RISK SHARING WITH EXPECTED AND DUAL UTILITIES

BY

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ABSTRACT

This paper analyzes optimal risk sharing among agents that are endowed with either expected utility preferences or with dual utility preferences. We find that Pareto optimal risk redistributions and the competitive equilibria can be obtained via bargaining with a hypothetical representative agent of expected utility maximizers and a hypothetical representative agent of dual utility maximizers. The representative agent of expected utility maximizers resembles an average risk-averse agent, whereas representative agent of dual utility maximizers resembles an agent that has lowest aversion to mean-preserving spreads. This bargaining leads to an allocation of the aggregate risk to both groups of agents. The optimal contract for the expected utility maximizers is proportional to their allocated risk, and the optimal contract for the dual utility maximizing agents is given by "tranching" of their allocated risk. We show a method to derive equilibrium prices. We identify a condition under which prices are locally independent of the expected utility functions, and given in closed form. Moreover, we characterize uniqueness of the competitive equilibrium.

KEYWORDS

Pareto optimal risk sharing, competitive equilibria, expected utility, dual utility.

1. INTRODUCTION

This paper studies risk sharing in markets with expected utility maximizers and with dual utility maximizers. Expected utility is often applied as preference relation for individuals, whereas dual utility is often applied to model the preferences of firms. Expected utility is characterized in the seminal work of Von Neumann and Morgenstern (1944) and is well studied in the economic literature. Dual utility is characterized by Yaari (1987) by a modification of the independence axiom in expected utility theory. For expected utility theory, the independence axiom requires independence with respect to probability mixtures of

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risks. For dual theory, the modified independence axiom requires independence with respect to direct mixing the realizations of the risks. The preferences towards risk are linear in the pay-offs but non-linear in the probabilities. Its main property is cash invariance. This means that cash payments do not affect risk preferences. Dual theory has applications in both actuarial science and finance, as it is related to the concept of coherent risk measures (Artzner *et al.*, 1999). It includes the expected shortfall that gained interest after the introduction of Basel III and Swiss Solvency Test regulations.

Risk sharing is a classical topic in actuarial science. There is a stream of papers that analyze optimal risk sharing in settings where all agents maximize expected utilities (Borch, 1962; Wilson, 1968; Bühlmann and Jewell, 1979; Raviv, 1979; Bühlmann, 1980, 1984; Aase, 1993, 2010). More recently, risk sharing in settings with dual utility maximizing agents is studied in the literature as well (Filipović and Kupper, 2008; Jouini *et al.*, 2008; Ludkovski and Young, 2009; Dana and Le Van, 2010; Boonen, 2015, 2017). To the best of our knowledge, we are the first to analyze markets in which both types of agents coexist. We do not argue that either expected utility or dual utility is better, and we do not find any clear arguments why not both types of agents might coexist in the market. In economic experiments, there is no clear consensus for one of these two preference relations as well (see, e.g., Hey and Orme, 1994). Heterogeneous agents models gained substantial interest in economics and finance (see Hommes, 2006, for an overview).

Our approach in this paper is twofold. First, we characterize all Pareto optimal risk redistributions. In this way, we generalize in this way the results of Borch (1962) for expected utilities and the results of Jouini *et al.* (2008) and Ludkovski and Young (2009) for dual utilities. Second, we select a specific Pareto optimal risk redistribution using the concept of competitive equilibria in a market where agents act as price-takers. We determine the equilibrium prices and corresponding risk redistributions and characterize uniqueness of the competitive equilibrium. Moreover, we illustrate the construction of the equilibrium in some special cases.

This paper is related to the equilibrium model of Chateauneuf *et al.* (2000) and Tsanakas and Christofides (2006). They all use rank-dependent utility (RDU) preferences in order to derive the Pareto optimal risk sharing contracts and the competitive equilibrium. RDU preferences are originally characterized by Quiggin (1982, 1993) and Schmeidler (1989) and generalize both expected and dual utility. In order to derive a solution, Chateauneuf *et al.* (2000) and Tsanakas and Christofides (2006) need all agents to have strictly concave expected utility and distortion functions. Pareto optimal risk redistributions are similar to the solution with regular expected utilities, but with heterogeneous distorted probability measures. This is in line with Wilson (1968), who studies markets with expected utility maximizers using subjective probabilities. Moreover, Tsanakas and Christofides (2006) obtain the competitive equilibrium by solving the first-order conditions where the comonotonicity conditions are slack. Strzalecki and Werner (2011) analyze the comonotonicity of Pareto optimal risk redistributions in the context of ambiguity. They show that all Pareto optimal risk redistributions are comonotone if agents use strictly convex preferences. Dual utility preferences are not strictly convex. In our model, either the utility function or the distorted probabilities are linear for every agent. Then, the comonotonicity of equilibrium risk redistributions is used as constraint in order to solve the solution numerically. We find that the corresponding equilibrium prices and risk redistributions are substantially different compared to Chateauneuf *et al.* (2000) and Tsanakas and Christofides (2006). Chateauneuf *et al.* (2000) and Tsanakas and Christofides (2006) find that the prices depend also on expected utility functions, whereas we find that prices may be locally independent of the expected utility functions.

This paper also contributes to the literature that study uniqueness of the competitive equilibrium. This is analyzed by Aase (1993, 2010) for expected utility preferences and by Boonen (2015, 2017) for dual utility maximizers. We derive a condition that characterizes uniqueness of the equilibrium. This condition is identical to one property of Boonen (2015), who states the condition for a market with only dual utility maximizers. Uniqueness of the equilibrium is relevant as it allows us to formalize the Capital Asset Pricing Model (CAPM) based on the unique prices. The prices follow from specific dual utility preferences in the market. Testing the equilibrium prices that we derive is mathematically equivalent to the test of De Giorgi and Post (2008) on the U.S. stock returns. They show a better fit than the classical CAPM model with mean-variance investors. Therefore, De Giorgi and Post (2008) provide an empirical motivation for the results in this paper as well.

This paper is set out as follows. Section 2 introduces the model. Section 3 analyzes the Pareto optimality. Section 4 characterizes the competitive equilibrium prices, as well as a characterization of uniqueness of the corresponding equilibrium risk redistribution. Section 5 illustrates the competitive equilibrium in case all expected utility maximizers use exponential utility functions. Finally, Section 6 concludes this paper.

2. Model outline

We consider a one-period model with a pre-determined future time. All random variables discussed in this paper are on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ such that:

- the state space Ω is finite. Let $\mathcal{F}$ the power set on Ω, and the cardinality of Ω equals $p > 1$;
- $\mathbb{P}(\{\omega\}) > 0$ for all $\omega \in \Omega$. The probability measure is common knowledge.

We denote $\mathbb{R}^{\Omega}$ as the set of all random variables on the state space Ω.

Dual utility is introduced by Yaari (1987). Moreover, it is characterized as a premium principle by Wang *et al.* (1997), which is called a distortion risk

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measure ρ. Dual utility is given by

$$V(X) = -\rho(X) = -\int_{-\infty}^{0} g(F_X(x))\,dx - \int_{0}^{\infty} (g(F_X(x)) - 1)\,dx, \text{ for all } X \in \mathbb{R}^{\Omega}, \tag{1}$$

for a continuous, concave and increasing distortion function $g : [0, 1] \to [0, 1]$ with $g(0) = 0$ and $g(1) = 1$, where F_X is the cumulative distribution function of random variable X. Here, X is interpreted as a future gain. We explicitly assume concavity of the distortion function g, which is equivalent to aversion to mean-preserving spreads (Yaari, 1987). Every distortion risk measure is coherent (see Wang *et al.*, 1997), which is later defined by Artzner *et al.* (1999). For a random variable $X \in \mathbb{R}^{\Omega}$ such that $X(\omega_1) \leq \cdots \leq X(\omega_p)$, it holds that

$$V(X) = E_{\mathbb{Q}}[X], \text{ for all } X \in \mathbb{R}^{\Omega}, \tag{2}$$

where $\mathbb{Q} : \mathcal{F} \to (0, 1]$ is the additive mapping such that

$$\mathbb{Q}(\{\omega_\ell\}) = g(\mathbb{P}(\{\omega_1, \ldots, \omega_\ell\})) - g(\mathbb{P}(\{\omega_1, \ldots, \omega_{\ell-1}\})), \text{ for all } \ell \in \{1, \ldots, p\}. \tag{3}$$

Throughout this paper, we assume that there exists:

- a finite collection of Von Neumann–Morgenstern expected utility maximizing agents that is given by $N_1 = \{1, \ldots, n_1\}$; the corresponding utility functions are given by u_i, $i \in N_1$. Moreover, we assume that $u_i'(\cdot) > 0$, $u_i''(\cdot) < 0$, and that the Inada conditions $\lim_{x\to-\infty} u_i'(x) = \infty$ and $\lim_{x\to\infty} u_i'(x) = 0$ are satisfied for all $i \in N_1$;
- a finite collection of dual utility maximizing agents that is given by $N_2 = \{n_1 + 1, \ldots, n_1 + n_2\}$; the corresponding distortion functions are strictly concave, and given by g_i, $i \in N_2$.

Later in this paper (Proposition 4.6), we will discuss results in case the distortion functions are concave instead of strictly concave. We define $N = N_1 \cup N_2$. Agent $i \in N$ holds a random variable $X_i \in \mathbb{R}^{\Omega}$ that we denote as *risk*. Generally, we define the utility of agent $i \in N$ as follows:

$$V_i(X) = \begin{cases} E_{\mathbb{P}}[u_i(X)] & \text{if } i \in N_1, \\ -\rho_i(X) & \text{if } i \in N_2, \end{cases} \tag{4}$$

for all $X \in \mathbb{R}^{\Omega}$.

For an overview of the differences between expected utility and dual utility, we refer to Wang and Young (1998). Dual utilities can be represented as Von Neumann–Morgenstern expected utilities if and only if the distortion function is given by $g(x) = x$ for all $x \in [0, 1]$, i.e., if the agent is risk neutral: $V(X) = E_{\mathbb{P}}[X]$ for all $X \in \mathbb{R}^{\Omega}$. This follows directly from the fact that the only class of expected utility functions satisfying positive homogeneity[1] is the class of linear utility functions.

