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Existence of solutions for critical Choquard equations via the concentration-compactness method

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In this paper, we consider the nonlinear Choquard equation

$$-\Delta u + V(x)u = \left(\int_{\mathbb{R}^N} \frac{G(u)}{|x-y|^{\mu}}\,\mathrm{d}y\right) g(u) \quad \text{in } \mathbb{R}^N,$$

where $0 < \mu < N$, $N \geqslant 3$, $g(u)$ is of critical growth due to the Hardy–Littlewood–Sobolev inequality and $G(u) = \int_0^u g(s)\,\mathrm{d}s$. Firstly, by assuming that the potential $V(x)$ might be sign-changing, we study the existence of Mountain-Pass solution via a nonlocal version of the second concentration-compactness principle. Secondly, under the conditions introduced by Benci and Cerami , we also study the existence of high energy solution by using a nonlocal version of global compactness lemma.

Keywords: Critical Choquard equation; Hardy–Littlewood–Sobolev inequality; Concentration-Compactness principle

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1. Introduction and main results

The nonlinear Choquard equation

$$- \Delta u + V(x)u = \left(|x|^{-\mu} * |u|^q\right) |u|^{q-2}u, \quad \text{in } \mathbb{R}^N \tag{1.1}$$

arises in various fields of mathematical physics, such as the description of the quantum theory of a polaron at rest by S. Pekar in 1954 [**33**] and the modelling of an electron trapped in its own hole in 1976 in the work of P. Choquard. It was also treated as a certain approximation to Hartree-Fock theory of one-component plasma [**21**]. Sometimes equation (1.1) was also known as the Schrödinger-Newton equation [**34**], since the convolution part might be treated as a coupling with a Newton equation.

Mathematically, for (1.1) with $\mu = 1$, $q = 2$ and V is a positive constant, Lieb [**21**] proved the existence and uniqueness, up to translations, of the ground state by using rearrangements technique. Later Lions [**23**] showed the existence of a sequence of radially symmetric solutions by variational methods. In the last decades, a great deal of mathematical efforts has been devoted to the study of existence, multiplicity and properties of the solutions of the nonlinear Choquard equation (1.1). In [**15**, **28**, **29**], the authors showed the regularity, positivity and radial symmetry of the ground states and derived decay property at infinity as well. For the Choquard equation with constant potential, Moroz and Van Schaftingen [**30**] considered the existence of ground states under the assumption of Berestycki-Lions type. If the periodic potential $V(x)$ changes sign and 0 lies in the gap of the spectrum of $-\Delta + V$, then the energy functional associated with the problem is strongly indefinite indeed. For this case, the existence of solution for $p = 2$ was considered in [**11**] there the authors developed reduction argument to obtain the existence of weak solution. Still, for the strongly indefinite case, Ackermann [**1**] established the splitting lemma for the nonlocal nonlinearities and proved the existence of infinitely many geometrically distinct weak solutions. If the nonlinear Choquard equation is equipped with deepening potential well $V(x) = \lambda a(x) + 1$ where $a(x)$ is a nonnegative continuous function such that $\Omega = \text{int } (a^{-1}(0))$ is a nonempty bounded open set with smooth boundary, Alves *et al.* [**5**] studied the existence and multiplicity of multi-bump shaped solutions. In quantum physics, to describe the transition from quantum mechanics to classical mechanics, people are lead to consider the existence and concentration behaviour of solutions for the singularly perturbed subcritical Choquard equation which was called semiclassical Problems, see for example [**2**–**4**, **6**, **14**, **32**, **37**]. Among these references, Wei and Winter [**37**] constructed families of solutions by a Lyapunov-Schmidt type reduction. Cingolani *et al.* [**14**] showed that there exists a family of solutions having multiple concentration regions which are located around the minimum points of the potential. Moroz and Van Schaftingen [**32**] developed a nonlocal penalization technique and showed the existence of a family of solutions concentrating around the local minimum of V. In [**2**, **3**], Alves and Yang proved the existence, multiplicity and concentration of solutions for the equation by penalization method and Lusternik-Schnirelmann theory.

To consider the nonlocal elliptic equation with Riesz type potential, it is necessary to recall the well–known Hardy–Littlewood–Sobolev inequality.

PROPOSITION 1.1. *(Hardy–Littlewood–Sobolev inequality). (See [22].) Let t, $r > 1$ and $0 < \mu < N$ with $1/t + \mu/N + 1/r = 2$, $f \in L^t(\mathbb{R}^N)$ and $h \in L^r(\mathbb{R}^N)$. There exists a sharp constant $C(t, N, \mu, r)$, independent of f, h, such that*

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{f(x)h(y)}{|x-y|^\mu}\,\mathrm{d}x\,\mathrm{d}y \leqslant C(t,N,\mu,r)|f|_t|h|_r, \tag{1.2}$$

where $|\cdot|_q$ for the $L^q(\mathbb{R}^N)$-norm for $q \in [1, \infty]$. If $t = r = 2N/(2N-\mu)$, then

$$C(t,N,\mu,r) = C(N,\mu) = \pi^{\mu/2}\frac{\Gamma\left(N/2-\mu/2\right)}{\Gamma\left(N-(\mu/2)\right)}\left\{\frac{\Gamma\left(N/2\right)}{\Gamma(N)}\right\}^{-1+(\mu/N)}.$$

In this case, there is equality in (1.2) if and only if $f \equiv Ch$ and

$$h(x) = A(\gamma^2 + |x-a|^2)^{-(2N-\mu)/2}$$

for some $A \in \mathbb{C}$, $0 \neq \gamma \in \mathbb{R}$ and $a \in \mathbb{R}^N$.

Let $H^1(\mathbb{R}^N)$ be the usual Sobolev spaces with norm

$$\|u\|_{H^1} := \left(\int_{\mathbb{R}^N}(|\nabla u|^2 + |u|^2)\,\mathrm{d}x\right)^{1/2},$$

$D^{1,2}(\mathbb{R}^N)$ be equipped with norm

$$\|u\| := \left(\int_{\mathbb{R}^N}|\nabla u|^2\,\mathrm{d}x\right)^{1/2}$$

and $L^s(\mathbb{R}^N)$, $1 \leqslant s \leqslant \infty$, denotes the Lebesgue space with norms

$$|u|_s := \left(\int_{\mathbb{R}^N}|u|^s\,\mathrm{d}x\right)^{1/s}.$$

By the Hardy–Littlewood–Sobolev inequality, for every $u \in H^1(\mathbb{R}^N)$, the integral

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)|^q|u(y)|^q}{|x-y|^\mu}\,\mathrm{d}x\,\mathrm{d}y$$

is well defined if

$$\frac{2N-\mu}{N} \leqslant q \leqslant \frac{2N-\mu}{N-2}.$$

Due to the Sobolev imbedding, $(2N-\mu)/N$ will be called the Hardy–Littlewood–Sobolev lower critical exponent and $2^*_\mu = (2N-\mu)/(N-2)$ the Hardy–Littlewood–Sobolev upper critical exponent. In [**12**, **31**], the authors considered the nonlinear Choquard equation (1.1) in $\mathbb{R}^N$ with lower critical exponent $(2N-\mu)/N$ and obtained some existence and nonexistence results. In order to study the critical nonlocal equation with upper critical exponent 2_μ^*, let S be the best Sobolev constant defined by:

$$S|u|_{2^*}^2 \leqslant \int_{\mathbb{R}^N} |\nabla u|^2 \, \mathrm{d}x \quad \text{for all } u \in D^{1,2}(\mathbb{R}^N),$$

we will use $S_{H,L}$ to denote the best constant defined by

$$S_{H,L} := \inf_{u \in D^{1,2}(\mathbb{R}^N)\setminus\{0\}} \frac{\int_{\mathbb{R}^N} |\nabla u|^2 \, \mathrm{d}x}{\left(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} [(|u(x)|^{2_\mu^*}|u(y)|^{2_\mu^*})/(|x-y|^\mu)] \, \mathrm{d}x \, \mathrm{d}y\right)^{(N-2)/(2N-\mu)}}. \tag{1.3}$$

In [18] it was observed that

Proposition 1.2 (See [18]). *The constant $S_{H,L}$ defined in (1.3) is achieved if and only if*

$$u(x) = C\left(\frac{b}{b^2 + |x-a|^2}\right)^{(N-2)/2},$$

where $C > 0$ is a fixed constant, $a \in \mathbb{R}^N$ and $b \in (0, \infty)$ are parameters. What's more,

$$S_{H,L} = \frac{S}{C(N,\mu)^{(N-2)/(2N-\mu)}},$$

where S is the best Sobolev constant and $C(N, \mu)$ is given in proposition 1.1.

Let $\widetilde{U}_{\delta,z}(x) := [N(N-2)\delta]^{(N-2)/4}/((\delta + |x-z|^2)^{(N-2)/2}$, $\delta > 0$, $z \in \mathbb{R}^N$. We know that $\widetilde{U}_{\delta,z}$ is a minimizer for S [38] and

$$U_{\delta,z}(x) := C(N,\mu)^{(2-N)/(2(N-\mu+2))} S^{((N-\mu)(2-N))/(4(N-\mu+2))} \widetilde{U}_{\delta,z}(x) \tag{1.4}$$

is the unique minimizer for $S_{H,L}$ that satisfies

$$-\Delta u = \left(\int_{\mathbb{R}^N} \frac{|u(y)|^{2_\mu^*}}{|x-y|^\mu} \, \mathrm{d}y\right) |u|^{2_\mu^*-2}u \quad \text{in } \mathbb{R}^N \tag{1.5}$$

and

$$\int_{\mathbb{R}^N} |\nabla U_{\delta,z}|^2 \, \mathrm{d}x = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|U_{\delta,z}(x)|^{2_\mu^*}|U_{\delta,z}(y)|^{2_\mu^*}}{|x-y|^\mu} \, \mathrm{d}x \, \mathrm{d}y = S_{H,L}^{(2N-\mu)/(N-\mu+2)}.$$

In [18, 19] the authors considered the Brézis-Nirenberg type problem

$$-\Delta u = \left(\int_{\Omega} \frac{|u(y)|^{2_\mu^*}}{|x-y|^\mu} \, \mathrm{d}y\right) |u|^{2_\mu^*-2}u + \lambda f(u) \quad \text{in } \Omega \tag{1.6}$$

and established the existence, multiplicity and nonexistence of solutions for the nonlinear Choquard equation in bounded domain. It is observed in [35] that equation

(1.6) can be regarded as a limit problem for a critical Choquard equation with deepening potential well, there the existence and asymptotic behaviour of the solutions were investigated. In [6], by investigating the ground states of the critical Choquard equation with constant coefficients, the authors studied the semiclassical limit problem for the singularly perturbed Choquard equation in $\mathbb{R}^3$ and characterized the concentration behaviour by variational methods. The planar case was considered in [4], there the authors established the existence of ground state for the limit problem with critical exponential growth which complemented those results for local case, and then they also studied the concentration around the global minimum set. Gao and Yang in [20] investigated the existence result for the strongly indefinite Choquard equation with upper critical exponent in the whole space.

In works [4, 6, 20], the method developed by Brezis and Nirenberg has been successfully adopt to study the Choquard equation with Hardy–Littlewood–Sobolev upper critical exponents. There the authors are able to prove the existence results by showing that the minimax value was below some critical criteria where the (PS) condition still holds. In the present paper, we continue to study the Choquard equation with upper critical exponents, but with different types of potential functions. We will see that the arguments in [4, 6, 20] does not apply for these new situations any longer.

On one hand, we are going to study the critical Choquard equation with subcritical perturbation and potential functions that might change sign

$$-\Delta u + V(x)u = \left(\int_{\mathbb{R}^N} \frac{|u(y)|^{2^*_\mu} + |u(y)|^p}{|x-y|^\mu}\,\mathrm{d}y\right)\left(|u|^{2^*_\mu-2}u + \frac{p}{2^*_\mu}|u|^{p-2}u\right) \quad \text{in } \mathbb{R}^N, \tag{1.7}$$

where $N \geqslant 3,\ 0<\mu<N,\ (2N-\mu)/N < p < (2N-\mu)/(N-2)$ and $2^*_\mu = (2N-\mu)/(N-2)$ is the upper critical exponent in the sense of the Hardy–Littlewood–Sobolev inequality. To obtain the existence result we are going to prove that the lack of compactness is recovered by using the second concentration compactness principle. As in [39], we assume that the functions $V(x)$ satisfies the following condition:

(V) There exists $\tau_0>0$ such that the set $\Omega_{\tau_0} = \{x\in\mathbb{R}^N : V(x)\leqslant \tau_0\}$ has finite Lebesgue measure. Moreover, $V \in L^\infty_{loc}(\mathbb{R}^N)\cap L^{N/2}(\mathbb{R}^N)$ and there holds

$$V_0 := |V_-(x)|_{N/2} < S,$$

where S is the best Sobolev constant and $V_- = \max\{-V(x), 0\}$.

