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ON COMPONENT FAILURE IN COHERENT SYSTEMS WITH APPLICATIONS TO MAINTENANCE STRATEGIES

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Abstract

Providing optimal strategies for maintaining technical systems in good working condition is an important goal in reliability engineering. The main aim of this paper is to propose some optimal maintenance policies for coherent systems based on some partial information about the status of components in the system. For this purpose, in the first part of the paper, we propose two criteria under which we compute the probability of the number of failed components in a coherent system with independent and identically distributed components. The first proposed criterion utilizes partial information about the status of the components with a single inspection of the system, and the second one uses partial information about the status of component failure under double monitoring of the system. In the computation of both criteria, we use the notion of the signature vector associated with the system. Some stochastic comparisons between two coherent systems have been made based on the proposed concepts. Then, by imposing some cost functions, we introduce new approaches to the optimal corrective and preventive maintenance of coherent systems. To illustrate the results, some examples are examined numerically and graphically.

Keywords: Preventive maintenance; corrective maintenance; minimal repair; order statistics; stochastic order; signature; totally positive of order 2

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1. Introduction

Nowadays, coherent systems are used in many areas of human life, such as industrial manufacturing lines, airplane systems, power supply systems, and telecommunication systems. In reliability engineering, an n-component system is called *coherent* if the structure function of the system is nondecreasing and the system has no irrelevant components (see Barlow and Proschan [7]). A well-known subclass of the class of coherent systems is that of k-out-of-n systems. Recall that an n-component system is said to be a k-out-of-n system if it operates when at least k components out of n operate.

In the last two decades, an extensive number of research works have been reported assessing the reliability and stochastic properties of coherent systems using various approaches. An approach which has recently received great attention is to use the notion of *signa-*

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ture. Let $X_1, X_2, \ldots, X_n$ denote the lifetimes of an n-component coherent system and let $T = T(X_1, \ldots, X_n)$ be the system lifetime. Under the assumption that the component lifetimes are independent and identically distributed (i.i.d.), Samaniego [31] defined the concept of signature to express the reliability function of the system lifetime as a mixture representation of the reliability function of ordered component lifetimes. To be more precise, let $X_{1:n}, X_{2:n}, \ldots, X_{n:n}$ denote the order statistics corresponding to the lifetimes $X_1, \ldots, X_n$. Then the reliability function of the system's lifetime, at time t, can be expressed as

$$\mathbb{P}(T > t) = \sum_{i=1}^{n} s_i \mathbb{P}(X_{i:n} > t), \tag{1}$$

where $s_i = \mathbb{P}(T = X_{i:n})$, $i = 1, 2, \ldots, n$. The probability vector $\boldsymbol{s} = (s_1, s_2, ..., s_n)$ is called the signature vector of the system. The ith element of the vector $\boldsymbol{s}$ is calculated as $s_i = n_i/n!$, where n_i denotes the number of permutations of components under which the ith component failure causes the system failure (see Samaniego [31]). It is known that the vector $\boldsymbol{s}$ depends only on the structure of the system. The representation (1) is valid under the weaker condition that the component lifetimes are exchangeable (see Navarro and Rychlik [23]). For references on the signature-based properties of system lifetime, we refer the reader to [5], [11], [17], [20], [24], [25], [32], [36], [38], and [41]. For a recent work on different methods and algorithms for computing system signature, see Reed [29] and references therein.

In assessing the reliability and stochastic characteristics of systems, a problem of interest for engineers and system designers is to maintain the system in optimum working condition and to determine the number of spares that should be available in the depot for this purpose. The importance of this problem arises from the fact that the failure and unavailability of the system may cause high unexpected costs to the users. In many complex coherent systems, such as k-out-of-n systems, the design of the structure of the system is such that the system operates even if a number of components have already failed. However, if the number of failed components passes a certain threshold, then the system does fail. Hence, the computation of the probability of the number of failed components in the system, under various conditions, is important for the system operators. These probabilities provide crucial information for preventing the system's failure and maintaining the system in optimal operating condition. The aim of maintenance schedules is mainly to diminish the occurrence of system failure or to change the status of a failed system to the working state. For this purpose, operators try to restore a failed component to an operative state. In the literature, this maintenance action is called *corrective maintenance* (CM). In a CM action, the failed components may undergo repair or may be replaced. Two other important actions in maintenance theory are (a) *minimal repair*, which eliminates the failure but does not change the failure rate, and (b) *preventive maintenance* (PM), which means performing a maintenance policy for an operating system (component) to bring the system (component) back to better working condition. Throughout the paper, we assume that the PM (CM) is perfect in the sense that an unfailed (failed) component is returned to 'as-good-as-new' condition.