3. Pareto optimality

In Section 2, we defined the preferences (expected and dual utilities) that are present in the market that we consider in this paper. In this section, we provide a full characterization of Pareto optimality in such markets. Moreover, we provide an algorithm to compute the Pareto optimal risk redistributions. In order to define the Pareto optimality properly, we first define the set of feasible risk redistributions as follows:

$$\mathcal{X} = \left\{ (\widetilde{X}_i)_{i\in N} \in (\mathbb{R}^{\Omega})^N : \sum_{i\in N} \widetilde{X}_i = \sum_{i\in N} X_i \right\}. \tag{5}$$

A risk redistribution is called the Pareto optimal if there does not exist another feasible redistribution that is weakly better for all agents, and strictly better for at least one agent. The set of Pareto optimal risk redistributions is given by

$$\mathcal{PO} = \left\{ (\widetilde{X}_i)_{i\in N} \in \mathcal{X} : \nexists (\hat{X}_i)_{i\in N} \in \mathcal{X} \text{ s.t. } (V_i(\hat{X}_i))_{i\in N} \gneqq (V_i(\widetilde{X}_i))_{i\in N} \right\}, \tag{6}$$

where for every $a, b \in \mathbb{R}^N$, $a \gneqq b$ means $a_i \geq b_i$ for every $i \in N$ and $a \neq b$. Here, the preferences V_i, $i \in N$, are given in (4).

From Kiesel and Rüschendorf (2007, Theorem 3.3 therein), we get that the Pareto optimal risk redistributions are obtained by maximizing

$$\sum_{i\in N} k_i V_i(\widetilde{X}_i), \tag{7}$$

over all $(\widetilde{X}_i)_{i\in N} \in \mathcal{X}$, where $k \in \mathbb{R}^N_{++}$.[2] We impose the normalization $k_{n_1+n_2} = 1$. Denote $\rho^*_{N_2}$ as the distortion risk measure generated by the strictly concave distortion function $g^*_{N_2}(x) = \min\{g_i(x) : i \in N_2\}$ for all $x \in [0, 1]$. It follows essentially from Jouini *et al.* (2008, Theorem 3.1 and Proposition 3.1 therein)[3] that for all $X \in \mathbb{R}^{\Omega}$, we have

$$\min \sum_{i\in N_2} \rho_i(\widetilde{X}_i) = \rho^*_{N_2}(X),$$

where the minimum is taken over all $(\widetilde{X}_i)_{i\in N_2}$ such that $\sum_{i\in N_2} \widetilde{X}_i = X$. Analogous to Jouini *et al.* (2008, Theorem 3.1 therein) for k not equal to the unit vector, we get that $\min \sum_{i\in N_2} k_i \rho_i(\widetilde{X}_i)$ does not exist for non-degenerate for all $i \in N_2$, where the minimum is taken over all $(\widetilde{X}_i)_{i\in N_2}$ such that $\sum_{i\in N_2} \widetilde{X}_i = X$. Therefore, we set $k_i = 1$ for all $i \in N_2$, and the objective function in (7) can be written as

$$\sum_{i\in N_1} k_i E_{\mathbb{P}}[u_i(\widetilde{X}_i)] - \rho^*_{N_2}\left(\sum_{i\in N_2} \widetilde{X}_i\right). \tag{8}$$

A risk redistribution $(\widetilde{X}_i)_{i\in N} \in \mathcal{X}$ is called comonotone with each other if there exists an ordering $(\omega_1, \dots, \omega_p)$ of the state space Ω such that $\widetilde{X}_i(\omega_1) \le \dots \le \widetilde{X}_i(\omega_p)$ for all $i \in N$. The existence of a Pareto optimal comonotone risk redistribution is shown by Landsberger and Meilijson (1994). They show that any allocation of an aggregate risk $\sum_{i\in N} X_i$ is dominated by a comonotone allocation in the sense of second-order stochastic dominance. Carlier *et al.* (2012, Theorem 3.1 therein) extend this result by showing that every strictly concave order preserving preference relation is such that for every non-comonotone risk redistribution, there exists a comonotone risk redistribution that Pareto dominates it. Moreover, Chew *et al.* (1987, Corollary 2 therein) show that dual utilities generated by a strictly concave distortion function strictly preserve the concave order. For strictly concave expected utility functions, this holds by definition (Rothschild and Stiglitz, 1970). From this follows directly the following lemma.

Lemma 3.1. *Every* $(\widetilde{X}_i)_{i\in N} \in \mathcal{PO}$ *is comonotone with each other, where the set* $\mathcal{PO}$ *is defined in* (6).

From Lemma 3.1, we get that all Pareto optimal risk redistributions are comonotone with the aggregate risk $\sum_{i\in N} X_i$. So, there exists an ordering of the finite probability space Ω such that for all $(\widetilde{X}_i)_{i\in N} \in \mathcal{PO}$ we have $\widetilde{X}_i(\omega_1) \le \dots \le \widetilde{X}_i(\omega_p)$ for all $i \in N$. Without loss of generality, we define the state space $\Omega = \{\omega_1, \dots, \omega_p\}$ such that

$$\sum_{i\in N} X_i(\omega_1) \le \dots \le \sum_{i\in N} X_i(\omega_p).$$

Moreover, we write $\Omega_\ell = \{\omega_1, \dots, \omega_\ell\}$ for all $\ell \in \{1, \dots, p\}$, and $\Omega_0 = \emptyset$. We will refer to the comonotonicity constraints as $\widetilde{X}_i(\omega_1) \le \dots \le \widetilde{X}_i(\omega_p)$ for all $i \in N$. From Lemma 3.1, (2), (3) and (8), we immediately derive that Pareto optimal risk redistributions $(\widetilde{X}_i)_{i\in N}$ are characterized as the ones that satisfy the following optimization problem:

$$\max \sum_{i\in N_1} k_i E_{\mathbb{P}}[u_i(\widetilde{X}_i)] + E_{\mathbb{Q}}\left[\sum_{i\in N_2} \widetilde{X}_i\right], \tag{9}$$

which is maximized over all $(\widetilde{X}_i)_{i\in N} \in \mathcal{X}$ such that the comonotonicity constraints are satisfied and

$$\sum_{i\in N_2} \rho_i(\widetilde{X}_i) = \rho^*_{N_2}\left(\sum_{i\in N_2} \widetilde{X}_i\right), \tag{10}$$

where $\mathbb{Q}(\{\omega_\ell\}) = g^*_{N_2}(\mathbb{P}(\Omega_\ell)) - g^*_{N_2}(\mathbb{P}(\Omega_{\ell-1}))$ for all $\ell \in \{1, \dots, p\}$. Since the objective function in (9) is concave, it follows from the Inada conditions that the objective in (9) is bounded from above. In this section, we solve this optimization problem using variational calculus.

From Lemma 3.1, we directly get the following corollary.

Corollary 3.2. *If* $\sum_{i\in N} X_i(\omega) = \sum_{i\in N} X_i(\omega')$ *for some* $\omega, \omega' \in \Omega$*, then* $\widetilde{X}_i(\omega) = \widetilde{X}_i(\omega')$ *for all* $i \in N$ *and for all Pareto optimal risk redistributions* $\widetilde{X}_i$, $i \in N$.

So, if $\sum_{i\in N} X_i(\omega) = \sum_{i\in N} X_i(\omega')$, we tread the set $\{\omega, \omega'\}$ as one state.

If $N = N_1$, we get from Borch (1962) that a risk redistribution $(\widetilde{X}_i)_{i\in N_1} \in \mathcal{X}$ is Pareto optimal if and only if there exists a $k \in \mathbb{R}^{N_1}_{++}$ such that

$$k_1 u_1'(\widetilde{X}_1(\omega)) = \cdots = k_{n_1} u_{n_1}'(\widetilde{X}_{n_1}(\omega)), \tag{11}$$

for all $\omega \in \Omega$. Moreover, if $N = N_2$, Jouini *et al.* (2008) show that a necessary and sufficient condition for a risk redistribution $(\widetilde{X}_i)_{i\in N} \in \mathcal{X}$ to be Pareto optimal is given by $\sum_{i\in N_2} V_i(\widetilde{X}_i) = -\rho^*_{N_2}\left(\sum_{i\in N_2} X_i\right)$. The following theorem extends these results for the case that there are some agents in the market maximizing utility and some agents that maximize dual utility. Since the objective function in (9) is concave and the constraints are all affine, we can use the Karush–Kuhn–Tucker (KKT) conditions to get that the Pareto optimal risk redistributions.

Theorem 3.3. *If* $N_2 \neq \emptyset$*, it holds that* $(\widetilde{X}_i)_{i\in N} \in \mathcal{PO}$ *if and only if there exists a* $k \in \mathbb{R}^{N_1}_{++}$ *such that*

$$\begin{aligned} &k_1 \mathbb{P}(\{\omega_\ell\}) u_1'(\widetilde{X}_1(\omega_\ell)) = \cdots = k_{n_1} \mathbb{P}(\{\omega_\ell\}) u_{n_1}'(\widetilde{X}_{n_1}(\omega_\ell)) \\ &\quad = g^*_{N_2}(\mathbb{P}(\Omega_\ell)) - g^*_{N_2}(\mathbb{P}(\Omega_{\ell-1})) + \begin{cases} -\hat{\gamma}_1 & \text{if } \ell = 1, \\ +\hat{\gamma}_{\ell-1} - \hat{\gamma}_\ell & \text{if } \ell = 2, \ldots, p-1, \\ \hat{\gamma}_{p-1} & \text{if } \ell = p, \end{cases} \end{aligned} \tag{12}$$

for all $\ell \in \{1, \ldots, p\}$*, and*

$$\sum_{i\in N_2} \rho_i(\widetilde{X}_i) = \rho^*_{N_2}\left(\sum_{i\in N_2} \widetilde{X}_i\right), \tag{13}$$

where $\sum_{i\in N_2} \widetilde{X}_i = \sum_{i\in N} X_i - \sum_{i\in N_1} \widetilde{X}_i$ *and* $\hat{\gamma}_\ell$ *is the Lagrangian multiplier of the constraint* $\sum_{i\in N_2} \widetilde{X}_i(\omega_{\ell+1}) \geq \sum_{i\in N_2} \widetilde{X}_i(\omega_\ell)$.

Proof. The objective function in (9) can be written as

$$\sum_{i\in N_1} k_i \sum_{\ell=1}^{p} \mathbb{P}(\{\omega_\ell\}) u_i(\widetilde{X}_i(\omega_\ell)) + \sum_{\ell=1}^{p} [g^*_{N_2}(\mathbb{P}(\Omega_\ell)) - g^*_{N_2}(\mathbb{P}(\Omega_{\ell-1}))] \sum_{i\in N_2} \widetilde{X}_i(\omega_\ell). \tag{14}$$

This function is maximized over all $\widetilde{X}_i \in \mathbb{R}^\Omega$, $i \in N$ such that $\sum_{i\in N} \widetilde{X}_i = \sum_{i\in N} X_i$, $\widetilde{X}_i(\omega_1) \leq \cdots \leq \widetilde{X}_i(\omega_p)$ for all $i \in N_1$, and $\sum_{i\in N_2} \widetilde{X}_i(\omega_1) \leq \cdots \leq$

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$\sum_{i\in N_2} \widetilde{X}_i(\omega_p)$, where (13) holds. We first leave out the conditions $\widetilde{X}_i(\omega_1) \leq \cdots \leq \widetilde{X}_i(\omega_p)$ for $i \in N_1$; we will later verify that these conditions are satisfied.

