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ELEMENTARY THEORY OF VALUED FIELDS WITH A VALUATION-PRESERVING AUTOMORPHISM

SALIH AZGIN[1] AND LOU VAN DEN DRIES[2]

[1] *Department of Mathematics and Statistics, McMaster University, 1280 Main Street, West Hamilton, Ontario L8S 4K1, Canada* (sazgin@math.mcmaster.ca)

[2] *Department of Mathematics, University of Illinois at Urbana-Champaign, 1409 W. Green Street, Urbana, IL 61801, USA* (vddries@math.uiuc.edu)

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Abstract We consider valued fields with a value-preserving automorphism and improve on model-theoretic results by Bélair, Macintyre and Scanlon on these objects by dropping assumptions on the residue difference field. In the equicharacteristic 0 case we describe the induced structure on the value group and the residue difference field.

Keywords: valued difference fields; model theory; elementary classification

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1. Introduction

Our goal in this paper is to contribute to the model theory of valued fields with a valuation-preserving automorphism. The key ideas are due to Scanlon [**9**, **10**] and to Bélair *et al*. [**3**]. We obtain the main result in [**3**] under weaker assumptions, and give a simpler proof.

Throughout we consider valued fields as three-sorted structures

$$\mathcal{K} = (K, \Gamma, \boldsymbol{k}; v, \pi),$$

where K is the underlying field, Γ is an ordered abelian group* (the *value group*), $\boldsymbol{k}$ is a field, the surjective map $v : K^\times \to \Gamma$ is the valuation, with valuation ring

$$\mathcal{O} = \mathcal{O}_v := \{a \in K : v(a) \geqslant 0\}$$

and maximal ideal $\mathfrak{m}_v := \{a \in K : v(a) > 0\}$ of $\mathcal{O}$, and $\pi : \mathcal{O} \to \boldsymbol{k}$ is a surjective ring morphism. Note that then π induces an isomorphism of fields,

$$a + \mathfrak{m} \mapsto \pi(a) : \mathcal{O}/\mathfrak{m} \to \boldsymbol{k} \quad (\mathfrak{m} := \mathfrak{m}_v),$$

* The ordering of an ordered abelian group is total, by convention.

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and there will be no harm in identifying the residue field $\mathcal{O}/\mathfrak{m}$ with $\boldsymbol{k}$ via this isomorphism. Accordingly, we refer to $\boldsymbol{k}$ as the *residue field*. To simplify notation we often write $\bar{a}$ instead of $\pi(a)$. We call the above $\mathcal{K}$ *unramified* if either

(i) $\operatorname{char} K = \operatorname{char} \boldsymbol{k} = 0$, or

(ii) $\operatorname{char} K = 0$, $\operatorname{char} \boldsymbol{k} = p > 0$ and $v(p)$ is the smallest positive element of Γ.

Ax and Kochen and, separately, Ershov proved the following classical result, which we shall refer to as the *AKE-principle*. (See Kochen [**8**] for a complete account.)

Let $\mathcal{K}$ and $\mathcal{K}'$ be unramified henselian valued fields with residue fields $\boldsymbol{k}$ and $\boldsymbol{k}'$ and value groups Γ and Γ' respectively. Then $\mathcal{K} \equiv \mathcal{K}'$ if and only if $\boldsymbol{k} \equiv \boldsymbol{k}'$, as fields, and $\Gamma \equiv \Gamma'$, as ordered abelian groups.

Thus the elementary theory of an unramified henselian valued field is determined by the elementary theories of its residue field and value group. Theorems 6.6 and 8.8 below are strong analogues of the AKE-principle in the presence of a valuation-preserving automorphism.

Compared to the standard way of proving the AKE-theorems as in Kochen [**8**], the two new tools we need are replacing a pseudo-Cauchy sequence by an equivalent one, in order to rescue pseudo-continuity, and, for positive residue characteristic, the use of a certain polynomial transformation, the D-transform. These two devices were introduced in [**3**], but we use them in combination with a simpler notion of *pc-sequence of σ-algebraic type*. The main improvement comes from a better understanding of purely residual extensions via Lemma 2 below. This allows us to drop a strong assumption, the *genericity axiom* of [**3**], about the automorphism induced on the residue field. Other differences with [**3**] will be indicated at various places. We assume familiarity with valuation theory, including henselization and pseudo-convergence.

This paper is a condensed version of [**2**] and is partly based on a chapter of the first author's thesis [**1**] with advice from the second author.

2. Preliminaries

Throughout, $\mathbb{N} = \{0, 1, 2, \dots\}$, and m, n range over $\mathbb{N}$. We let $K^\times = K \setminus \{0\}$ be the multiplicative group of a field K.

Difference fields

A *difference field* is a field equipped with a distinguished automorphism of the field, the *difference operator*. A difference field is considered in the obvious way as a structure for the language $\{0, 1, -, +, \cdot, \sigma\}$ of difference rings, with the unary function symbol σ to be interpreted in a difference field as its difference operator, which is accordingly also denoted by σ (unless specified otherwise). Let K be a difference field. The *fixed field* of K is its subfield

$$\operatorname{Fix}(K) := \{a \in K : \sigma(a) = a\}.$$

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We let σ^n denote the nth iterate of σ and let σ^{-n} denote the nth iterate of σ^{-1}. Let $K \subseteq K'$ be an extension of difference fields and $a \in K'$. We define $K\langle a\rangle$ to be the smallest difference subfield of K' containing K and a. The underlying field of $K\langle a\rangle$ is $K(\sigma^i(a) : i \in \mathbb{Z})$.

We now introduce difference polynomials in one variable over K. Each polynomial

$$f(x_0, \dots, x_n) \in K[x_0, \dots, x_n]$$

gives rise to a difference polynomial $F(x) = f(x, \sigma(x), \dots, \sigma^n(x))$ in the variable x over K; we put $\deg F := \deg f \in \mathbb{N} \cup \{-\infty\}$ (where $\deg f$ is the total degree of f), and refer to F as a *σ-polynomial* (over K). If F is not constant (that is, $F \notin K$), let $f(x_0, \dots, x_n)$ be as above with least possible n (which determines f uniquely), and put

$$\operatorname{order}(F) := n, \qquad \operatorname{complexity}(F) := (n, \deg_{x_n} f, \deg f) \in \mathbb{N}^3.$$

If $F \in K$, $F \neq 0$, then $\operatorname{order}(F) := -\infty$ and $\operatorname{complexity}(F) := (-\infty, 0, 0)$. Finally, $\operatorname{order}(0) := -\infty$ and $\operatorname{complexity}(0) := (-\infty, -\infty, -\infty)$. So in all cases we have $\operatorname{complexity}(F) \in (\mathbb{N} \cup \{-\infty\})^3$, and we order complexities lexicographically.

Let a be an element of a difference field extension of K. We say that a is *σ-transcendental* over K if there is no non-zero F as above with $F(a) = 0$, and otherwise a is said to be *σ-algebraic* over K. As an example, let $F(x) := \sigma(x) - x$. It has order 1, and $F(a) = 0$ for all a in the prime subfield of K, in particular, $F(a) = 0$ for infinitely many $a \in K$ if K has characteristic 0. If b is also an element in a difference field extension of K and a and b are σ-transcendental over K, then there is a unique difference field isomorphism $K\langle a\rangle \to K\langle b\rangle$ over K sending a to b.

A *minimal σ-polynomial of a over K* is a non-zero σ-polynomial $F(x)$ over K such that $F(a) = 0$ and $G(a) \neq 0$ for all non-zero σ-polynomials $G(x)$ over K of lower complexity than $F(x)$. So a has a minimal σ-polynomial over K if and only if a is σ-algebraic over K. *Suppose b is also an element in some difference field extension of K, and a and b have a common minimal σ-polynomial $F(x)$ over K. Is there a difference field isomorphism $K\langle a\rangle \to K\langle b\rangle$ over K sending a to b?* The answer is not always *yes*, but it is *yes* if F is of degree 1 in $\sigma^m(x)$ with F of order m. Another case in which the answer is *yes* is treated in Lemma 2 below.

A difference field extension L of K is said to be *σ-algebraic* over K if each $c \in L$ is σ-algebraic over K. For example, if a is σ-algebraic over K, then $K\langle a\rangle$ is σ-algebraic over K.

Let $x_0, \dots, x_n, y_0, \dots, y_n$ be distinct indeterminates, and put $\boldsymbol{x} = (x_0, \dots, x_n)$, $\boldsymbol{y} = (y_0, \dots, y_n)$. For a polynomial $f(\boldsymbol{x})$ over a field K we have a unique Taylor expansion in $K[\boldsymbol{x}, \boldsymbol{y}]$:

$$f(\boldsymbol{x} + \boldsymbol{y}) = \sum_{\boldsymbol{i}} f_{(\boldsymbol{i})}(\boldsymbol{x}) \cdot \boldsymbol{y}^{\boldsymbol{i}},$$

where the sum is over all $\boldsymbol{i} = (i_0, \dots, i_n) \in \mathbb{N}^{n+1}$, each $f_{(\boldsymbol{i})}(\boldsymbol{x}) \in K[\boldsymbol{x}]$, with $f_{(\boldsymbol{i})} = 0$ for $|\boldsymbol{i}| := i_0 + \dots + i_n > \deg F$, and $\boldsymbol{y}^{\boldsymbol{i}} := y_0^{i_0} \cdots y_n^{i_n}$. (Also, for a tuple $a = (a_0, \dots, a_n)$ with components a_i in any field we put $a^{\boldsymbol{i}} := a_0^{i_0} \cdots a_n^{i_n}$.) Thus $\boldsymbol{i}! f_{(\boldsymbol{i})}(\boldsymbol{x}) = \partial_{\boldsymbol{i}} f$ where $\partial_{\boldsymbol{i}}$ is

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the operator $(\partial/\partial x_0)^{i_0}\cdots(\partial/\partial x_n)^{i_n}$ on $K[\boldsymbol{x}]$, and $\boldsymbol{i}! = i_0!\cdots i_n!$. We construe $\mathbb{N}^{n+1}$ as a monoid under $+$ (componentwise addition), and let $\leqslant$ be the (partial) product ordering on $\mathbb{N}^{n+1}$ induced by the natural order on $\mathbb{N}$. Define $\binom{\boldsymbol{i}}{\boldsymbol{j}}$ as $\binom{i_0}{j_0}\cdots\binom{i_n}{j_n}\in\mathbb{N}$, when $\boldsymbol{j}\leqslant\boldsymbol{i}$ in $\mathbb{N}^{n+1}$. We then have the following lemma.

Lemma 2.1. *For $\boldsymbol{i},\boldsymbol{j}\in\mathbb{N}^{n+1}$ we have*

$$(f_{(\boldsymbol{i})})_{(\boldsymbol{j})} = \binom{\boldsymbol{i}+\boldsymbol{j}}{\boldsymbol{i}} f_{(\boldsymbol{i}+\boldsymbol{j})}.$$

In particular, $f_{(\boldsymbol{i})} = f$ for $|\boldsymbol{i}| = 0$, and if $|\boldsymbol{i}| = 1$ with $i_k = 1$, then

$$f_{(\boldsymbol{i})} = \frac{\partial f}{\partial x_k}.$$

Also, $\deg f_{(\boldsymbol{i})} < \deg f$ if $|\boldsymbol{i}|\geqslant 1$ and $f\neq 0$.

Let K be a difference field, and x an indeterminate. When n is clear from context we set $\boldsymbol{\sigma}(x) = (x,\sigma(x),\dots,\sigma^n(x))$, and also $\boldsymbol{\sigma}(a) = (a,\sigma(a),\dots,\sigma^n(a))$ for $a\in K$. Then for $f\in K[x_0,\dots,x_n]$ as above and $F(x) = f(\boldsymbol{\sigma}(x))$ we have the following identity in the ring of difference polynomials in the distinct indeterminates x and y over K:

$$\begin{aligned} F(x+y) &= f(\boldsymbol{\sigma}(x+y)) \\ &= f(\boldsymbol{\sigma}(x)+\boldsymbol{\sigma}(y)) \\ &= \sum_{\boldsymbol{i}} f_{(\boldsymbol{i})}(\boldsymbol{\sigma}(x))\cdot\boldsymbol{\sigma}(y)^{\boldsymbol{i}} \\ &= \sum_{\boldsymbol{i}} F_{(\boldsymbol{i})}(x)\cdot\boldsymbol{\sigma}(y)^{\boldsymbol{i}}, \end{aligned}$$

where $F_{(\boldsymbol{i})}(x) := f_{(\boldsymbol{i})}(\boldsymbol{\sigma}(x))$.

Valued fields

We consider valued fields as three-sorted structures

$$\mathcal{K} = (K,\Gamma,\boldsymbol{k};v,\pi)$$

as explained in the introduction. The three sorts are referred to as the *field sort* with variables ranging over K, the *value group sort* with variables ranging over Γ, and the *residue sort* with variables ranging over $\boldsymbol{k}$. We say that $\mathcal{K}$ is of *equal characteristic* 0 if $\mathrm{char}(K) = \mathrm{char}(\boldsymbol{k}) = 0$. If $\mathrm{char}(K) = 0$ and $\mathrm{char}(\boldsymbol{k}) = p > 0$, we say that $\mathcal{K}$ is of *mixed characteristic.*

In dealing with a valued field $\mathcal{K}$ as above we also let v denote the valuation of any valued field extension of $\mathcal{K}$ that gets mentioned, unless we indicate otherwise, and any subfield E of K is construed as a *valued* subfield of $\mathcal{K}$ in the obvious way.

A valued field extension $\mathcal{K}'$ of a valued field $\mathcal{K}$ is said to be *immediate* if the residue field and the value group of $\mathcal{K}'$ are the same as those of $\mathcal{K}$. A valued field is *maximal* if

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it has no proper immediate valued field extension and is *algebraically maximal* if it has no proper immediate algebraic valued field extension.

A key notion in the study of immediate extensions of valued fields is that of pseudo-cauchy sequence. First, a *well-indexed sequence* is a sequence $\{a_\rho\}$ indexed by the elements ρ of some non-empty well-ordered set without largest element; in this connection 'eventually' means 'for all sufficiently large ρ'.

Let $\mathcal{K}$ be a valued field. A *pseudo-Cauchy sequence* (henceforth *pc-sequence*) in $\mathcal{K}$ is a well-indexed sequence $\{a_\rho\}$ in K such that for some index ρ_0,

$$\rho'' > \rho' > \rho \geqslant \rho_0 \implies v(a_{\rho''} - a_{\rho'}) > v(a_{\rho'} - a_\rho).$$

In particular, a pc-sequence in $\mathcal{K}$ cannot be eventually constant. For a well-indexed sequence $\{a_\rho\}$ in $\mathcal{K}$ and a in some valued field extension of $\mathcal{K}$ we say that $\{a_\rho\}$ *pseudo-converges* to a, or a is a *pseudo-limit* of $\{a_\rho\}$ (notation: $a_\rho \rightsquigarrow a$) if the sequence $\{v(a-a_\rho)\}$ is eventually strictly increasing; note that then $\{a_\rho\}$ is a pc-sequence in $\mathcal{K}$.

Let $\{a_\rho\}$ be a pc-sequence in $\mathcal{K}$, pick ρ_0 as above, and put

$$\gamma_\rho := v(a_{\rho'} - a_\rho)$$

for $\rho' > \rho \geqslant \rho_0$; this depends only on ρ as the notation suggests. Then $\{\gamma_\rho\}_{\rho \geqslant \rho_0}$ is strictly increasing. The *width* of $\{a_\rho\}$ is the set

$$\{\gamma \in \Gamma \cup \{\infty\} : \gamma > \gamma_\rho \text{ for all } \rho \geqslant \rho_0\}.$$

Its significance is that if $a, b \in K$ and $a_\rho \rightsquigarrow a$, then $a_\rho \rightsquigarrow b$ if and only if $v(a-b)$ is in the width of $\{a_\rho\}$.

An old and useful observation by Macintyre is that if $\{a_\rho\}$ is a pc-sequence in an expansion of a valued field (for example, in a valued difference field), then $\{a_\rho\}$ has a pseudo-limit in an elementary extension of that expansion.

The following easy lemma will be useful in dealing with pc-sequences.

Lemma 2.2. *Let Γ be an ordered abelian group, let A be a subset of Γ, and let $\{\gamma_\rho\}$ be a well-indexed strictly increasing sequence in A. Let $f_1, \dots, f_n : A \to \Gamma$ be such that for all distinct $i, j \in \{1, \dots, n\}$ the function $f_i - f_j$ is either strictly increasing or strictly decreasing. Then there is a unique enumeration $i_1, \dots, i_n$ of $\{1, \dots, n\}$ such that*

$$f_{i_1}(\gamma_\rho) < \cdots < f_{i_n}(\gamma_\rho), \quad \textit{eventually}.$$

For this enumeration and $\delta \in \Gamma$ such that $\{\gamma \in \Gamma : 0 < \gamma < \delta\}$ is finite, if $1 \leqslant \mu < \nu \leqslant n$, then $f_{i_\nu}(\gamma_\rho) - f_{i_\mu}(\gamma_\rho) > \delta$, eventually.

For linear functions on Γ this was used by Kaplansky [**7**] in his work on immediate extensions of valued fields. The last part of the lemma is needed in dealing with finitely ramified valued fields of mixed characteristic. As in [**3**] we call the valued field $\mathcal{K}$ *finitely ramified* if the following two conditions are satisfied:

(i) K has characteristic 0;

(ii) $\{\gamma \in \Gamma : 0 < \gamma < v(p)\}$ is finite if $\boldsymbol{k}$ has characteristic $p > 0$.

In particular, $\mathcal{K}$ is finitely ramified if $\mathcal{K}$ is unramified as defined in the introduction.

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Let $\mathcal{K} = (K, \Gamma, \boldsymbol{k}; v, \pi)$ be a valued field. A *cross-section* on $\mathcal{K}$ is a group morphism $s : \Gamma \to K^\times$ such that $v(s\gamma) = \gamma$ for all $\gamma \in \Gamma$. The following is well known.

Lemma 2.3. *If $\mathcal{K}$ is $\aleph_1$-saturated, then there is a cross-section on $\mathcal{K}$. In particular, there is a cross-section on some elementary extension of $\mathcal{K}$.*

Proof. With $U(\mathcal{O})$ the multiplicative group of units of $\mathcal{O}$, the inclusion $U(\mathcal{O}) \to K^\times$ and $v : K^\times \to \Gamma$ yield the exact sequence of abelian groups

$$1 \to U(\mathcal{O}) \to K^\times \to \Gamma \to 0.$$

Suppose $\mathcal{K}$ is $\aleph_1$-saturated. Then the group $U(\mathcal{O})$ is $\aleph_1$-saturated, hence pure-injective by [**4**, p. 171]. It is also pure in $K^\times$ since Γ is torsion-free, and thus the above exact sequence splits. □

In the proof of Theorem 6.7 we need the following variant.

Lemma 2.4. *Let $\mathcal{K}$ be $\aleph_1$-saturated, let $\mathcal{E} = (E, \Gamma_E, \dots)$ be an $\aleph_1$-saturated valued subfield of $\mathcal{K}$ such that Γ_E is pure in Γ, and let s_E be a cross-section on $\mathcal{E}$. Then s_E extends to a cross-section on $\mathcal{K}$.*

Proof. By Lemma 2.3 we have a cross-section s on $\mathcal{K}$. Now Γ_E is pure-injective, and pure in Γ, so we have an internal direct sum decomposition $\Gamma = \Gamma_E \oplus \Delta$ with Δ a subgroup of Γ. This gives a cross-section on $\mathcal{K}$ that coincides with s_E on Γ_E and with s on Δ. □

An *angular component map* on $\mathcal{K}$ is a multiplicative group morphism $\mathrm{ac}\colon K^\times \to \boldsymbol{k}^\times$ such that $\mathrm{ac}(a) = \pi(a)$ whenever $v(a) = 0$; we extend it to $\mathrm{ac}\colon K \to \boldsymbol{k}$ by setting $\mathrm{ac}(0) = 0$ and also refer to this extension as an angular component map on $\mathcal{K}$. A cross-section s on $\mathcal{K}$ yields an angular component map ac on $\mathcal{K}$ by setting $\mathrm{ac}(x) = \pi(x/s(v(x)))$ for $x \in K^\times$. Thus Lemma 2.3 goes through with angular component maps instead of cross-sections.

