 . Also, $\alpha = \alpha(t)$ is defined as

$$\alpha = \left(\int_{S^n} H_g u^{2n/n-1} d\mu \right) / \left(\int_{S^n} f u^{((2n)/(n-1))} d\mu \right), \tag{2.4}$$

where $d\mu$ is the volume form of the standard metric g_{S^n} on S^n and

$$\int_{S^n} \varphi d\mu = \frac{1}{\omega_n} \int_{S^n} \varphi d\mu \quad \text{for all } \varphi \in C^\infty(S^n),$$

where $\omega_n = \int_{S^n} d\mu$ is the volume of S^n with respect to g_{S^n} .

We define the functionals

$$E[u] = \int_{S^n} \left(\frac{2}{n - 2} \frac{\partial u}{\partial \nu_{g_0}} + u \right) u d\mu = \int_{S^n} H_g u^{((2n)/(n-1))} d\mu \tag{2.5}$$

and

$$E_f[u] = E[u] / \left(\int_{S^n} f u^{((2n)/(n-1))} d\mu \right)^{((n-1)/(n))}. \tag{2.6}$$

It follows from lemma 2.2 in [45] that

$$\frac{d}{dt} E_f[u] \leq 0$$

along the flow (2.1). In particular, we have

$$E_f[u] \leq E_f[u_0] \quad \text{for all } t \geq 0. \tag{2.7}$$

It follows from §2.3 in [45] that (2.1)–(2.3) has a unique solution $u(t, x)$ on $B^{n+1} \times [0, \infty)$ such that, given $T > 0$, there exists a constant $C = C(T)$ such that

$$C^{-1} \leq u(x, t) \leq C \quad \text{for all } (x, t) \in B^{n+1} \times [0, T]. \tag{2.8}$$

LEMMA 2.1. *Given an isometry $\gamma : (S^n, g_{S^n}) \rightarrow (S^n, g_{S^n})$, we assume that*

$$f(\gamma(x)) = f(x) \quad \text{and } u_0(\gamma(x)) = u_0(x) \quad \text{for all } x \in S^n. \tag{2.9}$$

Let $u(x, t)$ be the solution of (2.1)–(2.3) with initial data u_0 . Then

$$u(\gamma(x), t) = u(x, t) \quad \text{for all } (x, t) \in S^n \times [0, \infty). \tag{2.10}$$

Proof. Since $\gamma : (S^n, g_{S^n}) \rightarrow (S^n, g_{S^n})$ is isometry, for any $\varphi \in C^\infty(B^{2n+1})$, we have

$$\left(\frac{\partial(\varphi \circ \gamma)}{\partial \nu_{g_0}} \right) (x) = \left(\frac{\partial \varphi}{\partial \nu_{g_0}} \right) (\gamma(x)) \quad \text{for all } x \in S^n. \tag{2.11}$$

Since the mean curvature of $g(x, t) = u(x, t)^{(4)/(n-1)} g_0$ satisfies

$$H_{g(x,t)}(x) = u(x, t)^{-((n+1)/(n-1))} \left(\frac{2}{n-1} \frac{\partial}{\partial \nu_{g_0}} u(x, t) + u(x, t) \right) \quad \text{for } x \in S^n,$$

it follows from (2.10) and (2.11) that

$$H_{g(x,t)}(\gamma(x)) = H_{g(\gamma(x),t)}(x) \quad \text{for all } x \in S^n. \tag{2.12}$$

By (2.1)–(2.4), the flow can be written as

$$\begin{aligned} \Delta_{g_0} u &= 0 \quad \text{in } B^{n+1}, \\ u_t &= -\frac{n-1}{4} u^{-((2)/(n-1))} \left(\frac{2}{n-1} \frac{\partial u}{\partial \nu_{g_0}} + u \right) \\ &\quad + \frac{n-1}{4} \left(\frac{\int_{S^n} H_g u^{((2n)/(n-1))} d\mu}{\int_{S^n} f u^{((2n)/(n-1))} d\mu} \right) f u \quad \text{on } S^n. \end{aligned} \tag{2.13}$$