Since the objective function in (14) is concave and the constraints are all affine, we get that the Pareto optimal risk redistributions $(\widetilde{X}_i)_{i\in N}$ are characterized by the KKT conditions. The KKT conditions are obtained by the first-order conditions of (14) with respect to $\widetilde{X}_i(\omega_\ell)$:

$$k_i\mathbb{P}(\{\omega_\ell\})u_i'(\widetilde{X}_i(\omega_\ell)) - \hat{\lambda}_\ell = 0, \tag{15}$$

if $i \in N_1$, and, otherwise, we get

$$g^*_{N_2}(\mathbb{P}(\omega_1)) - \hat{\lambda}_1 - \hat{\gamma}_1 = 0, \tag{16}$$

$$g^*_{N_2}(\mathbb{P}(\Omega_\ell)) - g^*_{N_2}(\mathbb{P}(\Omega_{\ell-1})) - \hat{\lambda}_\ell - \hat{\gamma}_\ell + \hat{\gamma}_{\ell-1} = 0, \text{ if } \ell = 2, \ldots, p-1, \tag{17}$$

$$1 - g^*_{N_2}(\mathbb{P}(\Omega_{p-1})) - \hat{\lambda}_p + \hat{\gamma}_{p-1} = 0, \tag{18}$$

and $\hat{\gamma}_\ell \left[\sum_{i\in N_2} \widetilde{X}_i(\omega_{\ell+1}) - \sum_{i\in N_2} \widetilde{X}_i(\omega_\ell)\right] = 0$ for all $\ell \in \{1, \ldots, p-1\}$ and $i \in N$, where $\hat{\lambda}_\ell \in \mathbb{R}$ and $\hat{\gamma}_\ell \geq 0$ are the Lagrangian multipliers of the constraints $\sum_{i\in N} \widetilde{X}_i(\omega_\ell) = \sum_{i\in N} X_i(\omega_\ell)$ and $\sum_{i\in N_2} \widetilde{X}_i(\omega_\ell) \leq \sum_{i\in N_2} \widetilde{X}_i(\omega_{\ell+1})$, respectively. For a given $\hat{\lambda}_\ell > 0$, the Equation (15) has a solution since $u_i'(\cdot) > 0$, $u_i''(\cdot) < 0$ and the Inada conditions are satisfied for every $i \in N_1$. This follows directly from the Intermediate Value Theorem. Hence, the result follows directly.

Next, we verify that $\widetilde{X}_i(\omega_1) \leq \cdots \leq \widetilde{X}_i(\omega_p)$ for $i \in N_1$. If $\sum_{i\in N_2} \widetilde{X}_i(\omega_\ell) = \sum_{i\in N_2} \widetilde{X}_i(\omega_{\ell+1})$, then $\sum_{i\in N_1} \widetilde{X}_i(\omega_\ell) \leq \sum_{i\in N_1} \widetilde{X}_i(\omega_{\ell+1})$. So, $\widetilde{X}_i(\omega_\ell) \leq \widetilde{X}_i(\omega_{\ell+1})$ for $i \in N_1$ follows directly from (15) and strict concavity of u_i, $i \in N_1$. Suppose $\sum_{i\in N_2} \widetilde{X}_i(\omega_\ell) < \sum_{i\in N_2} \widetilde{X}_i(\omega_{\ell+1})$. This leads to $\hat{\gamma}_\ell = 0$. Because the distortion functions g_i, $i \in N$ are strictly concave, it holds that the function $g^*_{N_2}$ is strictly concave as well. From this and $\mathbb{P}(\{\omega_\ell\}), \mathbb{P}(\{\omega_{\ell+1}\}) > 0$, it follows that

$$\frac{g^*_{N_2}(\mathbb{P}(\Omega_\ell)) - g^*_{N_2}(\mathbb{P}(\Omega_{\ell-1})) + \hat{\gamma}_{\ell-1}}{\mathbb{P}(\{\omega_\ell\})} > \frac{g^*_{N_2}(\mathbb{P}(\Omega_{\ell+1})) - g^*_{N_2}(\mathbb{P}(\Omega_\ell)) - \hat{\gamma}_{\ell+1}}{\mathbb{P}(\{\omega_{\ell+1}\})},$$

for any $\hat{\gamma}_{\ell-1}, \hat{\gamma}_{\ell+1} \geq 0$. Hence, the solution in (12) satisfies $\widetilde{X}_i(\omega_1) \leq \cdots \leq \widetilde{X}_i(\omega_p)$ for $i \in N_1$ due to strict concavity of the utility functions u_i, $i \in N_1$. This concludes the proof of Theorem 3.3. ■

We proceed with characterizing uniqueness of the Pareto optimal risk redistributions. We know from Jouini *et al.* (2008) that there are multiple solutions to (13). For the agents in N_1, we next show that the system (12) in Theorem 3.3 yields the same risks for a given value of k.

Proposition 3.4. *For a given* $k \in \mathbb{R}^{N_1}_{++}$*, all* $(\widetilde{X}_i)_{i\in N}, (\hat{X}_i)_{i\in N} \in \mathcal{PO}$ *solving* (9) *are such that* $\widetilde{X}_i = \hat{X}_i$ *for all* $i \in N_1$.

Proof. Let $k \in \mathbb{R}^{N_1}_{++}$. Given a positive value of (12), there is a unique solution of $\widetilde{X}_i$, $i \in N_1$ due to $u_i'(\cdot) > 0$, $u_i''(\cdot) < 0$ and that u_i satisfies the Inada conditions for all $i \in N_1$ (Intermediate Value Theorem). We tread the set N_2 as one representative agent with preferences $V(X) = -\rho^*_{N_2}(X)$, which will bear the risk $\sum_{i\in N_2} \widetilde{X}_i$. Next, we show that there is a unique value of (12) that satisfies the KKT conditions. Existence of a solution follows from existence of a solution of (9). Suppose there are two solutions $(\hat{X}_i)_{i\in N}, (\overline{X}_i)_{i\in N} \in \mathcal{X}$ solving the system in Theorem 3.3, and are not equal to each other for an agent $i \in N_1$. So, $(\hat{X}_i)_{i\in N}$ and $(\overline{X}_i)_{i\in N}$ both solve (9). Then, we see that $\frac{1}{2}\hat{X}_i + \frac{1}{2}\overline{X}_i$ is a strict improvement for agent i due to strict concavity of the utility function u_i. Moreover, we have for all $i \in N_1$ that

$$E_{\mathbb{P}}\left[u_i\left(\frac{1}{2}\hat{X}_i + \frac{1}{2}\overline{X}_i\right)\right] \geq \frac{1}{2}E_{\mathbb{P}}[u_i(\hat{X}_i)] + \frac{1}{2}E_{\mathbb{P}}[u_i(\overline{X}_i)],$$

due to concavity of u_i. Hence, we get from this and the fact that an expectation is additive that

$$\begin{aligned}&\sum_{i\in N_1} k_i E_{\mathbb{P}}\left[u_i\left(\frac{1}{2}\hat{X}_i + \frac{1}{2}\overline{X}_i\right)\right] + E_{\mathbb{Q}}\left[\sum_{i\in N_2}\frac{1}{2}\hat{X}_i + \frac{1}{2}\overline{X}_i\right]\\ &> \frac{1}{2}\left(\sum_{i\in N_1} k_i E_{\mathbb{P}}[u_i(\hat{X}_i)] + E_{\mathbb{Q}}\left[\sum_{i\in N_2}\hat{X}_i\right]\right)\\ &\quad + \frac{1}{2}\left(\sum_{i\in N_1} k_i E_{\mathbb{P}}[u_i(\overline{X}_i)] + E_{\mathbb{Q}}\left[\sum_{i\in N_2}\overline{X}_i\right]\right).\end{aligned}$$

So, not both $(\hat{X}_i)_{i\in N}, (\overline{X}_i)_{i\in N} \in \mathcal{X}$ solve (9). This is a contradiction, which concludes the proof. ■

Proposition 3.5. *For a given $k \in \mathbb{R}^{N_1}_{++}$, there is a unique vector $\hat{\gamma} \in \mathbb{R}^{p-1}_{+}$ that solves the system in Theorem 3.3.*

Proof. Suppose $\hat{\gamma}^1$ and $\hat{\gamma}^2$ both solve the system in Theorem 3.3 and $\hat{\gamma}^1 \neq \hat{\gamma}^2$. Then, the right-hand side of (12) is different for some state index $\ell \in \{1, \ldots, p\}$. Due to strict concavity of the expected utility functions $u_i, i \in N_1$, this leads to different solutions of $\widetilde{X}_i, i \in N_1$ for the choices of $\hat{\gamma}^1$ and $\hat{\gamma}^2$. Hence, these solutions do not both solve the system in Theorem 3.3 due to Proposition 3.4. This concludes the proof. ■

Suppose $N_2 \neq \emptyset$, and define

$$\mathcal{M} = \left\{ m : \{1, \ldots, p-1\} \to N_2 \,\middle|\, m(k) \in \operatorname*{argmin}_{j\in N_2}\left\{g_j\left(\mathbb{P}(\Omega_k)\right)\right\} \text{ for all } k \in \{1, \ldots, p-1\}\right\}. \tag{19}$$

Given $\sum_{i\in N_2}\widetilde{X}_i$, all Pareto optimal risk redistributions for agents in N_2 follow from (13). From Jouini *et al.* (2008, Proposition 3.1 therein), we get that for all $m \in \mathcal{M}$ and $d \in \mathbb{R}^{N_2}$ with $\sum_{i\in N_2} d_i = \sum_{i\in N_2}\widetilde{X}_i(\omega_p)$, it holds that $(\hat{X}_i)_{i\in N_2}$ is

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Pareto optimal, where

$$\hat{X}_i = \sum_{k=1}^{p-1} \left[\sum_{i \in N_2} \widetilde{X}_i(\omega_k) - \sum_{i \in N_2} \widetilde{X}_i(\omega_{k+1}) \right] \mathbb{1}_{m(k)=i}\, e_{\Omega_k} + d_i e_{\Omega}, \text{ for all } i \in N_2, \quad (20)$$

$\mathbb{1}_{m(k)=i} = 1$ if $m(k) = i$ and zero otherwise, and where the risk $e_A \in \mathbb{R}^{\Omega}$ for $A \in \mathcal{F}$ is given by

$$e_A(\omega) = \begin{cases} 1 & \text{if } \omega \in A, \\ 0 & \text{if } \omega \in \Omega \backslash A. \end{cases} \quad (21)$$

We refer to the structure of risk redistributions in (20) as tranching.