Valued difference fields

A *valued difference field* is a valued field $\mathcal{K}$ as above where K is not just a field, but a difference field whose difference operator σ satisfies $\sigma(\mathcal{O}) = \mathcal{O}$. It follows that σ induces an automorphism of the residue field:

$$\pi(a) \mapsto \pi(\sigma(a)) : \boldsymbol{k} \to \boldsymbol{k}, \quad a \in \mathcal{O}.$$

We denote this automorphism by $\bar{\sigma}$, and $\boldsymbol{k}$ equipped with $\bar{\sigma}$ is called the *residue difference field of* $\mathcal{K}$. (Likewise, σ induces an automorphism of the value group Γ; at a later stage we restrict attention to $\mathcal{K}$ where σ induces the identity on Γ.)

Let $\mathcal{K}$ be a valued difference field as above. The difference operator σ of K is also referred to as the *difference operator of* $\mathcal{K}$. By an *extension* of $\mathcal{K}$ we shall mean a valued difference field $\mathcal{K}' = (K', \dots)$ that extends $\mathcal{K}$ as a valued field and whose difference operator extends the difference operator of $\mathcal{K}'$. In this situation we also say that $\mathcal{K}$ is a *valued difference subfield of* $\mathcal{K}'$, and we indicate this by $\mathcal{K} \leqslant \mathcal{K}'$. Such an extension

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is called *immediate* if it is immediate as an extension of valued fields. In dealing with a valued difference field $\mathcal{K}$ as above v also denotes the valuation of any extension of $\mathcal{K}$ that gets mentioned (unless specified otherwise), and any difference subfield E of K is construed as a valued difference subfield of $\mathcal{K}$ in the obvious way. The residue field of the valued subfield $\mathrm{Fix}(K)$ of $\mathcal{K}$ is clearly a subfield of $\mathrm{Fix}(\boldsymbol{k})$.

Let $\mathcal{K}^{\mathrm{h}} = (K^{\mathrm{h}}, \Gamma, \boldsymbol{k}; \dots)$ be the henselization of the underlying valued field of $\mathcal{K}$. By the universal property of 'henselization' the operator σ extends uniquely to an automorphism σ^{h} of the field K^{h} such that $\mathcal{K}^{\mathrm{h}}$ with σ^{h} is a valued difference field. Accordingly we shall view $\mathcal{K}^{\mathrm{h}}$ as a valued difference field, making it thereby an immediate extension of the valued difference field $\mathcal{K}$.

Given an extension $\mathcal{K} \leqslant \mathcal{K}'$ of valued difference fields and $a \in K'$, we define $\mathcal{K}\langle a\rangle$ to be the smallest valued difference subfield of $\mathcal{K}'$ extending $\mathcal{K}$ and containing a in its underlying difference field; thus the underlying difference field of $\mathcal{K}\langle a\rangle$ is $K\langle a\rangle$.

Two lemmas

Suppose $\mathcal{K} = (K, \Gamma, \boldsymbol{k}; v, \pi)$ and $\mathcal{K}' = (K', \Gamma', \boldsymbol{k}'; v', \pi')$ are valued difference fields, put $\mathcal{O} := \mathcal{O}_v$, $\mathcal{O}' := \mathcal{O}_{v'}$, and let σ denote both the difference operator of $\mathcal{K}$ and of $\mathcal{K}'$. Let $\mathcal{E} = (E, \Gamma_E, \boldsymbol{k}_E; \dots)$ be a valued difference subfield of both $\mathcal{K}$ and $\mathcal{K}'$, that is, $\mathcal{E} \leqslant \mathcal{K}$ and $\mathcal{E} \leqslant \mathcal{K}'$. The next lemma is rather obvious, but Lemma 2 is more subtle. Our later use of it enables us to drop the Genericity Axiom of [**3**], which says that for all $n \geqslant 1$ and $a_0, \dots, a_n, b \in \boldsymbol{k}$ with $a_0 \neq 0$, $a_n \neq 0$ and all non-zero $F \in \boldsymbol{k}[x_0, \dots, x_n]$ there is $x \in \boldsymbol{k}$ such that

$$a_0 x + a_1 \bar{\sigma}(x) + \dots + a_n \bar{\sigma}^n(x) = b, \qquad F(\bar{\boldsymbol{\sigma}}(x)) \neq 0.$$

Lemma 2.5. *Let $a \in \mathcal{O}$ and assume $\alpha = \bar{a}$ is $\bar{\sigma}$-transcendental over $\boldsymbol{k}_E$. Then*

(i) *$v(P(a)) = \min_{\boldsymbol{l}}\{v(b_{\boldsymbol{l}})\}$ for each σ-polynomial $P(x) = \sum b_{\boldsymbol{l}} \boldsymbol{\sigma}^{\boldsymbol{l}}(x)$ over E;*

(ii) *$v(E\langle a\rangle^\times) = v(E^\times) = \Gamma_E$, and $\mathcal{E}\langle a\rangle$ has residue field $\boldsymbol{k}_E\langle\alpha\rangle$;*

(iii) *if $b \in \mathcal{O}'$ is such that $\beta = \bar{b}$ is $\bar{\sigma}$-transcendental over $\boldsymbol{k}_E$, then there is a valued difference field isomorphism $\mathcal{E}\langle a\rangle \to \mathcal{E}\langle b\rangle$ over $\mathcal{E}$ sending a to b.*

Proof. Let $P(x) = \sum b_{\boldsymbol{l}} \boldsymbol{\sigma}^{l}(x)$ be a non-zero σ-polynomial over E. Then $P(x) = cQ(x)$ where $c \in E^\times$ and $v(c) = \min_{\boldsymbol{l}}\{v(b_{\boldsymbol{l}})\}$, and $Q(x)$ is a σ-polynomial over the valuation ring of E with some coefficient equal to 1. Since $\alpha = \bar{a}$ is $\bar{\sigma}$-transcendental over $\boldsymbol{k}_E$, $\bar{Q}(\bar{a}) \neq 0$. Therefore, $v(Q(a)) = 0$, and thus

$$v(P(a)) = v(c) = \min_{\boldsymbol{l}}\{v(b_{\boldsymbol{l}})\}.$$

It follows that $v(E\langle a\rangle^\times) = v(E^\times)$. A similar argument shows that $E\langle a\rangle$ has residue field $\boldsymbol{k}_E\langle\alpha\rangle$. It also follows from (i) that a is σ-transcendental over E, and (iii) is an easy consequence of this fact and of (i). □

Recall that in the beginning of this section we defined the complexity of a difference polynomial over a difference field.

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Lemma 2.6.* *Assume that* $\mathrm{char}(\boldsymbol{k}) = 0$, *and let* $G(x)$ *be a non-constant* σ*-polynomial over the valuation ring of* E *whose reduction* $\bar{G}(x)$ *has the same complexity as* $G(x)$. *Let* $a \in \mathcal{O}$, $b \in \mathcal{O}'$, *and assume that* $G(a) = 0$, $G(b) = 0$, *and that* $\bar{G}(x)$ *is a minimal* $\bar{\sigma}$*-polynomial of* $\alpha := \bar{a}$ *and of* $\beta := \bar{b}$ *over* $\boldsymbol{k}_E$. *Then*

(i) $\mathcal{E}\langle a\rangle$ *has value group* $v(E^\times) = \Gamma_E$ *and residue field* $\boldsymbol{k}_E\langle\alpha\rangle$;

(ii) *if there is a difference field isomorphism* $\boldsymbol{k}_E\langle\alpha\rangle \to \boldsymbol{k}_E\langle\beta\rangle$ *over* $\boldsymbol{k}_E$ *sending* α *to* β, *then there is a valued difference field isomorphism* $\mathcal{E}\langle a\rangle \to \mathcal{E}\langle b\rangle$ *over* $\mathcal{E}$ *sending* a *to* b.

Proof. To simplify notation we set, for $k \in \mathbb{Z}$,

$$a_k := \sigma^k(a), \qquad \alpha_k := \bar{\sigma}^k(\alpha), \qquad b_k := \sigma^k(b), \qquad \beta_k := \bar{\sigma}^k(\beta).$$

As in the proof of Lemma 2.5 one shows that if $P(x) = \sum b_{\boldsymbol{l}}\boldsymbol{\sigma}(x)^{\boldsymbol{l}}$ is a σ-polynomial over E of lower complexity than $G(x)$, then $v(P(a)) = \min_{\boldsymbol{l}}\{v(b_{\boldsymbol{l}})\}$. It is also clear that G is a minimal σ-polynomial of a over E. Let G have order m and degree $d > 0$ with respect to $\sigma^m(x)$, so

$$G(x) = P_0(x) + P_1(x)\sigma^m(x) + \cdots + P_d(x)\sigma^m(x)^d,$$

where $P_0, \ldots, P_d$ are σ-polynomials over the valuation ring of E of order less than m, with $P_d \neq 0$. Then the valued subfield $E_{m-1} := E(a_0, \ldots, a_{m-1})$ of $\mathcal{K}$ has transcendence basis $a_0, \ldots, a_{m-1}$ over E, the residue field of E_{m-1} is $\boldsymbol{k}_E(\alpha_0, \ldots, \alpha_{m-1})$ with transcendence basis $\alpha_0, \ldots, \alpha_{m-1}$ over $\boldsymbol{k}_E$, and the value group of E_{m-1} is Γ_E. Also, $v(P_d(a)) = 0$ and $v(P_i(a)) \geqslant 0$ for $i = 0, \ldots, d-1$, and

$$g(T) := T^d + p_{d-1}T^{d-1} + \cdots + p_0, \quad \text{with } p_i := P_i(a)/P_d(a) \text{ for } i = 0, \ldots, d-1,$$

is the minimum polynomial of a_m over E_{m-1} and has its coefficients in the valuation ring of E_{m-1}, and the reduction $\bar{g}(T)$ of $g(T)$ is the minimum polynomial of α_m over $\boldsymbol{k}_E(\alpha_0, \ldots, \alpha_{m-1})$. For the rest of the proof we assume without loss that $\mathcal{K}$ and $\mathcal{K}'$ are henselian as valued fields.

For $n \geqslant m$ we set

$$E_n := E(a_0, \ldots, a_n), \quad \text{a valued subfield of } \mathcal{K},$$

we let E_n^{h} be the henselization of E_n in $\mathcal{K}$, and let E_{m-1}^{h} be the henselization of E_{m-1} in $\mathcal{K}$. For $n \geqslant m$, let $g_n(T)$ be the minimum polynomial of a_n over E_{n-1}^{h}.

Claim 1. For $n \geqslant m$ the polynomial g_n has its coefficients in the valuation ring of E_{n-1}^{h}, the residue field of E_n is $\boldsymbol{k}_E(\alpha_0, \ldots, \alpha_n)$, the value group of E_n is Γ_E, the reduction $\bar{g}_n$ of g_n is the minimum polynomial of α_n over $\boldsymbol{k}_E(\alpha_0, \ldots, \alpha_{n-1})$.

* We thank Martin Hils for pointing out a serious error in the proof of a related lemma in [**1**].

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Claim 1 holds for $n = m$: $\bar{g}(T)$ is the minimum polynomial of α_m over the residue field $\boldsymbol{k}_E(\alpha_0, \ldots, \alpha_{m-1})$ of E_{m-1}, and so the monic polynomial $g(T)$ is necessarily the minimum polynomial of a_m over E^{h}_{m-1}, that is, $g_m = g$. Assume inductively that the claim holds for a certain $n \geqslant m$. By applying σ to the coefficients of $g_n(T)$ we obtain a monic polynomial $g^{\sigma}_n(T)$ over the valuation ring of E^{h}_n with a_{n+1} as a zero. Thus $g_{n+1}(T)$ is a monic irreducible factor of $g^{\sigma}_n(T)$ in $E^{\mathrm{h}}_n[T]$, and has therefore coefficients in the valuation ring of E^{h}_n. Its reduction $\bar{g}_{n+1}$ divides the reduction of g^{σ}_n (in the polynomial ring $\boldsymbol{k}_E(\alpha_0, \ldots, \alpha_n)[T]$) and so α_{n+1}, being a simple zero of this last reduction, is a simple zero of $\bar{g}_{n+1}$. It only remains to show that $\bar{g}_{n+1}$ is irreducible in $\boldsymbol{k}_E(\alpha_0, \ldots, \alpha_n)[T]$. Suppose it is not. Then $\bar{g}_{n+1}(T) = \phi(T)\psi(T)$, where $\phi, \psi \in \boldsymbol{k}_E(\alpha_0, \ldots, \alpha_n)[T]$ are monic of degree at least 1, with ϕ irreducible in this polynomial ring and $\phi(\alpha_{n+1}) = 0$. Hence ϕ and ψ are coprime. Then the factorization $\bar{g}_{n+1} = \phi\psi$ can be lifted to a non-trivial factorization of g_{n+1} in $E^{\mathrm{h}}_n[T]$, a contradiction. Claim 1 is established.

It follows that $E(a_k : k \in \mathbb{N})$ has residue field $\boldsymbol{k}_E(\alpha_k : k \in \mathbb{N})$ and value group Γ_E. Applying the valued field automorphism σ^{-n} yields that the valued subfield $E(a_{k-n} : k \in \mathbb{N})$ of $\mathcal{K}$ has residue field $\boldsymbol{k}_E(\alpha_{k-n} : k \in \mathbb{N})$ and value group Γ_E. Hence $E\langle a \rangle$ has residue field $\boldsymbol{k}_E\langle \alpha \rangle$ and value group Γ_E. We have proved (i).

To prove (ii), let $\iota : \boldsymbol{k}_E\langle \alpha \rangle \to \boldsymbol{k}_E\langle \beta \rangle$ be a difference field isomorphism over $\boldsymbol{k}_E$ sending α to β. Let $F_{m-1} := E(b_0, \ldots, b_{m-1})$, a valued subfield of $\mathcal{K}'$. Then

$$h(T) := T^d + q_{d-1}T^{d-1} + \cdots + q_0, \quad \text{with } q_i := P_i(b)/P_d(b) \text{ for } i = 0, \ldots, d-1,$$

is the minimum polynomial of b_m over F_{m-1} and has its coefficients in the valuation ring of F_{m-1}, and the reduction $\bar{h}(T)$ of $h(T)$ is the minimum polynomial of β_m over $\boldsymbol{k}_E(\beta_0, \ldots, \beta_{m-1})$. Now $\bar{g}$ and $\bar{h}$ correspond under ι. For $n \geqslant m$, let

$$F_n := E(b_0, \ldots, b_n), \quad \text{a valued subfield of } \mathcal{K}',$$

and let F^{h}_n be the henselization of F_n in $\mathcal{K}'$. For $n \geqslant m$, let $h_n(T)$ be the minimum polynomial of b_n over F^{h}_{n-1}, so h_n has its coefficients in the valuation ring of F^{h}_{n-1}, the residue field of F_n is $\boldsymbol{k}_E(\beta_0, \ldots, \beta_n)$, the value group of F_n is Γ_E, the reduction $\bar{h}_n$ of h_n is the minimum polynomial of β_n over $\boldsymbol{k}_E(\beta_0, \ldots, \beta_{n-1})$. It follows that $\bar{g}_n$ and $\bar{h}_n$ correspond under ι, for each $n \geqslant m$.

Claim 2. For $n \geqslant m$ there is a (unique) valued field isomorphism $i_n : E_n \to F_n$ over E sending a_k to b_k for $k = 0, \ldots, n$.

From the remarks at the beginning of the proof it is clear that we have a unique valued field isomorphism $i_{m-1} : E_{m-1} \to F_{m-1}$ over E sending a_k to b_k for $k = 0, \ldots, m-1$. It follows that the minimum polynomials g and h correspond under i_{m-1}, and so we have a field isomorphism $E_m \to F_m$ extending i_{m-1} and sending a_m to b_m. This is a *valued* field isomorphism since the residue field $\boldsymbol{k}_E(\alpha_0, \ldots, \alpha_m)$ of E_m has the same degree over the residue field $\boldsymbol{k}_E(\alpha_0, \ldots, \alpha_{m-1})$ of E_{m-1} as E_m has over E_{m-1}, and likewise with F_m and F_{m-1}. This proves Claim 2 for $n = m$. Assume the claim holds for a certain $n \geqslant m$. Then g_n and h_n correspond under i_{n-1}, and so g^{σ}_n and h^{σ}_n correspond under i_n. From the unique lifting properties of henselian local rings it follows that g_{n+1} is the unique monic

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polynomial in $E_n^{\mathrm{h}}[T]$ that divides g_n^{σ}, has its coefficients in the valuation ring of E_n^{h}, and whose reduction is $\bar{g}_{n+1}$; likewise with h_{n+1}. Therefore, g_{n+1} and h_{n+1} correspond under i_n, and so we have a field isomorphism $E_n^{\mathrm{h}}(a_{n+1}) \to F_n^{\mathrm{h}}(b_{n+1})$ that extends i_n and sends a_{n+1} to b_{n+1}. Arguing as in the case $n = m$ we see that this field isomorphism is a valued field isomorphism; its restriction to E_{n+1} is the desired i_{n+1}. This proves Claim 2, and then it is easy to finish the proof of (ii). □

Lemma 2 and its proof go through if we replace the assumption $\operatorname{char} \boldsymbol{k} = 0$ by its consequence that α_m is a simple zero of its minimum polynomial over $\boldsymbol{k}_E(\alpha_0, \dots, \alpha_{m-1})$, where m is the order of G as in the proof. (Just add to Claim 1 in the proof that α_n is a simple zero of $\bar{g}_n$, for all $n \geqslant m$.)

Hahn difference fields and Witt difference fields

Let $\boldsymbol{k}$ be a field and Γ an ordered abelian group. This gives the Hahn field $\boldsymbol{k}((t^{\Gamma}))$ whose elements are the formal sums $a = \sum_{\gamma \in \Gamma} a_\gamma t^\gamma$ with $a_\gamma \in \boldsymbol{k}$ for all γ, with well-ordered *support* $\{\gamma : a_\gamma \neq 0\} \subseteq \Gamma$. With a as above, we define the valuation $v : \boldsymbol{k}((t^{\Gamma}))^{\times} \to \Gamma$ by $v(a) := \min\{\gamma : a_\gamma \neq 0\}$, and the surjective ring morphism $\pi : \mathcal{O}_v \to \boldsymbol{k}$ by $\pi(a) := a_0$. In this way we obtain the (maximal) valued field $\mathcal{K} = (\boldsymbol{k}((t^{\Gamma})), \Gamma, \boldsymbol{k}; v, \pi)$ to which we also just refer to as the *Hahn field* $\boldsymbol{k}((t^{\Gamma}))$.