We can draw the following conclusion.

Theorem 1.3. *Suppose that assumption (V) holds, $N\geqslant 3,\ 0<\mu<N$ and $(2N-\mu)/N < p < (2N-\mu)/(N-2)$. Then (1.7) admits a nontrivial solution.*

On the other hand, we are concerned with the existence of high energy solution for the critical Choquard equation. In [7], Benci and Cerami considered the following problem

$$-\Delta u + V(x)u = |u|^{2^*-2}u, \quad \text{in } \mathbb{R}^N, \tag{1.8}$$

where the potential $V(x)$ satisfies (V_1), (V_2) below and (V_3')

$$|V(x)|_{N/2} < S(2^{2/N} - 1).$$

They developed some global compactness lemma and proved that problem (1.8) has at least one positive high energy solution. Here we are quite interested if the same result still holds for the nonlocal Choquard equation

$$\begin{cases} -\Delta u + V(x)u = \left(\int_{\mathbb{R}^N} \frac{|u(y)|^{2^*_\mu}}{|x-y|^\mu} \, \mathrm{d}y\right) |u|^{2^*_\mu-2}u \quad \text{in } \mathbb{R}^N, \\ u \in D^{1,2}(\mathbb{R}^N), \end{cases} \tag{1.9}$$

here $0 < \mu < N$, $N \geqslant 3$, $2^*_\mu = (2N-\mu)/(N-2)$ and the potential V satisfies the assumptions

(V_1) $V \in \mathcal{C}(\mathbb{R}^N, \mathbb{R})$, $V \geqslant \nu > 0$ in a neighbourhood of 0.

(V_2) $\exists p_1 < N/2$, $p_2 > N/2$ and for $N = 3$, $p_2 < 3$, such that

$$V(x) \in L^p, \quad \forall\, p \in [p_1, p_2].$$

(V_3)

$$|V(x)|_{N/2} < C(N,\mu)^{(N-2)/(2N-\mu)} S_{H,L} (2^{(N+2-\mu)/(2N-\mu)} - 1),$$

where $S_{H,L}$ is defined in (1.3) and $C(N, \mu)$ is given in proposition 1.1. Under these assumptions, we have

THEOREM 1.4. *Suppose that assumptions (V_1), (V_2) and (V_3) hold, $0 < \mu < \min\{4, N\}$ and $N \geqslant 3$. Then equation (1.9) has at least one nontrivial solution u.*

An outline of this paper is as follows: In §2, we prove a version of Concentration-Compactness principle for the nonlocal type problem which complements the results in [**8**, **9**, **26**]. After that, we can use the compactness lemma to prove that the (PS) condition still holds below some criteria level and obtain the existence of solutions by Mountain-Pass Theorem. In §3, we prove a version of global compactness lemma for the nonlocal Choquard equation and then we show the existence of high energy solution for (1.9) following the linking arguments in [**7**].

2. Mountain-pass solution

In this section, we will study the existence of solutions for equation (1.7) under assumption (V). To prove the existence of solutions by variational methods, we introduce the Hilbert spaces

$$E := \left\{ u \in H^1(\mathbb{R}^N) : \int_{\mathbb{R}^N} V_+(x) u^2 \,\mathrm{d}x < \infty \right\}$$

with inner products

$$(u, v) := \int_{\mathbb{R}^N} \big(\nabla u \nabla v + V_+(x) uv \big) \,\mathrm{d}x$$

and the associated norms

$$\|u\|_V^2 = (u, u).$$

Obviously, E embeds continuously in $H^1(\mathbb{R}^N)$ (see [**17**]). Moreover,

LEMMA 2.1 ([**39**], lemma 2.3). *There exist $C_1, C_2 > 0$ depending only on the structural constants such that*

$$C_1 \|u\|_{H^1}^2 \leqslant C_2 \|u\|_V^2 \leqslant \int_{\mathbb{R}^N} (|\nabla u|^2 + V(x)|u|^2) \,\mathrm{d}x \leqslant \|u\|_V^2, \quad u \in E. \tag{2.1}$$

Denote

$$\|u\|_{NL} := \left(\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x)|^{2_\mu^*} |u(y)|^{2_\mu^*}}{|x - y|^\mu} \,\mathrm{d}x \,\mathrm{d}y \right)^{1/(2 \cdot 2_\mu^*)},$$

the following splitting Lemma was proved in lemma 2.2 of [**18**].

LEMMA 2.2. *Let $N \geqslant 3$ and $0 < \mu < N$. If $\{u_n\}$ is a bounded sequence in $L^{(2N)/(N-2)}(\mathbb{R}^N)$ such that $u_n \to u$ almost everywhere in $\mathbb{R}^N$ as $n \to \infty$, then the following hold,*

$$\|u_n\|_{NL}^{2 \cdot 2_\mu^*} - \|u_n - u\|_{NL}^{2 \cdot 2_\mu^*} \to \|u\|_{NL}^{2 \cdot 2_\mu^*}$$

as $n \to \infty$.

To study the problem variationally, we introduce the energy functional associated with equation (1.7) by

$$J(u) = \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla u|^2 + V(x)|u|^2) \,\mathrm{d}x - \frac{1}{2 \cdot 2_\mu^*} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(|u(x)|^{2_\mu^*} + |u(x)|^p)(|u(y)|^{2_\mu^*} + |u(y)|^p)}{|x - y|^\mu} \,\mathrm{d}x \,\mathrm{d}y.$$

The Hardy–Littlewood–Sobolev inequality implies that J is well defined on E and belongs to $\mathcal{C}^1$ with

$$\langle J'(u), \varphi \rangle = \int_{\mathbb{R}^N} (\nabla u \nabla \varphi + V(x) u \varphi) \,\mathrm{d}x - \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \times \frac{(|u(x)|^{2_\mu^*} + |u(x)|^p) \Big(|u(y)|^{2_\mu^* - 2} u(y) \varphi(y) + (p/(2_\mu^*)) |u(y)|^{p-2} u(y) \varphi(y) \Big)}{|x - y|^\mu} \,\mathrm{d}x \,\mathrm{d}y.$$

So u is a weak solution of (1.7) if and only if u is a critical point of the functional J.

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2.1. Concentration-compactness principle

To describe the lack of compactness of the injection from $D^{1,2}(\mathbb{R}^N)$ to $L^{2^*}(\mathbb{R}^N)$, P.L. Lions established the well-known Concentration-compactness principles [24–27]. Here we would like to recall the second concentration-compactness principle [26] for the convenience of the readers.

LEMMA 2.3. *Let $\{u_n\}$ be a bounded sequence in $D^{1,2}(\mathbb{R}^N)$ converging weakly and a.e. to some $u_0 \in D^{1,2}(\mathbb{R}^N)$. $|\nabla u_n|^2 \rightharpoonup \omega$, $|u_n|^{2^*} \rightharpoonup \zeta$ weakly in the sense of measures where ω and ζ are bounded nonnegative measures on $\mathbb{R}^N$. Then we have:*

(1) there exists some at most countable set I, a family $\{z_i : i \in I\}$ of distinct points in $\mathbb{R}^N$, and a family $\{\zeta_i : i \in I\}$ of positive numbers such that

$$\zeta = |u_0|^{2^*} + \sum_{i\in I} \zeta_i \delta_{z_i},$$

where δ_x is the Dirac-mass of mass 1 concentrated at $x \in \mathbb{R}^N$.

(2) In addition, we have

$$\omega \geqslant |\nabla u_0|^2 + \sum_{i\in I} \omega_i \delta_{z_i}$$

for some family $\{\omega_i : i \in I\}$, $\omega_i > 0$ satisfying

$$S\zeta_i^{2/2^*} \leqslant \omega_i, \quad \text{for all } i \in I.$$

In particular, $\sum_{i\in I} \zeta_i^{2/(2^*)} < \infty$.

The second concentration-compactness principle, roughly speaking, is only concerned with a possible concentration of a weakly convergent sequence at finite points and it does not provide any information about the loss of mass of a sequence at infinity. The following concentration-compactness principle at infinity was developed by Chabrowski [13], J. Bianchi, Chabrowski, Szulkin [9], Ben-Naoum, Troestler, Willem [8] which provided some quantitative information about the loss of mass of a sequence at infinity.

LEMMA 2.4. *Let $\{u_n\} \subset D^{1,2}(\mathbb{R}^N)$ be a sequence in lemma 2.3 and define*

$$\omega_\infty := \lim_{R\to\infty} \limsup_{n\to\infty} \int_{|x|\geqslant R} |\nabla u_n|^2 \,\mathrm{d}x, \quad \zeta_\infty := \lim_{R\to\infty} \limsup_{n\to\infty} \int_{|x|\geqslant R} |u_n|^{2^*} \,\mathrm{d}x.$$

Then it follows that

$$S\zeta_\infty^{2/2^*} \leqslant \omega_\infty,$$

$$\limsup_{n\to\infty} |\nabla u_n|_2^2 = \int_{\mathbb{R}^N} \mathrm{d}\omega + \omega_\infty,$$

$$\limsup_{n\to\infty} |u_n|_{2^*}^{2^*} = \int_{\mathbb{R}^N} \mathrm{d}\zeta + \zeta_\infty.$$

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The concentration-compactness principles [24–27] help not only to investigate the behaviour of the weakly convergent sequences in Sobolev spaces where the lack of compactness occurs either due to the appearance of a critical Sobolev exponent or due to the unboundedness of a domain, but also to find level sets of a given variational functional for which the Palais-Smale condition holds. It was mentioned in the famous paper by P.L. Lions [26] that the limit embeddings

$$\left(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)|^{2_\mu^*}|u(y)|^{2_\mu^*}}{|x-y|^\mu}\,\mathrm{d}x\,\mathrm{d}y\right)^{1/(2_\mu^*)} \leqslant C_0\int_{\mathbb{R}^N}|\nabla u|^2\,\mathrm{d}x$$

also cause the concentration of a weakly convergent sequence at finite points and the results in lemma 2.3 holds with $|u_n|^{2^*}$ replaced by

$$|u_n|^{2_\mu^*}\int_{\mathbb{R}^N}\frac{|u_n(y)|^{2_\mu^*}}{|x-y|^\mu}\,\mathrm{d}y.$$

Moreover, a version of concentration-compactness principle corresponding to lemma 2.3 was established in [27] to study the minimizing problem associated with the attainability of the best constant in the Hardy-Littlewood-Sobolev inequality of the form

$$\left|\frac{1}{|x|^\mu}*u\right|_q \leqslant C_0|u|_p$$

for some C_0 depending on N, μ, q, p, where $0<\mu<N$ and p, q satisfy

$$\frac{1}{p}+\frac{\mu}{n}=1+\frac{1}{q}.$$

In the present paper, we are interested in the existence of solutions for the critical Choquard equation due to the Hardy–Littlewood–Sobolev inequality. Since the lack of compactness also occurs when researchers consider the critical Choquard equation in unbounded domain, it is quite natural for people to turn to a possible use of the second concentration-compactness principle for the convolution type nonlinearities. However, to the best knowledge of the authors, there seems no such existing lemmas that describe the possible concentration of a weakly convergent sequence both at finite points and at infinity. And there also seems no application of such a second concentration-compactness principle in studying the critical Choquard equation. By taking some ideas from [26, 27], we prove a concentration-compactness principle involving the convolution nonlinearities to study the critical Choquard equation.

LEMMA 2.5. *Let $\{u_n\}$ be a bounded sequence in $D^{1,2}(\mathbb{R}^N)$ converging weakly and a.e. to some u_0 and $\omega, \omega_\infty, \zeta, \zeta_\infty$ be the bounded nonnegative measures in*

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lemmas 2.3 and 2.4. Assume that

$$\left(\int_{\mathbb{R}^N} \frac{|u_n(y)|^{2^*_\mu}}{|x-y|^\mu}\,\mathrm{d}y\right)|u_n(x)|^{2^*_\mu} \rightharpoonup \nu$$

weakly in the sense of measure where ν is a bounded positive measure on $\mathbb{R}^N$ and define

$$\nu_\infty := \lim_{R\to\infty}\limsup_{n\to\infty}\int_{|x|\geqslant R}\left(\int_{\mathbb{R}^N} \frac{|u_n(y)|^{2^*_\mu}}{|x-y|^\mu}\,\mathrm{d}y\right)|u_n(x)|^{2^*_\mu}\,\mathrm{d}x.$$

Then, there exists a countable sequence of points $\{z_i\}_{i\in I}\subset\mathbb{R}^N$ and families of positive numbers $\{\nu_i : i\in I\}$, $\{\zeta_i : i\in I\}$ and $\{\omega_i : i\in I\}$ such that

$$\nu = \left(\int_{\mathbb{R}^N} \frac{|u_0(y)|^{2^*_\mu}}{|x-y|^\mu}\,\mathrm{d}y\right)|u_0(x)|^{2^*_\mu} + \Sigma_{i\in I}\nu_i\delta_{z_i}, \quad \Sigma_{i\in I}\nu_i^{1/(2^*_\mu)}<\infty, \tag{2.2}$$

$$\omega \geqslant |\nabla u_0|^2 + \sum_{i\in I}\omega_i\delta_{z_i}, \tag{2.3}$$

$$\zeta \geqslant |u_0|^{2^*} + \sum_{i\in I}\zeta_i\delta_{z_i}, \tag{2.4}$$

and

$$S_{H,L}\nu_i^{1/(2^*_\mu)} \leqslant \omega_i, \quad \nu_i^{N/(2N-\mu)} \leqslant C(N,\mu)^{N/(2N-\mu)}\zeta_i, \tag{2.5}$$

where δ_x is the Dirac-mass of mass 1 concentrated at $x\in\mathbb{R}^N$.