From (2.16), (2.17), (2.19) and (2.20), we know that $u(\gamma(x), t)$ is also a solution of (2.13) with initial value u_0 . Now the assertion (2.18) follows from the uniqueness result stated in lemma 4.8 in [12]. □

For $r > 0$ and $x_0 \in S^n$, set $B_r^+(x_0) = \{x \in B^{n+1} : d_{g_0}(x, x_0) < r\}$ and $\partial' B_r^+(x_0) = \partial B_r^+(x_0) \cap S^n$. The following lemma was proved in [45]: (see §4.2 in [45])

LEMMA 2.2. Let $u(t)$ be the solution of the flow (2.1). Let $(t_k)_{k=1}^\infty$ be any time sequence with $t_k \rightarrow \infty$ as $k \rightarrow \infty$. Consider the sequence $u_k := u(t_k)$ and corresponding metrics $g_k = u(t_k)^{4/n-1}g_0$. Then, up to a subsequence, either

- (i) the sequence u_k is uniformly bounded in $W^{1,p}(S^n, g_{S^n}) \hookrightarrow L^\infty(S^n)$ for some $p > n$; or
- (ii) there exists a subsequence of u_k and finitely many points $x_1, \dots, x_L \in S^n$ such that for any $r > 0$ and any $l \in \{1, \dots, L\}$ there holds

$$\liminf_{k \rightarrow \infty} \left(\omega_n^{-1} \int_{\partial' B_r(x_l)} |H_{g_k}|^n d\mu_{g_k} \right)^{1/n} \geq 1. \tag{2.14}$$

In addition, if the alternative (ii) occurs, the sequence u_k is also uniformly bounded in L^p on any compact subset of $(S^n \setminus \{x_1, \dots, x_L\}, g_{S^n})$.

LEMMA 2.3. Suppose that lemma 2.2(ii) occurs. For a point $x_0 \in S^n$ with

$$\max_{S^n} f < 2^{((1)/(n-1))} f(x_0), \tag{2.15}$$

suppose that the initial data u_0 satisfies

$$E_f[u_0] \leq \frac{1}{f(x_0)^{((n-1)/(n))}} + \epsilon. \tag{2.16}$$

Then for $\epsilon > 0$ small enough, we have $L = 1$.

Proof. Let $x_1, \dots, x_L \in S^n$ be the blow-up points. If $L > 2$, let

$$r < \min_{1 \leq i < j \leq L} d_{g_0}(x_i, x_j). \tag{2.17}$$

For any $\epsilon > 0$, there exists k being sufficiently large such that

$$\begin{aligned} L - \epsilon &\leq \sum_{l=1}^L \left(\omega_n^{-1} \int_{\partial' B_r(x_l)} |H_{g_k}|^n d\mu_{g_k} \right)^{1/n} \\ &\leq L^{1-1/n} \left(\omega_n^{-1} \sum_{l=1}^L \int_{\partial' B_r(x_l)} |H_{g_k}|^n d\mu_{g_k} \right)^{1/n} \\ &\leq L^{1-1/n} \left(\int_{S^n} |H_{g_k}|^n d\mu_{g_k} \right)^{1/n} \\ &\leq L^{1-1/n} \alpha(t_k) \left(\int_{S^n} f^n d\mu_{g_k} \right)^{1/n} + L^{1-1/n} \left(\int_{S^n} |\alpha(t_k)f - H_{g_k}|^n d\mu_{g_k} \right)^{1/n}, \end{aligned} \tag{2.18}$$

where the first inequality follows from Hölder’s inequality, the second inequality follows from (2.14), and the third inequality follows from (2.17). By lemma 3.2 in

[45], for any $p \geq 2$, we have

$$\int_{S^n} |\alpha(t)f - H_g|^p d\mu_g \rightarrow 0 \quad \text{as } t \rightarrow \infty.$$