Let the field $\boldsymbol{k}$ also be equipped with an automorphism $\bar{\sigma}$. Then

$$\sum_\gamma a_\gamma t^\gamma \mapsto \sum_\gamma \bar{\sigma}(a_\gamma) t^\gamma$$

is an automorphism, to be denoted by σ, of the field $\boldsymbol{k}((t^{\Gamma}))$, with $\sigma(\mathcal{O}_v) = \mathcal{O}_v$. We consider the three-sorted structure $(\boldsymbol{k}((t^{\Gamma})), \Gamma, \boldsymbol{k}; v, \pi)$, with the field $\boldsymbol{k}((t^{\Gamma}))$ equipped with the automorphism σ as above, as a valued difference field, and also refer to it as the *Hahn difference field* $\boldsymbol{k}((t^{\Gamma}))$. Thus $\operatorname{Fix}(\boldsymbol{k}((t^{\Gamma}))) = \operatorname{Fix}(\boldsymbol{k})((t^{\Gamma}))$.

Now let $\boldsymbol{k}$ be a perfect field of characteristic $p > 0$. Then we have the ring $\mathrm{W}[\boldsymbol{k}]$ of Witt vectors over $\boldsymbol{k}$; it is a complete discrete valuation ring whose elements are the infinite sequences $(a_0, a_1, a_2, \dots)$ with all $a_n \in \boldsymbol{k}$; see, for example, [**11**] for how addition and multiplication are defined. The Frobenius automorphism $x \mapsto x^p$ of $\boldsymbol{k}$ induces the ring automorphism

$$(a_0, a_1, a_2, \dots) \mapsto (a_0^p, a_1^p, a_2^p, \dots)$$

of $\mathrm{W}[\boldsymbol{k}]$. This automorphism of $\mathrm{W}[\boldsymbol{k}]$ extends to a field automorphism, the *Witt Frobenius*, of the fraction field $\mathrm{W}(\boldsymbol{k})$ of $\mathrm{W}[\boldsymbol{k}]$. We consider $\mathrm{W}(\boldsymbol{k})$ as a valued difference field by taking the Witt Frobenius as its difference operator, by taking the valuation v to be the unique one with valuation ring $\mathrm{W}[\boldsymbol{k}]$, value group $\mathbb{Z}$ and $v(p) = 1$, and by letting $\pi : \mathrm{W}[\boldsymbol{k}] \to \boldsymbol{k}$ be the canonical map

$$(a_0, a_1, a_2, \dots) \mapsto a_0.$$

We refer to this valued difference field as the *Witt difference field* $\mathrm{W}(\boldsymbol{k})$. For any perfect subfield $\boldsymbol{k}'$ of $\boldsymbol{k}$ we consider $\mathrm{W}(\boldsymbol{k}')$ as a valued difference subfield of $\mathrm{W}(\boldsymbol{k})$ in the obvious way. In particular, with $\mathbb{F}_p$ the prime field of $\boldsymbol{k}$, we have $\operatorname{Fix}(\mathrm{W}(\boldsymbol{k})) = \mathrm{W}(\mathbb{F}_p)$, and the

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latter is identified with the valued field $\mathbb{Q}_p$ of p-adic numbers in the usual way. In the last section the functorial nature of W plays a role: any field embedding $\iota : \boldsymbol{k} \to \boldsymbol{k}'$ into a perfect field $\boldsymbol{k}'$ induces the ring embedding

$$\mathrm{W}[\iota]\colon\ \mathrm{W}[\boldsymbol{k}] \to \mathrm{W}[\boldsymbol{k}'], \quad (a_0, a_1, a_2, \dots) \mapsto (\iota a_0, \iota a_1, \iota a_2, \dots).$$

Three axioms

Let $\mathcal{K}$ be a valued difference field, and consider the following three conditions on $\mathcal{K}$. The first one says that σ preserves the valuation v.

Axiom 1. For all $a \in K^{\times}$, $v(\sigma(a)) = v(a)$.

Axiom 2. For all $\gamma \in \Gamma$ there is $a \in \mathrm{Fix}(K)$ such that $v(a) = \gamma$.

Axiom 3. For each integer $d > 0$ there is $y \in \boldsymbol{k}$ with $\bar{\sigma}^d(y) \neq y$. If $\mathrm{char}(\boldsymbol{k}) = p > 0$, then for any integers d, e with $d \neq 0$ and $e > 0$ there is $y \in \boldsymbol{k}$ with $\bar{\sigma}^d(y) \neq y^{p^e}$.

It is easy to see that Axiom 2 implies Axiom 1. If Γ is an ordered abelian group and $\boldsymbol{k}$ a difference field, then the Hahn difference field $\boldsymbol{k}(\!(t^{\Gamma})\!)$ satisfies Axiom 2. If $\boldsymbol{k}$ is a perfect field of characteristic $p > 0$, then the Witt difference field $\mathrm{W}(\boldsymbol{k})$ satisfies Axiom 2. If $\mathcal{K}$ satisfies Axiom 1, so does any valued difference subfield of $\mathcal{K}$, and any extension of $\mathcal{K}$ with the same value group. If $\mathcal{K}$ satisfies Axiom 2, so does any extension with the same value group.

Note that Axiom 3 is actually an axiom scheme. By [**5**, p. 201] Axiom 3 implies that there are no residual σ-identities at all, that is, for every non-zero polynomial $f \in \boldsymbol{k}[x_0, \dots, x_n]$, there is a y in $\boldsymbol{k}$ with $f(\bar{\boldsymbol{\sigma}}(y)) \neq 0$ (and thus the set $\{y \in \boldsymbol{k} : f(\bar{\boldsymbol{\sigma}}(y)) \neq 0\}$ is infinite).

From now on we assume that all our valued difference fields satisfy Axiom 1. By this convention, whenever we refer to an extension of a valued difference field, this extension is also assumed to satisfy Axiom 1.

3. Pseudo-convergence and $\boldsymbol{\sigma}$-polynomials

If $\{a_\rho\}$ is a pc-sequence in a valued field K and $a_\rho \rightsquigarrow a$ with $a \in K$, then for an ordinary non-constant polynomial $f(x) \in K[x]$ we have $f(a_\rho) \rightsquigarrow f(a)$ (see [**7**]). This fails in general for non-constant σ-polynomials over valued difference fields. We do, however, have a variant of this pseudo-continuity for σ-polynomials using *equivalent pc-sequences*, a key device from [**3**]. This section presents the relevant definitions and facts but omits most proofs. Some of these facts vary slightly from corresponding results in [**3**]. Readers can either adapt proofs in [**3**] or consult [**2**]. Theorems 3.3, 3.5, 3.9 and 3.10 below correspond to Theorems 5.6, 5.8 and 5.9 in [**3**].

Definition 3.1. Two pc-sequences $\{a_\rho\}$, $\{b_\rho\}$ in a valued field are equivalent if for all a in all valued field extensions, $a_\rho \rightsquigarrow a \Leftrightarrow b_\rho \rightsquigarrow a$.

This is an equivalence relation on the set of pc-sequences in a given valued field with given index set.

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Lemma 3.2. *Two pc-sequences $\{a_\rho\}$ and $\{b_\rho\}$ in a valued field are equivalent if and only if they have the same width and a common pseudo-limit in some valued field extension.*

Theorem 3.3. *Let $\mathcal{K}$ be a valued difference field satisfying Axioms 2 and 3. If $\{a_\rho\}$ is a pc-sequence in K with $a_\rho \rightsquigarrow a$ in an extension, and Σ is a finite set of σ-polynomials $G(x)$ over K, then there is a pc-sequence $\{b_\rho\}$ from K, equivalent to $\{a_\rho\}$, such that $G(b_\rho) \rightsquigarrow G(a)$ for each non-constant G in Σ.*

Corollary 3.4. *The same with a removed, and one only asks that $\{G(b_\rho)\}$ is a pc-sequence for each non-constant G in Σ.*

Thus σ-polynomials can be made to preserve pseudo-continuity by passing to equivalent pc-sequences. Moreover, we have the following theorem.

Theorem 3.5. *Let $\mathcal{K}$ be a valued difference field satisfying Axioms 2 and 3. Let $\{a_\rho\}$ be a pc-sequence from K and let a in some extension of $\mathcal{K}$ be such that $a_\rho \rightsquigarrow a$. Let $G(x)$ be a σ-polynomial over K such that*

(i) $G(a_\rho) \rightsquigarrow 0$,

(ii) $G_{(\boldsymbol{l})}(b_\rho) \not\rightsquigarrow 0$ *whenever $|\boldsymbol{l}| \geqslant 1$ and $\{b_\rho\}$ is a pc-sequence in K equivalent to $\{a_\rho\}$.*

Let Σ be a finite set of σ-polynomials $H(x)$ over K. Then there is a pc-sequence $\{b_\rho\}$ in K, equivalent to $\{a_\rho\}$, such that $G(b_\rho) \rightsquigarrow 0$, and $H(b_\rho) \rightsquigarrow H(a)$ for every non-constant H in Σ.

The Witt case

Let $\mathcal{K} = (K, \Gamma, \boldsymbol{k}; v, \pi)$ be a valued difference field, satisfying Axiom 1 as usual. Assume that $\mathrm{char}(K) = 0$, $\mathrm{char}(\boldsymbol{k}) = p > 0$, $\boldsymbol{k}$ is perfect, that Γ has a least positive element 1 with $v(p) = 1$, and, finally, that $\bar{\sigma}(y) = y^p$ on $\boldsymbol{k}$. We call this the *Witt case* (for p). These assumptions are satisfied by the Witt difference field $\mathrm{W}(\boldsymbol{k})$.

Axiom 3 fails in the Witt case but we indicate below how to obtain analogues of Theorems 3.3 and 3.5. As in [**3**] we use the formalism of ∂-rings from [**6**].

∂-rings

Let $\partial_0 : \mathcal{O} \to \mathcal{O}$ be the identity map, and define

$$\partial_1 : \mathcal{O} \to \mathcal{O}, \qquad \partial_1(x) := \frac{\sigma(x) - x^p}{p}.$$

Usually ∂_1 is written as ∂; it satisfies the axioms for a p-derivation on $\mathcal{O}$, namely

$$\partial(1) = 0,$$
$$\partial(x+y) = \partial(x) + \partial(y) - \sum_{i=1}^{p-1} a(p,i) x^i y^{p-i}, \quad a(p,i) := \binom{p}{i} \Big/ p,$$
$$\partial(xy) = x^p \partial(y) + y^p \partial(x) + p\partial(x)\partial(y).$$

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A ∂-ring is a commutative ring with 1 equipped with a unary operation ∂ satisfying the above identities. For the basic facts on ∂-rings used below, see [**6**]. Because $\mathcal{O}$ is a ∂-ring, there is a unique sequence of unary operations $\partial_0, \partial_1, \partial_2, \dots : \mathcal{O} \to \mathcal{O}$ with ∂_0, ∂_1 as above such that for all $a \in \mathcal{O}$ and all n,

$$\sigma^n(a) = W_n(\partial_0(a), \dots, \partial_n(a)),$$
$$W_n(x_0, \dots, x_n) := x_0^{p^n} + px_1^{p^{n-1}} + \dots + p^n x_n \in \mathbb{Z}[x_0, \dots, x_n].$$

Recall that addition of Witt vectors [**11**] is given in terms of polynomials

$$S_n \in \mathbb{Z}[y_0, \dots, y_n, z_0, \dots, z_n]$$

such that

$$W_n(y_0, \dots, y_n) + W_n(z_0, \dots, z_n) = W_n(S_0, \dots, S_n)$$

and, accordingly, $\partial_n(a + b) = S_n(\partial_0(a), \dots, \partial_n(a), \partial_0(b), \dots, \partial_n(b))$ for all $a, b \in \mathcal{O}$.

In $\mathrm{W}[\boldsymbol{k}]$, the $\overline{\partial_n}$ yield the *components* of Witt vectors, namely, each $a \in \mathrm{W}[\boldsymbol{k}]$ equals $(\overline{\partial_0(a)}, \overline{\partial_1(a)}, \overline{\partial_2(a)}, \dots)$. In our Witt case, $\mathcal{O}/p^{n+1}\mathcal{O} \cong \mathrm{W}[\boldsymbol{k}]/(p^{n+1})$.

Lemma 3.6. *Identifying the vectors* $(a_0, \dots, a_n) \in \boldsymbol{k}^{n+1}$ *with the elements of* $\mathrm{W}[\boldsymbol{k}]/(p^{n+1})$ *in the usual way, we have a surjective ring morphism*

$$\mathcal{O} \to \mathrm{W}[\boldsymbol{k}]/(p^{n+1}), \qquad a \mapsto (\overline{\partial_0(a)}, \overline{\partial_1(a)}, \dots, \overline{\partial_n(a)})$$

with kernel $p^{n+1}\mathcal{O}$.

The analogue of Theorem 3.3 for the Witt case is Theorem 3.9 below. A difference with the treatment in [**3**] is that the proof in [**2**] uses the following lemma.

Lemma 3.7. *Let* $g \in \mathcal{O}[y_0, \dots, y_n]$ *be such that its image* $\bar{g} \in \boldsymbol{k}[y_0, \dots, y_n]$ *is non-zero. Then there is a* $g^* \in \mathcal{O}[y_0, \dots, y_n, z_0, \dots, z_n]$ *such that, for all* $a, b \in \mathcal{O}$,

$$g(\partial_0(a + b), \dots, \partial_n(a + b)) = g^*(\partial_0(a), \dots, \partial_n(a), \partial_0(b), \dots, \partial_n(b)),$$

and the image of $g^*(y_0, \dots, y_n, \partial_0(b), \dots, \partial_n(b))$ *in* $\boldsymbol{k}[y_0, \dots, y_n]$ *is non-zero.*

Proof. With the S_n as above, put $g^* := g(S_0, \dots, S_n)$. Then the displayed identity holds. Let $b \in \mathcal{O}$ and put $h := g^*(y_0, \dots, y_n, \partial_0(b), \dots, \partial_n(b)) \in \mathcal{O}[y_0, \dots, y_n]$. In order to show that its image $\bar{h}$ in $\boldsymbol{k}[y_0, \dots, y_n]$ is non-zero, we can assume that $\boldsymbol{k}$ is infinite (passing to a suitable Witt extension of K if necessary). Take $c_0, \dots, c_n \in \boldsymbol{k}$ such that $\bar{g}(c_0, \dots, c_n) \neq 0$. By Lemma 3.6, $(c_0, \dots, c_n) = (\overline{\partial_0(x)}, \overline{\partial_1(x)}, \dots, \overline{\partial_n(x)})$ for a suitable $x \in \mathcal{O}$. Let $a := x - b$. Then by the above,

$$g(\partial_0(x), \dots, \partial_n(x)) = h(\partial_0(a), \dots, \partial_n(a)),$$

with image $\bar{g}(c_0, \dots, c_n) \neq 0$ in $\boldsymbol{k}$. Thus $\bar{h} \neq 0$. □

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The $\boldsymbol{D}$-transform

In analogy with $\boldsymbol{\sigma}$ and $\bar{\boldsymbol{\sigma}}$, we sometimes write $\boldsymbol{\partial}(a)$ for $(\partial_0(a),\dots,\partial_n(a))$, and $\bar{\boldsymbol{\partial}}(a)$ for $(\overline{\partial_0(a)},\dots,\overline{\partial_n(a)})$ for a in the valuation ring of a Witt extension. Thus $\boldsymbol{\sigma}(a)=D(\boldsymbol{\partial}(a))$ for all such a, where

$$D(y_0,\dots,y_n)=(y_0,y_0^p+py_1,\dots,y_0^{p^n}+py_1^{p^{n-1}}+\dots+p^ny_n).$$

Let $F\in K[x_0,\dots,x_n]$ be homogeneous of degree $m>0$, and consider its D-transform $F(D(y_0,\dots,y_n))\in K[y_0,\dots,y_n]$. This D-transform is not in general homogeneous, but its constant term is zero and it has total degree at most mp^n. Write

$$F(x_0,\dots,x_n)=\sum_{|\boldsymbol{l}|=m}a_{\boldsymbol{l}}\boldsymbol{x}^{\boldsymbol{l}} \qquad (\text{all } a_{\boldsymbol{l}}\in K),$$

$$F(D(y_0,\dots,y_n))=\sum_{1\leqslant|\boldsymbol{j}|\leqslant mp^n}b_{\boldsymbol{j}}\boldsymbol{y}^{\boldsymbol{j}} \quad (\text{all } b_{\boldsymbol{j}}\in K).$$

To express how the $b_{\boldsymbol{j}}$ depend on the $a_{\boldsymbol{l}}$ we introduce a tuple $(x_{\boldsymbol{l}})$ of new variables, indexed by the $\boldsymbol{l}$ with $|\boldsymbol{l}|=m$.

Lemma 3.8. *$b_{\boldsymbol{j}}=\Lambda_{\boldsymbol{j},m}((a_{\boldsymbol{l}}))$ where $\Lambda_{\boldsymbol{j},m}\in\mathbb{Z}[(x_{\boldsymbol{l}})]$ is homogeneous of degree 1 and depends only on $\boldsymbol{j}$, m and p, not on K or F.*

Theorem 3.9. *Let $\mathcal{K}$ be a Witt case valued difference field that satisfies Axiom 2 and has infinite residue field. If $\{a_\rho\}$ is a pc-sequence from K, and $a_\rho\rightsquigarrow a$ in a Witt case extension, then the conclusion of Theorem 3.3 holds. Also the corollary to Theorem 3.3 goes through.*

Let $\mathcal{K}$ be a Witt case with $\boldsymbol{k}$ of characteristic p. For a σ-polynomial $G(x)$ over K of order at most n and $a\in K$ we set

$$G(m,x):=(G_{(\boldsymbol{l})}(x))_{|\boldsymbol{l}|=m}, \qquad G(m,a):=(G_{(\boldsymbol{l})}(a))_{|\boldsymbol{l}|=m}.$$

Note that if G is non-constant, then $\Lambda_{\boldsymbol{j},m}(G(m,x))$ has lower complexity than G for $1\leqslant m\leqslant\deg G$, $\boldsymbol{j}\in\mathbb{N}^{n+1}$, $1\leqslant|\boldsymbol{j}|\leqslant mp^n$, where $\Lambda_{\boldsymbol{j},m}$ is as in Lemma 3.8.

Theorem 3.10. *Suppose that $\mathcal{K}$ satisfies Axiom 2, and is a Witt case with infinite $\boldsymbol{k}$ of characteristic p. Let $\{a_\rho\}$ be a pc-sequence from K and $a_\rho\rightsquigarrow a$ with a in a Witt case extension. Let $G(x)$ be a σ-polynomial over K of order at most n so that*

(i) *$G(a_\rho)\rightsquigarrow 0$;*

(ii) *$\Lambda_{\boldsymbol{j},m}(G(m,b_\rho))\not\rightsquigarrow 0$ whenever $1\leqslant m\leqslant\deg G$, $\boldsymbol{j}\in\mathbb{N}^{n+1}$, $1\leqslant|\boldsymbol{j}|\leqslant mp^n$, and $\{b_\rho\}$ is a pc-sequence in K equivalent to $\{a_\rho\}$.*

Let Σ be a finite set of σ-polynomials $H(x)$ over K. Then there is a pc-sequence $\{b_\rho\}$ from K, equivalent to $\{a_\rho\}$, such that $G(b_\rho)\rightsquigarrow 0$, and $H(b_\rho)\rightsquigarrow H(a)$ for each non-constant $H\in\Sigma$.

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4. Newton–Hensel approximation

Let $\mathcal{K} = (K, \Gamma, \boldsymbol{k}; v, \pi)$ be a valued difference field, satisfying Axiom 1 of course. Until Definition 4.4 we fix a σ-polynomial G over $\mathcal{O}$ of order at most n, and let $a \in \mathcal{O}$.

Definition 4.1. G is *σ-henselian at a* if $v(G(a)) > 0$ and $\min_{|\boldsymbol{i}|=1} v(G_{(\boldsymbol{i})}(a)) = 0$.