For the energy at infinity, we have

$$\limsup_{n\to\infty}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u_n(y)|^{2^*_\mu}|u_n(x)|^{2^*_\mu}}{|x-y|^\mu}\,\mathrm{d}y\,\mathrm{d}x = \nu_\infty + \int_{\mathbb{R}^N}\mathrm{d}\nu, \tag{2.6}$$

and

$$\begin{aligned} &C(N,\mu)^{(-2N)/(2N-\mu)}\nu_\infty^{(2N)/(2N-\mu)} \leqslant \zeta_\infty\left(\int_{\mathbb{R}^N}\mathrm{d}\zeta + \zeta_\infty\right), \\ &S_{H,L}^2\nu_\infty^{2/(2^*_\mu)} \leqslant \omega_\infty\left(\int_{\mathbb{R}^N}\mathrm{d}\omega + \omega_\infty\right). \end{aligned} \tag{2.7}$$

Moreover, if $u=0$ and $\int_{\mathbb{R}^N}\mathrm{d}\omega = S_{H,L}\left(\int_{\mathbb{R}^N}\mathrm{d}\nu\right)^{1/(2^*_\mu)}$, then ν is concentrated at a single point.

Proof. Since $\{u_n\}$ is a bounded sequence in $D^{1,2}(\mathbb{R}^N)$ converging weakly to u, denote by $v_n := u_n - u_0$, we have $v_n(x)\to 0$ a.e. in $\mathbb{R}^N$ and v_n converges weakly

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to 0 in $D^{1,2}(\mathbb{R}^N)$. Applying lemma 2.2, in the sense of measure, we have

$$\begin{aligned} &|\nabla v_n|^2 \rightharpoonup \varpi := \omega - |\nabla u_0|^2,\\ &\left(\int_{\mathbb{R}^N}(|v_n(y)|^{2^*_\mu}\right)/(|x-y|^\mu)\,\mathrm{d}y)|v_n(x)|^{2^*_\mu} \rightharpoonup \kappa\\ &\quad := \nu - \left(\int_{\mathbb{R}^N}(|u_0(y)|^{2^*_\mu}\right)/(|x-y|^\mu)\,\mathrm{d}y)|u_0(x)|^{2^*_\mu}\\ \text{and}\quad &|v_n|^{2^*} \rightharpoonup \varsigma := \zeta - |u_0|^{2^*}. \end{aligned}$$

To prove the possible concentration at finite points, we first show that

$$\left|\int_{\mathbb{R}^N}(|x|^{-\mu} * |\phi v_n(x)|^{2^*_\mu})|\phi v_n(x)|^{2^*_\mu}\,\mathrm{d}x - \int_{\mathbb{R}^N}(|x|^{-\mu} * |v_n(x)|^{2^*_\mu})|\phi(x)|^{2^*_\mu}|\phi v_n(x)|^{2^*_\mu}\,\mathrm{d}x\right| \to 0, \tag{2.8}$$

where $\phi \in \mathcal{C}_0^\infty(\mathbb{R}^N)$. In fact, we denote

$$\Phi_n(x) := (|x|^{-\mu} * |\phi v_n(x)|^{2^*_\mu})|\phi v_n(x)|^{2^*_\mu} - (|x|^{-\mu} * |v_n(x)|^{2^*_\mu})|\phi(x)|^{2^*_\mu}|\phi v_n(x)|^{2^*_\mu}.$$

Since $\phi \in \mathcal{C}_0^\infty(\mathbb{R}^N)$, we have for every $\delta > 0$ there exists $M > 0$ such that

$$\int_{|x|\geqslant M} |\Phi_n(x)|\mathrm{d}x < \delta \quad (\forall n \geqslant 1). \tag{2.9}$$

Since the Riesz potential defines a linear operator, from the fact that $v_n(x) \to 0$ a.e. in $\mathbb{R}^N$ we know that

$$\int_{\mathbb{R}^N} \frac{|v_n(y)|^{2^*_\mu}}{|x-y|^\mu}\,\mathrm{d}y \to 0$$

a.e. in $\mathbb{R}^N$ and so we have $\Phi_n(x) \to 0$ a.e. in $\mathbb{R}^N$. Notice that

$$\begin{aligned} \Phi_n(x) &= \int_{\mathbb{R}^N} \frac{(|\phi(y)|^{2^*_\mu} - |\phi(x)|^{2^*_\mu})|v_n(y)|^{2^*_\mu}}{|x-y|^\mu}\,\mathrm{d}y|\phi v_n(x)|^{2^*_\mu}\\ &:= \int_{\mathbb{R}^N} L(x,y)|v_n(y)|^{2^*_\mu}\,\mathrm{d}y|\phi v_n(x)|^{2^*_\mu}. \end{aligned}$$

For almost all x, there exists $R > 0$ large enough such that

$$\int_{\mathbb{R}^N} L(x,y)|v_n(y)|^{2^*_\mu}\,\mathrm{d}y = \int_{|y|\leqslant R} L(x,y)|v_n(y)|^{2^*_\mu}\,\mathrm{d}y - |\phi(x)|^{2^*_\mu}\int_{|y|\geqslant R}\frac{|v_n(y)|^{2^*_\mu}}{|x-y|^\mu}\,\mathrm{d}y.$$

As observed in [27] that $L(x, y) \in L^r(B_R)$ for each x, where $r < N/(\mu-1)$ if $\mu > 1$, $r \leqslant +\infty$ if $0 < \mu \leqslant 1$. By the Young inequality, there exists $s > (2N)/(\mu)$ such that

$$\left(\int_{B_M}\left(\int_{B_R} L(x,y)|v_n(y)|^{2^*_\mu}\,\mathrm{d}y\right)^s \mathrm{d}x\right)^{1/s} \leqslant C_\phi |L(x,y)|_r ||v_n|^{2^*_\mu}|_{2N/(2N-\mu)} \leqslant C'_\phi,$$

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where M is given in (2.9). It is easy to see that for $R>0$ large enough

$$\left(\int_{B_M}\left(|\phi(x)|^{2_\mu^*}\int_{|y|\geqslant R}\frac{|v_n(y)|^{2_\mu^*}}{|x-y|^\mu}\,\mathrm{d}y\right)^s\mathrm{d}x\right)^{1/s}\leqslant C$$

and so, we have

$$\left(\int_{B_M}\left(\int_{\mathbb{R}^N}L(x,y)|v_n(y)|^{2_\mu^*}\,\mathrm{d}y\right)^s\mathrm{d}x\right)^{1/s}\leqslant C''_\phi.$$

Denote $\tau=(\mu s-2N)/(2N+2Ns-\mu s)$, we can get

$$\int_{B_M}|\Phi_n(x)|^{1+\tau}\,\mathrm{d}x\leqslant\left(\int_{B_M}\left(\int_{\mathbb{R}^N}L(x,y)|v_n(y)|^{2_\mu^*}\,\mathrm{d}y\right)^s\mathrm{d}x\right)^{(1+\tau)/s}\times\left(\int_{B_M}|\phi v_n|^{2^*}\,\mathrm{d}x\right)^{(2N-\mu)(\tau+1)/2N}\leqslant C''_\phi.$$

Combining this and $\Phi_n(x)\to 0$ a.e. in $\mathbb{R}^N$, we can get

$$\int_{B_M}|\Phi_n(x)|\mathrm{d}x\to 0\quad(n\to\infty).$$

Together with (2.9), we have

$$\int_{\mathbb{R}^N}|\Phi_n(x)|\mathrm{d}x\to 0.$$

For all $\phi\in\mathcal{C}_0^\infty(\mathbb{R}^N)$, by the Hardy–Littlewood–Sobolev inequality, we have

$$\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{|\phi v_n(y)|^{2_\mu^*}}{|x-y|^\mu}\,\mathrm{d}y\right)|\phi v_n(x)|^{2_\mu^*}\,\mathrm{d}x\leqslant C(N,\mu)|\phi v_n|_{2^*}^{2\cdot 2_\mu^*}.$$

By (2.8), we have

$$\int_{\mathbb{R}^N}|\phi(x)|^{2\cdot 2_\mu^*}\left(\int_{\mathbb{R}^N}\frac{|v_n(y)|^{2_\mu^*}}{|x-y|^\mu}\,\mathrm{d}y\right)|v_n(x)|^{2_\mu^*}\,\mathrm{d}x\leqslant C(N,\mu)|\phi v_n|_{2^*}^{2\cdot 2_\mu^*}+o(1).$$

Passing to the limit as $n\to+\infty$ we obtain

$$\int_{\mathbb{R}^N}|\phi(x)|^{2\cdot 2_\mu^*}\,\mathrm{d}\kappa\leqslant C(N,\mu)\left(\int_{\mathbb{R}^N}|\phi|^{2^*}\,\mathrm{d}\varsigma\right)^{(2N-\mu)/N}.\tag{2.10}$$

Applying lemma 1.2 in [**26**] we know (2.4) holds.

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Taking $\phi = \chi_{\{z_i\}}$, $i \in I$, in (2.10), we get

$$\nu_i^{N/(2N-\mu)} \leqslant C(N,\mu)^{N/(2N-\mu)}\zeta_i, \quad \forall\, i \in I.$$

By the definition of $S_{H,L}$, we also have

$$S_{H,L}\left(\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{|\phi v_n(y)|^{2_\mu^*}}{|x-y|^\mu}\,\mathrm{d}y\right)|\phi v_n(x)|^{2_\mu^*}\,\mathrm{d}x\right)^{(N-2)/(2N-\mu)} \leqslant \int_{\mathbb{R}^N}|\nabla(\phi v_n)|^2\,\mathrm{d}x.$$

Using (2.8) and the fact that $v_n \to 0$ in $L^2_{loc}(\mathbb{R}^N)$, we have

$$\begin{aligned}S_{H,L}&\left(\int_{\mathbb{R}^N}|\phi(x)|^{2\cdot 2_\mu^*}\left(\int_{\mathbb{R}^N}\frac{|v_n(y)|^{2_\mu^*}}{|x-y|^\mu}\,\mathrm{d}y\right)|v_n(x)|^{2_\mu^*}\,\mathrm{d}x\right)^{(N-2)/(2N-\mu)}\\ &\leqslant \int_{\mathbb{R}^N}\phi^2|\nabla v_n|^2\,\mathrm{d}x + o(1).\end{aligned}$$

Passing to the limit as $n \to +\infty$ we obtain

$$S_{H,L}\left(\int_{\mathbb{R}^N}|\phi(x)|^{2\cdot 2_\mu^*}\,\mathrm{d}\kappa\right)^{(N-2)/(2N-\mu)} \leqslant \int_{\mathbb{R}^N}\phi^2\,\mathrm{d}\varpi. \tag{2.11}$$

Applying lemma 1.2 in [**26**] again we know (2.6) holds. Now by taking $\phi = \chi_{\{z_i\}}$, $i \in I$, in (2.11), we get

$$S_{H,L}\nu_i^{1/(2_\mu^*)} \leqslant \omega_i, \quad \forall\, i \in I.$$

Thus we have (2.2) and (2.5).