In particular,

$$\int_{S^n} |\alpha(t_k) - H_{g_k}|^n d\mu_{g_k} \rightarrow 0 \quad \text{as } k \rightarrow \infty. \tag{2.19}$$

Conversely, by (2.4)–(2.6), we have

$$\begin{aligned} \alpha &= \left(\int_{S^n} H_g u^{((2n)/(n-1))} d\mu \right) / \left(\int_{S^n} f u^{((2n)/(n-1))} d\mu \right) \\ &= E_f[u] / \left(\int_{S^n} f u^{((2n)/(n-1))} d\mu \right)^{1/n}. \end{aligned} \tag{2.20}$$

Combining (2.18)–(2.20), we obtain

$$\begin{aligned} L - \epsilon &\leq L^{1-1/n} E_f[u_k] \left(\int_{S^n} f^n d\mu_{g_k} \right)^{1/n} / \left(\int_{S^n} f d\mu_{g_k} \right)^{1/n} + o(1) \\ &\leq L^{1-1/n} E_f[u_k] (\max_{S^n} f)^{n-1/n} + o(1) \\ &\leq L^{1-1/n} \left[\left(\frac{\max_{S^n} f}{f(x_0)} \right)^{n-1/n} + O(\epsilon) \right] + o(1) \\ &\leq L^{1-1/n} (2^{1/n} + O(\epsilon)) + o(1), \end{aligned}$$

where we have used (2.16) in the second last inequality, and (2.15) in the last inequality. This implies that $L = 1$ when $\epsilon > 0$ is small enough. \square

We have the following lemma regarding the blow-up point. Its proof can be found in [45].

LEMMA 2.4. *Under the assumptions of lemma 2.2(ii) and 2.3, there is a point $Q \in S^n$ such that the following statements hold:*

- (a) *As $k \rightarrow \infty$, the metrics g_k concentrate at Q in the sense described by (ii) in lemma 4.2 in [45]. As a consequence, for any positive number ρ , $\max_{\partial' B_\rho(Q)} u_k$ cannot be uniformly bounded from above for all $k \gg 1$,*
- (b) *Q is a critical point of f ,*
- (c) *$\Delta_{S^n} f(Q) \leq 0$, and*
- (d) *$\lim_{k \rightarrow \infty} E_f[u_k] = \frac{1}{f(Q)^{n-1/n}}$.*

3. Proof of Theorem 1.2

As before, the south pole is denoted by $S = (0, \dots, 0, -1) \in S^n$. Let

$$\mathbb{R}_+^{n+1} = \{(z_1, \dots, z_n, z_{n+1}) \in \mathbb{R}^{n+1} : z_{n+1} \geq 0\}$$

be the $(n + 1)$ -dimensional upper half Euclidean space equipped with the Euclidean metric g_E . Consider the following map $\Sigma : B^{n+1} \setminus \{S\} \rightarrow \mathbb{R}_+^{n+1}$ given by

$$\Sigma(x_1, \dots, x_n, x_{n+1}) = \left(\frac{4\bar{x}}{|\bar{x}|^2 + (1 + x_{n+1})^2}, \frac{2(1 - |\bar{x}|^2 - x_{n+1}^2)}{|\bar{x}|^2 + (1 + x_{n+1})^2} \right) \tag{3.1}$$

where $\bar{x} = (x_1, \dots, x_n)$. Note that

$$\Sigma(x) = \left(\frac{2\bar{x}}{1 + x_{n+1}}, 0 \right) \text{ for } x \in S^n \setminus \{S\}. \tag{3.2}$$