The coefficients of all $G_{(\boldsymbol{i})}$ are in $\mathcal{O}$. Hence, if G is σ-henselian at a, and $b \in \mathcal{O}$ satisfies $v(a-b) > 0$, then G is also σ-henselian at b. If G is σ-henselian at a and $G(a) \neq 0$, does there exist $b \in \mathcal{O}$ such that $v(a-b) > 0$ and $v(G(b)) > v(G(a))$? To get a positive answer we use an additional assumption on $\boldsymbol{k}$.

Axiom 4_n. Each inhomogeneous linear $\bar{\sigma}$-equation

$$1 + \alpha_0 x + \cdots + \alpha_n \bar{\sigma}^n(x) = 0 \quad (\text{all } \alpha_i \in \boldsymbol{k},\ \text{some } \alpha_i \neq 0),$$

has a solution in $\boldsymbol{k}$. (And we say that $\mathcal{K}$ satisfies Axiom 4_n if $\boldsymbol{k}$ does.)

Lemma 4.2. *Suppose $\mathcal{K}$ satisfies Axiom 4_n and G is σ-henselian at a, with $G(a) \neq 0$. Then there is $b \in \mathcal{O}$ such that $v(a-b) \geqslant v(G(a))$ and $v(G(b)) > v(G(a))$. For any such b we have $v(a-b) = v(G(a))$ and G is σ-henselian at b.*

Proof. Let $b = a + G(a)u$ where $u \in \mathcal{O}$ is to be determined later. Then

$$G(b) = G(a) + \sum_{|\boldsymbol{i}| \geqslant 1} G_{(\boldsymbol{i})}(a) \cdot \boldsymbol{\sigma}(G(a)u)^{\boldsymbol{i}}.$$

Extracting a factor $G(a)$ and using Axiom 1 it follows that

$$G(b) = G(a) \cdot \left(1 + \sum_{|\boldsymbol{i}|=1} c_{\boldsymbol{i}} \boldsymbol{\sigma}(u)^{\boldsymbol{i}} + \sum_{|\boldsymbol{j}|>1} c_{\boldsymbol{j}} \boldsymbol{\sigma}(u)^{\boldsymbol{j}}\right),$$

where $\min_{|\boldsymbol{i}|=1} v(c_{\boldsymbol{i}}) = 0$ and $v(c_{\boldsymbol{j}}) > 0$ for $|\boldsymbol{j}| > 1$. Using Axiom 4_n, we can pick $u \in \mathcal{O}$ such that $\bar{u}$ is a solution of

$$1 + \sum_{|\boldsymbol{i}|=1} \overline{c_{\boldsymbol{i}}} \cdot \bar{\boldsymbol{\sigma}}(x)^{\boldsymbol{i}} = 0.$$

Then $v(b-a) = v(G(a))$, and $v(G(b)) > v(G(a))$. It is clear that any $b \in \mathcal{O}$ with $v(a-b) \geqslant v(G(a))$ and $v(G(b)) > v(G(a))$ is obtained in this way. □

Lemma 4.3. *Suppose $\mathcal{K}$ satisfies Axiom 4_n and $G(x)$ is σ-henselian at a. Suppose also that there is no $b \in K$ with $G(b) = 0$ and $v(a-b) = v(G(a))$. Then there is a pc-sequence $\{a_\rho\}$ in K with the following properties:*

1. *$a_0 = a$ and $\{a_\rho\}$ has no pseudo-limit in K;*
2. *$\{v(G(a_\rho))\}$ is strictly increasing, and thus $G(a_\rho) \rightsquigarrow 0$;*
3. *$v(a_{\rho'} - a_\rho) = v(G(a_\rho))$ whenever $\rho < \rho'$;*
4. *for any extension $\mathcal{K}' = (K', \dots)$ of $\mathcal{K}$ and $b, c \in K'$ such that $a_\rho \rightsquigarrow b$, $G(c) = 0$ and $v(b-c) \geqslant v(G(b))$, we have $a_\rho \rightsquigarrow c$.*

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Proof. Let $\{a_\rho\}_{\rho<\lambda}$ be a sequence in $\mathcal{O}$ with λ an ordinal greater than 0, $a_0 = a$, and

(i) G is σ-henselian at a_ρ, for all $\rho < \lambda$,

(ii) $v(a_{\rho'} - a_\rho) = v(G(a_\rho))$ whenever $\rho < \rho' < \lambda$,

(iii) $v(G(a_{\rho'})) > v(G(a_\rho))$ whenever $\rho < \rho' < \lambda$.

(Note that for $\lambda = 1$ we have such a sequence.) Suppose $\lambda = \mu + 1$ is a successor ordinal. Then Lemma 4.2 yields $a_\lambda \in K$ such that $v(a_\lambda - a_\mu) = v(G(a_\mu))$ and $v(G(a_\lambda)) > v(G(a_\mu))$. Then the extended sequence $\{a_\rho\}_{\rho<\lambda+1}$ has the above properties with $\lambda + 1$ instead of λ.

Suppose λ is a limit ordinal. Then $\{a_\rho\}$ is a pc-sequence and $G(a_\rho) \rightsquigarrow 0$. If $\{a_\rho\}$ has no pseudo-limit in K we are done. Assume otherwise, and take a pseudo-limit $a_\lambda \in K$ of $\{a_\rho\}$. The extended sequence $\{a_\rho\}_{\rho<\lambda+1}$ clearly satisfies the conditions (i) and (ii) with $\lambda + 1$ instead of λ. Since G is over $\mathcal{O}$ we have

$$v(G(a_\lambda) - G(a_\rho)) \geqslant v(a_\lambda - a_\rho) = v(a_{\rho+1} - a_\rho) = v(G(a_\rho))$$

for $\rho < \lambda$. Therefore, $v(G(a_\lambda)) \geqslant v(G(a_\rho))$ for $\rho < \lambda$, and by (iii) this yields $v(G(a_\lambda)) > v(G(a_\rho))$ for $\rho < \lambda$. So the extended sequence also satisfies (iii) with $\lambda + 1$ instead of λ. For cardinality reasons this building process must come to an end and thus yield a pc-sequence $\{a_\rho\}$ satisfying (1)–(3).

Let b, c in an extension of $\mathcal{K}$ be such that $a_\rho \rightsquigarrow b$, $G(c) = 0$ and $v(b - c) \geqslant v(G(b))$. Then G is σ-henselian at b, and for $\rho < \rho'$,

$$\gamma_\rho := v(a_{\rho'} - a_\rho) = v(G(a_\rho)) = v(b - a_\rho),$$

so

$$v(b - c) \geqslant v(G(b)) = v(G(b) - G(a_\rho) + G(a_\rho)) \geqslant \gamma_\rho,$$

since $v(G(b) - G(a_\rho)) \geqslant v(b - a_\rho) = \gamma_\rho$. Thus $a_\rho \rightsquigarrow c$, as claimed. □

Definition 4.4. We say $\mathcal{K}$ is σ-*henselian* if for each σ-polynomial $G(x)$ over $\mathcal{O}$ and $a \in \mathcal{O}$ such that G is σ-henselian at a, there exists $b \in \mathcal{O}$ such that $G(b) = 0$ and $v(a - b) \geqslant v(G(a))$. (*By the arguments above, any such b will actually satisfy $v(a - b) = v(G(a))$.*)

Corollary 4.5. *If $\mathcal{K}$ is σ-henselian, then the residue field of* $\mathrm{Fix}(K)$ *is* $\mathrm{Fix}(\boldsymbol{k})$.

Proof. Suppose $\mathcal{K}$ is σ-henselian, and let $\alpha \in \mathrm{Fix}(\boldsymbol{k})$; we shall find $b \in \mathrm{Fix}(K)$ such that $v(b) = 0$ and $\bar{b} = \alpha$. Take $a \in K$ with $v(a) = 0$ and $\bar{a} = \alpha$. Then $v(\sigma(a) - a) > 0$, so $\sigma(x) - x$ is σ-henselian at a. So there is a b as promised. □

By 'Axiom 4' we mean the axiom scheme $\{\text{Axiom } 4_n : n = 0, 1, 2, \dots\}$.

Remark 4.6. If $\Gamma = \{0\}$, then $\mathcal{K}$ is σ-henselian. Suppose $\Gamma \neq \{0\}$, $\mathcal{K}$ satisfies Axiom 2 and is σ-henselian. Then $\mathcal{K}$ satisfies Axiom 4 by [**9**, Proposition 5.3], so $\bar{\sigma}^n \neq \mathrm{id}_{\boldsymbol{k}}$ for all $n \geqslant 1$. Hence, if also $\mathrm{char}(\boldsymbol{k}) = 0$, then $\mathcal{K}$ satisfies Axiom 3.

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From part (1) of Lemma 4.3 we obtain the following corollary.

Corollary 4.7. *If $\mathcal{K}$ is maximal as valued field and satisfies Axiom 4, then $\mathcal{K}$ is σ-henselian. In particular, if $\mathcal{K}$ is complete with discrete valuation and satisfies Axiom 4, then $\mathcal{K}$ is σ-henselian.*

Thus if the difference field $\boldsymbol{k}$ satisfies Axiom 4, then the Hahn difference field $\boldsymbol{k}((t^{\Gamma}))$ is σ-henselian. Suppose $\boldsymbol{k}$ has characteristic $p > 0$ and every equation

$$1 + \alpha_0 x + \alpha_1 x^p + \cdots + \alpha_n x^{p^n} = 0 \quad (\text{all } \alpha_i \in \boldsymbol{k},\ \text{some } \alpha_i \neq 0)$$

is solvable in $\boldsymbol{k}$. Then by Corollary 4.7 the Witt difference field $\mathrm{W}(\boldsymbol{k})$ is σ-henselian, where σ is the Witt Frobenius. As noted in [**3**], this condition on the residue field $\boldsymbol{k}$ is *Hypothesis A* in [**7**], where it is related to uniqueness of maximal immediate extensions of valued fields. It is equivalent to $\boldsymbol{k}$ not having any field extension of finite degree divisible by p (see [**12**]).

Note that if $\mathcal{K}$ is σ-henselian, then it is henselian as a valued field. We have the following analogue of an important result about henselian valued fields:

Theorem 4.8. *Suppose that $\mathcal{K}$ is σ-henselian and* $\mathrm{char}(\boldsymbol{k}) = 0$. *Let $K_0 \subseteq \mathcal{O}$ be a σ-subfield of K. Then there is a σ-subfield K_1 of K such that $K_0 \subseteq K_1 \subseteq \mathcal{O}$ and $\bar{K}_1 = \boldsymbol{k}$.*

Proof. Suppose that $\bar{K}_0 \neq \boldsymbol{k}$. Take $a \in \mathcal{O}$ such that $\bar{a} \notin \bar{K}_0$. If $v(G(a)) = 0$ for all non-zero $G(x)$ over K_0, then $K_0\langle a \rangle$ is a proper σ-field extension of K_0 contained in $\mathcal{O}$. Next, consider the case that $v(G(a)) > 0$ for some non-zero $G(x)$ over K_0. Pick such G of minimal complexity. So $v(H(a)) = 0$ for all non-zero $H(x)$ over K_0 of lower complexity. It follows that G is σ-henselian at a. So there is $b \in \mathcal{O}$ with $G(b) = 0$ and $v(a-b) = v(G(a))$, so $\bar{a} = \bar{b}$. We claim that $K_0\langle b \rangle$ is a proper σ-field extension of K_0 contained in $\mathcal{O}$. To prove the claim, let G have order m. Then the $\sigma^k(b)$ with $k \in \mathbb{Z}$ are algebraic over $K_0(b, \ldots, \sigma^{m-1}(b))$ and thus

$$K_0\langle b \rangle = K_0(\sigma^k(b) : k \in \mathbb{Z}) = K_0(b, \ldots, \sigma^{m-1}(b))[\sigma^k(b) : k \in \mathbb{Z}] \subseteq \mathcal{O},$$

which establishes the claim. We finish the proof by Zorn's Lemma. □

The notion 'σ-henselian at a' applies only to σ-polynomials over $\mathcal{O}$ and $a \in \mathcal{O}$. It will be convenient to extend it a little. Let $G(x)$ be over K of order at most n and $a \in K$.

Definition 4.9. We say (G, a) is in *σ-hensel configuration* if $G_{(\boldsymbol{i})}(a) \neq 0$ for some $\boldsymbol{i} \in \mathbb{N}^{n+1}$ with $|\boldsymbol{i}| = 1$, and either $G(a) = 0$ or there is $\gamma \in \Gamma$ such that

$$v(G(a)) = \min_{|\boldsymbol{i}|=1} v(G_{(\boldsymbol{i})}(a)) + \gamma < v(G_{(\boldsymbol{j})}(a)) + |\boldsymbol{j}| \cdot \gamma$$

for all $\boldsymbol{j}$ with $|\boldsymbol{j}| > 1$. For (G, a) in σ-hensel configuration we put

$$\gamma(G, a) := v(G(a)) - \min_{|\boldsymbol{i}|=1} v(G_{(\boldsymbol{i})}(a)).$$

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Let (G, a) be in σ-hensel configuration, $G(a) \neq 0$, and take $c \in K$ with $v(c) = \gamma(G, a)$ and put $H(x) := G(cx)/G(a)$, $\alpha := a/c$. Then

$$H(\alpha) = 1, \qquad \min_{|\boldsymbol{i}|=1} v(H_{(\boldsymbol{i})}(\alpha)) = 0, \qquad v(H_{(\boldsymbol{j})}(\alpha)) > 0 \quad \text{for } |\boldsymbol{j}| > 1,$$

as is easily verified. In particular, $H(\alpha + x)$ is over $\mathcal{O}$. Now assume that $\mathcal{K}$ satisfies also Axiom 4. This gives a unit $u \in \mathcal{O}$ such that $v(H(\alpha + u)) > 0$. We claim that $H(\alpha + x)$ is σ-henselian at u. This is because for $P(x) := H(\alpha + x)$ we have $v(P(u)) > 0$, and, for each i,

$$P_{(\boldsymbol{i})}(u) = H_{(\boldsymbol{i})}(\alpha + u) = H_{(\boldsymbol{i})}(\alpha) + \sum_{|\boldsymbol{j}| \geqslant 1} H_{(\boldsymbol{i})(\boldsymbol{j})}(\alpha)\boldsymbol{\sigma}(u)^{\boldsymbol{j}},$$

so $\min_{|\boldsymbol{i}|=1} v(P_{(\boldsymbol{i})}(u)) = 0$. If $\mathcal{K}$ is σ-henselian, we can take u as above such that $H(\alpha+u) = 0$, and then $b := a + cu$ satisfies $G(b) = 0$ and $v(a - b) = \gamma$. Summarizing, we have the following lemma.

Lemma 4.10. *Assume $\mathcal{K}$ satisfies Axiom 4 and is σ-henselian. Let (G, a) be in σ-hensel configuration. Then there is $b \in K$ such that $G(b) = 0$ and $v(a - b) = \gamma(G, a)$.*

Up to this point we treated the Witt and non-Witt case separately, but from now on it makes sense to handle both cases at once. We say that $\mathcal{K}$ is *workable* if either it satisfies Axioms 2 and 3 (as in Theorem 3.5) or it satisfies Axiom 2 and is a Witt case with infinite $\boldsymbol{k}$ (as in Theorem 3.10). An *extension* of a workable $\mathcal{K}$ is an extension as before (and does not have to be workable), but in the Witt case we also require the extension to be a Witt case.

In the next definition we assume that $\mathcal{K}$ is workable, and that $\{a_\rho\}$ is a pc-sequence from K.

Definition 4.11. We say $\{a_\rho\}$ is of *σ-algebraic type over K* if $G(b_\rho) \rightsquigarrow 0$ for some σ-polynomial $G(x)$ over K and an equivalent pc-sequence $\{b_\rho\}$ in K.

If $\{a_\rho\}$ is of σ-algebraic type over K, then a *minimal σ-polynomial of $\{a_\rho\}$ over K* is a σ-polynomial $G(x)$ over K with the following properties:

(i) $G(b_\rho) \rightsquigarrow 0$ for some pc-sequence $\{b_\rho\}$ in K, equivalent to $\{a_\rho\}$;

(ii) $H(b_\rho) \not\rightsquigarrow 0$ whenever $H(x)$ is a σ-polynomial over K of lower complexity than G and $\{b_\rho\}$ is a pc-sequence in K equivalent to $\{a_\rho\}$.

If $\{a_\rho\}$ is of σ-algebraic type over K, then $\{a_\rho\}$ clearly has a minimal σ-polynomial over K. The next lemma is used to study immediate extensions in the next section. Its finite ramification hypothesis is satisfied by all Witt cases. ('Finitely ramified' is defined right after Lemma 2.2.) The next lemma is close to Theorem 6.10 in [**3**].

Lemma 4.12. *Suppose $\mathcal{K}$ is workable and finitely ramified. Let $\{a_\rho\}$ from K be a pc-sequence of σ-algebraic type over K with minimal σ-polynomial $G(x)$ over K, and with pseudo-limit a in some extension. Let Σ be a finite set of σ-polynomials $H(x)$ over K. Then there is a pc-sequence $\{b_\rho\}$ in K, equivalent to $\{a_\rho\}$, such that, with $\gamma_\rho := v(a - a_\rho)$:*

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(1) $v(a - b_\rho) = \gamma_\rho$, *eventually, and* $G(b_\rho) \rightsquigarrow 0$;

(2) *if* $H \in \Sigma$ *and* $H \notin K$, *then* $H(b_\rho) \rightsquigarrow H(a)$;

(3) (G, a) *is in* σ*-hensel configuration, and* $\gamma(G, a) > \gamma_\rho$, *eventually.*

Proof. Let G have order n. We can assume that Σ includes all $G_{(\boldsymbol{i})}$. In the rest of the proof $\boldsymbol{i}$, $\boldsymbol{j}$, $\boldsymbol{l}$ range over $\mathbb{N}^{n+1}$. Theorems 3.5 and 3.10 and their proofs yield an equivalent pc-sequence $\{b_\rho\}$ in K such that (1) and (2) hold. The proof of Theorem 3.3 shows that we can arrange that in addition there is a unique m_0 with $1 \leqslant m_0 \leqslant \deg G$ such that, eventually,

$$v(G(b_\rho) - G(a)) = \min_{|\boldsymbol{i}|=m_0} v(G_{(\boldsymbol{i})}(a)) + m_0\gamma_\rho < v(G_{(\boldsymbol{j})}(a)) + |\boldsymbol{j}| \cdot \gamma_\rho,$$

for each $\boldsymbol{j}$ with $|\boldsymbol{j}| \geqslant 1$ and $|\boldsymbol{j}| \neq m_0$. Now $\{v(G(b_\rho))\}$ is strictly increasing, eventually, so $v(G(a)) > v(G(b_\rho))$ eventually, and for $|\boldsymbol{j}| \geqslant 1$, $|\boldsymbol{j}| \neq m_0$:

$$v(G(b_\rho)) = \min_{|\boldsymbol{i}|=m_0} v(G_{(\boldsymbol{i})}(a)) + m_0 \cdot \gamma_\rho < v(G_{(\boldsymbol{j})}(a)) + |\boldsymbol{j}| \cdot \gamma_\rho, \quad \text{eventually.}$$

We claim that $m_0 = 1$. Let $|\boldsymbol{i}| = 1$ with $G_{(\boldsymbol{i})} \neq 0$, and let $\boldsymbol{j} > \boldsymbol{i}$; our claim will then follow by deriving

$$v(G_{(\boldsymbol{i})}(a)) + \gamma_\rho < v(G_{(\boldsymbol{j})}(a)) + |\boldsymbol{j}|\gamma_\rho, \quad \text{eventually.}$$

The proof of Theorem 3.3 with $G_{(\boldsymbol{i})}$ in the role of G shows that we can arrange that our sequence $\{b_\rho\}$ also satisfies

$$v(G_{(\boldsymbol{i})}(b_\rho) - G_{(\boldsymbol{i})}(a)) \leqslant v(G_{(\boldsymbol{i})(\boldsymbol{l})}(a)) + |\boldsymbol{l}| \cdot \gamma_\rho, \quad \text{eventually,}$$

for all $\boldsymbol{l}$ with $|\boldsymbol{l}| \geqslant 1$. Since $v(G_{(\boldsymbol{i})}(b_\rho)) = v(G_{(\boldsymbol{i})}(a))$ eventually, this yields

$$v(G_{(\boldsymbol{i})}(b_\rho)) \leqslant v(G_{(\boldsymbol{i})(\boldsymbol{l})}(a)) + |\boldsymbol{l}| \cdot \gamma_\rho, \quad \text{eventually,}$$

for all $\boldsymbol{l}$ with $|\boldsymbol{l}| \geqslant 1$, hence for all such $\boldsymbol{l}$,

$$v(G_{(\boldsymbol{i})}(b_\rho)) \leqslant v\binom{\boldsymbol{i}+\boldsymbol{l}}{\boldsymbol{i}} + v(G_{(\boldsymbol{i}+\boldsymbol{l})}(a)) + |\boldsymbol{l}| \cdot \gamma_\rho, \quad \text{eventually.}$$

For $\boldsymbol{l}$ with $\boldsymbol{i} + \boldsymbol{l} = \boldsymbol{j}$, this yields

$$v(G_{(\boldsymbol{i})}(a)) \leqslant v\binom{\boldsymbol{j}}{\boldsymbol{i}} + v(G_{(\boldsymbol{j})}(a)) + (|\boldsymbol{j}| - 1) \cdot \gamma_\rho, \quad \text{eventually.}$$

Now K is finitely ramified, so

$$v(G_{(\boldsymbol{i})}(a)) < v(G_{(\boldsymbol{j})}(a)) + (|\boldsymbol{j}| - 1) \cdot \gamma_\rho, \quad \text{eventually,}$$

hence

$$v(G_{(\boldsymbol{i})}(a)) + \gamma_\rho < v(G_{(\boldsymbol{j})}(a)) + |\boldsymbol{j}| \cdot \gamma_\rho, \quad \text{eventually.}$$

Thus $m_0 = 1$, as claimed. The above inequalities then yield (3). □

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5. Immediate extensions

Throughout this section $\mathcal{K} = (K, \Gamma, \boldsymbol{k}; v, \pi)$ is a workable valued difference field. The immediate extensions of $\mathcal{K}$ are then workable as well, and we prove here the basic facts on these immediate extensions. To avoid heavy handed notation we often let K stand for $\mathcal{K}$ when the context permits.