Next we are going to prove the possible loss of mass at infinity. For $R > 1$, let $\psi_R \in \mathcal{C}^\infty(\mathbb{R}^N)$ be such that $\psi_R(x) = 1$ for $|x| > R+1$, $\psi_R(x) = 0$ for $|x| < R$ and $0 \leqslant \psi_R(x) \leqslant 1$ on $\mathbb{R}^N$. For every $R > 1$, we have

$$\begin{aligned}&\limsup_{n\to\infty}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u_n(y)|^{2_\mu^*}|u_n(x)|^{2_\mu^*}}{|x-y|^\mu}\,\mathrm{d}y\,\mathrm{d}x\\ &\quad= \limsup_{n\to\infty}\left(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u_n(y)|^{2_\mu^*}|u_n(x)|^{2_\mu^*}\psi_R(x)}{|x-y|^\mu}\,\mathrm{d}y\,\mathrm{d}x\right.\\ &\qquad\left.+\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u_n(y)|^{2_\mu^*}|u_n(x)|^{2_\mu^*}(1-\psi_R(x))}{|x-y|^\mu}\,\mathrm{d}y\,\mathrm{d}x\right)\\ &\quad= \limsup_{n\to\infty}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u_n(y)|^{2_\mu^*}|u_n(x)|^{2_\mu^*}\psi_R(x)}{|x-y|^\mu}\,\mathrm{d}y\,\mathrm{d}x + \int_{\mathbb{R}^N}(1-\psi_R)\,\mathrm{d}\nu.\end{aligned}$$

When $R \to \infty$, we obtain, by Lebesgue's theorem,

$$\limsup_{n\to\infty}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u_n(y)|^{2_\mu^*}|u_n(x)|^{2_\mu^*}}{|x-y|^\mu}\,\mathrm{d}y\,\mathrm{d}x = \nu_\infty + \int_{\mathbb{R}^N}\mathrm{d}\nu.$$

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By the Hardy–Littlewood–Sobolev inequality, we have

$$\begin{aligned}\nu_\infty &= \lim_{R\to\infty}\limsup_{n\to\infty}\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{|u_n(y)|^{2^*_\mu}}{|x-y|^\mu}\,\mathrm{d}y\right)|\psi_R u_n(x)|^{2^*_\mu}\,\mathrm{d}x\\ &\leqslant C(N,\mu)\lim_{R\to\infty}\limsup_{n\to\infty}\left(\int_{\mathbb{R}^N}|u_n|^{2^*}\,\mathrm{d}x\int_{\mathbb{R}^N}|\psi_R u_n|^{2^*}\,\mathrm{d}x\right)^{(2N-\mu)/2N}\\ &= C(N,\mu)\left(\zeta_\infty\left(\int_{\mathbb{R}^N}\mathrm{d}\zeta+\zeta_\infty\right)\right)^{(2N-\mu)/2N},\end{aligned}$$

which means

$$C(N,\mu)^{(-2N)/(2N-\mu)}\nu_\infty^{(2N)/(2N-\mu)}\leqslant\zeta_\infty\left(\int_{\mathbb{R}^N}\mathrm{d}\zeta+\zeta_\infty\right).$$

Similarly, by the definition of $S_{H,L}$ and ν_∞, we have

$$\begin{aligned}\nu_\infty &= \lim_{R\to\infty}\limsup_{n\to\infty}\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{|u_n(y)|^{2^*_\mu}}{|x-y|^\mu}\,\mathrm{d}y\right)|\psi_R u_n(x)|^{2^*_\mu}\,\mathrm{d}x\\ &\leqslant C(N,\mu)\lim_{R\to\infty}\limsup_{n\to\infty}\left(\int_{\mathbb{R}^N}|u_n|^{2^*}\,\mathrm{d}x\int_{\mathbb{R}^N}|\psi_R u_n|^{2^*}\,\mathrm{d}x\right)^{(2N-\mu)/2N}\\ &\leqslant S_{H,L}^{-2^*_\mu}\lim_{R\to\infty}\limsup_{n\to\infty}\left(\int_{\mathbb{R}^N}|\nabla u_n|^2\,\mathrm{d}x\int_{\mathbb{R}^N}|\nabla(\psi_R u_n)|^2\,\mathrm{d}x\right)^{2^*_\mu/2}\\ &= S_{H,L}^{-2^*_\mu}\left(\omega_\infty\left(\int_{\mathbb{R}^N}\mathrm{d}\omega+\omega_\infty\right)\right)^{2^*_\mu/2},\end{aligned}$$

which implies

$$S_{H,L}^2\nu_\infty^{2/(2^*_\mu)}\leqslant\omega_\infty\left(\int_{\mathbb{R}^N}\mathrm{d}\omega+\omega_\infty\right).$$

Moreover, if $u=0$ then $\kappa=\nu$ and $\varpi=\omega$. Then the Hölder inequality and (2.11) imply that, for $\phi\in\mathcal{C}_0^\infty(\mathbb{R}^N)$,

$$\begin{aligned}&S_{H,L}\left(\int_{\mathbb{R}^N}|\phi(x)|^{2\cdot 2^*_\mu}\,\mathrm{d}\nu\right)^{(N-2)/(2N-\mu)}\\ &\quad\leqslant\left(\int_{\mathbb{R}^N}\mathrm{d}\omega\right)^{(N-\mu+2)/(2N-\mu)}\left(\int_{\mathbb{R}^N}\phi^{2\cdot 2^*_\mu}\,\mathrm{d}\omega\right)^{(N-2)/(2N-\mu)}.\end{aligned}$$

Thus we can deduce that $\nu=S_{H,L}^{-2^*_\mu}(\int_{\mathbb{R}^N}\mathrm{d}\omega)^{(N-\mu+2)/(N-2)}\omega$. It follows from (2.11) that, for $\phi\in\mathcal{C}_0^\infty(\mathbb{R}^N)$,

$$\left(\int_{\mathbb{R}^N}|\phi(x)|^{2\cdot 2^*_\mu}\,\mathrm{d}\nu\right)^{(N-2)/(2N-\mu)}\left(\int_{\mathbb{R}^N}\mathrm{d}\nu\right)^{(N-\mu+2)/(2N-\mu)}\leqslant\int_{\mathbb{R}^N}|\phi|^2\,\mathrm{d}\nu.$$

And so, for each open set Ω,

$$\nu(\Omega)^{(N-2)/(2N-\mu)}\nu(\mathbb{R}^N)^{(N-\mu+2)/(2N-\mu)} \leqslant \nu(\Omega).$$

It follows that ν is concentrated at a single point. □

2.2. Convergence of (PS) sequences

Let $\{u_n\}$ be a (PS) sequence of J at level c, it is easy to see that $\{u_n\}$ is bounded in E. Hence, without loss of generality, we may assume that $\{u_n\}$ converges weakly and a.e. to some $u_0 \in E$. Then we are able to recover the lack of compactness by applying the second concentration-compactness principle to the nonlocal Choquard equation. In fact, we have the following proposition which was inspired by [39].

PROPOSITION 2.6. *There exists a positive number $c_0 > 0$ such that every $(PS)_c$ sequence $\{u_n\}$ of J with $c < c_0$ satisfies*

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|(u_n-u_0)(x)|^{2^*_\mu}|(u_n-u_0)(y)|^{2^*_\mu}}{|x-y|^\mu}\,\mathrm{d}x\,\mathrm{d}y = 0,$$

where $u_0 \in E$ is the weak limit of $\{u_n\}$.

Proof. Let $\eta \in \mathcal{C}_0^\infty([0,\infty))$ be a standard cut-off function on $[0, 1]$, that is,

$$\eta(t) \equiv 1, \quad t \in [0,1]; \quad \eta(t) \equiv 0, \quad t > 2; \quad |\eta'(t)| \leqslant C, \quad 0 \leqslant \eta(t) \leqslant 1$$

for some $C > 0$. Fix $i \in I$. For $\varepsilon > 0$, put

$$\eta_\varepsilon = \eta\left(\frac{x-z_i}{\varepsilon}\right), \quad B = B_{2\varepsilon}(z_i). \tag{2.12}$$

It follows from the Hölder inequality and the Sobolev inequality that for all $\sigma \in [0, 2^*)$,

$$\begin{aligned}\int_B |u_n|^\sigma\,\mathrm{d}x &\leqslant |B|^{1-(\sigma/2^*)}\left(\int_B |u_n|^{2^*}\,\mathrm{d}x\right)^{(\sigma)/(2^*)}\\ &\leqslant C|B|^{1-(\sigma/2^*)}\left(\int_B |\nabla u_n|^2\,\mathrm{d}x\right)^{\sigma/2} = o(1), \quad \text{as } \varepsilon \to 0^+.\end{aligned}$$

Hence, by the Hardy–Littlewood–Sobolev inequality, as $\varepsilon \to 0^+$, there holds

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u_n(x)|^{2^*_\mu}|u_n(y)|^p\eta_\varepsilon(y)}{|x-y|^\mu}\,\mathrm{d}x\,\mathrm{d}y = o(1), \tag{2.13}$$

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u_n(x)|^p|u_n(y)|^p\eta_\varepsilon(y)}{|x-y|^\mu}\,\mathrm{d}x\,\mathrm{d}y = o(1) \tag{2.14}$$

and

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u_n(x)|^p|u_n(y)|^{2^*_\mu}\eta_\varepsilon(y)}{|x-y|^\mu}\,\mathrm{d}x\,\mathrm{d}y = o(1). \tag{2.15}$$

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Using proposition 2.5, and passing to the limit by first letting $n \to \infty$ and then letting $\varepsilon \to 0^+$, we have

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u_n(x)|^{2^*_\mu} |u_n(y)|^{2^*_\mu} \eta_\varepsilon(y)}{|x-y|^\mu} \, \mathrm{d}x \, \mathrm{d}y = \nu_i. \tag{2.16}$$

For $R > 0$, put

$$\eta_R = \eta\left(\frac{2R}{|x|}\right) \tag{2.17}$$

and denote

$$\begin{aligned} V_\infty &:= \lim_{R\to\infty} \limsup_{n\to\infty} \int_{\mathbb{R}^N} V|u_n|^2 \eta_R \, \mathrm{d}x, \\ F_\infty &:= \lim_{R\to\infty} \limsup_{n\to\infty} \left(\frac{p}{2^*_\mu} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u_n(x)|^{2^*_\mu} |u_n(y)|^p \eta_R(y)}{|x-y|^\mu} \, \mathrm{d}x \, \mathrm{d}y \right. \\ &\quad + \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u_n(x)|^p |u_n(y)|^{2^*_\mu} \eta_R(y)}{|x-y|^\mu} \, \mathrm{d}x \, \mathrm{d}y \\ &\quad \left. + \frac{p}{2^*_\mu} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u_n(x)|^p |u_n(y)|^p \eta_R(y)}{|x-y|^\mu} \, \mathrm{d}x \, \mathrm{d}y \right). \end{aligned}$$

Multiplying $J'(u_n)$ with the test function $u_n \eta_R$, we obtain by the definition of ω_∞, ν_∞ that

$$\omega_\infty + V_\infty = \nu_\infty + F_\infty. \tag{2.18}$$

It follows that

$$\begin{aligned} c + o(1) &= J(u_n) - \frac{1}{2}\langle J'(u_n), u_n \rangle \\ &= \frac{p + 2^*_\mu - 2}{2 \cdot 2^*_\mu} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u_n(x)|^{2^*_\mu} |u_n(y)|^p}{|x-y|^\mu} \, \mathrm{d}x \, \mathrm{d}y \\ &\quad + \frac{p-1}{2 \cdot 2^*_\mu} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u_n(x)|^p |u_n(y)|^p}{|x-y|^\mu} \, \mathrm{d}x \, \mathrm{d}y \\ &\quad + \frac{N+2-\mu}{4N-2\mu} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u_n(x)|^{2^*_\mu} |u_n(y)|^{2^*_\mu}}{|x-y|^\mu} \, \mathrm{d}x \, \mathrm{d}y \\ &\geqslant \alpha' \left[\left(\frac{p}{2^*_\mu} + 1 \right) \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u_n(x)|^{2^*_\mu} |u_n(y)|^p}{|x-y|^\mu} \, \mathrm{d}x \, \mathrm{d}y \right. \\ &\quad \left. + \frac{p}{2^*_\mu} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u_n(x)|^p |u_n(y)|^p}{|x-y|^\mu} \, \mathrm{d}x \, \mathrm{d}y \right] \\ &\quad + \frac{N+2-\mu}{4N-2\mu} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u_n(x)|^{2^*_\mu} |u_n(y)|^{2^*_\mu}}{|x-y|^\mu} \, \mathrm{d}x \, \mathrm{d}y \\ &\geqslant \alpha' F_\infty + \frac{N+2-\mu}{4N-2\mu} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u_0(x)|^{2^*_\mu} |u_0(y)|^{2^*_\mu}}{|x-y|^\mu} \, \mathrm{d}x \, \mathrm{d}y \end{aligned}$$

$$
\begin{aligned}
&+\frac{N+2-\mu}{4N-2\mu}\left(\nu_\infty+\sum_{i\in I}\nu_i\right)+o(1)\\
&\geqslant \frac{N+2-\mu}{4N-2\mu}\sum_{i\in I}\nu_i+\alpha'(\nu_\infty+F_\infty)+o(1)\\
&\geqslant \frac{N+2-\mu}{4N-2\mu}\sum_{i\in I}\nu_i+\alpha'\omega_\infty+o(1) \qquad (2.19)
\end{aligned}
$$

for some $0<\alpha'<p-1/2p$, where we have used lemma 2.5,

$$
\lim_{n\to\infty}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u_n(x)|^{2^*_\mu}|u_n(y)|^{2^*_\mu}}{|x-y|^\mu}\,\mathrm{d}x\,\mathrm{d}y=\nu_\infty + \sum_{i\in I}\nu_i+\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u_0(x)|^{2^*_\mu}|u_0(y)|^{2^*_\mu}}{|x-y|^\mu}\,\mathrm{d}x\,\mathrm{d}y
$$

and the fact $\nu_\infty+F_\infty=\omega_\infty+V_\infty\geqslant\omega_\infty$.