If the sets N_1 and N_2 are both non-empty, one can use for a given $k \in \mathbb{R}^{N_1}_{++}$ the following two-step procedure to solve for the Pareto optimal risk redistributions:

- solve the system in Theorem 3.3; we get the Pareto optimal risk profiles $\widetilde{X}_i, i \in N_1$. By Proposition 3.4, these risk profiles are unique given vector k;
- we compute

$$\sum_{i \in N_2} \widetilde{X}_i = \sum_{i \in N} X_i - \sum_{i \in N_1} \widetilde{X}_i,$$

  and determine non-unique Pareto optimal risk redistributions $\widetilde{X}_i, i \in N_2$ from (20).

For every risk redistribution problem, there is an allocation of the aggregate risk $\sum_{i \in N} X_i$ to group N_1 and group N_2. Given $\sum_{i \in N_1} \widetilde{X}_i$ for group N_1, the Pareto optimal risk redistribution follows from (11). If the agents in N_1 all use an equi-cautious Hyperbolic Absolute Risk Aversion expected utility function, the Pareto optimal risk redistribution is an affine contract on $\sum_{i \in N_1} \widetilde{X}_i$ (Wilson, 1968). Given $\sum_{i \in N_2} \widetilde{X}_i$, the Pareto optimal risk redistribution is given by tranching of this risk $\sum_{i \in N_2} \widetilde{X}_i$.

We next show an algorithm to solve the Pareto optimal risk redistributions via Theorem 3.3 for any given $k \in \mathbb{R}^{N_1}_{++}$. Corresponding to vector k, we aim to find the unique Pareto optimal $\widetilde{X}_i, i \in N_1$, as all Pareto optimal $\widetilde{X}_i, i \in N_2$, given $\sum_{i \in N_2} \widetilde{X}_i$, are given in (20).

1. Set $\hat{\gamma}_\ell = 0$ for all $\ell \in \{1, \ldots, p-1\}$. Solve the system (12) for given $\hat{\gamma}$ to obtain $(\widetilde{X}_i)_{i \in N}$.
2. If $\sum_{i \in N_2} \widetilde{X}_i(\omega_1) \leq \cdots \leq \sum_{i \in N_2} \widetilde{X}_i(\omega_p)$, then it holds that $\widetilde{X}_i(\omega_1) \leq \cdots \leq \widetilde{X}_i(\omega_p)$ for all $i \in N$ (see Theorem 3.3); stop here. Otherwise, go to the next step.
3. Find the first $\ell \in \{1, \ldots, p-1\}$ for which there exists an $z \in \{0, 1, \ldots\}$ such that $\sum_{i \in N_2} \widetilde{X}_i(\omega_\ell) \geq \cdots \geq \sum_{i \in N_2} \widetilde{X}_i(\omega_{\ell+z+1})$ with one strict inequality. Take this largest z for which this series of inequalities hold. Then,

determine $\hat{\gamma}_a \geq 0, a \in \{\ell, \dots, \ell+z\}$ such that $\sum_{i \in N_2} \widetilde{X}_i(\omega_\ell) = \cdots = \sum_{i \in N_2} \widetilde{X}_i(\omega_{\ell+z+1})$ for solutions of (12) with given $\hat{\gamma}$. Then, go back to Step 2.

In this algorithm, it may be tedious to solve Step 3 when the size z is large. At least, we know from Theorem 3.3 that it yields a Pareto optimal risk redistribution. From Proposition 3.4, we get for every Pareto optimal risk redistribution $(\widetilde{X}_i)_{i \in N}$ that the risks $\widetilde{X}_i, i \in N_1$ are the same for given $k \in \mathbb{R}^{N_1}_{++}$. So, the algorithm selects this unique risks $\widetilde{X}_i, i \in N_1$ corresponding to Pareto optimal risk redistributions $(\widetilde{X}_i)_{i \in N}$.

We proceed this paper with characterizing specific Pareto optimal risk redistributions, namely the competitive equilibria. We discuss this topic in the next section.

4. Competitive equilibria

4.1. Definition and characterization

Let the pricing function be linear, i.e., we have $\pi(\hat{p}, X) = \sum_{\omega \in \Omega} \hat{p}_\omega X(\omega)$ for all $X \in \mathbb{R}^\Omega$. To avoid arbitrage, we assume that the price vector $\hat{p}$ is strictly positive, i.e., $\hat{p} \in \mathbb{R}^\Omega_{++}$. The risk-free rate is set equal to zero, i.e., $\pi(\hat{p}, e_\Omega) = 1$. This assumption will serve as a normalization, as the equilibrium risk redistributions do not depend on it. The economy is in equilibrium when every agent $i \in N$ solves

$$\max_{\widetilde{X}_i \in \mathbb{R}^\Omega} V_i(\widetilde{X}_i) \tag{22}$$

$$\text{s.t. } \pi(\hat{p}, \widetilde{X}_i) \leq \pi(\hat{p}, X_i), \tag{23}$$

where the price vector $\hat{p}$ induces market clearing, i.e.,

$$(\widetilde{X}_i)_{i \in N} \in \mathcal{X}. \tag{24}$$

Existence of competitive equilibria follows from Arrow and Debreu (1954) and Werner (1987). The First Fundamental Welfare Theorem states that any equilibrium is Pareto optimal. This theorem applies to our setting as the preferences are non-satiated (Arrow, 1963). If $N = N_1$ or $N = N_2$, competitive equilibria are studied by, e.g., Borch (1962), Aase (1993, 2010), Filipović and Kupper (2008) and Dana and Le Van (2010).

Theorem 4.1. *Let $N_1, N_2 \neq \emptyset$, and recall the definition of a competitive equilibrium in* (22)–(24). *Then, $(\hat{p}, (\widetilde{X}_i)_{i \in N})$ is an equilibrium if and only if we have*

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$(\widetilde{X}_i)_{i\in N} \in \mathcal{X}$,

$$\mathbb{P}(\{\omega_\ell\})u_i'(\widetilde{X}_i(\omega_\ell)) = \lambda_i \hat{p}_\ell, \qquad \textit{for all } i \in N_1, \ell \in \{1,\ldots,p\}, \tag{25}$$

$$\pi(\hat{p}, \widetilde{X}_i) = \pi(\hat{p}, X_i), \qquad \textit{for all } i \in N_1, \tag{26}$$

$$-\rho_i(\widetilde{X}_i) = \pi(\hat{p}, X_i), \qquad \textit{for all } i \in N_2, \tag{27}$$

with $\lambda \in \mathbb{R}^N_{++}$*, and the price vector* $\hat{p}$ *is given by*

$$\hat{p}_\ell = g^*_{N_2}(\mathbb{P}(\Omega_\ell)) - g^*_{N_2}(\mathbb{P}(\Omega_{\ell-1})) + \begin{cases} -\gamma_\ell & \textit{if } \ell = 1, \\ +\gamma_{\ell-1} - \gamma_\ell & \textit{if } \ell = 2,\ldots,p-1, \\ +\gamma_{\ell-1} & \textit{if } \ell = p. \end{cases} \tag{28}$$

Here, $\gamma_\ell \geq 0, \ell \in \{1,\ldots,p-1\}$ *are the Lagrangian parameters of*

$$\sum_{i\in N} X_i(\omega_{\ell+1}) - \sum_{i\in N_1} \widetilde{X}_i(\omega_{\ell+1}) \geq \sum_{i\in N} X_i(\omega_\ell) - \sum_{i\in N_1} \widetilde{X}_i(\omega_\ell),$$

where $\widetilde{X}_i, i \in N_1$ *follow from* (25)*,* (26)*.*

Proof. From the First Fundamental Welfare Theorem (Arrow, 1963), we get that any competitive equilibrium is Pareto optimal. Therefore, according to Lemma 3.1, all equilibrium risk redistributions must be comonotone with each other.

Then, if we explicitly impose that $\widetilde{X}_i$ is comonotone with each other, we can write $V_i(\widetilde{X}_i) = -E_{\mathbb{Q}}[X_i]$ for all $\widetilde{X}_i$, where $\mathbb{Q}(\{\omega_\ell\}) = g^*_{N_2}(\mathbb{P}(\Omega_\ell)) - g^*_{N_2}(\mathbb{P}(\Omega_{\ell-1}))$ for all $\ell \in \{1,\ldots,p\}$. Since the objective function is concave and the constraints are all affine, we get that the equilibrium risk redistributions $(\widetilde{X}_i)_{i\in N}$ are characterized by the KKT conditions one-to-one. The KKT conditions are obtained by the first-order conditions of the following function with respect to $\widetilde{X}_i(\omega_\ell)$:

$$V_i(\widetilde{X}_i) + \lambda_i \left(\sum_{\omega\in\Omega} \hat{p}_\omega X_i(\omega) - \sum_{\omega\in\Omega} \hat{p}_\omega \widetilde{X}_i(\omega) \right) + \sum_{\hat{\ell}=1}^{p-1} \gamma_{i,\hat{\ell}} (\widetilde{X}_i(\omega_{\hat{\ell}+1}) - \widetilde{X}_i(\omega_{\hat{\ell}})),$$

where $\lambda_i \in \mathbb{R}, i \in N$ are the Lagrangian parameters of the price constraint and $\gamma_{i,\ell} \geq 0, i \in N, \ell \in \{1,\ldots,p-1\}$ are the Lagrangian parameters of the comonotonicity constraints. For agents in N_1, we now assume that $\widetilde{X}_i, i \in N_1$ are such that $\gamma_{i,\ell} = 0$ for all $i \in N_1, \ell \in \{1,\ldots,p-1\}$. We check at the end of this proof that the risks $\widetilde{X}_i, i \in N_1$ are indeed comonotone with $\sum_{i\in N} X_i$. Then, for $\ell \in \{1, \ldots, p\}$, the KKT conditions are given by

$$\mathbb{P}(\{\omega_\ell\})u_i'(\widetilde{X}_i(\omega_\ell)) = \lambda_i \hat{p}_\ell, \text{ for all } i \in N_1, \tag{29}$$

$$g_i(\mathbb{P}(\Omega_\ell)) - g_i(\mathbb{P}(\Omega_{\ell-1})) = \lambda_i \hat{p}_\ell + \begin{cases} -\gamma_{i,\ell} & \text{if } \ell = 1, \\ -\gamma_{i,\ell} + \gamma_{i,\ell-1} & \text{if } \ell = 2, \ldots, p-1, \\ \gamma_{i,\ell-1} & \text{if } \ell = p, \end{cases} \tag{30}$$

for all $i \in N_2$. Since $g_i(0) = 0$ and $g_i(1) = 1$, it holds that

$$\sum_{\ell=1}^{p}[g_i(\mathbb{P}(\Omega_\ell)) - g_i(\mathbb{P}(\Omega_{\ell-1}))] = 1, \tag{31}$$

and, moreover, it holds that

$$\sum_{\omega\in\Omega} \hat{p}_\omega = 1, \tag{32}$$

since $\pi(\hat{p}, e_\Omega) = 1$, and

$$\gamma_{i,1} + \sum_{\ell=2}^{p-1} \left(\gamma_{i,\ell} - \gamma_{i,\ell-1}\right) - \gamma_{i,p-1} = 0. \tag{33}$$

From (30)–(33), it follows that $\lambda_i = 1$ for all $i \in N_2$. Note that due to Pareto optimality of the equilibrium (First Fundamental Welfare Theorem), the price constraint in (23) is binding. This leads to constraint (26). From the equilibrium risks $\widetilde{X}_i$, $i \in N_1$ in equilibrium, we derive a value of $\sum_{i\in N_2} \widetilde{X}_i = \sum_{i\in N} X_i - \sum_{i\in N_1} \widetilde{X}_i$. The equilibrium risk redistributions are Pareto optimal and, so, comonotone. Then, for a given total risk $\sum_{i\in N_2} \widetilde{X}_i$ and equilibrium prices, the equilibrium risk redistributions are characterized one-to-one by (27) due to Filipović and Kupper (2008, Theorem 3.2 therein). Hence, $(\hat{p}, (\widetilde{X}_i)_{i\in N})$ is an equilibrium if and only if we have $(\widetilde{X}_i)_{i\in N} \in \mathcal{X}$ and the system (25)–(27) holds for some $\lambda \in \mathbb{R}^N_{++}$ and price vector $\hat{p}$.