Also, $\Sigma^{-1} : \mathbb{R}_+^{n+1} \rightarrow B^{n+1} \setminus \{S\}$ is given by

$$\Sigma^{-1}(\bar{z}, z_{n+1}) = \left(\frac{4\bar{z}}{|\bar{z}|^2 + (2 + z_{n+1})^2}, \frac{4 - z_{n+1}^2 - |\bar{z}|^2}{|\bar{z}|^2 + (2 + z_{n+1})^2} \right) \tag{3.3}$$

where $(\bar{z}, z_{n+1}) \in \mathbb{R}_+^{n+1}$ and $\bar{z} = (z_1, \dots, z_n)$. Note that $\Sigma^{-1} : (\mathbb{R}_+^{n+1}, g_E) \rightarrow (B^{n+1} \setminus \{S\}, g_0)$ is a conformal map such that

$$\left[\left(\frac{4}{(2 + z_{n+1})^2 + |\bar{z}|^2} \right)^{n-1/2} \right]^{4/n-1} g_E = (\Sigma^{-1})^*(g_0). \tag{3.4}$$

For $u \in C^\infty(B^{n+1})$, let

$$v(z) = \left(\frac{4}{(2 + z_{n+1})^2 + |\bar{z}|^2} \right)^{n-1/2} u(\Sigma^{-1}(z)).$$

It follows from (3.4) that

$$\begin{aligned} & \left[\left(\frac{4}{(2 + z_{n+1})^2 + |\bar{z}|^2} \right)^{n-1/2} \right]^{((n+1)/(n-1))} \left(\frac{2}{n-1} \frac{\partial}{\partial \nu_{g_0}} u + u \right) (\Sigma^{-1}(z)) \\ &= \frac{2}{n-1} \frac{\partial}{\partial \nu_{g_E}} v(z). \end{aligned}$$

This implies that the mean curvatures of $g = u^{4/n-1} g_0$ and $\tilde{g} = v^{4/n-1} g_E$ are related by

$$\begin{aligned} u^{-((n+1)/(n-1))} \left(\frac{2}{n-1} \frac{\partial u}{\partial \nu_{g_0}} + u \right) &= H_g(\Sigma^{-1}(z)) = H_{\tilde{g}}(z) \\ &= \frac{2}{n-1} v^{-((n+1)/(n-1))} \frac{\partial}{\partial \nu_{g_E}} v, \end{aligned} \tag{3.5}$$

Therefore, by (2.5), (2.6) and (3.5), we have

$$E_f[u] = \left(\omega_n^{-1} \int_{\partial\mathbb{R}_+^{n+1}} H_{\tilde{g}}(z)v(z)^{(2n)/(n-1)} dz \right) / \left(\omega_n^{-1} \int_{\partial\mathbb{R}_+^{n+1}} f(\Sigma^{-1}(z))v(z)^{(2n)/(n-1)} dz \right)^{n-1/n}. \tag{3.6}$$

LEMMA 3.1. *For any point $x_0 \in S^n$ and any positive number $\epsilon > 0$, there exists a function $u_0 \in C^\infty(S^n)$ such that*

$$E_f[u_0] \leq \frac{1}{f(x_0)^{n-1/n}} + \epsilon.$$

Moreover, we can choose u_0 to be invariant under the reflection upon a hyperplane passing through x_0 and the origin $0 \in R^{n+1}$, and invariant under rotations with the axis passing through x_0 and 0 .

Proof. As the situation is unchanged after a rotation of S^n , we may assume that $x_0 = N = (0, \dots, 0, 1)$ the north pole. For $\lambda > 0$, we let

$$v_0(z) = \left(\frac{2\lambda}{|\bar{z}|^2 + (z_{n+1} + \lambda)^2} \right)^{n-1/2} \text{ for } z = \Sigma(x).$$

Then v_0 satisfies the equation:

$$\begin{aligned} \Delta v_0 &= 0 \quad \text{in } \mathbb{R}_+^{n+1}, \\ \frac{2}{n-1} \frac{\partial v_0}{\partial z_{n+1}} + v_0^{((n+1)/(n-1))} &= 0 \quad \text{in } \partial\mathbb{R}_+^{n+1}, \end{aligned} \tag{3.7}$$

for some $c_0 > 0$. We choose λ such that

$$\int_{\partial\mathbb{R}_+^{n+1}} v_0(z)^{(2n)/(n-1)} dz = \omega_n.$$

If $\tilde{g} = v_0(z)^{4/n-1}g_E$, then it follows from (3.7) that

$$\int_{\partial\mathbb{R}_+^{n+1}} H_{\tilde{g}}(z)v_0(z)^{(2n)/(n-1)} dz = \omega_n. \tag{3.8}$$