Now we want to show that there exists $c_0>0$ such that if $c<c_0$ then the singular part and escaping part of the energy of the $(PS)_c$ sequence $\{u_n\}$ are trivial. First, we claim that

$$
I=\emptyset. \qquad (2.20)
$$

On the contrary, assume that $I\neq\emptyset$, then there holds

$$
\nu_i\geqslant S_{H,L}^{(2N-\mu)/(N+2-\mu)},\quad i\in I. \qquad (2.21)
$$

In particular, the set I is finite. In fact, let η_ε be the cut-off function defined in (2.12). By definition, a direct computation yields

$$
\|u_n\eta_\varepsilon\|_V=\left(\int_{\mathbb{R}^N}|\nabla(u_n\eta_\varepsilon)|^2\,\mathrm{d}x+\int_{\mathbb{R}^N}V_+|u_n\eta_\varepsilon|^2\,\mathrm{d}x\right)^{1/2}\leqslant C\|u_n\|_V=O(1).
$$

Multiplying $J'(u_n)$ with the test function $u_n\eta_\varepsilon$, we obtain

$$
\begin{aligned}
o(1)&=\langle J'(u_n),u_n\eta_\varepsilon\rangle\\
&=\int_{\mathbb{R}^N}|\nabla u_n|^2\eta_\varepsilon\,\mathrm{d}x+\int_{\mathbb{R}^N}u_n\nabla u_n\nabla\eta_\varepsilon\,\mathrm{d}x+\int_{\mathbb{R}^N}Vu_n^2\eta_\varepsilon\,\mathrm{d}x\\
&\quad-\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{(|u_n(x)|^{2^*_\mu}+|u_n(x)|^p)(|u_n(y)|^{2^*_\mu}\eta_\varepsilon(y)+(p/(2^*_\mu))|u_n(y)|^p\eta_\varepsilon(y))}{|x-y|^\mu}\,\mathrm{d}x\,\mathrm{d}y, \qquad (2.22)
\end{aligned}
$$

since $\{u_n\}$ is a (PS) sequence. By lemma 2.5, we know

$$
\int_{\mathbb{R}^N}|\nabla u_n|^2\eta_\varepsilon\,\mathrm{d}x\to\omega_i'\geqslant\omega_i. \qquad (2.23)
$$

By lemma 3.1 in [39], we have as $\varepsilon\to0^+$

$$
\int_{\mathbb{R}^N}u_n\nabla u_n\nabla\eta_\varepsilon\,\mathrm{d}x=o(1) \qquad (2.24)
$$

and

$$\Big|\int_{\mathbb{R}^N} V u_n^2 \eta_\varepsilon \, \mathrm{d}x\Big| = o(1). \tag{2.25}$$

From (2.13) to (2.15) and (2.23)–(2.25), we infer that for each fixed $i \in I$

$$\omega_i - \nu_i \leqslant 0.$$

Utilizing (2.5), we finally arrive at

$$S_{H,L}\nu_i^{1/(2_\mu^*)} - \nu_i \leqslant 0$$

and thus (2.21) follows. Now (2.19) leads to a contradiction if $c_0 \leqslant (N+2-\mu)/(4N-2\mu)S_{H,L}^{(2N-\mu)/(N+2-\mu)}$ and thus the singular part is empty.

Next, we prove that

$$\nu_\infty = \omega_\infty = F_\infty = V_\infty = 0. \tag{2.26}$$

To prove that the escaping part is trivial, let

$$a_p := \frac{2^* - (2Np/(2N-\mu))}{2^*-2} \quad \text{and} \quad b_p := \frac{(2Np/(2N-\mu)) - 2}{2^*-2},$$

then a_p, $b_p \in (0,\, 1)$ and $a_p + b_p = 1$. By lemma 3.2 in [**39**] we know that

$$V_\infty = \lim_{R\to\infty}\lim_{n\to\infty}\int_{\mathbb{R}^N} V|u_n|^2\eta_R \, \mathrm{d}x \geqslant \tau_0 \lim_{R\to\infty}\lim_{n\to\infty}\int_{\mathbb{R}^N} |u_n|^2\eta_R \, \mathrm{d}x. \tag{2.27}$$

With this fact, applying the Hardy–Littlewood–Sobolev inequality and the Hölder inequality, we have

$$\begin{aligned}
&\lim_{R\to\infty}\limsup_{n\to\infty}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u_n(x)|^{2_\mu^*}|u_n(y)|^p\eta_R(y)}{|x-y|^\mu}\,\mathrm{d}x\,\mathrm{d}y\\
&\quad\leqslant C(N,\mu)\lim_{R\to\infty}\limsup_{n\to\infty}|u_n|_{2^*}^{2_\mu^*}||u_n|^p\eta_R|_{2N/(2N-\mu)}\\
&\quad\leqslant C_1\lim_{R\to\infty}\limsup_{n\to\infty}\left(\int_{\mathbb{R}^N}|u_n|^2\eta_R\,\mathrm{d}x\right)^{a_p}\left(\int_{\mathbb{R}^N}|u_n|^{2^*}\eta_R\,\mathrm{d}x\right)^{b_p}\\
&\quad\leqslant C_1\left(\frac{V_\infty}{\tau_0}\right)^{a_p}\zeta_\infty^{b_p},
\end{aligned} \tag{2.28}$$

$$\begin{aligned}
&\lim_{R\to\infty}\limsup_{n\to\infty}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u_n(x)|^p|u_n(y)|^{2_\mu^*}\eta_R(y)}{|x-y|^\mu}\,\mathrm{d}x\,\mathrm{d}y\\
&\quad\leqslant C(N,\mu)\lim_{R\to\infty}\limsup_{n\to\infty}|u_n|_{2Np/(2N-\mu)}^p\\
&\quad\times\left(\int_{\mathbb{R}^N}|u_n|^{2^*}|\eta_R|^{(2N)/(2N-\mu)}\,\mathrm{d}x\right)^{(2N-\mu)/2N}\\
&\quad\leqslant C_2\zeta_\infty^{(2N-\mu)/2N}
\end{aligned} \tag{2.29}$$

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and

$$\begin{aligned}
&\lim_{R\to\infty}\limsup_{n\to\infty}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u_n(x)|^p|u_n(y)|^p\eta_R(y)}{|x-y|^\mu}\,\mathrm{d}x\,\mathrm{d}y\\
&\leqslant C(N,\mu)\lim_{R\to\infty}\limsup_{n\to\infty}|u_n|^p_{2Np/(2N-\mu)}||u_n|^p\eta_R|_{2N/(2N-\mu)}\\
&\leqslant C_3\lim_{R\to\infty}\limsup_{n\to\infty}\left(\int_{\mathbb{R}^N}|u_n|^2\eta_R\,\mathrm{d}x\right)^{a_p}\left(\int_{\mathbb{R}^N}|u_n|^{2^*}\eta_R\,\mathrm{d}x\right)^{b_p}\\
&\leqslant C_3\left(\frac{V_\infty}{\tau_0}\right)^{a_p}\zeta_\infty^{b_p},
\end{aligned}\tag{2.30}$$

where C_1, C_2, C_3 depend only on the embedding constant and the best constant $S_{H,L}$, since lemma 2.1 holds and

$$\begin{aligned}
\left(\frac{1}{2}-\frac{1}{2p}\right)&\int_{\mathbb{R}^N}(|\nabla u_n|^2+V(x)|u_n|^2)\,\mathrm{d}x\leqslant c<c_0\\
&\leqslant\frac{N+2-\mu}{4N-2\mu}S_{H,L}^{(2N-\mu)/(N+2-\mu)}+o_n(1).
\end{aligned}$$

Now, by the definition of F_∞, from (2.28) to (2.30) we know

$$F_\infty\leqslant C\left(\frac{V_\infty}{\tau_0}\right)^{a_p}\zeta_\infty^{b_p}+C\zeta_\infty^{(2N-\mu)/2N}.\tag{2.31}$$

Similarly, we have

$$\begin{aligned}
\nu_\infty&=\lim_{R\to\infty}\limsup_{n\to\infty}\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{|u_n(y)|^{2^*_\mu}}{|x-y|^\mu}\,\mathrm{d}y\right)|u_n(x)|^{2^*_\mu}\eta_R\,\mathrm{d}x\\
&\leqslant C(N,\mu)\lim_{R\to\infty}\limsup_{n\to\infty}|u_n|^{2^*_\mu}_{2^*}\left(\int_{\mathbb{R}^N}|u_n|^{2^*}\eta_R\,\mathrm{d}x\right)^{(2N-\mu)/2N}\\
&=C_4\zeta_\infty^{(2N-\mu)/2N}.
\end{aligned}\tag{2.32}$$

Substituting (2.31) and (2.32) into (2.18) we obtain that

$$\omega_\infty+V_\infty\leqslant C_5\left(\frac{V_\infty}{\tau_0}\right)^{a_p}\zeta_\infty^{b_p}+C_6\zeta_\infty^{(2N-\mu)/2N}.\tag{2.33}$$

Now, if $\zeta_\infty=0$ then it is easy to see the conclusion

$$\nu_\infty=\omega_\infty=F_\infty=V_\infty=0.$$

Otherwise, if $\zeta_\infty>0$ then applying the Young inequality to (2.33) we know that there exists $\Lambda_0>0$ such that

$$\zeta_\infty\geqslant\Lambda_0.$$

Thus applying lemma 2.4, we know that

$$\omega_\infty\geqslant S\Lambda_0^{2/2^*}.$$

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Thus we know this is a contradiction if

$$c < \alpha' S \Lambda_0^{2/2^*}.$$

From the arguments above, let

$$c_0 = \min\left\{\frac{N+2-\mu}{4N-2\mu} S_{H,L}^{(2N-\mu)/(N+2-\mu)}, \alpha' S \Lambda_0^{2/(2^*)}\right\},$$

if $c < c_0$ then we have

$$\nu_\infty = \omega_\infty = F_\infty = V_\infty = 0, \quad I = \emptyset. \tag{2.34}$$

By using lemma 2.5 to derive

$$\lim_{n\to\infty} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u_n(x)|^{2^*_\mu}|u_n(y)|^{2^*_\mu}}{|x-y|^\mu}\,\mathrm{d}x\,\mathrm{d}y = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u_0(x)|^{2^*_\mu}|u_0(y)|^{2^*_\mu}}{|x-y|^\mu}\,\mathrm{d}x\,\mathrm{d}y,$$

which together with lemma 2.2 imply

$$\lim_{n\to\infty} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|(u_n-u_0)(x)|^{2^*_\mu}|(u_n-u_0)(y)|^{2^*_\mu}}{|x-y|^\mu}\,\mathrm{d}x\,\mathrm{d}y = 0.$$

□

2.3. Proof of theorem 1.3

We can verify that the functional J satisfies the Mountain-Pass geometry. By lemma 2.1 we have

$$\begin{aligned} J(u) &= \frac{1}{2}\int_{\mathbb{R}^N} (|\nabla u|^2 + V(x)|u|^2)\,\mathrm{d}x \\ &\quad - \frac{1}{2\cdot 2^*_\mu}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{(|u(x)|^{2^*_\mu} + |u(x)|^p)(|u(y)|^{2^*_\mu} + |u(y)|^p)}{|x-y|^\mu}\,\mathrm{d}x\,\mathrm{d}y \\ &\geqslant C\|u\|_V^2 - C_1\|u\|_V^{2\cdot 2^*_\mu} - C_2\|u\|_V^{2p}. \end{aligned}$$