Definition 5.1. A pc-sequence $\{a_\rho\}$ from K is said to be of *σ-transcendental type over K* if it is not of σ-algebraic type over K, that is, $G(b_\rho) \not\rightsquigarrow 0$ for each σ-polynomial $G(x)$ over K and each equivalent pc-sequence $\{b_\rho\}$ from K.

In particular, such a pc-sequence cannot have a pseudo-limit in K. The next two lemmas are σ-analogues of familiar results for valued fields, and correspond to Lemmas 7.1 and 7.2 in [**3**], where 'σ-algebraic' and 'σ-transcendental' are defined in a slightly different way. (For proofs, see [**3**] or [**2**].)

Lemma 5.2. *Let $\{a_\rho\}$ from K be a pc-sequence of σ-transcendental type over K. Then $\mathcal{K}$ has an immediate extension $(K\langle a\rangle, \Gamma, \boldsymbol{k}; v_a, \pi_a)$ such that:*

(1) *a is σ-transcendental over K and $a_\rho \rightsquigarrow a$;*

(2) *for any extension $(K_1, \Gamma_1, \boldsymbol{k}_1; v_1, \pi_1)$ of $\mathcal{K}$ and any $b \in K_1$ with $a_\rho \rightsquigarrow b$ there is a unique embedding*

$$(K\langle a\rangle, \Gamma, \boldsymbol{k}; v_a, \pi_a) \to (K_1, \Gamma_1, \boldsymbol{k}_1; v_1, \pi_1)$$

over $\mathcal{K}$ that sends a to b.

Lemma 5.3. *Suppose $\mathcal{K}$ is finitely ramified. Let $\{a_\rho\}$ from K be a pc-sequence of σ-algebraic type over K, with no pseudo-limit in K. Let $G(x)$ be a minimal σ-polynomial of $\{a_\rho\}$ over K. Then $\mathcal{K}$ has an immediate extension $(K\langle a\rangle, \Gamma, \boldsymbol{k}; v_a, \pi_a)$ such that*

(1) *$G(a) = 0$ and $a_\rho \rightsquigarrow a$;*

(2) *for any extension $(K_1, \Gamma_1, \boldsymbol{k}_1; v_1, \pi_1)$ of $\mathcal{K}$ and any $b \in K_1$ with $G(b) = 0$ and $a_\rho \rightsquigarrow b$ there is a unique embedding*

$$(K\langle a\rangle, \Gamma, \boldsymbol{k}; v_a, \pi_a) \to (K_1, \Gamma_1, \boldsymbol{k}_1; v_1, \pi_1)$$

over $\mathcal{K}$ that sends a to b.

We note the following consequences of Lemmas 5.2 and 5.3.

Corollary 5.4. *Let a from some extension of $\mathcal{K}$ be σ-algebraic over K and let $\{a_\rho\}$ be a pc-sequence in K such that $a_\rho \rightsquigarrow a$. Then $\{a_\rho\}$ is of σ-algebraic type over K.*

Corollary 5.5. *Suppose $\mathcal{K}$ is finitely ramified. Then $\mathcal{K}$ as a valued field has a proper immediate extension if and only if $\mathcal{K}$ as a valued difference field has a proper immediate extension.*

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We say that $\mathcal{K}$ is *σ-algebraically maximal* if it has no proper immediate σ-algebraic extension, and we say it is *maximal* if it has no proper immediate extension. Corollary 5.4 and Lemmas 5.3 and 4.3 yield the following corollary.

Corollary 5.6. *Suppose $\mathcal{K}$ is finitely ramified. Then*

(1) *$\mathcal{K}$ is σ-algebraically maximal if and only if each pc-sequence in K of σ-algebraic type over K has a pseudo-limit in K;*

(2) *if $\mathcal{K}$ satisfies Axiom 4 and is σ-algebraically maximal, then $\mathcal{K}$ is σ-henselian.*

It is clear that $\mathcal{K}$ has σ-algebraically maximal immediate σ-algebraic extensions, and also maximal immediate extensions. If $\mathcal{K}$ satisfies Axiom 4 both kinds of extensions are unique up to isomorphism, but for this we need one more lemma.

Lemma 5.7. *Suppose $\mathcal{K}$ is finitely ramified and $\mathcal{K}'$ is a workable finitely ramified σ-algebraically maximal extension of $\mathcal{K}$ satisfying Axiom 4. Let $\{a_\rho\}$ from K be a pc-sequence of σ-algebraic type over K, with no pseudo-limit in K, and with minimal σ-polynomial $G(x)$ over K. Then there exists $b \in K'$ such that $a_\rho \rightsquigarrow b$ and $G(b) = 0$.*

Proof. Lemma 5.3 provides a pseudo-limit $a \in K'$ of $\{a_\rho\}$. Take a pc-sequence $\{b_\rho\}$ in K equivalent to $\{a_\rho\}$ with the properties listed in Lemma 4.12. Since $\mathcal{K}'$ is σ-henselian and satisfies Axiom 4, Lemma 4.10 yields $b \in K'$ such that

$$v'(a-b) = \gamma(G,a) \quad \text{and} \quad G(b) = 0.$$

Note that $a_\rho \rightsquigarrow b$ since $\gamma(G,a) > v(a - a_\rho) = \gamma_\rho$ eventually. □

Together with Lemmas 5.2 and 5.3 this yields the following theorem.

Theorem 5.8. *Suppose $\mathcal{K}$ is finitely ramified and satisfies Axiom 4. Then all its maximal immediate extensions are isomorphic over $\mathcal{K}$, and all its σ-algebraically maximal immediate σ-algebraic extensions are isomorphic over $\mathcal{K}$.*

We now state minor variants of these results using the notion of saturation from model theory, as needed in the proof of the embedding theorem in the next section. Let $|X|$ denote the cardinality of a set X, and let κ be a cardinal.

Lemma 5.9. *Suppose $\mathcal{E} = (E, \Gamma_E, \dots) \leqslant \mathcal{K}$ is workable and $\mathcal{K}$ is finitely ramified, σ-henselian, and κ-saturated with $\kappa > |\Gamma_E|$. Let $\{a_\rho\}$ from E be a pc-sequence of σ-algebraic type over E, with no pseudo-limit in E, and with minimal σ-polynomial $G(x)$ over E. Then there exists $b \in K$ such that $a_\rho \rightsquigarrow b$ and $G(b) = 0$.*

Proof. By the saturation assumption we have a pseudo-limit $a \in K$ of $\{a_\rho\}$. Let $\gamma_\rho = v(a - a_\rho)$. By Lemma 4.12, (G,a) is in σ-hensel configuration with $\gamma(G,a) > \gamma_\rho$, eventually. Since $\mathcal{K}$ is σ-henselian, it satisfies Axiom 4, so Lemma 4.10 yields $b \in K$ such that $v(a-b) = \gamma(G,a)$ and $G(b) = 0$. Note that $a_\rho \rightsquigarrow b$ since $\gamma(G,a) > \gamma_\rho$ eventually. □

In combination with Lemmas 5.2 and 5.3 this yields the following corollary.

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Corollary 5.10. *If $\mathcal{E} = (E, \Gamma_E, \dots) \leqslant \mathcal{K}$ is workable and satisfies Axiom 4, and $\mathcal{K}$ is finitely ramified, σ-henselian, and κ-saturated with $\kappa > |\Gamma_E|$, then any maximal immediate extension of $\mathcal{E}$ can be embedded in $\mathcal{K}$ over $\mathcal{E}$.*

6. The equivalence theorem

Theorem 6.6, the main result of the paper, tells us when two workable σ-henselian valued difference fields of equal characteristic zero are elementarily equivalent over a common substructure. In § 8 we derive from it in the usual way some attractive consequences on the elementary theories of such valued difference fields and on the induced structure on value group and residue difference field. In § 9 we use coarsening to obtain analogues in the mixed characteristic case.

We begin with a short subsection on angular component maps. The presence of such maps simplifies the proof of the Equivalence Theorem, but in the aftermath we can often discard these maps again, by Corollary 6.2.

Angular components

Let $\mathcal{K} = (K, \Gamma, \boldsymbol{k}; v, \pi)$ be a valued difference field. An *angular component map* on $\mathcal{K}$ is an angular component map ac on $\mathcal{K}$ as valued field such that in addition $\bar{\sigma}(\mathrm{ac}(a)) = \mathrm{ac}(\sigma(a))$ for all $a \in K$. Examples are the Hahn difference fields $\boldsymbol{k}(\!(t^\Gamma)\!)$ with angular component map given by $\mathrm{ac}(a) = a_{\gamma_0}$ for non-zero $a = \sum a_\gamma t^\gamma \in \boldsymbol{k}(\!(t^\Gamma)\!)$ and $\gamma_0 = v(a)$, and also the Witt difference fields $\mathrm{W}(\boldsymbol{k})$ with angular component map determined by $\mathrm{ac}(p) = 1$. (To see this, use the next lemma and the fact that $\mathrm{Fix}(\mathrm{W}(\boldsymbol{k})) = \mathrm{W}(\mathbb{F}_p) = \mathbb{Q}_p$.)

Lemma 6.1. *Suppose $\mathcal{K}$ satisfies Axiom 2. Then each angular component map on the valued subfield* $\mathrm{Fix}(K)$ *of $\mathcal{K}$ extends uniquely to an angular component map on $\mathcal{K}$. If in addition $\mathcal{K}$ is σ-henselian, then every angular component map on $\mathcal{K}$ is obtained in this way from an angular component map on* $\mathrm{Fix}(K)$.

Proof. Given an angular component map ac on $\mathrm{Fix}(K)$ the claimed extension to $\mathcal{K}$, also denoted by ac, is obtained as follows: for $x \in K^\times$ we have $x = uy$ with $u, y \in K^\times$, $v(u) = 0$, $\sigma(y) = y$; then $\mathrm{ac}(x) = \bar{u}\,\mathrm{ac}(y)$. The second claim of the lemma follows from Corollary 4.5. □

Here is an immediate consequence of Lemmas 2.3 and 6.1.

Corollary 6.2. *Suppose $\mathcal{K}$ satisfies Axiom 2. Then there is an angular component map on some elementary extension of $\mathcal{K}$.*

The main result

In this subsection we consider 3-sorted structures

$$\mathcal{K} = (K, \Gamma, \boldsymbol{k}; v, \pi, \mathrm{ac}),$$

where $(K, \Gamma, \boldsymbol{k}; v, \pi)$ is a valued difference field (satisfying Axiom 1 of course) and where $\mathrm{ac} : K \to \boldsymbol{k}$ is an angular component map on $(K, \Gamma, \boldsymbol{k}; v, \pi)$. Such a structure will be

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called an *ac-valued difference field*. Any subfield E of K is viewed as a valued subfield of $\mathcal{K}$ with valuation ring $\mathcal{O}_E := \mathcal{O} \cap E$.

If $\operatorname{char}(\boldsymbol{k}) = 0$ and $\mathcal{K}$ is σ-henselian, then by Theorem 4.8 there is a difference ring morphism $i : \boldsymbol{k} \to \mathcal{O}$ such that $\pi(i(a)) = a$ for all $a \in K$; we call such i a *σ-lifting* of $\boldsymbol{k}$ to $\mathcal{K}$. This will play a minor role in the proof of the Equivalence Theorem.

A *good substructure* of $\mathcal{K} = (K, \Gamma, \boldsymbol{k}; v, \pi, \mathrm{ac})$ is a triple $\mathcal{E} = (E, \Gamma_{\mathcal{E}}, \boldsymbol{k}_{\mathcal{E}})$ such that

(1) E is a difference subfield of K,

(2) $\Gamma_{\mathcal{E}}$ is an ordered abelian subgroup of Γ with $v(E^\times) \subseteq \Gamma_{\mathcal{E}}$,

(3) $\boldsymbol{k}_{\mathcal{E}}$ is a difference subfield of $\boldsymbol{k}$ with $\mathrm{ac}(E) \subseteq \boldsymbol{k}_{\mathcal{E}}$ (hence $\pi(\mathcal{O}_E) \subseteq \boldsymbol{k}_{\mathcal{E}}$).

For good substructures $\mathcal{E}_1 = (E_1, \Gamma_1, \boldsymbol{k}_1)$ and $\mathcal{E}_2 = (E_2, \Gamma_2, \boldsymbol{k}_2)$ of $\mathcal{K}$, we define $\mathcal{E}_1 \subseteq \mathcal{E}_2$ to mean that $E_1 \subseteq E_2$, $\Gamma_1 \subseteq \Gamma_2$, $\boldsymbol{k}_1 \subseteq \boldsymbol{k}_2$. If E is a difference subfield of K with $\mathrm{ac}(E) = \pi(\mathcal{O}_E)$, then $(E, v(E^\times), \pi(\mathcal{O}_E))$ is a good substructure of $\mathcal{K}$, and if in addition $F \supseteq E$ is a difference subfield of K such that $v(F^\times) = v(E^\times)$, then $\mathrm{ac}(F) = \pi(\mathcal{O}_F)$. Throughout this subsection

$$\mathcal{K} = (K, \Gamma, \boldsymbol{k}; v, \pi, \mathrm{ac}), \qquad \mathcal{K}' = (K', \Gamma', \boldsymbol{k}'; v', \pi', \mathrm{ac}')$$

are ac-valued difference fields, with valuation rings $\mathcal{O}$ and $\mathcal{O}'$, and

$$\mathcal{E} = (E, \Gamma_{\mathcal{E}}, \boldsymbol{k}_{\mathcal{E}}), \qquad \mathcal{E}' = (E', \Gamma_{\mathcal{E}'}, \boldsymbol{k}_{\mathcal{E}'})$$

are good substructures of $\mathcal{K}$, $\mathcal{K}'$ respectively. To avoid too many accents we let σ denote the difference operator of each of K, K', E, E', and put $\mathcal{O}_{E'} := \mathcal{O}' \cap E'$.

A *good map* $\boldsymbol{f} : \mathcal{E} \to \mathcal{E}'$ is a triple $\boldsymbol{f} = (f, f_{\mathrm{v}}, f_{\mathrm{r}})$ consisting of a difference field isomorphism $f : E \to E'$, an ordered group isomorphism $f_{\mathrm{v}} : \Gamma_{\mathcal{E}} \to \Gamma_{\mathcal{E}'}$ and a difference field isomorphism $f_{\mathrm{r}} : \boldsymbol{k}_{\mathcal{E}} \to \boldsymbol{k}_{\mathcal{E}'}$ such that

(i) $f_{\mathrm{v}}(v(a)) = v'(f(a))$ for all $a \in E^\times$, and f_{v} is elementary as a partial map between the ordered abelian groups Γ and Γ';

(ii) $f_{\mathrm{r}}(\mathrm{ac}(a)) = \mathrm{ac}'(f(a))$ for all $a \in E$, and f_{r} is elementary as a partial map between the difference fields $\boldsymbol{k}$ and $\boldsymbol{k}'$.

Let $\boldsymbol{f} : \mathcal{E} \to \mathcal{E}'$ be a good map as above. Then the field part $f : E \to E'$ of $\boldsymbol{f}$ is a valued difference field isomorphism, and f_{v} and f_{r} agree on $v(E^\times)$ and $\pi(\mathcal{O}_E)$ with the maps $v(E^\times) \to v'(E'^\times)$ and $\pi(\mathcal{O}_E) \to \pi'(\mathcal{O}_{E'})$ induced by f. We say that a good map $\boldsymbol{g} = (g, g_{\mathrm{v}}, g_{\mathrm{r}}) : \mathcal{F} \to \mathcal{F}'$ *extends* $\boldsymbol{f}$ if $\mathcal{E} \subseteq \mathcal{F}$, $\mathcal{E}' \subseteq \mathcal{F}'$, and g, g_{v}, g_{r} extend f, f_{v}, f_{r}, respectively. The *domain* of $\boldsymbol{f}$ is $\mathcal{E}$.

The next two lemmas show that condition (ii) above is automatically satisfied by certain extensions of good maps.

Lemma 6.3. *Let $\boldsymbol{f} : \mathcal{E} \to \mathcal{E}'$ be a good map, and $F \supseteq E$ and $F' \supseteq E'$ subfields of K and K', respectively, such that $v(F^\times) = v(E^\times)$ and $\pi(\mathcal{O}_F) \subseteq \boldsymbol{k}_{\mathcal{E}}$. Let $g : F \to F'$ be a valued field isomorphism such that g extends f and $f_{\mathrm{r}}(\pi(u)) = \pi'(g(u))$ for all $u \in \mathcal{O}_F$. Then $\mathrm{ac}(F) \subseteq \boldsymbol{k}_{\mathcal{E}}$ and $f_{\mathrm{r}}(\mathrm{ac}(a)) = \mathrm{ac}'(g(a))$ for all $a \in F$.*

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Proof. Let $a \in F$. Then $a = a_1 u$, where $a_1 \in E$ and $u \in \mathcal{O}_F$, $v(u) = 0$, so $\mathrm{ac}(a) = \mathrm{ac}(a_1)\pi(u) \in \boldsymbol{k}_{\mathcal{E}}$. It follows easily that $f_{\mathrm{r}}(\mathrm{ac}(a)) = \mathrm{ac}'(g(a))$. $\square$

In the same way we obtain the following lemma.