We estimate

$$\begin{aligned} &\int_{\partial\mathbb{R}_+^{n+1}} v_0(z)^{(2n)/(n-1)} f(\Sigma^{-1}(z)) dz \\ &= \int_{\partial\mathbb{R}_+^{n+1}} v_0(z)^{(2n)/(n-1)} [f(\Sigma^{-1}(z)) - f(\Sigma^{-1}(0))] dz \\ &\quad + f(\Sigma^{-1}(0)) \int_{\partial\mathbb{R}_+^{n+1}} v_0(z)^{(2n)/(n-1)} dz \end{aligned}$$

$$\begin{aligned}
 &= \int_{\partial\mathbb{R}_+^{n+1}} v_0(z)^{((2n)/(n-1))} [f(\Sigma^{-1}(z)) - f(\Sigma^{-1}(0))] dz \\
 &\quad + f(\Sigma^{-1}(0)) \int_{\partial\mathbb{R}_+^{n+1}} v_0(z)^{((2n)/(n-1))} dz \\
 &= \int_{B_\delta(0)} v_0(z)^{((2n)/(n-1))} [f(\Sigma^{-1}(z)) - f(\Sigma^{-1}(0))] dz \\
 &\quad + \int_{\partial\mathbb{R}_+^{n+1} \setminus B_\delta(0)} v_0(z)^{((2n)/(n-1))} [f(\Sigma^{-1}(z)) - f(\Sigma^{-1}(0))] dz \\
 &\quad + f(\Sigma^{-1}(0)) \omega_n \\
 &= O(\epsilon_1) + O\left(\left[\frac{\lambda}{\delta_1}\right]^n\right) + f(N)\omega_n, \tag{3.9}
 \end{aligned}$$

since

$$|f(\Sigma^{-1}(z)) - f(\Sigma^{-1}(0))| \leq \epsilon_1 \quad \text{whenever } z \in B_{\delta_1}(0)$$

and

$$|v_0(z)| \leq C \left(\frac{\lambda}{\delta_1}\right)^n \quad \text{whenever } z \in \partial\mathbb{R}_+^{n+1} \setminus B_{\delta_1}(0)$$

for some uniform constant C . By first choosing $\delta_1 > 0$ to be small enough so that $\epsilon_1 > 0$ is small and then choosing a small $\lambda > 0$ so that $\lambda \cdot \delta_1^{-1}$ to be small, we obtain from (3.6), (3.8) and (3.9) that

$$E_f[u_0] \leq \frac{1}{f(x_0)^{n-1/n}} + \epsilon.$$

Here,

$$\begin{aligned}
 u_0(x) &= \left(\frac{1}{1 + |\bar{z}|^2}\right)^{n-1/2} \\
 v_0(z) &= \left(\frac{1}{1 + |\bar{z}|^2}\right)^{n-1/2} \left(\frac{2\lambda}{|\bar{z}|^2 + \lambda^2}\right)^{n-1/2} \quad \text{for } z = \Sigma(x), \tag{3.10}
 \end{aligned}$$

where $x \in S^n$. By (3.1)–(3.3), we have

$$|\bar{z}|^2 = \frac{2|\bar{x}|^2}{|1 + x_{n+1}|^2} = \frac{2(1 - |x_{n+1}|^2)}{|1 + x_{n+1}|^2} \quad \text{for } z = \Sigma(x),$$

where $x \in S^n$. That is, u_0 defined in (3.10) depends only on x_{n+1} . One can verify the claimed symmetries directly. □

We are ready to prove theorem 1.2.