Since $2 < 2p < 2^*_\mu + p < 2\cdot 2^*_\mu$, we can choose some α, $\rho > 0$ such that $J(u) \geqslant \alpha$ for $\|u\|_V = \rho$. For any $u \in E\setminus\{0\}$, we have

$$J(tu) \leqslant \frac{t^2}{2}\int_{\mathbb{R}^N} (|\nabla u|^2 + V(x)u^2\,\mathrm{d}x - \frac{t^{2\cdot 2^*_\mu}}{2\cdot 2^*_\mu}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)|^{2^*_\mu}|u(y)|^{2^*_\mu}}{|x-y|^\mu}\,\mathrm{d}x\,\mathrm{d}y < 0$$

for $t > 0$ large enough. Hence, we can apply the mountain pass theorem without (PS) condition (cf. [**38**]) to get a bounded (PS) sequence $\{u_n\}$ such that $J(u_n) \to$

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$c^\star$ and $J'(u_n) \to 0$ in E^{-1} at the minimax level

$$c^\star = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} J(\gamma(t)) > 0,$$

where

$$\Gamma := \{\gamma \in C([0,1], E) : \gamma(0) = 0, J(\gamma(1)) < 0\}.$$

We claim that

$$\inf \left\{ \int_{\mathbb{R}^N} |\nabla \varphi|^2 \,\mathrm{d}x : \ \varphi \in \mathcal{C}_0^\infty(\mathbb{R}^N), \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|\varphi(x)|^p |\varphi(y)|^p}{|x-y|^\mu} \,\mathrm{d}x \,\mathrm{d}y = 1 \right\} = 0. \tag{2.35}$$

In fact, for all fixed φ satisfying

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|\varphi(x)|^p |\varphi(y)|^p}{|x-y|^\mu} \,\mathrm{d}x \,\mathrm{d}y = 1,$$

let us define

$$\varphi_t = t^{(2N-\mu)/2p} \varphi(tx), \quad t > 0,$$

then we have

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|\varphi_t(x)|^p |\varphi_t(y)|^p}{|x-y|^\mu} \,\mathrm{d}x \,\mathrm{d}y = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|\varphi(x)|^p |\varphi(y)|^p}{|x-y|^\mu} \,\mathrm{d}x \,\mathrm{d}y = 1$$

and

$$\int_{\mathbb{R}^N} |\nabla \varphi_t|^2 \,\mathrm{d}x = t^{((2N-\mu)/p)-N+2} \int_{\mathbb{R}^N} |\nabla \varphi|^2 \,\mathrm{d}x.$$

Since $((2N-\mu)/p) > N-2$, we know

$$\int_{\mathbb{R}^N} |\nabla \varphi_t|^2 \,\mathrm{d}x \to 0$$

as $t \to 0$, the claim is thus proved. Now, for any $\delta > 0$ one can choose $\varphi_\delta \in \mathcal{C}_0^\infty(\mathbb{R}^N)$ such that

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|\varphi_\delta(x)|^p |\varphi_\delta(y)|^p}{|x-y|^\mu} \,\mathrm{d}x \,\mathrm{d}y = 1$$

and

$$|\nabla \varphi_\delta|_2^2 < \delta.$$

Since $\varphi_\delta \in \mathcal{C}_0^\infty(\mathbb{R}^N)$ and $V(x) \in L^{N/2}(\mathbb{R}^N)$, we have

$$\begin{aligned} \left| \int_{\mathbb{R}^N} V(x) |\varphi_\delta|^2 \,\mathrm{d}x \right| &\leqslant \left(\int_{\mathbb{R}^N} |V(x)|^{N/2} \,\mathrm{d}x \right)^{2/N} \left(\int_{\mathbb{R}^N} |\varphi_\delta|^{2^*} \,\mathrm{d}x \right)^{(N-2)/N} \\ &\leqslant CS \int_{\mathbb{R}^N} |\nabla \varphi_\delta|^2 \,\mathrm{d}x, \end{aligned}$$

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where S is the best Sobolev constant. And so,

$$J(\varphi_\delta) \leqslant \frac{1+CS}{2}\int_{\mathbb{R}^N} \left|\nabla \varphi_\delta\right|^2 \mathrm{d}x - \frac{1}{2\cdot 2_\mu^*}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|\varphi_\delta(x)|^p|\varphi_\delta(y)|^p}{|x-y|^\mu}\,\mathrm{d}x\,\mathrm{d}y.$$

Denote

$$\Psi(\varphi_\delta) := \frac{1+CS}{2}\int_{\mathbb{R}^N} \left|\nabla \varphi_\delta\right|^2 \mathrm{d}x - \frac{1}{2\cdot 2_\mu^*}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|\varphi_\delta(x)|^p|\varphi_\delta(y)|^p}{|x-y|^\mu}\,\mathrm{d}x\,\mathrm{d}y.$$

Then, we know

$$\begin{aligned}
&\max_{t\in\mathbb{R}^+}\Psi(t\varphi_\delta)\\
&\quad= \max_{t\in\mathbb{R}^+}\left\{t^2\frac{1+CS}{2}\int_{\mathbb{R}^N} \left|\nabla \varphi_\delta\right|^2 \mathrm{d}x - \frac{t^{2p}}{2\cdot 2_\mu^*}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|\varphi_\delta(x)|^p|\varphi_\delta(y)|^p}{|x-y|^\mu}\,\mathrm{d}x\,\mathrm{d}y\right\}\\
&\quad= \left(\frac{2\cdot 2_\mu^*}{p}\right)^{1/(p-1)}\left(1-\frac{1}{p}\right)\left(\frac{1+CS}{2}\int_{\mathbb{R}^N}|\nabla\varphi_\delta|^2\,\mathrm{d}x\right)^{p/(p-1)}\\
&\quad< \left(\frac{2\cdot 2_\mu^*}{p}\right)^{1/(p-1)}\left(1-\frac{1}{p}\right)\left(\frac{1+CS}{2}\right)^{p/(p-1)}\delta^{p/(p-1)}.
\end{aligned}\tag{2.36}$$

Thus, for $c_0 > 0$ be the number given in proposition 2.6, there exists $\delta_0 > 0$ such that, for any $0 < \delta < \delta_0$

$$\max_{t\in\mathbb{R}^+}\Psi(t\varphi_\delta) < c_0$$

that is,

$$\max_{t\in\mathbb{R}^+}J(t\varphi_\delta) < c_0,$$

and so

$$c^\star < c_0.$$

Assume that $\{u_n\}$ converges weakly and a.e. to some weak solution $u_0 \in E$ of (1.7). In particular,

$$\begin{aligned}
&\int_{\mathbb{R}^N}(|\nabla u_0|^2 + V(x)u_0^2)\,\mathrm{d}x\\
&\quad= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{(|u_0(x)|^{2_\mu^*}+|u_0(x)|^p)\left(|u_0(y)|^{2_\mu^*}+(p/(2_\mu^*))|u_0(y)|^p\right)}{|x-y|^\mu}\,\mathrm{d}x\,\mathrm{d}y.
\end{aligned}\tag{2.37}$$

Since $c^\star < c_0$, by proposition 2.6, we have

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|(u_n-u_0)(x)|^{2_\mu^*}|(u_n-u_0)(y)|^{2_\mu^*}}{|x-y|^\mu}\,\mathrm{d}x\,\mathrm{d}y = 0$$

and

$$\nu_\infty = \omega_\infty = F_\infty = V_\infty = 0, \quad I = \emptyset.$$

So, we have

$$\lim_{n\to\infty}\int_{\mathbb{R}^N} V|u_n-u_0|^2\,\mathrm{d}x=0,$$

and

$$\begin{aligned}\lim_{n\to\infty}\Bigg(\left(\frac{p}{2^*_\mu}+1\right)\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u_n(x)|^{2^*_\mu}|u_n(y)|^p}{|x-y|^\mu}\,\mathrm{d}x\,\mathrm{d}y\\ +\frac{p}{2^*_\mu}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u_n(x)|^p|u_n(y)|^p}{|x-y|^\mu}\,\mathrm{d}x\,\mathrm{d}y\Bigg)\\ =\left(\frac{p}{2^*_\mu}+1\right)\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u_0(x)|^{2^*_\mu}|u_0(y)|^p}{|x-y|^\mu}\,\mathrm{d}x\,\mathrm{d}y\\ +\frac{p}{2^*_\mu}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u_0(x)|^p|u_0(y)|^p}{|x-y|^\mu}\,\mathrm{d}x\,\mathrm{d}y.\end{aligned}$$

It follows that

$$\begin{aligned}0&=\lim_{n\to\infty}\langle J'(u_n),u_n\rangle\\ &=\lim_{n\to\infty}\Bigg(\int_{\mathbb{R}^N}(|\nabla u_n|^2+Vu_n^2)\,\mathrm{d}x\\ &\quad-\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{(|u_n(x)|^{2^*_\mu}+|u_n(x)|^p)\left(|u_n(y)|^{2^*_\mu}+(p/(2^*_\mu))|u_n(y)|^p\right)}{|x-y|^\mu}\,\mathrm{d}x\,\mathrm{d}y\Bigg)\\ &=\lim_{n\to\infty}\int_{\mathbb{R}^N}|\nabla u_n|^2\,\mathrm{d}x+\int_{\mathbb{R}^N}Vu_0^2\,\mathrm{d}x\\ &\quad-\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{(|u_0(x)|^{2^*_\mu}+|u_0(x)|^p)\left(|u_0(y)|^{2^*_\mu}+(p/(2^*_\mu))|u_0(y)|^p\right)}{|x-y|^\mu}\,\mathrm{d}x\,\mathrm{d}y.\end{aligned}$$

Combining this with (2.37), we have

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}|\nabla u_n|^2\,\mathrm{d}x=\int_{\mathbb{R}^N}|\nabla u_0|^2\,\mathrm{d}x.$$

Thus,

$$J(u_0)=\lim_{n\to\infty}J(u_n)=c^\star\geqslant\alpha>0,$$

which leads to the conclusion $u_0\neq 0$.

3. High energy solution

In this section, we assume that conditions (V_1), (V_2) and (V_3) hold, $0<\mu<\min\{4,\,N\}$ and $N\geqslant 3$. We introduce the energy functional associated with equation

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(1.9) by

$$J_V(u) = \frac{1}{2}\int_{\mathbb{R}^N} (|\nabla u|^2 + V(x)|u|^2)\,\mathrm{d}x - \frac{1}{2\cdot 2^*_\mu}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)|^{2^*_\mu}|u(y)|^{2^*_\mu}}{|x-y|^\mu}\,\mathrm{d}x\,\mathrm{d}y.$$

The Hardy–Littlewood–Sobolev inequality implies that J_V is well defined on $D^{1,2}(\mathbb{R}^N)$ and belongs to $\mathcal{C}^1$. And so u is a weak solution of (1.9) if and only if u is a critical point of the functional J_V. To continue the proof, we need to consider the energy functional associated with equation (1.5) defined by

$$J_0(u) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2\,\mathrm{d}x - \frac{1}{2\cdot 2^*_\mu}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)|^{2^*_\mu}|u(y)|^{2^*_\mu}}{|x-y|^\mu}\,\mathrm{d}x\,\mathrm{d}y.$$

3.1. A nonlocal global compactness lemma

Let $u \to u_{r,x_0} = r^{(N-2)/2}u(rx+x_0)$ be the rescaling, where $r \in \mathbb{R}^+$ and $x_0 \in \mathbb{R}^N$. The following lemma is taken from [**16**] which is inspired by [**36**, **38**], we sketch the proof here for the readers' convenience.

LEMMA 3.1. *Suppose that conditions (V_1), (V_2) and (V_3) hold and $N \geqslant 3$, $0 < \mu < \min\{4, N\}$. Assume that $\{u_n\} \subset D^{1,2}(\mathbb{R}^N)$ is a (PS) sequence for J_V. Then there exists a number $k \in \mathbb{N}$, a solution u^0 of (1.9), solutions $u^1, \ldots, u^k$ of (1.5), sequences of points $x_n^1, \ldots, x_n^k \in \mathbb{R}^N$ and radii $r_n^1, \ldots, r_n^k > 0$ such that for some subsequence $n \to \infty$*

$$u_n^0 \equiv u_n \rightharpoonup u^0 \quad \textit{weakly in } D^{1,2}(\mathbb{R}^N),$$
$$u_n^j \equiv (u_n^{j-1} - u^{j-1})_{r_n^j, x_n^j} \rightharpoonup u^j \quad \textit{weakly in } D^{1,2}(\mathbb{R}^N), \quad j = 1, \ldots, k.$$

Moreover, as $n \to \infty$

$$\|u_n\|^2 \to \Sigma_{j=0}^k \|u^j\|^2,$$
$$J_V(u_n) \to J_V(u^0) + \Sigma_{j=1}^k J_0(u^j).$$

Proof. Since $\{u_n\}$ is a (PS) sequence for J_V, we know easily that it is bounded in $D^{1,2}(\mathbb{R}^N)$. Hence we may assume that $u_n \rightharpoonup u^0$ weakly in $D^{1,2}(\mathbb{R}^N)$ as $n \to \infty$ and that u^0 is a weak solution of (1.9). So if we put

$$v_n^1(x) = (u_n - u^0)(x),$$

then v_n^1 is a (PS) sequence for J_V satisfying

$$v_n^1 \rightharpoonup 0 \quad \text{weakly in } D^{1,2}(\mathbb{R}^N).$$

Then, together with the Brézis-Lieb Lemma [**10**] and (2.16) in [**7**] that

$$\int_{\mathbb{R}^N} V(x)|v_n^1|^2\,\mathrm{d}x \to 0, \tag{3.1}$$

we have

$$J_0(v_n^1) = J_V(v_n^1) + o(1) = J_V(u_n) - J_V(u^0) + o(1), \tag{3.2}$$

$$J_0'(v_n^1) = J_V'(v_n^1) + o(1) = o(1). \tag{3.3}$$

If $v_n^1 \to 0$ strongly in $D^{1,2}(\mathbb{R}^N)$ we are done. Now suppose that

$$v_n^1 \nrightarrow 0 \quad \text{strongly in } D^{1,2}(\mathbb{R}^N)$$

and there exists $\gamma \in (0, \infty)$ such that

$$J_0(v_n^1) \geqslant \gamma > 0 \tag{3.4}$$

for n large enough.