Lemma 6.4. *Suppose $\pi(\mathcal{O}_E) = \boldsymbol{k}_{\mathcal{E}}$, let $\boldsymbol{f} : \mathcal{E} \to \mathcal{E}'$ be a good map, and let $F \supseteq E$ and $F' \supseteq E'$ be subfields of K and K', respectively, such that $v(F^\times) = v(E^\times)$. Let $g : F \to F'$ be a valued field isomorphism extending f. Then $\mathrm{ac}(F) = \pi(\mathcal{O}_F)$ and $g_{\mathrm{r}}(\mathrm{ac}(a)) = \mathrm{ac}'(g(a))$ for all $a \in F$, where the map $g_{\mathrm{r}} : \pi(\mathcal{O}_F) \to \pi'(\mathcal{O}_{F'})$ is induced by g (and thus extends f_{r}).*

The following is useful in connection with Axiom 2.

Lemma 6.5. *Let $b \in K^\times$. Then the following are equivalent:*

(1) *there is $c \in \mathrm{Fix}(K)$ such that $v(c) = v(b)$;*

(2) *there is $d \in K$ such that $v(d) = 0$ and $\sigma(d) = (b/\sigma(b)) \cdot d$.*

Proof. For c as in (1), $d = cb^{-1}$ is as in (2). For d as in (2), $c = bd$ is as in (1). $\square$

We say that $\mathcal{E}$ satisfies Axiom 2 (respectively, Axiom 3, Axiom 4) if the valued difference subfield $(E, v(E^\times), \pi(\mathcal{O}_E); \dots)$ of $\mathcal{K}$ does. Likewise, we say that $\mathcal{E}$ is workable (respectively, σ-henselian) if this valued difference subfield of $\mathcal{K}$ is.

Theorem 6.6. *Suppose $\mathrm{char}(\boldsymbol{k}) = 0$, $\mathcal{K}$, $\mathcal{K}'$ satisfy Axiom 2 and are σ-henselian. Then any good map $\mathcal{E} \to \mathcal{E}'$ is a partial elementary map between $\mathcal{K}$ and $\mathcal{K}'$.*

Proof. The theorem holds trivially for $\Gamma = \{0\}$, so assume that $\Gamma \neq \{0\}$. Then $\mathcal{K}$ and $\mathcal{K}'$ are workable. Let $\boldsymbol{f} = (f, f_{\mathrm{v}}, f_{\mathrm{r}}) : \mathcal{E} \to \mathcal{E}'$ be a good map. By passing to suitable elementary extensions of $\mathcal{K}$ and $\mathcal{K}'$ we arrange that $\mathcal{K}$ and $\mathcal{K}'$ are κ-saturated, where κ is an uncountable cardinal such that $|\boldsymbol{k}_{\mathcal{E}}|, |\Gamma_{\mathcal{E}}| < \kappa$. Call a good substructure $\mathcal{E}_1 = (E_1, \boldsymbol{k}_1, \Gamma_1)$ of $\mathcal{K}$ *small* if $|\boldsymbol{k}_1|, |\Gamma_1| < \kappa$. We shall prove that the good maps with small domain form a back-and-forth system between $\mathcal{K}$ and $\mathcal{K}'$. (This clearly suffices to obtain the theorem.) In other words, we shall prove that under the present assumptions on $\mathcal{E}$, $\mathcal{E}'$ and $\boldsymbol{f}$, there is for each $a \in K$ a good map $\boldsymbol{g}$ extending $\boldsymbol{f}$ such that $\boldsymbol{g}$ has small domain $\mathcal{F} = (F, \dots)$ with $a \in F$.

In addition to Corollary 5.10, we have several basic extension procedures.

(1) *Given $\alpha \in \boldsymbol{k}$, arranging that $\alpha \in \boldsymbol{k}_{\mathcal{E}}$.* By saturation and the definition of 'good map' this can be achieved without changing f, f_{v}, E, $\Gamma_{\mathcal{E}}$ by extending f_{r} to a partial elementary map between $\boldsymbol{k}$ and $\boldsymbol{k}'$ with α in its domain.

(2) *Given $\gamma \in \Gamma$, arranging that $\gamma \in \Gamma_{\mathcal{E}}$.* This follows in the same way.

(3) *Arranging $\boldsymbol{k}_{\mathcal{E}} = \pi(\mathcal{O}_E)$.* Suppose $\alpha \in \boldsymbol{k}_{\mathcal{E}}, \alpha \notin \pi(\mathcal{O}_E)$; set $\alpha' := f_{\mathrm{r}}(\alpha)$.

If α is $\bar{\sigma}$-transcendental over $\pi(\mathcal{O}_E)$, we pick $a \in \mathcal{O}$ and $a' \in \mathcal{O}'$ such that $\bar{a} = \alpha$ and $\bar{a}' = \alpha'$, and then Lemmas 2.5 and 6.3 yield a good map $\boldsymbol{g} = (g, f_{\mathrm{v}}, f_{\mathrm{r}})$ with small domain $(E\langle a\rangle, \Gamma_{\mathcal{E}}, \boldsymbol{k}_{\mathcal{E}})$ such that $\boldsymbol{g}$ extends $\boldsymbol{f}$ and $g(a) = a'$.

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Next, assume that α is $\bar{\sigma}$-algebraic over $\pi(\mathcal{O}_E)$. Let $G(x)$ be a σ-polynomial over $\mathcal{O}_E$ such that $\bar{G}(x)$ is a minimal $\bar{\sigma}$-polynomial of α over $\pi(\mathcal{O}_E)$ and has the same complexity as $G(x)$. Pick $a \in \mathcal{O}$ such that $\bar{a} = \alpha$. Then G is σ-henselian at a. So we have $b \in \mathcal{O}$ such that $G(b) = 0$ and $\bar{b} = \bar{a} = \alpha$. Likewise, we obtain $b' \in \mathcal{O}'$ such that $f(G)(b') = 0$ and $\bar{b}' = \alpha'$, where $f(G)$ is the difference polynomial over E' that corresponds to G under f. By Lemmas 2 and 6.3 we obtain a good map extending $\boldsymbol{f}$ with small domain $(E\langle b\rangle, \Gamma_{\mathcal{E}}, \boldsymbol{k}_{\mathcal{E}})$ and sending b to b'.

By iterating these steps we can arrange $\boldsymbol{k}_{\mathcal{E}} = \pi(\mathcal{O}_E)$; this condition is actually preserved in the extension procedures (4)–(6) below, as the reader may easily verify. We do assume in the rest of the proof that $\boldsymbol{k}_{\mathcal{E}} = \pi(\mathcal{O}_E)$, and so we can refer from now on to $\boldsymbol{k}_{\mathcal{E}}$ as the *residue difference field* of E.

(4) *Extending $\boldsymbol{f}$ to a good map whose domain satisfies Axiom 2.* Let $\delta \in v(E^\times)$. Pick $b \in E^\times$ such that $v(b) = \delta$. Since Axiom 2 holds in $\mathcal{K}$, we can use Lemma 6.5 to get $d \in K$ such that $v(d) = 0$ and $G(d) = 0$ where

$$G(x) := \sigma(x) - \frac{b}{\sigma(b)} \cdot x.$$

Note that $v(qd) = 0$ and $G(qd) = 0$ for all $q \in \mathbb{Q}^\times \subseteq E^\times$. Hence by saturation we can assume that $v(d) = 0$, $G(d) = 0$ and $\bar{d}$ is transcendental over $\boldsymbol{k}_{\mathcal{E}}$. We set $\alpha = \bar{d}$, so $\bar{G}(x)$ is a minimal $\bar{\sigma}$-polynomial of α over $\boldsymbol{k}_{\mathcal{E}}$. By Lemma 2,

$$E\langle d\rangle = E(d), \qquad v(E(d)^\times) = v(E^\times), \qquad \pi(\mathcal{O}_{E(d)}) = \boldsymbol{k}_{\mathcal{E}}(\alpha), \qquad \sigma(E(d)) = E(d).$$

We shall find a good map extending $\boldsymbol{f}$ with domain $(E(d), \Gamma_{\mathcal{E}}, \boldsymbol{k}_{\mathcal{E}}(\alpha))$. Consider the σ-polynomial $H := f(G)$, that is,

$$H(x) = \sigma(x) - \frac{f(b)}{\sigma(f(b))} \cdot x.$$

By saturation we can find $\alpha' \in \boldsymbol{k}'$ with $\bar{H}(\alpha') = 0$ and a difference field isomorphism $g_{\mathrm{r}} : \boldsymbol{k}_{\mathcal{E}}(\alpha) \to \boldsymbol{k}_{\mathcal{E}'}(\alpha')$ that extends f_{r}, sends α to α' and is elementary as a partial map between the difference fields $\boldsymbol{k}$ and $\boldsymbol{k}'$. Using again Lemma 6.5 we find $d' \in K'$ such that $v'(d') = 0$ and $H(d') = 0$. Since $\bar{H}(\bar{d}') = \bar{H}(\alpha') = 0$, we can multiply d' by an element in K' of valuation zero and fixed by σ to assume further that $\bar{d}' = \alpha'$. Then Lemmas 2 and 6.4 yield a good map $\boldsymbol{g} = (g, f_{\mathrm{v}}, g_{\mathrm{r}})$ where $g : E(d) \to E'(d')$ extends f and sends d to d'. The domain $(E(d), \Gamma_{\mathcal{E}}, \boldsymbol{k}_{\mathcal{E}}(\alpha))$ of $\boldsymbol{g}$ is small.

In the extension procedures (3) and (4) the value group $v(E^\times)$ does not change, so if the domain $\mathcal{E}$ of $\boldsymbol{f}$ satisfies Axiom 2, then so does the domain of any extension of $\boldsymbol{f}$ constructed as in (3) or (4). Also $\Gamma_{\mathcal{E}}$ does not change in (3) and (4), but at this stage we can have $\Gamma_{\mathcal{E}} \neq v(E^\times)$. By repeated application of (1)–(4) we can arrange that $\mathcal{E}$ is workable and satisfies Axiom 4. Then by Corollary 5.10 we can arrange that in addition $\mathcal{E}$ is σ-henselian. (Any use of this in what follows will be explicitly indicated.)

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(5) *Towards arranging $\Gamma_{\mathcal{E}} = v(E^\times)$; the case of no torsion modulo $v(E^\times)$.*

Suppose $\gamma \in \Gamma_{\mathcal{E}}$ has no torsion modulo $v(E^\times)$, that is, $n\gamma \notin v(E^\times)$ for all $n > 0$. Take $a \in \mathrm{Fix}(K)$ such that $v(a) = \gamma$. Let i be a σ-lifting of the residue difference field $\boldsymbol{k}$ to $\mathcal{K}$. Since $\mathrm{ac}(a)$ is fixed by $\bar{\sigma}$, $a/i(\mathrm{ac}(a)) \in \mathrm{Fix}(K)$ and $v(a/i(\mathrm{ac}(a))) = \gamma$. So replacing a by $a/i(\mathrm{ac}(a))$ we arrange that $v(a) = \gamma$ and $\mathrm{ac}(a) = 1$. In the same way we obtain $a' \in \mathrm{Fix}(K')$ such that $v'(a') = \gamma' := f_{\mathrm{v}}(\gamma)$ and $\mathrm{ac}'(a') = 1$. Then by a familiar fact from the valued field context we have an isomorphism of valued fields $g : E(a) \to E'(a')$ extending f with $g(a) = a'$. Then $(g, f_{\mathrm{v}}, f_{\mathrm{r}})$ is a good map with small domain $(E(a), \Gamma_{\mathcal{E}}, \boldsymbol{k}_{\mathcal{E}})$; this domain satisfies Axiom 2 if $\mathcal{E}$ does.

(6) *Towards arranging $\Gamma_{\mathcal{E}} = v(E^\times)$; the case of prime torsion modulo $v(E^\times)$.* Here we assume that $\mathcal{E}$ satisfies Axiom 2 and is σ-henselian.

Let $\gamma \in \Gamma_{\mathcal{E}} \setminus v(E^\times)$ with $\ell\gamma \in v(E^\times)$, where ℓ is a prime number. As $\mathcal{E}$ satisfies Axiom 2 we can pick $b \in \mathrm{Fix}(E)$ such that $v(b) = \ell\gamma$. Since $\mathcal{E}$ is σ-henselian we have a σ-lifting of its difference residue field $\boldsymbol{k}_{\mathcal{E}}$ to $\mathcal{E}$ and we can use this as in (5) to arrange that $\mathrm{ac}(b) = 1$. We shall find $c \in \mathrm{Fix}(K)$ such that $c^\ell = b$ and $\mathrm{ac}(c) = 1$. As in (5) we have $a \in \mathrm{Fix}(K)$ such that $v(a) = \gamma$ and $\mathrm{ac}(a) = 1$. Then the polynomial $P(x) := x^\ell - b/a^\ell$ over K is henselian at 1. This gives $u \in K$ such that $P(u) = 0$ and $\bar{u} = 1$. Now let $c = au$. Clearly, $c^\ell = b$ and $\mathrm{ac}(c) = 1$. Note that $\sigma(c)^\ell = b$, hence $\sigma(c) = \omega c$ where ω is an ℓth root of unity. Using $\mathrm{ac}(c) = 1$ we get $\mathrm{ac}(\omega) = 1$, so $\omega = 1$, that is, $c \in \mathrm{Fix}(K)$, as promised. Likewise we find $c' \in \mathrm{Fix}(K')$ such that $c'^\ell = f(b)$ and $\mathrm{ac}'(c') = 1$. Then $\boldsymbol{f}$ extends easily to a good map with domain $(E(c), \Gamma_{\mathcal{E}}, \boldsymbol{k}_{\mathcal{E}})$ sending c to c'; this domain satisfies Axiom 2.

By iterating (5) and (6) we can assume in the rest of the proof that $\Gamma_{\mathcal{E}} = v(E^\times)$, and we shall do so. This condition is actually preserved in the earlier extension procedures (3) and (4), as the reader may easily verify. Anyway, we can refer from now on to $\Gamma_{\mathcal{E}}$ as the *value group* of E. Note also that in the extension procedures (5) and (6) the residue difference field does not change.

Now let $a \in K$ be given. We want to extend $\boldsymbol{f}$ to a good map whose domain is small and contains a. At this stage we can assume $\boldsymbol{k}_{\mathcal{E}} = \pi(\mathcal{O}_E)$, $\Gamma_{\mathcal{E}} = v(E^\times)$, and $\mathcal{E}$ is workable. Appropriately iterating and alternating the above extension procedures we arrange in addition that $\mathcal{E}$ satisfies Axiom 4 and $E\langle a\rangle$ is an immediate extension of E. Let $\mathcal{E}\langle a\rangle$ be the valued difference subfield of $\mathcal{K}$ that has $E\langle a\rangle$ as underlying difference field. By Corollary 5.10, $\mathcal{E}\langle a\rangle$ has a maximal immediate valued difference field extension $\mathcal{E}_1 \leqslant \mathcal{K}$. Then $\mathcal{E}_1$ is a maximal immediate extension of $\mathcal{E}$ as well. Applying Corollary 5.10 to $\mathcal{E}'$ and using Theorem 5.8, we can extend $\boldsymbol{f}$ to a good map with domain $\mathcal{E}_1$, construed here as a good substructure of $\mathcal{K}$ in the obvious way. It remains to note that a is in the underlying difference field of $\mathcal{E}_1$. □

A variant

At the cost of a purity assumption we can eliminate angular component maps in the Equivalence Theorem. More precisely, let $\mathcal{K}$, $\mathcal{K}'$ be as before except that we do not require

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angular component maps as part of these structures. The notion of *good substructure* of $\mathcal{K}$ is similarly modified by changing clause (3) in its definition to '$\boldsymbol{k}_{\mathcal{E}}$ is a difference subfield of $\boldsymbol{k}$ with $\pi(\mathcal{O}_E) \subseteq \boldsymbol{k}_{\mathcal{E}}$'. In defining good maps, condition (ii) on f_{r} is to be changed to '$f_{\mathrm{r}}(\pi(a)) = \pi'(f(a))$ for all $a \in \mathcal{O}_E$', and f_{r} is elementary as a partial map between the difference fields $\boldsymbol{k}$ and $\boldsymbol{k}'$.

Theorem 6.7. *Suppose* $\mathrm{char}(\boldsymbol{k}) = 0$, $\mathcal{K}$, $\mathcal{K}'$ *satisfy Axiom 2 and are* σ*-henselian, and* $v(E^\times)$ *is pure in* Γ*. Then any good map* $\mathcal{E} \to \mathcal{E}'$ *is a partial elementary map between* $\mathcal{K}$ *and* $\mathcal{K}'$*.*

Proof. The case $\Gamma = \{0\}$ being trivial, let $\Gamma \neq \{0\}$, and let $\boldsymbol{f} : \mathcal{E} \to \mathcal{E}'$ be a good map; our task is to show that $\boldsymbol{f}$ is a partial elementary map between $\mathcal{K}$ and $\mathcal{K}'$. We first arrange that the valued difference subfield $(E, v(E^\times), \pi(\mathcal{O}_E); \dots)$ of $\mathcal{K}$ is $\aleph_1$-saturated by passing to an elementary extension of a suitable many-sorted structure with $\mathcal{K}$, $\mathcal{K}'$, $\mathcal{E}$, $\mathcal{E}'$ and $\boldsymbol{f}$ as ingredients. As in the beginning of the proof of Theorem 6.6 we arrange next that $\mathcal{K}$ and $\mathcal{K}'$ are κ-saturated, where κ is an uncountable cardinal such that $|\boldsymbol{k}_{\mathcal{E}}|, |\Gamma_{\mathcal{E}}| < \kappa$. Then we apply the extension procedures (2) and (3) in the proof of Theorem 6.6 to arrange that $\boldsymbol{k}_{\mathcal{E}} = \pi(\mathcal{O}_E)$ and $\mathcal{E}$ satisfies Axiom 2, without changing $v(E^\times)$. To simplify notation we identify $\mathcal{E}$ and $\mathcal{E}'$ via $\boldsymbol{f}$; we have to show that then $\mathcal{K} \equiv_{\mathcal{E}} \mathcal{K}'$. Since $(E, v(E^\times), \pi(\mathcal{O}_E); \dots)$ is $\aleph_1$-saturated, Lemmas 2.3 and 2.4 yield cross-sections

$$s_E : v(E^\times) \to \mathrm{Fix}(E)^\times, \qquad s : \Gamma \to \mathrm{Fix}(K)^\times, \qquad s' : \Gamma' \to \mathrm{Fix}(K')^\times$$

such that s and s' extend s_E. These cross-sections induce angular component maps ac_E on $\mathrm{Fix}(E)$, ac on $\mathrm{Fix}(K)$, and ac' on $\mathrm{Fix}(K')$, which by Lemma 6.1 extend uniquely to angular component maps on $\mathcal{E}$, $\mathcal{K}$ and $\mathcal{K}'$. (Here we use that $\mathcal{E}$ satisfies Axiom 2.) This allows us to apply Theorem 6.6 to obtain the desired conclusion. $\square$

7. Relative quantifier elimination

Here we derive various consequences of the Equivalence Theorem of § 6. We use the symbols $\equiv$ and $\preceq$ for the relations of elementary equivalence and being an elementary submodel, in the setting of many-sorted structures, and 'definable' means 'definable with parameters from the ambient structure'. Let $\mathcal{L}$ be the 3-sorted language of valued fields, with sorts f (the field sort), v (the value group sort), and r (the residue sort). We view a valued field $(K, \Gamma, \boldsymbol{k}; \dots)$ as an $\mathcal{L}$-structure, with f-variables ranging over K, v-variables over Γ and r-variables over $\boldsymbol{k}$. Augmenting $\mathcal{L}$ with a function symbol σ of sort (f, f) gives the language $\mathcal{L}(\sigma)$ of valued difference fields, and augmenting it further with a function symbol ac of sort (f, r) gives the language $\mathcal{L}(\sigma, \mathrm{ac})$ of ac-valued difference fields. In this section,

$$\mathcal{K} = (K, \Gamma, \boldsymbol{k}; \dots), \qquad \mathcal{K}' = (K', \Gamma', \boldsymbol{k}'; \dots)$$

are ac-valued difference fields of equicharacteristic 0 that satisfy Axiom 2 and are σ-henselian; they are considered as $\mathcal{L}(\sigma, \mathrm{ac})$-structures.