We proceed with showing the equilibrium price vector $\hat{p}$. The risk redistribution $(\widetilde{X}_i)_{i\in N_1}$ in equilibrium follows directly from (25). From this, we get $\sum_{i\in N_2} \widetilde{X}_i = \sum_{i\in N} X_i - \sum_{i\in N_1} \widetilde{X}_i$. Suppose $\sum_{i\in N_2} \widetilde{X}_i(\omega_1) < \sum_{i\in N_2} \widetilde{X}_i(\omega_2)$, so that $\gamma_1 = 0$. Then, it follows that there exists at least one $i_0 \in N_2$ such that $\widetilde{X}_{i_0}(\omega_1) < \widetilde{X}_{i_0}(\omega_2)$, and so $\gamma_{i_0,1} = 0$. From this and $\gamma_{j,1} \geq 0$ for all $j \in N_2$ it follows from (30), with $\lambda_i = 1$, that

$$\hat{p}_1 = g_{i_0}(\mathbb{P}(\Omega_1)) = g^*_{N_2}(\mathbb{P}(\Omega_1)) \text{ and } \gamma_{i,1} = g_i(\mathbb{P}(\Omega_1)) - g^*_{N_2}(\mathbb{P}(\Omega_1)), \text{ for all } i \in N_2. \tag{34}$$

If the equilibrium prices yield $\sum_{i\in N_2} \widetilde{X}_i(\omega_1) = \sum_{i\in N_2} \widetilde{X}_i(\omega_2)$, it follows from comonotonicity of Pareto optimal risk redistributions that $\widetilde{X}_i(\omega_1) = \widetilde{X}_i(\omega_2)$ for all $i \in N_2$. Therefore, it holds that $\gamma_{i,1} \geq 0$ for all $i \in N_2$. From this, the fact

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that $g_i(x) \geq g^*_{N_2}(x)$ for all $x \in [0, 1]$ and all $i \in N_2$ and from the fact that for all $x \in [0, 1]$ there exists a $j \in N_2$ such that $g_j(x) = g^*_{N_2}(x)$, we get for $\gamma_1 \geq 0$ that

$$\hat{p}_1 = g^*_{N_2}(\mathbb{P}(\Omega_1)) - \gamma_1, \text{ and } \gamma_{i,1} = g_i(\mathbb{P}(\Omega_1)) - g^*_{N_2}(\mathbb{P}(\Omega_1)) + \gamma_1. \quad (35)$$

If $p > 2$ and if $\sum_{i \in N_2} \widetilde{X}_i(\omega_2) < \sum_{i \in N_2} \widetilde{X}_i(\omega_3)$, it follows from (30), with $\lambda_i = 1$, and (35) that:

$$g_i(\mathbb{P}(\Omega_2)) - g^*_{N_2}(\mathbb{P}(\Omega_1)) = \hat{p}_2 - \gamma_1 + \gamma_{i,2}, \text{ for all } i \in N_2, \quad (36)$$

and, in line with (35), we get

$$\hat{p}_2 = g^*_{N_2}(\mathbb{P}(\Omega_2)) - g^*_{N_2}(\mathbb{P}(\Omega_1)) + \gamma_1 \text{ and } \gamma_{i,2} = g_i(\mathbb{P}(\Omega_2)) - g^*_{N_2}(\mathbb{P}(\Omega_2)), \quad (37)$$

for all $i \in N_2$. If the equilibrium prices yield $\sum_{i \in N_2} \widetilde{X}_i(\omega_2) = \sum_{i \in N_2} \widetilde{X}_i(\omega_3)$, we get for $\gamma_2 \geq 0$ that

$$\hat{p}_2 = g^*_{N_2}(\mathbb{P}(\Omega_2)) - g^*_{N_2}(\mathbb{P}(\Omega_1)) + \gamma_1 - \gamma_2, \text{ and } \gamma_{i,1} = g_i(\mathbb{P}(\Omega_1)) - g^*_{N_2}(\mathbb{P}(\Omega_1)) + \gamma_2. \quad (38)$$

Continuing this procedure for all states $\ell \in \{1, \ldots, p\}$ leads to equilibrium price vectors expressed as the function of $\gamma_1, \ldots, \gamma_{p-1}$ as in (28).

Finally, we show that for solutions of (25) that the risks $\widetilde{X}_i, i \in N_1$ are indeed comonotone with $\sum_{i \in N} X_i$. Random variables X and Y are called anti-comonotone when X and $-Y$ are comonotone. Then, we show that any equilibrium price vector $\hat{p}$ is such that the random variable $\frac{\hat{p}_\ell}{\mathbb{P}(\{\omega_\ell\})}, \ell \in \{1, \ldots, p\}$ is anti-comonotone with $\sum_{i \in N} X_i$. Suppose this is not true, and there exists a state ω_ℓ such that $\frac{\hat{p}_\ell}{\mathbb{P}(\{\omega_\ell\})} < \frac{\hat{p}_{\ell+1}}{\mathbb{P}(\{\omega_{\ell+1}\})}$. Then, it follows from (29) and the fact that the function $u'_i(\cdot)$ is continuous and strictly decreasing that $\widetilde{X}_i(\omega_\ell) > \widetilde{X}_i(\omega_{\ell+1})$ for all $i \in N_1$. From this and $\sum_{i \in N} X_i(\omega_\ell) \leq \sum_{i \in N} X_i(\omega_{\ell+1})$, it follows that $\sum_{i \in N_2} \widetilde{X}_i(\omega_\ell) < \sum_{i \in N_2} \widetilde{X}_i(\omega_{\ell+1})$. This is a contradiction with Lemma 3.1. Hence, it holds that $\frac{\hat{p}_\ell}{\mathbb{P}(\{\omega_\ell\})} \geq \frac{\hat{p}_{\ell+1}}{\mathbb{P}(\{\omega_{\ell+1}\})}$. This concludes the proof that the random variable $\frac{\hat{p}_\ell}{\mathbb{P}(\{\omega_\ell\})}, \ell \in \{1, \ldots, p\}$ is anti-comonotone with the risk $\sum_{i \in N} X_i$.

From the result that $\frac{\hat{p}_\ell}{\mathbb{P}(\{\omega_\ell\})}, \ell \in \{1, \ldots, p\}$ is anti-comonotone with the risk $\sum_{i \in N} X_i$ and the fact that the functions $u'_i(\cdot), i \in N_1$, are continuous and strictly decreasing, we get for the solution of (25) that the risks $\widetilde{X}_i, i \in N_1$ are indeed comonotone with $\sum_{i \in N} X_i$. ■

Theorem 4.1 characterizes the competitive equilibrium. For all $i \in N_2$ and for equilibrium price vector $\hat{p}$, we readily get that $\rho_i(X) \geq \rho^*_{N_2}(X) \geq -\pi(\hat{p}, X)$ for all $X \in \mathbb{R}^\Omega$, and $\rho_i(\widetilde{X}_i) = \rho^*_{N_2}(\widetilde{X}_i) = -\pi(\hat{p}, \widetilde{X}_i)$ if $(\hat{p}, (\widetilde{X}_i)_{i \in N})$ is a competitive equilibrium.

We proceed with providing some characteristics of the competitive equilibrium. To do so, we first define the following condition:

Condition [C]: the solution $(\widetilde{X}_i)_{i\in N} \in \mathcal{X}$ of (25)–(27) with

$$\hat{p}^*_\ell = g^*_{N_2}(\mathbb{P}(\Omega_\ell)) - g^*_{N_2}(\mathbb{P}(\Omega_{\ell-1})), \text{ for all } \ell \in \{1, \dots, p\}, \tag{39}$$

is such that

$$\sum_{i\in N_2} \widetilde{X}_i(\omega_1) \leq \cdots \leq \sum_{i\in N_2} \widetilde{X}_i(\omega_p).$$

The next corollary follows directly from Theorem 4.1.

Corollary 4.2. *If condition [C] holds, equilibrium prices are given by* (39).

Condition [C] has no direct interpretation, but allows us to compute the equilibrium prices directly from the preferences of the agents in N_2, and in closed form (Corollary 4.2).

Proposition 4.3. *Condition [C] implies that* $\sum_{i\in N} X_i(\omega_1) < \cdots < \sum_{i\in N} X_i(\omega_p)$.

Proof. Let condition [C] hold, and suppose that $\sum_{i\in N} X_i(\omega_1) < \cdots < \sum_{i\in N} X_i(\omega_p)$ does not hold. Then, there exists $\ell \in \{1, \dots, p-1\}$ such that $\sum_{i\in N} X_i(\omega_\ell) = \sum_{i\in N} X_i(\omega_{\ell+1})$. Let (25)–(27) hold. For price vector $\hat{p}$ as in (39), it holds that $\frac{\hat{p}_\ell}{\mathbb{P}(\{\omega_\ell\})}$ is strictly decreasing in ℓ due to Theorem 4.1. Since the functions $u_i, i \in N_1$ are strictly concave, we get $\widetilde{X}_i(\omega_\ell) < \widetilde{X}_i(\omega_{\ell+1})$ for all $i \in N_1$, and so $\sum_{i\in N_1} \widetilde{X}_i(\omega_\ell) < \sum_{i\in N_1} \widetilde{X}_i(\omega_{\ell+1})$. This implies that $\sum_{i\in N} X_i(\omega_\ell) - \sum_{i\in N_1} \widetilde{X}_i(\omega_\ell) > \sum_{i\in N} X_i(\omega_{\ell+1}) - \sum_{i\in N_1} \widetilde{X}_i(\omega_{\ell+1})$, which is a contradiction due to Lemma 3.1. Hence, condition [C] does not hold. This concludes the proof. ■

If there exist states $\omega_\ell, \omega_{\ell+1} \in \Omega$ such that $\sum_{i\in N} X_i(\omega_\ell) = \sum_{i\in N} X_i(\omega_{\ell+1})$, then we get from Corollary 3.2 that for every Pareto optimal redistribution (and so for every competitive equilibrium) it holds that $\widetilde{X}_i(\omega_\ell) = \widetilde{X}_i(\omega_{\ell+1})$ for all $i \in N$. Hence, the probability-weighted equilibrium prices in both states are the same, i.e., $\frac{\hat{p}_\ell}{\mathbb{P}(\{\omega_\ell\})} = \frac{\hat{p}_{\ell+1}}{\mathbb{P}(\{\omega_{\ell+1}\})}$. This implies that we can adjust the problem without loss of generality such that $\sum_{i\in N} X_i(\omega_1) < \cdots < \sum_{i\in N} X_i(\omega_p)$ holds.