Claim: there exist sequences $\{r_n\}$ and $\{y_n\}$ of points in $\mathbb{R}^N$ such that

$$h_n = (v_n^1)_{r_n, y_n} \rightharpoonup h \not\equiv 0 \quad \text{weakly in } D^{1,2}(\mathbb{R}^N) \tag{3.5}$$

as $n \to \infty$.

In fact, by (3.3), we obtain

$$J_0(v_n^1) = \frac{N-\mu+2}{2(2N-\mu)} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|v_n^1(x)|^{2_\mu^*}|v_n^1(y)|^{2_\mu^*}}{|x-y|^\mu}\,\mathrm{d}x\,\mathrm{d}y + o(1).$$

So, by the Hardy–Littlewood–Sobolev inequality, (3.4) and the boundedness of $\{u_n\}$, we know that $0 < a_1 < |v_n^1|_{2^*}^{2_\mu^*} < A_1$ for some $a_1,\ A_1 > 0$. Let us define the Levy concentration function:

$$Q_n(r) := \sup_{z\in\mathbb{R}^N} \int_{B_r(z)} |v_n^1(x)|^{2^*}\,\mathrm{d}x.$$

Since $Q_n(0) = 0$ and $Q_n(\infty) > a_1^{(2N)/(2N-\mu)}$, we may assume there exists sequences $\{r_n\}$ and $\{y_n\}$ of points in $\mathbb{R}^N$ such that $r_n > 0$ and

$$\sup_{z\in\mathbb{R}^N} \int_{B_{r_n}(z)} |v_n^1(x)|^{2^*}\,\mathrm{d}x = \int_{B_{r_n}(y_n)} |v_n^1(x)|^{2^*}\,\mathrm{d}x = b$$

for some

$$0 < b < \min\left\{\frac{S^{2N/(4-\mu)}}{(2C(N,\mu)A_1)^{(2N)/(4-\mu)}}, a_1^{(2N)/(2N-\mu)}\right\}.$$

Let us define $h_n := (v_n^1)_{r_n,\, y_n}$. We may assume that $h_n \rightharpoonup h$ weakly in $D^{1,2}(\mathbb{R}^N)$ and $h_n \to h$ a.e. on $\mathbb{R}^N$. It is easy to see that

$$\sup_{z\in\mathbb{R}^N} \int_{B_1(z)} |h_n(x)|^{2^*}\,\mathrm{d}x = \int_{B_1(0)} |h_n(x)|^{2^*}\,\mathrm{d}x = b.$$

By invariance of the $D^{1,2}(\mathbb{R}^N)$ norms under translation and dilation, we get

$$\|v_n^1\| = \|h_n\|, \quad |v_n^1|_{2^*} = |h_n|_{2^*}$$

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and

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|v_n^1(x)|^{2_\mu^*}|v_n^1(y)|^{2_\mu^*}}{|x-y|^\mu}\,\mathrm{d}x\,\mathrm{d}y=\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|h_n(x)|^{2_\mu^*}|h_n(y)|^{2_\mu^*}}{|x-y|^\mu}\,\mathrm{d}x\,\mathrm{d}y.$$

By direct calculation, we have

$$J_0(h_n)=J_0(v_n^1)=J_V(u_n)-J_V(u^0)+o(1) \tag{3.6}$$

and

$$J_0'(h_n)=J_0'(v_n^1)=o(1). \tag{3.7}$$

If $h=0$ then $h_n\to 0$ strongly in $L^2_{loc}(\mathbb{R}^N)$. Let $\psi\in\mathcal{C}_0^\infty(\mathbb{R}^N)$ be such that $\mathrm{Supp}\psi\subset B_1(y)$ for some $y\in\mathbb{R}^N$. Then, we have

$$\begin{aligned}
&\int_{\mathbb{R}^N}|\nabla(\psi h_n)|^2\,\mathrm{d}x\\
&\quad=\int_{\mathbb{R}^N}\nabla h_n\nabla(\psi^2 h_n)\,\mathrm{d}x+o(1)\\
&\quad=\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|h_n(x)|^{2_\mu^*}|\psi(y)|^2|h_n(y)|^{2_\mu^*}}{|x-y|^\mu}\,\mathrm{d}x\,\mathrm{d}y+o(1)\\
&\quad\leqslant C(N,\mu)|h_n|_{2^*}^{2_\mu^*}\left(\int_{\mathbb{R}^N}(|\psi|^2|h_n|^{2_\mu^*})^{(2N)/(2N-\mu)}\,\mathrm{d}x\right)^{(2N-\mu)/2N}+o(1)\\
&\quad=C(N,\mu)|h_n|_{2^*}^{2_\mu^*}\\
&\qquad\times\left(\int_{\mathbb{R}^N}|\psi h_n|^{(4N)/(2N-\mu)}|h_n|^{(2N(4-\mu))/((2N-\mu)(N-2))}\,\mathrm{d}x\right)^{(2N-\mu)/2N}+o(1)\\
&\quad\leqslant C(N,\mu)|h_n|_{2^*}^{2_\mu^*}|h_n|_{L^{2^*}(B_1(y))}^{2_\mu^*-2}\frac{1}{S}\int_{\mathbb{R}^N}|\nabla(\psi h_n)|^2\,\mathrm{d}x+o(1)\\
&\quad\leqslant C(N,\mu)b^{(2_\mu^*-2)/(2^*)}\frac{A_1}{S}\int_{\mathbb{R}^N}|\nabla(\psi h_n)|^2\,\mathrm{d}x+o(1)\\
&\quad\leqslant\frac{1}{2}\int_{\mathbb{R}^N}|\nabla(\psi h_n)|^2\,\mathrm{d}x+o(1),
\end{aligned}$$

thanks to $0<\mu<\min\{4,N\}$. We obtain $\nabla h_n\to 0$ strongly in $L^2_{loc}(\mathbb{R}^N)$ and $h_n\to 0$ strongly in $L^{2^*}_{loc}(\mathbb{R}^N)$, which contradicts with $\int_{B_1(0)}|h_n(x)|^{2^*}\,\mathrm{d}x=b>0$. So, $h\neq 0$. By (3.3) and weakly sequentially continuous J_0', we know h solves (1.5) weakly. The sequences $\{h_n\}$, $\{r_n^1\}$, and $\{y_n^1\}$ are the wanted sequences.

By iteration, we obtain sequences $v_n^j=u_n^{j-1}-u^{j-1}$, $j\geqslant 2$, and the rescaled functions $u_n^j=(v_n^j)_{r_n^j,y_n^j}\rightharpoonup u^j$ weakly in $D^{1,2}(\mathbb{R}^N)$, where each u^j solves (1.5). By

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induction we know that

$$\|u_n^j\|^2 = \|u_n\|^2 - \Sigma_{i=0}^{j-1}\|u^i\|^2 + o(1) \tag{3.8}$$

and

$$J_0(u_n^j) = J_V(u_n) - J_V(u^0) - \Sigma_{i=1}^{j-1} J_0(u^i) + o(1). \tag{3.9}$$

Furthermore, from the estimate

$$\begin{aligned} 0 = \langle J_0'(u^j), u^j\rangle &= \|u^j\|^2 - \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u^j(x)|^{2_\mu^*}|u^j(y)|^{2_\mu^*}}{|x-y|^\mu}\,\mathrm{d}x\,\mathrm{d}y \\ &\geqslant \|u^j\|^2(1 - S_{H,L}^{-2_\mu^*}\|u^j\|^{2\cdot 2_\mu^*-2}), \end{aligned}$$

we see that $\|u^j\|^2 \geqslant S_{H,L}^{(2N-\mu)/(N-\mu+2)}$ and the iteration must terminate at some index $k \geqslant 0$ due to (3.8). □

Let

$$P(u) = \int_{\mathbb{R}^N} (|\nabla u|^2 + V(x)|u|^2)\,\mathrm{d}x$$

and

$$\mathcal{M} = \{u \in D^{1,2}(\mathbb{R}^N) : \|u\|_{NL} = 1\}.$$

PROPOSITION 3.2. *Suppose that conditions* (V_1), (V_2) *and* (V_3) *hold. Then the minimization problem*

$$\inf\{P(u) : u \in \mathcal{M}\} \tag{3.10}$$

has no solution.

Proof. Let us denote by $S_{\mathcal{M}}$ the infimum defined by (3.10). Obviously, $S_{\mathcal{M}} \geqslant S_{H,L}$. First, we shall show that actually the equality holds. Let us consider the sequence

$$\begin{aligned} \varphi_{(1/n),0}(x) &= S_{H,L}^{(2-N)/(2(N-\mu+2))} U_{(1/n),0}(x) \\ &= S_{H,L}^{2-N/4} C(N,\mu)^{(N(2-N))/(4(2N-\mu))} \frac{[N(N-2)1/n]^{(N-2)/4}}{((1/n)+|x|^2)^{(N-2)/2}}, \end{aligned}$$

then $\forall p \in (N/(N-2), (2N)/(N-2))$, $|\varphi_{1/n,0}(x)|_p \to 0$ (see (2.4), [7]), in fact,

$$\begin{aligned} |\varphi_{(1/n),0}(x)|_p^p &= \left[\frac{N(N-2)}{S_{H,L}}\right]^{(N-2)p/4} C(N,\mu)^{(N(2-N)p)/(4(2N-\mu))} \\ &\quad \times \int_{\mathbb{R}^N} \frac{(1/n)^{(N-2)p/4}}{((1/n)+|x|^2)^{(N-2)p/2}}\,\mathrm{d}x \\ &= \left[\frac{N(N-2)}{S_{H,L}}\right]^{(N-2)p/4} C(N,\mu)^{(N(2-N)p)/(4(2N-\mu))} \\ &\quad \times \left(\frac{1}{n}\right)^{(N/2)-(((N-2)p)/4)} \int_{\mathbb{R}^N} \frac{1}{(1+|x|^2)^{(N-2)p/2}}\,\mathrm{d}x. \end{aligned}$$

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Moreover, using the definition of $S_{H,L}$ and the fact that $U_{1/n,0}$ solves (1.5) it is easy to verify that

$$\varphi_{(1/n),0} \in \mathcal{M}, \quad \text{i.e.} \quad \|\varphi_{(1/n),0}\|_{NL} = 1.$$

Now using the Hölder inequality with $p \in (N/2,\, p_2)$ we get

$$\begin{aligned} P\left(\varphi_{(1/n),0}\right) &= \int_{\mathbb{R}^N} |\nabla \varphi_{(1/n),0}|^2 \,\mathrm{d}x + \int_{\mathbb{R}^N} V(x)|\varphi_{(1/n),0}|^2 \,\mathrm{d}x \\ &\leqslant S_{H,L} + |V(x)|_p |\varphi_{(1/n),0}(x)|^2_{2p'}. \end{aligned}$$

Since $2p' \in (N/(N-2),\, (2N)/(N-2))$, we can obtain $S_{\mathcal{M}} = S_{H,L}$.