Corollary 7.1. $\mathcal{K} \equiv \mathcal{K}'$ *if and only if* $\boldsymbol{k} \equiv \boldsymbol{k}'$ *as difference fields and* $\Gamma \equiv \Gamma'$ *as ordered abelian groups.*

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Proof. The 'only if' direction is obvious. Suppose $\boldsymbol{k} \equiv \boldsymbol{k}'$ as difference fields, and $\Gamma \equiv \Gamma'$ as ordered groups. This gives good substructures $\mathcal{E} := (\mathbb{Q}, \{0\}, \mathbb{Q})$ of $\mathcal{K}$, and $\mathcal{E}' := (\mathbb{Q}, \{0\}, \mathbb{Q})$ of $\mathcal{K}'$, and a trivial good map $\mathcal{E} \to \mathcal{E}'$. Now apply Theorem 6.6. □

Thus $\mathcal{K}$ is elementarily equivalent to the Hahn difference field $\boldsymbol{k}((t^\Gamma))$ with angular component map defined in the beginning of § 6.

Corollary 7.2. *Let $\mathcal{E} = (E, \Gamma_E, \boldsymbol{k}_E; \dots)$ be a σ-henselian ac-valued difference subfield of $\mathcal{K}$ satisfying Axiom 2 such that $\boldsymbol{k}_E \preceq \boldsymbol{k}$ as difference fields, and $\Gamma_E \preceq \Gamma$ as ordered abelian groups. Then $\mathcal{E} \preceq \mathcal{K}$.*

Proof. Take an elementary extension $\mathcal{K}'$ of $\mathcal{E}$. Then $\mathcal{K}'$ satisfies Axiom 2, $(E, \Gamma_E, \boldsymbol{k}_E)$ is a good substructure of both $\mathcal{K}$ and $\mathcal{K}'$, and the identity on $(E, \Gamma, \boldsymbol{k}_E)$ is a good map. Hence by Theorem 6.6 we have $\mathcal{K} \equiv_{\mathcal{E}} \mathcal{K}'$. Since $\mathcal{E} \preceq \mathcal{K}'$, this gives $\mathcal{E} \preceq \mathcal{K}$. □

The proofs of these corollaries use only weak forms of the Equivalence Theorem, but now we turn to a result that uses its full strength: a relative elimination of quantifiers for the $\mathcal{L}(\sigma, \mathrm{ac})$-theory T of σ-henselian ac-valued difference fields of equicharacteristic 0 that satisfy Axiom 2. We specify that the function symbols v and π of $\mathcal{L}(\sigma, \mathrm{ac})$ are to be interpreted as *total* functions in any $\mathcal{K}$ as follows: extend $v : K^\times \to \Gamma$ to $v : K \to \Gamma$ by $v(0) = 0$, and extend $\pi : \mathcal{O} \to \boldsymbol{k}$ to $\pi : K \to \boldsymbol{k}$ by $\pi(a) = 0$ for $a \notin \mathcal{O}$.

Let $\mathcal{L}_{\mathrm{r}}$ be the sublanguage of $\mathcal{L}(\sigma, \mathrm{ac})$ involving only the sort r, that is, $\mathcal{L}_{\mathrm{r}}$ is a copy of the language of difference fields, with $\bar{\sigma}$ as the symbol for the difference operator. Let $\mathcal{L}_{\mathrm{v}}$ be the sublanguage of $\mathcal{L}(\sigma, \mathrm{ac})$ involving only the sort v, that is, $\mathcal{L}_{\mathrm{v}}$ is the language of ordered abelian groups.

Let $x = (x_1, \dots, x_l)$ be a tuple of distinct f-variables, $y = (y_1, \dots, y_m)$ a tuple of distinct r-variables, and $z = (z_1, \dots, z_n)$ a tuple of distinct v-variables. Define a *special* r*-formula in* (x, y) to be an $\mathcal{L}(\sigma, \mathrm{ac})$-formula

$$\psi(x, y) := \psi'(\mathrm{ac}(q_1(x)), \dots, \mathrm{ac}(q_k(x)), y),$$

where $k \in \mathbb{N}$, $\psi'(u_1, \dots, u_k, y)$ is an $\mathcal{L}_{\mathrm{r}}$-formula, and $q_1(x), \dots, q_k(x) \in \mathbb{Z}[x]$. Also, a *special* v*-formula in* (x, z) is an $\mathcal{L}(\sigma, \mathrm{ac})$-formula

$$\theta(x, z) := \theta'(v(q_1(x)), \dots, v(q_k(x)), z),$$

where $k \in \mathbb{N}$, $\theta'(v_1, \dots, v_k, y)$ is an $\mathcal{L}_{\mathrm{v}}$-formula, and $q_1(x), \dots, q_k(x) \in \mathbb{Z}[x]$. Note that these special formulae do not have quantified f-variables. We can now state our relative quantifier elimination.

Corollary 7.3. *Every $\mathcal{L}(\sigma, \mathrm{ac})$-formula $\phi(x, y, z)$ is T-equivalent to a boolean combination of special* r*-formulae in (x, y) and special* v*-formulae in (x, z).*

Proof. Let $\psi(x, y)$ and $\theta(x, z)$ range over special formulae as described above. For a model $\mathcal{K} = (K, \Gamma, \boldsymbol{k}; \dots)$ of T and $a \in K^l$, $r \in \boldsymbol{k}^m$, $\gamma \in \Gamma^n$, let

$$\mathrm{tp}_{\mathrm{r}}^{\mathcal{K}}(a, r) := \{\psi(x, y) : \mathcal{K} \models \psi(a, r)\},$$
$$\mathrm{tp}_{\mathrm{v}}^{\mathcal{K}}(a, \gamma) := \{\theta(x, z) : \mathcal{K} \models \theta(a, \gamma)\}.$$

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Let $\mathcal{K}$ and $\mathcal{K}'$ be any models of T, and let

$$(a, r, \gamma) \in K^l \times \boldsymbol{k}^m \times \Gamma^n, \qquad (a', r', \gamma') \in K'^l \times \boldsymbol{k}'^m \times \Gamma'^n$$

be such that

$$\mathrm{tp}_{\mathrm{r}}^{\mathcal{K}}(a, r) = \mathrm{tp}_{\mathrm{r}}^{\mathcal{K}'}(a', r')$$

and

$$\mathrm{tp}_{\mathrm{v}}^{\mathcal{K}}(a, \gamma) = \mathrm{tp}_{\mathrm{v}}^{\mathcal{K}'}(a', \gamma').$$

It suffices to show that under these assumptions we have

$$\mathrm{tp}^{\mathcal{K}}(a, r, \gamma) = \mathrm{tp}^{\mathcal{K}'}(a', r', \gamma').$$

Let $\mathcal{E} := (E, \Gamma_{\mathcal{E}}, \boldsymbol{k}_{\mathcal{E}})$, where $E := \mathbb{Q}\langle a \rangle$, $\Gamma_{\mathcal{E}}$ is the ordered subgroup of Γ generated by γ over $v(E^\times)$, and $\boldsymbol{k}_{\mathcal{E}}$ is the difference subfield of $\boldsymbol{k}$ generated by $\mathrm{ac}(E)$ and r, so $\mathcal{E}$ is a good substructure of $\mathcal{K}$. Likewise we define the good substructure $\mathcal{E}'$ of $\mathcal{K}'$. For each $q(x) \in \mathbb{Z}[x]$ we have $q(a) = 0$ if and only if $\mathrm{ac}(q(a)) = 0$, and also $q(a') = 0$ if and only if $\mathrm{ac}'(q(a')) = 0$. In view of this fact, the assumptions give us a good map $\mathcal{E} \to \mathcal{E}'$ sending a to a', γ to γ' and r to r'. It remains to apply Theorem 6.6. □

In the proof above it is important that our notion of a good substructure $\mathcal{E} = (E, \Gamma_{\mathcal{E}}, \boldsymbol{k}_{\mathcal{E}})$ did not require $\Gamma_{\mathcal{E}} = v(E^\times)$ or $\boldsymbol{k}_{\mathcal{E}} = \pi(\mathcal{O}_E)$. This is a difference with the treatment in [**3**]. Related to it is that in Corollary 7.3 we have a separation of r- and v-variables; this makes the next result almost obvious.

Corollary 7.4. *Each subset of $\boldsymbol{k}^m \times \Gamma^n$ definable in $\mathcal{K}$ is a finite union of rectangles $X \times Y$ with $X \subseteq \boldsymbol{k}^m$ definable in the difference field $\boldsymbol{k}$ and $Y \subseteq \Gamma^n$ definable in the ordered abelian group Γ.*

Proof. By Corollary 7.3 and using its notation it is enough to observe that, for $a \in K^l$, a special r-formula $\psi(x, y)$ in (x, y), and a special v-formula $\theta(x, z)$ in (x, z), the set $\{r \in \boldsymbol{k}^m : \mathcal{K} \models \psi(a, r)\}$ is definable in the difference field $\boldsymbol{k}$ and the set $\{\gamma \in \Gamma^n : \mathcal{K} \models \theta(a, \gamma)\}$ is definable in the ordered abelian group Γ. □

Corollary 7.4 says in particular that the relations on $\boldsymbol{k}$ definable in $\mathcal{K}$ are definable in the difference field $\boldsymbol{k}$, and likewise, the relations on Γ definable in $\mathcal{K}$ are definable in the ordered abelian group Γ. Thus $\boldsymbol{k}$ and Γ are stably embedded in $\mathcal{K}$. The corollary says in addition that $\boldsymbol{k}$ and Γ are orthogonal in $\mathcal{K}$.

By Corollary 6.2 we can get rid of angular component maps in Corollaries 7.1 and 7.4: these go through if we replace 'ac-valued' by 'valued'. Also Corollary 7.2 goes through with this change, but for this we need Theorem 6.7. In particular, any σ-henselian valued difference field satisfying Axiom 2, with residue difference field $\boldsymbol{k}$ of characteristic 0 and value group Γ, is elementarily equivalent to the Hahn difference field $\boldsymbol{k}(\!(t^\Gamma)\!)$.

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8. The unramified mixed characteristic case

We now aim for mixed characteristic analogues of §§ 6 and 7. Kochen [**8**] has a clear account how a result like Corollary 7.1 for henselian valued fields can be obtained in mixed characteristic from the equicharacteristic zero case by coarsening. We follow here the same track, but to get the mixed characteristic Equivalence Theorem 8.8 we use an elementary fact (Lemma 8.6) in a way that may be new and yields a proof that differs from the rather complicated treatment in [**3**].

A better equivalence theorem

We first improve Theorem 6.6 by allowing extra structure on the residue difference field and on the value group.

Let $\mathcal{L}$ be the 3-sorted language of valued fields and $\mathcal{L}(\sigma, \mathrm{ac})$ the language of ac-valued difference fields, as introduced in § 7. Consider now a language $\mathcal{L}^* \supseteq \mathcal{L}(\sigma, \mathrm{ac})$ such that every symbol of $\mathcal{L}^* \setminus \mathcal{L}(\sigma, \mathrm{ac})$ is a relation symbol of some sort $(\mathrm{v}, \dots, \mathrm{v})$ or $(\mathrm{r}, \dots, \mathrm{r})$. Let $\mathcal{L}^*_{\mathrm{v}}$ be the sublanguage of $\mathcal{L}^*$ involving only the sort v, that is, the language of ordered abelian groups together with the new relation symbols of sort $(\mathrm{v}, \dots, \mathrm{v})$. Also, let $\mathcal{L}^*_{\mathrm{r}}$ be the sublanguage of $\mathcal{L}^*$ involving only the sort r, that is, (a copy of) the language of difference fields together with the new relation symbols of sort $(\mathrm{r}, \dots, \mathrm{r})$. (The difference operator symbol of $\mathcal{L}^*_{\mathrm{r}}$ is $\bar{\sigma}$, to avoid confusion with the difference operator symbol σ of sort (f, f).) By a $*$-valued difference field we mean an $\mathcal{L}^*$-structure whose $\mathcal{L}(\sigma, \mathrm{ac})$-reduct is an ac-valued difference field.

Let $\mathcal{K} = (K, \Gamma, \boldsymbol{k}; \dots)$ be a $*$-valued difference field. Then we shall view Γ as an $\mathcal{L}^*_{\mathrm{v}}$-structure and $\boldsymbol{k}$ as an $\mathcal{L}^*_{\mathrm{r}}$-structure, in the obvious way. Any subfield E of K is viewed as a valued subfield of $\mathcal{K}$ with valuation ring $\mathcal{O}_E := \mathcal{O} \cap E$.

A *good substructure* of $\mathcal{K} = (K, \Gamma, \boldsymbol{k}; \dots)$ is a triple $\mathcal{E} = (E, \Gamma_{\mathcal{E}}, \boldsymbol{k}_{\mathcal{E}})$ such that

(1) E is a difference subfield of K,

(2) $\Gamma_{\mathcal{E}} \subseteq \Gamma$ as $\mathcal{L}^*_{\mathrm{v}}$-structures with $v(E^{\times}) \subseteq \Gamma_{\mathcal{E}}$,

(3) $\boldsymbol{k}_{\mathcal{E}} \subseteq \boldsymbol{k}$ as $\mathcal{L}^*_{\mathrm{r}}$-structures with $\mathrm{ac}(E) \subseteq \boldsymbol{k}_{\mathcal{E}}$.

In the rest of this subsection $\mathcal{K} = (K, \Gamma, \boldsymbol{k}; \dots)$ and $\mathcal{K}' = (K', \Gamma', \boldsymbol{k}'; \dots)$ are $*$-valued difference fields, and $\mathcal{E} = (E, \Gamma_{\mathcal{E}}, \boldsymbol{k}_{\mathcal{E}})$, $\mathcal{E}' = (E', \Gamma_{\mathcal{E}'}, \boldsymbol{k}_{\mathcal{E}'})$ are good substructures of $\mathcal{K}$, $\mathcal{K}'$ respectively.

A *good map* $\boldsymbol{f} : \mathcal{E} \to \mathcal{E}'$ is a triple $\boldsymbol{f} = (f, f_{\mathrm{v}}, f_{\mathrm{r}})$ consisting of an isomorphism $f : E \to E'$ of difference fields, an isomorphism $f_{\mathrm{v}} : \Gamma_{\mathcal{E}} \to \Gamma_{\mathcal{E}'}$ of $\mathcal{L}^*_{\mathrm{v}}$-structures and an isomorphism $f_{\mathrm{r}} : \boldsymbol{k}_{\mathcal{E}} \to \boldsymbol{k}_{\mathcal{E}'}$ of $\mathcal{L}^*_{\mathrm{r}}$-structures such that

(i) $f_{\mathrm{v}}(v(a)) = v'(f(a))$ for all $a \in E^{\times}$, and f_{v} is elementary as a partial map between the $\mathcal{L}^*_{\mathrm{v}}$-structures Γ and Γ';

(ii) $f_{\mathrm{r}}(\mathrm{ac}(a)) = \mathrm{ac}'(f(a))$ for all $a \in E$, and f_{r} is elementary as a partial map between the $\mathcal{L}^*_{\mathrm{r}}$-structures $\boldsymbol{k}$ and $\boldsymbol{k}'$.

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Theorem 6.6 goes through in this enriched setting, with the same proof except for obvious changes.*

Theorem 8.1. *If* $\operatorname{char}(\boldsymbol{k}) = 0$ *and* $\mathcal{K}$, $\mathcal{K}'$ *satisfy Axiom 2 and are* σ*-henselian, then any good map* $\mathcal{E} \to \mathcal{E}'$ *is a partial elementary map between* $\mathcal{K}$ *and* $\mathcal{K}'$*.*

The four corollaries of § 8 also go through in this enriched setting, with residue difference fields and value groups replaced by their $\mathcal{L}_{\mathrm{r}}^*$-expansions and $\mathcal{L}_{\mathrm{v}}^*$-expansions, respectively. In the notions used in Corollary 7.3 the roles of $\mathcal{L}_{\mathrm{r}}$ and $\mathcal{L}_{\mathrm{v}}$ are of course taken over by $\mathcal{L}_{\mathrm{r}}^*$ and $\mathcal{L}_{\mathrm{v}}^*$, respectively. Except for obvious changes the proofs are the same as in § 8, using Theorem 8.1 in place of Theorem 6.6.

A variant

In dealing with the mixed characteristic case it is useful to eliminate angular component maps in Theorem 8.1. So let $\mathcal{K}$, $\mathcal{K}'$ be as in the previous subsection except that we do not require angular component maps as part of these structures. The notion of *good substructure* of $\mathcal{K}$ is then modified by replacing in clause (3) of its definition the condition $\operatorname{ac}(E) \subseteq \boldsymbol{k}_{\mathcal{E}}$ by $\pi(\mathcal{O}_E) \subseteq \boldsymbol{k}_{\mathcal{E}}$. In defining the notion of a good map $\boldsymbol{f} = (f, f_{\mathrm{v}}, f_{\mathrm{r}}) : \mathcal{E} \to \mathcal{E}'$ the condition on f_{r} is to be changed to $f_{\mathrm{r}}(\pi(a)) = \pi(f(a))$ for all $a \in \mathcal{O}_E$, and f_{r} is elementary as a partial map between the $\mathcal{L}_{\mathrm{r}}^*$-structures $\boldsymbol{k}$ and $\boldsymbol{k}'$. Then the same arguments as we used in proving Theorem 6.7 yield the following.

Theorem 8.2. *If* $\operatorname{char}(\boldsymbol{k}) = 0$ *and* $\mathcal{K}$ *and* $\mathcal{K}'$ *satisfy Axiom 2 and are* σ*-henselian, and* $\mathcal{E}$ *and* $\mathcal{E}'$ *are good substructures of* $\mathcal{K}$ *and* $\mathcal{K}'$*, respectively, with* $v(E^\times)$ *pure in* Γ*, then any good map* $\mathcal{E} \to \mathcal{E}'$ *is a partial elementary map between* $\mathcal{K}$ *and* $\mathcal{K}'$*.*

Coarsening

To reduce the mixed characteristic case to the equal characteristic zero case we use coarsening. In this subsection $\mathcal{K} = (K, \Gamma, \boldsymbol{k}; \dots)$ is a valued difference field. Let Δ be a convex subgroup of Γ, let $\dot{\Gamma} := \Gamma/\Delta$ be the ordered quotient group, and let $\dot{v} : K^\times \to \dot{\Gamma}$ be the composition $K^\times \to \Gamma \to \dot{\Gamma}$ of v with the canonical map $\Gamma \to \dot{\Gamma}$, so $\dot{v}$ is again a valuation. Let $\dot{\mathcal{O}}$ be the valuation ring of $\dot{v}$, and $\dot{\mathfrak{m}}$ its maximal ideal, so

$$\begin{aligned} \dot{\mathcal{O}} &= \{x \in K : v(x) \geqslant \delta, \text{ for some } \delta \in \Delta\} \supseteq \mathcal{O} := \mathcal{O}_v, \\ \dot{\mathfrak{m}} &= \{x \in K : v(x) > \Delta\} \subseteq \mathfrak{m}. \end{aligned}$$

Let $\dot{\boldsymbol{k}} = \dot{\mathcal{O}}/\dot{\mathfrak{m}}$ be the residue field for $\dot{v}$ and let $\dot{\pi} : \dot{\mathcal{O}} \to \dot{\boldsymbol{k}}$ be the canonical map. This gives a valued difference field $\dot{\mathcal{K}} := (K, \dot{\Gamma}, \dot{\boldsymbol{k}}; \dot{v}, \dot{\pi})$ satisfying Axiom 1. Some other axioms are also preserved.