Let $\hat{\lambda} \in \mathbb{R}^{N_1}_{++}$. For every $X \in \mathbb{R}^\Omega$, there is a unique $(\widetilde{X}_i)_{i\in N_1}$ such that $\sum_{i\in N_1} \widetilde{X}_i = X$ and $\hat{\lambda}_i^{-1} u_i'(\widetilde{X}_i) = \hat{\lambda}_j^{-1} u_j'(\widetilde{X}_j)$ for all $i, j \in N_1$ (Proposition 3.4). In this way, we define the function $u'_{\hat{\lambda}}$ as $u'_{\hat{\lambda}}(X) = \hat{\lambda}_i^{-1} u_i'(\widetilde{X}_i)$ for any $i \in N_1$, where $(\widetilde{X}_i)_{i\in N_1}$ is such that $\sum_{i\in N_1} \widetilde{X}_i = X$ and $\hat{\lambda}_i^{-1} u_i'(\widetilde{X}_i) = \hat{\lambda}_j^{-1} u_j'(\widetilde{X}_j)$ for all $i, j \in N_1$. The function $u'_{\hat{\lambda}}$ represents the preferences of every agent in N_1. Here, the prime on $u'_{\hat{\lambda}}$ is just a matter of notation. In line with Aase (1993), we have that $u'_{\hat{\lambda}}$ is a derivative of some utility function for a representative agent of the set N_1.

Let $\lambda \in \mathbb{R}^{N_1}_{++}$ be a vector as in the solution of (25)–(28) in Theorem 4.1. It follows from the proof of Theorem 4.1 that equilibrium prices can be determined as if there are just two hypothetical agents in the market: one is endowed with

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marginal expected utility function u'_λ and one is endowed with $-\rho^*_{N_2}$. Moreover, if condition [C] is satisfied, the equilibrium risk redistribution for agents in N_1 depends locally not on the aggregate risk $\sum_{i\in N} X_i$, but also on the preferences of the "representative" least risk-averse agent of the set N_2. Here, we denote risk aversion in dual utility as aversion to mean-preserving spreads.

Next, we focus on condition [C]. From (25), we get

$$u'_\lambda\left(\sum_{i\in N_1}\widetilde{X}_i\right)\cdot\mathbb{P}=\hat{p}, \tag{40}$$

where the probability measure $\mathbb{P}$, random variables and prices are written as vectors, and $\cdot$ is the is the Hadamard (element-wise) product operator. From $u''_i(\cdot)<0$ for all $i\in N_1$, it follows that $u'_\lambda(\cdot)$ is continuous and strictly decreasing. From $u'_i(\cdot)>0, u''_i(\cdot)<0$ and that u_i satisfies the Inada conditions for all $i\in N_1$, it follows that $u'_\lambda(\cdot)$ has range $(0,\infty)$. Hence, u'^{-1}_λ exists. We get

$$\sum_{i\in N_1}\widetilde{X}_i=u'^{-1}_\lambda\left(\frac{\hat{p}}{\mathbb{P}}\right).$$

Hence, condition [C] is equivalent to

$$\sum_{i\in N}X_i(\omega_{\ell+1})-\sum_{i\in N}X_i(\omega_\ell)\geq u'^{-1}_\lambda\left(\frac{\hat{p}^*_{\ell+1}}{\mathbb{P}(\{\omega_{\ell+1}\})}\right)-u'^{-1}_\lambda\left(\frac{\hat{p}^*_\ell}{\mathbb{P}(\{\omega_\ell\})}\right), \tag{41}$$

for all $\ell\in\{1,\ldots,p-1\}$, where $\hat{p}^*$ is given in (39). Since the function u'_λ is strictly decreasing and $g^*_{N_2}$ is strictly concave, we find that the right-hand side of (41) is strictly positive. The more risk averse the representative (average risk averse) agent in N_1 and the representative (least risk averse) agent in N_2 are, the more strong condition [C] is on the aggregate risk.

4.2. Equilibrium prices

In this section, we provide an algorithm that yields an equilibrium price vector. Equilibrium prices in (28) follow from the following algorithm.

1. Set $\hat{\gamma}_\ell=0$ for all $\ell\in\{1,\ldots,p-1\}$. Solve the system (25)–(26) for given $\hat{\gamma}$ to obtain $(\widetilde{X}_i)_{i\in N}$.
2. If $\sum_{i\in N_2}\widetilde{X}_i(\omega_1)\leq\cdots\leq\sum_{i\in N_2}\widetilde{X}_i(\omega_p)$, then we found the equilibrium price vector; stop here. Otherwise, go to the next step.
3. Find the first $\ell\in\{1,\ldots,p-1\}$ for which there exists $z\in\{0,1,\ldots\}$ such that $\sum_{i\in N_2}\widetilde{X}_i(\omega_\ell)\geq\cdots\geq\sum_{i\in N_2}\widetilde{X}_i(\omega_{\ell+z+1})$ with one strict inequality. Take this largest z for which this series of inequalities hold. Then, we need to determine $\hat{\gamma}_a>0, a\in\{\ell,\ldots,\ell+z\}$ such that $\sum_{i\in N_2}\widetilde{X}_i(\omega_\ell)=\cdots=\sum_{i\in N_2}\widetilde{X}_i(\omega_{\ell+z+1})$ for solutions of (25), (26) with given $\hat{\gamma}$. Solve the system (25), (26) for given $\hat{\gamma}$ to obtain $(\widetilde{X}_i)_{i\in N}$. Then, go back to Step 2.

This procedure has a finite number of iterations due to a finite state space.

In the next proposition, we show that the values $\gamma_1, \ldots, \gamma_{p-1}$ solving this algorithm are unique.

Proposition 4.4. *The algorithm above leads to a unique equilibrium price vector $\hat{p}$.*

Proof. We show that Step 3 above has a unique solution for $\hat{\gamma}$. Let $\ell \in \{1, \ldots, p-1\}$ be the first index for which there exists $z \in \{0, 1, \ldots\}$ such that the corresponding risk redistribution is such that $\sum_{i \in N_2} \widetilde{X}_i(\omega_\ell) \geq \cdots \geq \sum_{i \in N_2} \widetilde{X}_i(\omega_{\ell+z+1})$ with one strict inequality. Therefore, we have $\sum_{i \in N_2} \widetilde{X}_i(\omega_{\ell-1}) < \sum_{i \in N_2} \widetilde{X}_i(\omega_\ell)$ so that $\hat{\gamma}_{\ell-1} = 0$. From (25), we get that $\widetilde{X}_i(\omega_\ell), \ldots, \widetilde{X}_i(\omega_{\ell+z+1})$, $i \in N_1$ are solutions of

$$\frac{u_i'(\hat{X}_i(\omega_{\ell+s}))}{u_i'(\hat{X}_i(\omega_{\ell+s+1}))} = \frac{(g^*_{N_2}(\mathbb{P}(\Omega_{\ell+s})) + \hat{\gamma}_{\ell+s-1} - \hat{\gamma}_{\ell+s})/\mathbb{P}(\{\omega_{\ell+s}\})}{(g^*_{N_2}(\mathbb{P}(\Omega_{\ell+s+1})) + \hat{\gamma}_{\ell+s} - \hat{\gamma}_{\ell+s+1})/\mathbb{P}(\{\omega_{\ell+s+1}\})}, \tag{42}$$

$s \in \{0, \ldots, z\}$, where $\hat{\gamma}_{\ell-1} = 0$, and $\hat{\gamma}_{\ell+z+1}$ is fixed and such that $\hat{\gamma}_{\ell+z+1} = 0$ if $\ell + z = p$. We solve uniqueness of such Lagrangian parameters by mathematical induction, where we vary z. We define Event s, with $s \in \{0, \ldots, z-1\}$, as follows.

For given $\hat{\gamma}_{\ell+s+1} \geq 0$, the values of $\hat{\gamma}_\ell, \ldots, \hat{\gamma}_{\ell+s}$ such that $\sum_{i \in N_2} \hat{X}_i(\omega_\ell) = \cdots = \sum_{i \in N_2} \hat{X}_i(\omega_{\ell+z+1})$ are unique, and non-negative. Moreover, $\hat{\gamma}_\ell$ is continuous and strictly increasing in the value of $\hat{\gamma}_{\ell+s+1}$.

Step 1: first, we show the result for Event $s = 0$. Fix $\hat{\gamma}_{\ell+1}$. Then, the right-hand side of (42) only depends on $\hat{\gamma}_\ell$. This equation is continuous and strictly decreasing in $\hat{\gamma}_\ell \geq 0$. Moreover, if $\hat{\gamma}_\ell = 0$, then we get that $\sum_{i \in N_2} \hat{X}_i(\omega_\ell) \geq \sum_{i \in N_2} \hat{X}_i(\omega_{\ell+1})$. Moreover, there is a unique $\hat{\gamma}_\ell > 0$ for which $\frac{\hat{p}_\ell/\mathbb{P}(\{\omega_\ell\})}{\hat{p}_{\ell+1}/\mathbb{P}(\{\omega_{\ell+1}\})} = 1$, i.e., $u_i'(\hat{X}_i(\omega_\ell)) = u_i'(\hat{X}_i(\omega_{\ell+1}))$ for all $i \in N_1$, and so, due to $u_i''(\cdot) < 0$, we then have $\sum_{i \in N_2} \hat{X}_i(\omega_\ell) \leq \sum_{i \in N_2} \hat{X}_i(\omega_{\ell+1})$. Due to $u_i''(\cdot) < 0, i \in N_1$ and the Intermediate Value Theorem, we get that there is a unique $\hat{\gamma}_\ell \geq 0$ such that $\sum_{i \in N_2} \hat{X}_i(\omega_\ell) = \sum_{i \in N_{\ell+1}} \hat{X}_i(\omega_{\ell=1})$. This $\hat{\gamma}_\ell$ is continuous and strictly increasing in $\hat{\gamma}_{\ell+1}$.