Now it is easy to prove the nonexistence result arguing by contradiction. Let $u \in \mathcal{M}$ be a function such that

$$P(u) = S_{H,L}.$$

If $\int_{\mathbb{R}^N} V(x)|u|^2 \,\mathrm{d}x > 0$, then we have

$$\int_{\mathbb{R}^N} |\nabla u|^2 \,\mathrm{d}x < \int_{\mathbb{R}^N} |\nabla u|^2 \,\mathrm{d}x + \int_{\mathbb{R}^N} V(x)|u|^2 \,\mathrm{d}x = S_{H,L}$$

contradicting the definition of $S_{H,L}$. If $\int_{\mathbb{R}^N} V(x)|u|^2 \,\mathrm{d}x = 0$, then

$$\int_{\mathbb{R}^N} |\nabla u|^2 \,\mathrm{d}x = S_{H,L}.$$

Recall that the minimizer of (1.3) is unique, then we know

$$u = C(N,\mu)^{(2-N)/(2(2N-\mu))} S^{2-N/4} \frac{[N(N-2)\delta^1]^{(N-2)/4}}{(\delta^1 + |x - z^1|^2)^{(N-2)/2}}$$

for some $\delta^1 > 0$ and $z^1 \in \mathbb{R}^N$. Since $V(x) \geqslant 0$ on $\mathbb{R}^N$ and $V(x) > 0$ in a positive measure set, we have

$$\int_{\mathbb{R}^N} V(x)|u|^2 \,\mathrm{d}x > 0,$$

which contradicts with $\int_{\mathbb{R}^N} V(x)|u|^2 \,\mathrm{d}x = 0$.

So in conclusion, we know that $S_{\mathcal{M}}$ is not attained. □

COROLLARY 3.3. *The functional $P|_{\mathcal{M}}$ satisfies the $(PS)_c$-condition for $c \in (S_{H,L},\, 2^{(N+2-\mu)/(2N-\mu)} S_{H,L})$.*

Proof. Let $\{u_n\} \subset D^{1,2}(\mathbb{R}^N)$ be a $(PS)_c$-sequence for $P|_{\mathcal{M}}$ with $c \in (S_{H,L},\, 2^{(N+2-\mu)/(2N-\mu)} S_{H,L})$. Then, $\{w_n\}$ is a $(PS)_c$-sequence for J_V with

$$\frac{N+2-\mu}{4N-2\mu} S_{H,L}^{(2N-\mu)/(N+2-\mu)} < c < \frac{N+2-\mu}{2N-\mu} S_{H,L}^{(2N-\mu)/(N+2-\mu)},$$

where $w_n = P(u_n)^{(N-2)/(2(N+2-\mu))} u_n$. We know from lemma 3.1 that there exists a number $k \in \mathbb{N}$, a solution w^0 of (1.9) and solutions $w^1, \ldots, w^k$ of (1.5), such that

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for some subsequence $n \to \infty$

$$\|w_n\|^2 \to \Sigma_{j=0}^k \|w^j\|^2,$$
$$J_V(w_n) \to J_V(w^0) + \Sigma_{j=1}^k J_0(w^j).$$

By proposition 3.2, if w is a nontrivial solution of (1.9), then

$$J_V(w) > \frac{N+2-\mu}{2(2N-\mu)} S_{H,L}^{(2N-\mu)/(N+2-\mu)}.$$

While for every nontrivial solution v of (1.5)

$$J_0(v) \geqslant \frac{N+2-\mu}{2(2N-\mu)} S_{H,L}^{(2N-\mu)/(N+2-\mu)}.$$

Since

$$c < \frac{N+2-\mu}{2N-\mu} S_{H,L}^{(2N-\mu)/(N+2-\mu)},$$

we have $k = 0$ or $k = 1$ with $w^0 = 0$. In conclusion, $\{w_n\}$ is relatively compact in $D^{1,2}(\mathbb{R}^N)$.

So, the functional $P|_{\mathcal{M}}$ satisfies the $(PS)_c$-condition for $c \in (S_{H,L}, 2^{(N+2-\mu)/(2N-\mu)} S_{H,L})$. □

3.2. Proof of theorem 1.4

We now consider the functions

$$\varphi_{\delta,z}(x) = \frac{U_{\delta,z}(x)}{\|U_{\delta,z}(x)\|_{NL}} = S_{H,L}^{2-N/4} C(N,\mu)^{(N(2-N))/(4(2N-\mu))} \times \frac{[N(N-2)\delta]^{(N-2)/4}}{(\delta+|x-z|^2)^{(N-2)/2}}, \quad \delta > 0, \quad z \in \mathbb{R}^N.$$

Note that $\forall \delta > 0, \quad z \in \mathbb{R}^N$

$$\|\varphi_{\delta,z}\|^2 = S_{H,L}, \quad \|\varphi_{\delta,z}\|_{NL} = 1$$

and so $\varphi_{\delta,z} \in \mathcal{M}$. Moreover, $|\varphi_{\delta,z}|_p$, $p \in (N/(N-2), (2N)/(N-2))$, for any fixed p depends only on δ because of the invariance by translation of the $L^p(\mathbb{R}^N)$ norm.

Lemma 3.4. *Suppose that $V(x)$ satisfies (V_3). Then*

$$P(\varphi_{\delta,z}) < 2^{(N+2-\mu)/(2N-\mu)} S_{H,L}, \quad \forall\, \delta > 0, \quad \forall\, z \in \mathbb{R}^N.$$

Proof. Using (V_3), the Hölder inequality and the Hardy–Littlewood–Sobolev inequality, we get

$$\begin{aligned} P(\varphi_{\delta,z}) &= \int_{\mathbb{R}^N} |\nabla\varphi_{\delta,z}|^2\,\mathrm{d}x + \int_{\mathbb{R}^N} V(x)|\varphi_{\delta,z}|^2\,\mathrm{d}x \\ &\leqslant S_{H,L} + |V(x)|_{N/2}\left(\int_{\mathbb{R}^N} |\varphi_{\delta,z}|^{(2N)/(N-2)}\,\mathrm{d}x\right)^{(N-2)/N} \\ &= S_{H,L} + |V(x)|_{(N/2)}\frac{1}{C(N,\mu)^{(N-2)/(2N-\mu)}} \\ &\quad\times\left(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|\varphi_{\delta,z}(x)|^{2^*_\mu}|\varphi_{\delta,z}(y)|^{2^*_\mu}}{|x-y|^\mu}\,\mathrm{d}x\,\mathrm{d}y\right)^{(N-2)/(2N-\mu)} \\ &< S_{H,L} + (2^{(N+2-\mu)/(2N-\mu)}-1)S_{H,L} = 2^{(N+2-\mu)/(2N-\mu)}S_{H,L}. \end{aligned}$$ □

Now put

$$\phi(x) = \begin{cases} 0 & \text{if} \quad |x| < 1, \\ 1 & \text{if} \quad |x| \geqslant 1, \end{cases}$$

and define

$$\alpha : D^{1,2}(\mathbb{R}^N) \to \mathbb{R}^{N+1}$$

$$\alpha(u) = \frac{1}{S_{H,L}}\int_{\mathbb{R}^N}\left(\frac{x}{|x|},\phi(x)\right)|\nabla u|^2\,\mathrm{d}x = (\beta(u),\gamma(u)),$$

where

$$\beta(u) = \frac{1}{S_{H,L}}\int_{\mathbb{R}^N}\frac{x}{|x|}|\nabla u|^2\,\mathrm{d}x$$

and

$$\gamma(u) = \frac{1}{S_{H,L}}\int_{\mathbb{R}^N}\phi(x)|\nabla u|^2\,\mathrm{d}x.$$

Denote

$$\mathcal{A} := \left\{u \in \mathcal{M} : \alpha(u) = \left(0,\frac{1}{2}\right)\right\},$$

and

$$c^\star = \inf_{u\in\mathcal{A}} P(u).$$

The following proposition is due to Benci and Cerami [7] with S replaced by $S_{H,L}$.

Proposition 3.5.

(1). $c^\star > S_{H,L}$;

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(2). There is a $\delta_1 : 0 < \delta_1 < 1/2$ *such that*

$$P(\varphi_{\delta_1,z}) < \frac{S_{H,L} + c^\star}{2}, \quad \forall z \in \mathbb{R}^N,$$
$$\gamma(\varphi_{\delta_1,z}) < \frac{1}{2}, \quad \forall z : |z| < \frac{1}{2},$$
$$\left|\beta(\varphi_{\delta_1,z}) - \frac{z}{|z|}\right| < \frac{1}{4}, \quad \forall z : |z| \geqslant \frac{1}{2};$$

(3). There is a $\delta_2 : \delta_2 > 1/2$ *such that*

$$P(\varphi_{\delta_2,z}) < \frac{S_{H,L} + c^\star}{2}, \quad \forall z \in \mathbb{R}^N,$$
$$\gamma(\varphi_{\delta_2,z}) > \frac{1}{2}, \quad \forall z \in \mathbb{R}^N;$$

(4). There exists $R \in \mathbb{R}^+$ *such that*

$$P(\varphi_{\delta,z}) < \frac{S_{H,L} + c^\star}{2}, \quad \forall z : |z| \geqslant R \quad \text{and} \quad \delta \in [\delta_1, \delta_2],$$
$$(\beta(\varphi_{\delta,z})|z)_{\mathbb{R}^N} > 0, \quad \forall z : |z| \geqslant R \quad \text{and} \quad \delta \in [\delta_1, \delta_2].$$

Now let

$$Z = \{(z, \delta) \in \mathbb{R}^{N+1} : |z| < R, \delta \in [\delta_1, \delta_2]\},$$

and let Φ be the operator

$$\Phi : [\mathbb{R}^N \times (0, +\infty)] \to D^{1,2}(\mathbb{R}^N)$$

given by

$$\Phi(z, \delta) = \varphi_{\delta,z}(x).$$

Note that Φ is continuous. Call Σ the subset of $\mathcal{M}$ defined by

$$\Sigma = \{\Phi(z, \delta) : (z, \delta) \in \overline{Z}\}.$$

Consider then the family

$$\mathbb{A} := \left\{h \in L(\mathcal{M}, \mathcal{M}) : h(u) = u, \forall u \in P^{-1}\left(\left(-\infty, \frac{S_{H,L} + c^\star}{2}\right)\right)\right\}$$

and define

$$\Gamma = \{B \subset \mathcal{M} : B = h(\Sigma), h \in \mathbb{A}\}.$$

Similar to the proof of lemma 3.12 [7], we know that

LEMMA 3.6. *If* $B \in \Gamma$*, then* $B \cap \mathcal{A} \neq \emptyset$.

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Now we set

$$c = \inf_{B\in\Gamma}\sup_{u\in B} P(u) \tag{3.11}$$

$$K_c = \{u \in \mathcal{M} : P(u) = c \text{ and } P'|_{\mathcal{M}}(u) = 0\}.$$

Moreover, for $d \in \mathbb{R}$, P^d will be

$$P^h = \{u \in \mathcal{M} : P(u) \leqslant d\}.$$

Proof of theorem 1.4. We shall prove the theorem showing that $K_c \neq \emptyset$, that is, that c defined by (3.11) is a critical level and there is a critical point u such that $P(u) = c$. By $\Sigma \in \Gamma$ and lemma 3.4, we know

$$c \leqslant \sup_{u\in\Sigma} P(u) \leqslant \sup_{z\in\mathbb{R}^N,\delta\in\mathbb{R}^+} P(\varphi_{\delta,z}) < 2^{(N+2-\mu)/(2N-\mu)} S_{H,L}.$$

Also by lemma 3.6, $B \cap \mathcal{A} \neq \emptyset$, $\forall B \in \Gamma$, so

$$c \geqslant \inf_{\mathcal{A}} P(u) = c^\star > S_{H,L}.$$

Hence

$$S_{H,L} < c^\star < 2^{(N+2-\mu)/(2N-\mu)} S_{H,L}.$$

Suppose now $K_c = \emptyset$. By proposition 3.3 the Palais-Smale condition holds in

$$\{u \in \mathcal{M} : S_{H,L} < P(u) < 2^{(N+2-\mu)/(2N-\mu)} S_{H,L}\},$$

then using a variant of a well-known deformation Lemma (see [**38**]) we find a continuous map

$$\eta : [0, 1] \times \mathcal{M} \to \mathcal{M}$$

and a positive number ε_0 such that

$$P^{c+\varepsilon_0} \backslash P^{c-\varepsilon_0} \subset P^{2^{(N+2-\mu)/(2N-\mu)} S_{H,L}} \backslash P^{S_{H,L}+c^\star/2},$$

$$\eta(0, u) = u,$$

$$\eta(t, u) = u, \quad \forall\, u \in P^{c-\varepsilon_0} \cup \{\mathcal{M}\backslash P^{c+\varepsilon_0}\}, \quad \forall\, t \in (0, 1)$$

and

$$\eta(1, P^{c+\varepsilon_0/2}) \subset P^{c-\varepsilon_0/2}.$$

Now let $\widetilde{B} \in \Gamma$ be such that

$$c \leqslant \sup_{\widetilde{B}} P(u) < c + \frac{\varepsilon_0}{2}.$$

Then $\eta(1, \widetilde{B}) \in \Gamma$ and

$$\sup_{u\in\eta(1,\widetilde{B})} P(u) < c - \frac{\varepsilon_0}{2}$$

contradicting with the definition of c. □

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