Lemma 8.3. *If* $\mathcal{K}$ *satisfies Axiom 2, so does* $\dot{\mathcal{K}}$*. If* $\mathcal{K}$ *satisfies Axiom 2 and is* σ*-henselian, then* $\dot{\mathcal{K}}$ *is* σ*-henselian.*

* The referee informed us that Theorem 8.1 is also a formal consequence of Theorem 6.6 and the stable embeddedness and orthogonality coming from Corollary 7.4.

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Proof. The claim about Axiom 2 is obvious. Assume $\mathcal{K}$ satisfies Axiom 2 and is σ-henselian. Let $G(x)$ over $\dot{\mathcal{O}}$ (of order at most n) be σ-henselian at $a \in \dot{\mathcal{O}}$, with respect to $\dot{\mathcal{K}}$. It is easy to check that then G, a is in σ-hensel configuration with respect to $\mathcal{K}$. Lemma 4.10 and Remark 4.6 then yield $b \in K$ such that $v(a-b) = v(G(a)) - \min_{|\boldsymbol{i}|=1} v(G_{(\boldsymbol{i})}(a))$, and thus $\dot{v}(a-b) = \dot{v}(G(a))$, as desired. □

Let $\dot{\sigma}$ be the automorphism of the field $\dot{\boldsymbol{k}}$ induced by the difference operator σ of $\dot{\mathcal{K}}$. The field $\dot{\boldsymbol{k}}$ carries the valuation $v_\Delta : \dot{\boldsymbol{k}}^\times \to \Delta$ given by $v_\Delta(x + \dot{\mathfrak{m}}) = v(x)$ for x a unit of $\dot{\mathcal{O}}$. The valuation ring of v_Δ is $\dot{\pi}(\mathcal{O})$, and we have the surjective ring morphism $\pi_\Delta : \dot{\pi}(\mathcal{O}) \to \boldsymbol{k}$ given by $\pi_\Delta(\dot{\pi}(a)) = \pi(a)$ for all $a \in \mathcal{O}$. Note that

$$((\dot{\boldsymbol{k}}, \dot{\sigma}), \Delta, \boldsymbol{k}; v_\Delta, \pi_\Delta)$$

is a valued difference field satisfying Axiom 1 with $\dot{\sigma}$ inducing on the residue field $\boldsymbol{k}$ the same automorphism $\bar{\sigma}$ as the difference operator σ of $\mathcal{K}$ does. The following is now immediate.

Lemma 8.4. *If $\mathcal{K}$ satisfies Axiom 3, so does $\dot{\mathcal{K}}$.*

Let $\dot{\boldsymbol{k}}(*)$ be the expansion $(\dot{\boldsymbol{k}}, \dot{\sigma}, \dot{\pi}(\mathcal{O}))$ of the difference field $(\dot{\boldsymbol{k}}, \dot{\sigma})$, and let $\dot{\mathcal{K}}(*)$ be the corresponding expansion $(K, \dot{\Gamma}, \dot{\boldsymbol{k}}(*); \dot{v}, \dot{\pi})$ of $\dot{\mathcal{K}}$. Note that $\mathcal{O}$ is definable in the structure $\dot{\mathcal{K}}(*)$ by a formula that does not depend on $\mathcal{K}$:

$$\mathcal{O} = \{a \in \dot{\mathcal{O}} : \dot{\pi}(a) \in \dot{\pi}(\mathcal{O})\}.$$

In this way we reconstruct $\mathcal{K}$ from $\dot{\mathcal{K}}(*)$. The advantage of working with $\dot{\mathcal{K}}(*)$ is that it has equicharacteristic 0 if $\mathcal{K}$ has mixed characteristic and $v(p) \in \Delta$.

Now let $\mathcal{K}$ be unramified with $\operatorname{char} K = 0$, $\operatorname{char} \boldsymbol{k} = p > 0$. Then $\mathbb{Z} \cdot v(p)$ is a convex subgroup of Γ. We set $\Delta := \mathbb{Z} \cdot v(p)$ and note that then $\operatorname{char} \dot{\boldsymbol{k}} = 0$. With these assumptions we have the following.

Lemma 8.5. *If $\mathcal{K}$ is workable, so is $\dot{\mathcal{K}}$.*

Proof. Suppose $\mathcal{K}$ is workable. Then either $\mathcal{K}$ satisfies Axioms 2 and 3 or it satisfies Axiom 2 and is a Witt case with infinite $\boldsymbol{k}$. In the first case, $\dot{\mathcal{K}}$ also satisfies Axioms 2 and 3 by Lemmas 8.3 and 8.4, and is thus workable. It remains to consider the case that $\mathcal{K}$ satisfies Axiom 2 and is a Witt case with infinite $\boldsymbol{k}$. Then $\dot{\mathcal{K}}$ satisfies Axiom 2 by Lemma 8.3, and because $\boldsymbol{k}$ is infinite and $\bar{\sigma}$ is the Frobenius map, we have $\bar{\sigma}^d \neq \mathrm{id}$ for all $d > 0$, and thus $\dot{\sigma}^d \neq \mathrm{id}$ for all $d > 0$. So $\dot{\mathcal{K}}$ satisfies Axiom 3 as well. □

Keeping the assumptions preceding Lemma 8.5, assume also that $\mathcal{K}$ is $\aleph_1$-saturated and $\boldsymbol{k}$ is perfect. Then the saturation assumption guarantees that $\dot{\pi}(\mathcal{O})$ is a complete discrete valuation ring of $\dot{\boldsymbol{k}}$. Since $\boldsymbol{k}$ is perfect, this gives a unique ring isomorphism $\iota : \mathrm{W}[\boldsymbol{k}] \cong \dot{\pi}(\mathcal{O})$ such that $\pi_\Delta \circ \iota : \mathrm{W}[\boldsymbol{k}] \to \boldsymbol{k}$ is the projection map $(a_0, a_1, a_2, \dots) \mapsto a_0$. Denote the extension of ι to a field isomorphism $\mathrm{W}(\boldsymbol{k}) \cong \dot{\boldsymbol{k}}$ also by ι. If $\mathcal{K}$ is a Witt case, this gives an isomorphism $(\iota, \dots)$ of the Witt difference field $\mathrm{W}(\boldsymbol{k})$ onto $(\dot{\boldsymbol{k}}, \Delta, \boldsymbol{k}; v_\Delta, \pi_\Delta)$.

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Two lemmas

The proof of the next lemma uses mainly the functoriality of W.

Lemma 8.6. *Let $\boldsymbol{k}_0$ be a perfect field with* $\operatorname{char}(\boldsymbol{k}_0) = p > 0$, *let $\boldsymbol{k}$ and $\boldsymbol{k}'$ be perfect extension fields of $\boldsymbol{k}_0$, and let σ and σ' be automorphisms of $\boldsymbol{k}$ and $\boldsymbol{k}'$, respectively. Let κ be an uncountable cardinal such that the difference fields $(\boldsymbol{k}, \sigma)$ and $(\boldsymbol{k}', \sigma')$ are κ-saturated, $|\boldsymbol{k}_0| < \kappa$, and $(\boldsymbol{k}, \sigma) \equiv_{\boldsymbol{k}_0} (\boldsymbol{k}', \sigma')$. Then, as difference rings,*

$$(\mathrm{W}[\boldsymbol{k}], \mathrm{W}[\sigma]) \equiv_{\mathrm{W}[\boldsymbol{k}_0]} (\mathrm{W}[\boldsymbol{k}'], \mathrm{W}[\sigma']).$$

Proof. We just apply the functor W to a suitable back-and-forth system between $(\boldsymbol{k}, \sigma)$ and $(\boldsymbol{k}', \sigma')$. In detail, let $(\boldsymbol{k}_1, \sigma_1)$ range over the difference subfields of $(\boldsymbol{k}, \sigma)$ such that $\boldsymbol{k}_0 \subseteq \boldsymbol{k}_1$ and $|\boldsymbol{k}_1| < \kappa$, and let $(\boldsymbol{k}_2, \sigma_2)$ range over the difference subfields of $(\boldsymbol{k}', \sigma')$ such that $\boldsymbol{k}_0 \subseteq \boldsymbol{k}_2$ and $|\boldsymbol{k}_2| < \kappa$. Let Φ be the set of all difference field isomorphisms $\phi : (\boldsymbol{k}_1, \sigma_1) \to (\boldsymbol{k}_2, \sigma_2)$ that are the identity on $\boldsymbol{k}_0$ and are partial elementary maps between $(\boldsymbol{k}, \sigma)$ and $(\boldsymbol{k}', \sigma')$. Note that some $\phi \in \Phi$ maps the definable closure of $\boldsymbol{k}_0$ in $(\boldsymbol{k}, \sigma)$ onto the definable closure of $\boldsymbol{k}_0$ in $(\boldsymbol{k}', \sigma')$, so $\Phi \neq \emptyset$ and Φ is a back-and-forth system between $(\boldsymbol{k}, \sigma)$ and $(\boldsymbol{k}', \sigma')$. The functorial properties of W and κ-saturation yield a back-and-forth system $\mathrm{W}[\Phi]$ between $(\mathrm{W}[\boldsymbol{k}], \mathrm{W}[\sigma])$ and $(\mathrm{W}[\boldsymbol{k}'], \mathrm{W}[\sigma'])$ consisting of the

$$\mathrm{W}[\phi] : (\mathrm{W}[\boldsymbol{k}_1], \mathrm{W}[\sigma_1]) \to (\mathrm{W}[\boldsymbol{k}_2], \mathrm{W}[\sigma_2]),$$

with $\phi : (\boldsymbol{k}_1, \sigma_1) \to (\boldsymbol{k}_2, \sigma_2)$ an element of Φ. $\square$

A similar use of functoriality gives the following.

Lemma 8.7. *Let Γ_0 be an ordered abelian group with smallest positive element* 1 *and let Γ and Γ' be ordered abelian extension groups of Γ_0 with the same smallest positive element* 1. *Let κ be an uncountable cardinal such that Γ and Γ' are κ-saturated, $|\Gamma_0| < \kappa$, and $\Gamma \equiv_{\Gamma_0} \Gamma'$. Let Δ be the common convex subgroup $\mathbb{Z} \cdot 1$ of Γ_0, Γ and Γ'. Then the ordered quotient groups $\dot{\Gamma} := \Gamma/\Delta$ and $\dot{\Gamma}' := \Gamma'/\Delta$ are elementarily equivalent over their common ordered subgroup $\dot{\Gamma}_0 := \Gamma_0/\Delta$.*

Equivalence in mixed characteristic

In this final subsection we fix a prime number p, and $\mathcal{K} = (K, \Gamma, \boldsymbol{k}; v, \pi)$ is a σ-henselian valued difference field such that $\operatorname{char}(K) = 0$, $\boldsymbol{k}$ is perfect with $\operatorname{char}(\boldsymbol{k}) = p$, and $v(p)$ is the smallest positive element of Γ. Moreover, assume either that $\boldsymbol{k}$ is infinite and $\bar{\sigma}(x) = x^p$ for all $x \in \boldsymbol{k}$ (the Witt case), or that $\boldsymbol{k}$ satisfies Axiom 2. In particular, $\mathcal{K}$ is workable and $\mathcal{K}$ is not equipped here with an angular component map.

We make the corresponding assumptions about $\mathcal{K}' = (K', \Gamma', \boldsymbol{k}'; v', \pi')$. Also, assume that $\mathcal{E} = (E, \Gamma_{\mathcal{E}}, \boldsymbol{k}_{\mathcal{E}})$ and $\mathcal{E}' = (E', \Gamma_{\mathcal{E}'}, \boldsymbol{k}_{\mathcal{E}'})$ are good substructures of $\mathcal{K}$ and $\mathcal{K}'$, respectively, in the ac-free sense specified at the end of § 6, where we defined the corresponding ac-free notion of a *good map* $\mathcal{E} \to \mathcal{E}'$. Theorem 6.7 goes through in the present setting.

Theorem 8.8. *Suppose that $v(E^\times)$ is pure in Γ and $\boldsymbol{f} : \mathcal{E} \to \mathcal{E}'$ is a good map. Then $\boldsymbol{f}$ is a partial elementary map between $\mathcal{K}$ and $\mathcal{K}'$.*

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Proof. We first arrange that $\mathcal{K}$ and $\mathcal{K}'$ are κ-saturated, where κ is an uncountable cardinal such that $|\boldsymbol{k}_{\mathcal{E}}|, |\Gamma_{\mathcal{E}}| < \kappa$. To simplify notation we identify $\mathcal{E}$ and $\mathcal{E}'$ via $\boldsymbol{f}$, so $\boldsymbol{f}$ becomes the identity on $\mathcal{E}$. We have to show that then $\mathcal{K} \equiv_{\mathcal{E}} \mathcal{K}'$. With $v(p) = 1$ as the smallest positive element of $\Gamma_0 := \Gamma_{\mathcal{E}}$ and of Γ and Γ' and using the notations of Lemma 8.7 we have $\dot{\Gamma} \equiv_{\dot{\Gamma}_0} \dot{\Gamma}'$ by that same lemma. From the purity of $v(E^\times)$ in Γ it follows that $\dot{v}(E^\times)$ is pure in $\dot{\Gamma}$. Since $\mathcal{K}$ and $\mathcal{K}'$ are $\aleph_1$-saturated, it is harmless to identify $\dot{\boldsymbol{k}}$ and $\dot{\boldsymbol{k}}'$ with the fields $\mathrm{W}(\boldsymbol{k})$ and $\mathrm{W}(\boldsymbol{k}')$, respectively. Then the respective valuation rings $\dot{\pi}(\mathcal{O})$ and $\dot{\pi}'(\mathcal{O}')$ of $\dot{\boldsymbol{k}}$ and $\dot{\boldsymbol{k}}'$ are $\mathrm{W}[\boldsymbol{k}]$ and $\mathrm{W}[\boldsymbol{k}']$, and we have the common subfield $\dot{\boldsymbol{k}}_{\mathcal{E}} := \mathrm{W}(\boldsymbol{k}_{\mathcal{E}})$ of $\dot{\boldsymbol{k}}$ and $\dot{\boldsymbol{k}}'$. It now follows from Lemma 8.6 that $\dot{\boldsymbol{k}}(*) \equiv_{\dot{\boldsymbol{k}}_{\mathcal{E}}} \dot{\boldsymbol{k}}'(*)$. Hence the assumptions of Theorem 8.2 are satisfied with $\dot{\mathcal{K}}(*)$, and $\dot{\mathcal{K}}'(*)$ in the role of $\mathcal{K}$ and $\mathcal{K}'$, and $\dot{\mathcal{E}}(*) := (E, \dot{\Gamma}_0, \dot{\boldsymbol{k}}_{\mathcal{E}})$ in the role of both $\mathcal{E}$ and $\mathcal{E}'$. and with the identity on $\dot{\mathcal{E}}(*)$ as a good map. This theorem therefore gives

$$\dot{\mathcal{K}}(*) \equiv_{\dot{\mathcal{E}}(*)} \dot{\mathcal{K}}'(*).$$

This yields $\mathcal{K} \equiv_{\mathcal{E}} \mathcal{K}'$ by what we observed just after Lemma 8.4. ☐

Corollary 8.9. *$\mathcal{K} \equiv \mathcal{K}'$ if and only if $\boldsymbol{k} \equiv \boldsymbol{k}'$ as difference fields and $\Gamma \equiv \Gamma'$ as ordered abelian groups.*

Proof. The 'only if' direction is obvious. Suppose $\boldsymbol{k} \equiv \boldsymbol{k}'$ as difference fields, and $\Gamma \equiv \Gamma'$ as ordered groups. Then we have good substructures $\mathcal{E} := (\mathbb{Q}, \mathbb{Z}, \mathbb{F}_p)$ of $\mathcal{K}$, and $\mathcal{E}' := (\mathbb{Q}, \mathbb{Z}, \mathbb{F}_p)$ of $\mathcal{K}'$, and an obviously good map $\mathcal{E} \to \mathcal{E}'$. Now apply Theorem 8.8. ☐

In particular, any σ-henselian Witt case valued difference field satisfying Axiom 2, with infinite residue field $\boldsymbol{k}$ and value group $\Gamma \equiv \mathbb{Z}$ as ordered abelian groups, is elementarily equivalent to the Witt difference field $\mathrm{W}(\boldsymbol{k})$. The next result follows from Theorem 8.8 in the same way as Corollary 7.2 from Theorem 6.6.

Corollary 8.10. *Let $\mathcal{E} = (E, \Gamma_E, \boldsymbol{k}_E; \dots)$ be a σ-henselian valued difference subfield of $\mathcal{K}$ satisfying Axiom 2 such that $\boldsymbol{k}_E \preceq \boldsymbol{k}$ as difference fields, and $\Gamma_E \preceq \Gamma$ as ordered abelian groups. Then $\mathcal{E} \preceq \mathcal{K}$.*

Theorem 8.8 does not seem to give a nice relative quantifier elimination such as Corollary 7.3 but it does yield the following analogue of Corollary 7.4.

Corollary 8.11. *Each subset of $\boldsymbol{k}^m \times \Gamma^n$ that is definable in $\mathcal{K}$ is a finite union of rectangles $X \times Y$ with $X \subseteq \boldsymbol{k}^m$ definable in the difference field $\boldsymbol{k}$ and $Y \subseteq \Gamma^n$ definable in the ordered abelian group Γ.*

Proof. By standard arguments we can reduce to the following situation: $\mathcal{K}$ is $\aleph_1$-saturated, $\mathcal{E} = (E, \Gamma_E, \boldsymbol{k}_E; \dots) \preceq \mathcal{K}$ is countable, $r, r' \in \boldsymbol{k}^m$ have the same type over $\boldsymbol{k}_E$, and $\gamma, \gamma' \in \Gamma^n$ have the same type over Γ_E:

$$\mathrm{tp}(r|\boldsymbol{k}_E) = \mathrm{tp}(r'|\boldsymbol{k}_E) \quad \text{(in the difference field } \boldsymbol{k}\text{)},$$
$$\mathrm{tp}(\gamma|\Gamma_E) = \mathrm{tp}(\gamma'|\Gamma_E) \quad \text{(in the ordered abelian group } \Gamma\text{)}.$$

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It suffices to show that then (r,γ) and (r',γ') have the same type over $\mathcal{E}$ in $\mathcal{K}$. Let $\boldsymbol{k}_1$ and $\boldsymbol{k}_1'$ be the definable closures of $\boldsymbol{k}_E(r)$ and $\boldsymbol{k}_E(r')$ in the difference field $\boldsymbol{k}$, and let Γ_1 and Γ_1' be the ordered subgroups of Γ generated over Γ_E by γ and γ'. Then $(E,\Gamma_1,\boldsymbol{k}_1)$ and $(E,\Gamma_1',\boldsymbol{k}_1')$ are good substructures of $\mathcal{K}$, and the assumption on types yields a good map $(E,\Gamma_1,\boldsymbol{k}_1)\to(E,\Gamma_1',\boldsymbol{k}')$ that is the identity on $(E,\Gamma_E,\boldsymbol{k}_E)$, sends γ to γ' and r to r'. Note also that $v(E^\times)=\Gamma_E$ is pure in Γ. It remains to apply Theorem 8.8. □

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