Step 2: suppose that Event s^* holds with $s^* \in \{0, \ldots, z-2\}$, i.e., $\hat{\gamma}_\ell, \ldots, \hat{\gamma}_{\ell+s^*}$ are given functions of $\hat{\gamma}_{\ell+s^*+1}$, and $\hat{\gamma}_\ell$ is strictly increasing in $\hat{\gamma}_{\ell+s^*+1}$. We show that Event $s^* + 1$ holds, i.e., for every given $\hat{\gamma}_{\ell+s^*+2} \geq 0$, $\hat{\gamma}_\ell, \ldots, \hat{\gamma}_{\ell+s^*+1}$ are unique, and $\hat{\gamma}_\ell$ is continuous and strictly increasing in $\hat{\gamma}_{\ell+s^*+2}$. Fix $\hat{\gamma}_{\ell+s^*+2}$. From (42), we get via iterative multiplications that

$$\frac{u_i'(\hat{X}_i(\omega_\ell))}{u_i'(\hat{X}_i(\omega_{\ell+s^*+2}))} = \frac{(g^*_{N_2}(\mathbb{P}(\Omega_\ell)) - \hat{\gamma}_\ell)/\mathbb{P}(\{\omega_\ell\})}{(g^*_{N_2}(\mathbb{P}(\Omega_{\ell+s^*+2})) + \hat{\gamma}_{\ell+s^*+1} - \hat{\gamma}_{\ell+s^*+2})/\mathbb{P}(\{\omega_{\ell+s^*+2}\})}. \tag{43}$$

The right-hand side of this equation is continuous and strictly increasing in $\hat{\gamma}_\ell \geq 0$. If $\hat{\gamma}_\ell = 0$, then we get that $\sum_{i\in N_2} \widetilde{X}_i(\omega_\ell) \geq \sum_{i\in N_2} \widetilde{X}_i(\omega_{\ell+s^*+2})$. Moreover, there exists unique $\hat{\gamma}_\ell > 0$ for which $\frac{\hat{p}_\ell/\mathbb{P}(\{\omega_\ell\})}{\hat{p}_{\ell+s^*+2}/\mathbb{P}(\{\omega_{\ell+s^*+2}\})} = 1$ so that $\sum_{i\in N_2} \widetilde{X}_i(\omega_\ell) \leq \sum_{i\in N_2} \widetilde{X}_i(\omega_{\ell+s^*+2})$. Due to $u_i''(\cdot) < 0, i \in N_1$ and the Intermediate Value Theorem, we get that there is a unique $\hat{\gamma}_\ell \geq 0$ such that $\sum_{i\in N_2} \hat{X}_i(\omega_\ell) = \sum_{i\in N_2} \hat{X}_i(\omega_{\ell+s^*+2})$. Then, $\hat{\gamma}_{\ell+1}, \ldots, \hat{\gamma}_{\ell+s^*}$ follow because we assumed that Event s^* holds. From (43), we readily see that the value of $\hat{\gamma}_\ell$ is continuous and strictly increasing in $\hat{\gamma}_{\ell+s^*+2}$. Hence, Event s^*+1 holds. This concludes the proof, where we show uniqueness of finding the Lagrangian parameters $\hat{\gamma}_\ell, \ldots, \hat{\gamma}_{\ell+z}$. ■

We conjecture that the price vector obtained from the algorithm of this section is the only equilibrium price vector that constitutes a competitive equilibrium. In the sequel of this paper, we discuss competitive equilibria only with the equilibrium price vector as in this section.

4.3. Uniqueness of the competitive equilibrium, and its capital asset pricing model

In this section, we characterize uniqueness of the competitive equilibrium. From (20), we get that every Pareto optimal risk redistribution for agents depends on the functions $m \in \mathcal{M}$ and the side-payments d. For the competitive equilibria, the side-payments d are determined. The following condition specifies whether the set $\mathcal{M}$ is small enough to guarantee uniqueness of the competitive equilibrium:

Condition [U]: for all $\ell \in \{1, \ldots, p-1\}$ such that $\sum_{i\in N_2} \widetilde{X}_i(\omega_\ell) < \sum_{i\in N_2} \widetilde{X}_i(\omega_{\ell+1})$, there exists exactly one agent $i \in N_2$ such that $g_i(\mathbb{P}(\Omega_\ell))$ is minimal.

Note that condition [U] is satisfied when the set $\mathcal{M}$ in (19) or N_2 is single valued. If there exists a globally least risk-averse agent in N_2, then this agent bears the risk $\sum_{i\in N_2} \widetilde{X}_i$. If there does not exist a globally least risk-averse agent, then it is Pareto optimal that a locally least risk-averse agent bears the risk $\sum_{i\in N_2} \widetilde{X}_i$ locally (see (20)). Condition [U] holds if there is a unique locally least risk-averse agent everywhere.

Theorem 4.5. *If $N_1 \neq \emptyset$, the competitive equilibrium is unique if and only if condition [U] is satisfied, where the competitive equilibrium is defined in* (22)–(24) *with price vector as Section 4.2.*

Proof. The price vector as Section 4.2 is unique (see Proposition 4.4). First, we show that there are unique equilibrium risk profiles $(\widetilde{X}_i)_{i\in N_1}$ and $\sum_{i\in N_2} \widetilde{X}_i$. Let $\hat{p}$ be the unique equilibrium price vector from (28). From $u_i'(\cdot) > 0$,

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$u_i''(\cdot) < 0$ and that u_i satisfies the Inada conditions for all $i \in N_1$, it follows that $u_\lambda'(\cdot)$ is continuous, strictly decreasing and has range $(0, \infty)$. Therefore, we get that the inverse function $u_i'^{-1}$ exists. From (25), we define $\widetilde{X}_i^{\lambda_i} = u_i'^{-1}(\lambda_i \frac{\hat{p}}{\mathbb{P}})$. Due to $u_i''(\cdot) < 0$ and that u_i satisfies the Inada conditions, it follows that $\widetilde{X}_i^{\lambda_i}(\omega)$ is strictly decreasing and continuous in λ_i for every $\omega \in \Omega$, with $\lim_{\lambda_i \downarrow 0} \widetilde{X}_i^{\lambda_i}(\omega) = \infty$ and $\lim_{\lambda_i \to \infty} \widetilde{X}_i^{\lambda_i}(\omega) = -\infty$ for all $\omega \in \Omega$. Since $\hat{p} > 0$, the function $\pi(\hat{p}, \cdot)$ in (23) is continuous and strictly increasing. Hence, there is a unique λ_i solving the budget constraint in (26). So, the risk redistribution $(\widetilde{X}_i)_{i \in N_1}$ is the same in every equilibrium. Then, so are the risks $\sum_{i \in N_1} \widetilde{X}_i$ and $\sum_{i \in N_2} \widetilde{X}_i$. For a given $\sum_{i \in N_2} \widetilde{X}_i = \sum_{i \in N} X_i - \sum_{i \in N_1} \widetilde{X}_i$, the equilibrium risk redistribution for the group N_2 is determined by Pareto optimal risk redistributions satisfying the price constraint (27). By Boonen (2015, Theorem 3.8 therein), this is unique if and only if condition [U] holds. This concludes the proof. ■

If $N = N_2$, then it follows from Boonen (2015) that condition [U] and $X(\omega_1) < \cdots < X(\omega_p)$ are jointly sufficient to have uniqueness of the competitive equilibrium. If $N = N_1$, Aase (1993, 2010) proposes conditions for uniqueness of the competitive equilibrium. His conditions are either assumed in the setting of this paper, or are irrelevant since we assume that the state space Ω is finite.

If condition [C] is not satisfied, we get from Section 4.2 an algorithm to determine equilibrium prices. After we determine the equilibrium price vector, the equilibrium risk redistribution follows from (25)–(27). This method is analogous to the algorithm in Section 3 for Pareto optima, where the vector k is not fixed, but implicitly given by λ^{-1} which follows from (25), (26).

If condition [C] holds, the corresponding pricing kernel is given by the following Radon–Nikodym derivative:

$$\frac{d\mathbb{Q}}{d\mathbb{P}}(\{\omega_\ell\}) = \frac{g_{N_2}^*(\mathbb{P}(\Omega_\ell)) - g_{N_2}^*(\mathbb{P}(\Omega_{\ell-1}))}{\mathbb{P}(\{\omega_\ell\})} = \frac{dg_{N_2}^* \circ \mathbb{P}}{d\mathbb{P}}(\{\omega_\ell\}), \tag{44}$$

for all $\ell \in \{1, \ldots, p\}$. We can test this pricing kernel empirically via the corresponding CAPM. This is done by De Giorgi and Post (2008) for the setting where every agent in the market is endowed with the same distortion risk measure.[4] Using U.S. stock returns, they find a better fit than the classical CAPM with mean-variance investors. If we would test the equilibrium prices, we would assume a functional form of the distortion function $g_{N_2}^*$, and test (44) empirically. So, testing our model is analogous to the test of De Giorgi and Post (2008). Hence, De Giorgi and Post (2008) show that our model with dual utilities has a better fit than the CAPM with mean-variance investors.

Even if condition [C] does not hold, we get from Theorem 4.1 the prices in any competitive equilibrium. However, computing any competitive equilibrium may be tedious. It requires to compute equilibrium prices where some

Lagrangian parameters are strictly positive (see Theorem 3.3), and thus the corresponding constraints binding.

4.4. Competitive equilibrium with expected shortfall

As dual utility is related to coherent risk measures, dual utility preferences may be deduced from regulation. For instance, agents (firms) may aim to minimize their risk-adjusted value of the liabilities (for more detailed information, see, e.g., Chi, 2012). Expected shortfall (see, e.g., Acerbi and Tasche, 2002) is a popular risk measure as it is used in Basel III and Swiss Solvency Test regulations. Expected shortfall is a distortion risk measure, with distortion function $g(x) = \min\{\frac{x}{1-\alpha}, 1\}$ for all $x \in [0, 1]$, where $\alpha \in (0, 1)$ is the parameter used (Kusuoka, 2001). This function is concave but not strictly concave. For this reason, we focus competitive equilibria in the setting that only differs from the setting in Sections 4.1–4.3 by allowing the distortion functions g_i, $i \in N_2$ to be concave and non-decreasing.

If we focus on comonotone equilibrium risk redistributions only, we can use a result of Landsberger and Meilijson (1994). They show that for every risk redistribution, there exists a comonotone risk redistribution that dominates it in the sense of second-order stochastic dominance. Since dual utilities with concave distortion functions are preserving second-order stochastic dominance (Chew *et al.*, 1987), there exist competitive equilibria with comonotone risk redistributions. The following result follows directly from this and the proof of Theorem 4.1.