Proof of Theorem 1.2. Without loss of generality, we may assume that

$$S \in \mathcal{F} \quad \text{and} \quad f(S) = \max_{\mathcal{F}} f$$

where $S = (0, \dots, 0, -1) \in S^n$ is the south pole. We claim that there is $\delta > 0$ such that

$$f(S) \geq f(x_c) + \delta \quad \text{for any point } x_c \in \mathcal{F} \text{ with } \Delta_{S^n} f(x_c) \leq 0. \tag{3.11}$$

If not, then there exists a sequence of points $x_{c_i} \in \mathcal{F}$ such that

$$\lim_{i \rightarrow \infty} f(x_{c_i}) = f(S) = \max_{\mathcal{F}} f \quad \text{and} \quad \Delta_{S^n} f(x_{c_i}) > 0.$$

By passing to subsequence, we assume that $x_{c_i} \rightarrow x_m \in \mathcal{F}$ as $i \rightarrow \infty$ such that

$$\lim_{i \rightarrow \infty} f(x_m) = \max_{\mathcal{F}} f \quad \text{and} \quad \Delta_{S^n} f(x_m) \geq 0,$$

which contradicts (1.2). This proves (3.11).

It follows from (3.11) that

$$\frac{1}{f(S)^{n-1/n}} + \epsilon < \frac{1}{f(x_c)^{n-1/n}} \quad \text{for } x_c \in \mathcal{F} \quad \text{with } \Delta_{S^n} f(x_c) \leq 0, \tag{3.12}$$

where $\epsilon > 0$ is a small positive number. Let u_0 be the positive smooth function constructed in lemma 3.1. We claim that, with this choice of initial data, lemma 2.1(i) occurs. Suppose not, lemma 2.1(ii) occurs. It follows from lemma 2.3 that $L = 1$. Let Q be the blow-up point.

We are going to show that $Q \in \mathcal{F}$. Suppose $Q \notin \mathcal{F}$. Then there exists an isometry γ described in assumptions 1.1 or 1.2 such that

$$\gamma(Q) \neq Q,$$

which implies that

$$\max_{\partial' B_r(\gamma(Q))} u_k \leq C \quad \text{for all } k \gg 0 \tag{3.13}$$

whenever r is small enough, since $\{u_k\}$ is uniformly bounded on any compact subsets of $S^n \setminus \{Q\}$ by lemma 2.2. But lemma 2.1 implies that

$$u_k(\gamma(x)) = u(\gamma(x), t_k) = u(x, t_k) = u_k(x) \quad \text{for all } x \in S^n.$$

This together with (3.13) implies that $\max_{\partial' B_r(Q)} u_k$ is uniformly bounded, which contradicts lemma 2.4(a). This proves that $Q \in \mathcal{F}$.

Hence, by lemma 2.4(c), $Q \in \mathcal{F}$ and $\Delta_{S^n} f(Q) \leq 0$. This together with (3.12) implies that

$$\frac{1}{f(S)^{n-1/n}} + \epsilon < \frac{1}{f(Q)^{n-1/n}}.$$

Combining this with lemma 2.4(d) and lemma 3.1, we obtain

$$E_f[u_0] \leq \frac{1}{f(S)^{n-1/n}} + \epsilon < \frac{1}{f(Q)^{n-1/n}} = \lim_{k \rightarrow \infty} E_f[u_k],$$

which contradicts (2.7).

Therefore, lemma 2.1(i) occurs. By lemma 4.2(i) in [45], u_k converges to u_∞ along the flow (2.1)–(2.3) such that

$$\begin{aligned} \Delta_{g_0} u_\infty &= 0 \quad \text{in } B^{n+1}, \\ \frac{2}{n-1} \frac{\partial u_\infty}{\partial \nu_{g_0}} + u_\infty &= \alpha_\infty f u_\infty^{((n+1)/(n-1))} \quad \text{on } S^n \end{aligned}$$

for some $\alpha_\infty > 0$. This proves theorem 1.2. \square

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