Proposition 4.6. *Let the set N_2 contain dual utility maximizing agents such that the distortion functions g_i, $i \in N_2$, are all concave and non-decreasing. Then, every equilibrium $(\hat{p}, (\widetilde{X}_i)_{i\in N})$ such that $(\widetilde{X}_i)_{i\in N_2}$ is comonotone, is a solution of* (25)–(28)*, where the equilibria are defined in* (22)–(24).

Remark. Suppose there exists an agent i that uses the preference relation $V_i(X) = E_{\mathbb{P}}[X]$, i.e., it is risk neutral. Note that this is the only expected utility function that is a dual utility function as well. Then, in every Pareto optimum, this agent will bear all risk, i.e., every Pareto optimum is such that $\widetilde{X}_j$ is deterministic for all $j \neq i$. This observation is consistent with results on expected utility (Borch, 1962) and dual utility (Jouini *et al.*, 2008). Since the distortion function is linear, it is concave. Therefore, we obtain some competitive equilibria from Proposition 4.6 with $i \in N_2$.

Remark. Proposition 4.6 cannot be generalized to non-concave distortion functions. It is possible that every comonotone risk redistribution is not Pareto optimal (see, e.g., Theorem 4.3 from Embrechts *et al.*, 2016), and so does not constitute a competitive equilibrium due to the First Fundamental Welfare Theorem (see Arrow, 1963). Therefore, we do not discuss this case in more detail.

5. SPECIAL CASE WITH EXPONENTIAL UTILITIES

In this section, we restrict the expected utility maximizers in N_1 to use exponential expected utility functions, i.e., agent $i \in N_1$ maximizes

$$V_i(X) = E_{\mathbb{P}}[u_i(X)] = E_{\mathbb{P}}\left[\exp\left(\frac{X}{\alpha_i}\right)\right], \text{ for all } X \in \mathbb{R}^{\Omega},$$

where $\alpha_i > 0$. It follows from Aase (1993, equation (4.1) therein) that

$$u'_{\lambda}\left(\sum_{i\in N_1} \widetilde{X}_i\right) = \exp\left\{\frac{K - \sum_{i\in N_1} \widetilde{X}_i}{\sum_{i\in N_1} \alpha_i}\right\}, \text{ where } K = -\sum_{i\in N_1} \alpha_i \log \lambda_i.$$

From this and (40), we derive

$$\sum_{i\in N_1} \widetilde{X}_i = -\sum_{i\in N_1} \alpha_i \log\left(\frac{\hat{p}}{\mathbb{P}}\right) + K, \tag{45}$$

where $\hat{p}$ is defined in (28). Moreover, any Pareto optimal risk redistribution $(\widetilde{X}_i)_{i\in N}$ with exponential utilities in N_1 is such that

$$\widetilde{X}_i = \frac{\alpha_i}{\sum_{i\in N_1} \alpha_i} \sum_{i\in N_1} \widetilde{X}_i + K_i \tag{46}$$

$$= -\alpha_i \log\left(\frac{\hat{p}}{\mathbb{P}}\right) + \hat{K}_i, \tag{47}$$

for all $i \in N_1$, where $K_i, i \in N_1$ are such that $\sum_{i\in N_1} K_i = 0$ and $\hat{K}_i = K + K_i, i \in N_1$ are constants. Here, (46) follows from (12) and Bühlmann and Jewell (1979), and (47) follows from substituting (45) in (46). From this, we derive that condition [C] can be written as

$$\sum_{i\in N} X_i(\omega_{\ell+1}) - \sum_{i\in N} X_i(\omega_\ell) \geq \sum_{i\in N_1} \alpha_i \left(\log\left(\frac{\hat{p}_\ell}{\mathbb{P}(\{\omega_\ell\})}\right) - \log\left(\frac{\hat{p}_{\ell+1}}{\mathbb{P}(\{\omega_{\ell+1}\})}\right)\right),$$

for all $\ell \in \{1, \ldots, p-1\}$. So, the condition [C] is satisfied whenever the average agent of the set N_1 is relatively risk averse (small value of $\sum_{i\in N_1} \alpha_i$) and the least risk-averse agent in N_2 is relatively little risk averse (small values of $\log(\frac{\hat{p}_\ell}{\mathbb{P}(\{\omega_\ell\})}) - \log(\frac{\hat{p}_{\ell+1}}{\mathbb{P}(\{\omega_{\ell+1}\})})$).

If $N = N_1$, the equilibrium prices are given by Bühlmann (1980). When the risks X_i and $\sum_{j\neq i} X_j$ are independent, the premium for agent i equals the Esscher premium principle. Even if there is just one agent in N_2 with a small risk, we get that equilibrium prices may be very different.

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Example 5.1. *In this example, we consider a market with four agents, where* $N_1 = \{1, 2\}$ *and* $N_2 = \{3, 4\}$. *The agents in* N_1 *use an exponential utility function with* $\alpha_1 = 1$ *and* $\alpha_2 = 2$. *Agents 3 and 4 use distortion functions* $g_3(x) = \sqrt{x}$ *and* $g_4(x) = 1\frac{1}{4}(1 - \frac{1}{5}^x)$ *for all* $x \in [0, 1]$. *The state space is given by* $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}$, *with* $\mathbb{P}(\{\omega\}) = \frac{1}{4}$ *for all* $\omega \in \Omega$. *Moreover, let* $X_i = \frac{1}{4}X$ *for all* $i \in N$, *with* $X(\omega_k) = 4k$ *for* $k \in \{1, 2, 3, 4\}$. *For our convenience, we write stochastic variables as vectors.*

First, we determine the equilibrium risks $\widetilde{X}_1$, $\widetilde{X}_2$ *and* $\widetilde{X}_3 + \widetilde{X}_4$ *via the representative agent of the set* N_2, $\rho^*_{N_2}$. *We find that conditions [C] and [U] are satisfied. So, from Corollary 4.2, we get that equilibrium prices follow from* (39): $\hat{p} \approx (0.5, 0.207, 0.169, 0.124)$. *From this,* (25) *and* (27), *we get that* $(\hat{p}, (\widetilde{X}_i)_{i \in N})$ *is a competitive equilibrium, where*

$$\widetilde{X}_1 \approx (0.96, 2.31, 2.83, 3.87),$$
$$\widetilde{X}_2 \approx (1.41, 2.19, 2.42, 2.83),$$
$$\widetilde{X}_3 + \widetilde{X}_4 \approx (1.63, 3.50, 6.74, 9.30).$$

We also find that $\mathcal{M}$ *is single-valued, and its unique element* $m \in \mathcal{M}$ *is such that* $m(1) = m(2) = 3$ *and* $m(3) = 4$, *where* $\mathcal{M}$ *is defined in* (19). *From this, and equations* (20) *and* (26), *we find that* $\widetilde{X}_3 \approx (0.03, 1.90, 5.14, 5.14)$ *and* $\widetilde{X}_4 \approx (1.60, 1.60, 1.60, 4.16)$. $\triangledown$

Example 5.2. *In this example, we consider the same problem as in Example 5.1, but we vary the value of* $X(\omega_3)$. *We get that condition [C] is not satisfied anymore when* $X(\omega_3) \in [8, 8.70)$. *Let* $X(\omega_3) = 8.5$. *We apply the KKT conditions that are derived in the proof of Theorem 4.1. We obtain that the constraint* $\sum_{i \in N} X_i(\omega_2) - \sum_{i \in N_1} \widetilde{X}_i(\omega_2) \leq \sum_{i \in N} X_i(\omega_3) - \sum_{i \in N_1} \widetilde{X}_i(\omega_3)$ *is binding. From* (28) *in Theorem 4.1, we derive* $\gamma_1 = \gamma_3 = 0$ *and* $\gamma_2 \approx 0.0053$. *This leads to pricing vector* $\hat{p} \approx (0.5, 0.207, 0.169, 0.124) + (0, -0.0053, 0.0053, 0) \approx (0.5, 0.202, 0.174, 0.124)$. *Moreover, we derive that the risk redistribution given by*

$$\widetilde{X}_1 \approx (0.88, 2.19, 2.54, 3.57),$$
$$\widetilde{X}_2 \approx (1.30, 2.05, 2.20, 2.61),$$
$$\widetilde{X}_3 \approx (0.80, 2.74, 2.74, 2.74),$$
$$\widetilde{X}_4 \approx (1.02, 1.02, 1.02, 7.07),$$

constitutes a competitive equilibrium. Note that from $\gamma_2 > 0$ *it follows by construction that* $\widetilde{X}_3(\omega_2) = \widetilde{X}_3(\omega_3)$ *and* $\widetilde{X}_4(\omega_2) = \widetilde{X}_4(\omega_3)$. $\triangledown$

6. Conclusion

This paper studies optimal risk redistributions in markets with expected and dual theory maximizers. In contrast to the previous literature, we study markets where both types of agents are present. Pareto optimal contracts are characterized in a way that extends both the result of Borch (1962) for expected utilities and the result of Jouini *et al.* (2008) for dual utility maximizers. We derive that under some circumstances, equilibrium prices do not depend on the expected utility maximizers in the market. Moreover, we characterize uniqueness of the competitive equilibrium.

The Pareto optimal and equilibrium risk redistributions follow from the preferences of two hypothetical representative agents. This is an average risk-averse expected utility maximizing agent and dual utility maximizing agent that has lowest aversion to mean-preserving spreads. Given a (non-trivial) allocation of the total risk to both groups, the solution to expected utility maximizers is in line with the well-known result of Borch (1962) applied to their allocated risk as if it were to be the aggregate risk. Moreover, the solution to dual utility maximizing agents is given by a particular tranching of their allocated risk.

An important question that we leave open for future research is what the Pareto optima and competitive equilibria are in the case of a continuous state space. This paper characterizes the competitive equilibrium using a finite dimensional optimization problem. This approach cannot be used in case the state space is continuous. Moreover, the equilibrium prices that we characterize in this paper have no trivial translation to the setting with a continuous state space.

Notes

1. A preference relation $V : \mathbb{R}^{\Omega} \to \mathbb{R}$ is positive homogeneous if $V(\alpha X) = \alpha V(X)$ for all $\alpha > 0$ and all $X \in \mathbb{R}^{\Omega}$.

2. This representation is already shown by Borch (1962) for expected utilities. Kiesel and Rüschendorf (2007) extend this result to cases that include expected and dual utilities.

3. For a precise derivation, see Boonen (2015, Equation (10) and Proposition 3.6 therein).

4. Note that our setting is more general than the setting of De Giorgi and Post (2008) since we allow for heterogenous distortion risk measures and include expected utility maximizers. However, the corresponding pricing Radon-Nikodym derivative in (44) has the same structure as in De Giorgi and Post (2008).

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