In the literature, many research papers and books have been devoted to various maintenance schedules. We refer the reader to [4], [6], [9], [12], [13], [14], [18], [21], [27], [34], [39], and [42]. Recently, some comparisons of policies for minimal repair of systems have been studied in Belzunce *et al.* [8] and Arriaza *et al.* [1].

The main objective of the present research is to propose some maintenance policies for a coherent system under some partial information on the number of failures in the system. So far, only a small portion of the literature has considered maintenance of a multi-component system.

Most of the works on this topic have been limited to a one-unit system, or to an entire system treated as a single-unit system. Finkelstein and Gertsbakh [12], [13] studied PM for networks (systems) where the components fail based on shock models. Cha *et al.* [9] considered PM of items operating in a random environment. Zarezadeh and Asadi [40] studied PM scheduling for systems under multiple external shocks. We introduce two new optimal strategies for the maintenance of an n-component coherent system, with i.i.d. component lifetimes and signature vector $\boldsymbol{s}$, under the condition that there is some partial information on the number of destroyed components in the system. The information is collected under two scenarios: single inspection and double inspection of the system. In the first strategy, before a predetermined time τ the system undergoes minimal repair, and after τ the system is equipped with a warning light that turns on at the time of the kth component failure. Then the system is inspected at time t, $t > \tau$. The operator decides to perform CM on the entire system when the system fails, or to perform distinct PM actions (depending on whether the light turns on or not) when the total operating time reaches t, whichever occurs first. In the second strategy, the system is inspected at two times t_1 and t_2, $t_1 < t_2$, and depending on the information obtained at t_1, the operator performs different maintenance actions at t_2.

Suppose the system starts to operate at time $t = 0$ and each component may fail over time. Assume that the system is functioning at time t and at least k components have failed before t. Under these assumptions, in Section 2, we compute the probability of the number of failed components in the system. In other words, if N_t is the number of failed components up to time t, then we calculate the following conditional probabilities:

$$p^t_{k,n}(i) = \mathbb{P}(N_t = i \mid X_{k:n} \le t < T), \quad i = k, \ldots, n-1. \tag{2}$$

The second scenario considers the condition that exactly k components have failed by time t_1, and at time t_2 ($t_2 > t_1$) the system is still operating. Under this condition, we calculate the probability of the number of failed components N_{t_2}; i.e., we calculate

$$p^{t_1,t_2}_{k,n}(i) = \mathbb{P}(N_{t_2} = i \mid X_{k:n} \le t_1 < X_{k+1:n},\, T > t_2), \quad i = k, \ldots, n-1. \tag{3}$$

We investigate the properties of $p^t_{k,n}(i)$ and $p^{t_1,t_2}_{k,n}(i)$ in Section 2. In particular, some stochastic ordering results for these conditional probabilities are established. In Section 3, we propose some optimal PM policies for coherent systems with n components as applications of $p^t_{k,n}(i)$ and $p^{t_1,t_2}_{k,n}(i)$. The criteria which will be employed to obtain the optimal PM time for the system are the minimal long-run expected cost per unit of time and stationary availability of the system. We examine the results of the paper by considering the well-known bridge system consisting of 5 components. Using the presented bridge system, the robustness of the proposed approaches is also analyzed numerically. The graphical and computational results of the paper are obtained using MATHEMATICA®, Version 10.

The following auxiliary concepts and definitions are useful in our derivations.

Definition 1.1. Assume that X and Y are nonnegative random variables with cumulative distribution functions (CDFs) F and G, probability density functions f and g, and reliability functions $\bar{F} = 1 - F$ and $\bar{G} = 1 - G$, respectively.

(a) If $\frac{g(t)}{f(t)}$ is increasing in $t \ge 0$, then X is said to be less than Y in the likelihood ratio ordering (denoted by $X \le_{lr} Y$).

(b) If $\frac{\bar{G}(t)}{\bar{F}(t)}$ is increasing in $t \ge 0$, then X is said to be less than Y in the hazard rate ordering (denoted by $X \le_{hr} Y$).

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(c) If $\bar{G}(t) \geq \bar{F}(t)$ for all $t \geq 0$, then X is said to be less than Y in the usual stochastic ordering (denoted by $X \leq_{st} Y$).

The implications among these orderings are as follows:

$$X \leq_{lr} Y \quad \Rightarrow \quad X \leq_{hr} Y \quad \Rightarrow \quad X \leq_{st} Y.$$

Definition 1.2.

(a) A probability vector $\boldsymbol{p} = (p_1, p_2, \ldots, p_n)$ is said to have increasing failure rate (IFR) if $p_k / \sum_{i=k}^{n} p_i$ is increasing in $k = 1, 2, \ldots, n$.

(b) For two probability vectors $\boldsymbol{p} = (p_1, p_2, \ldots, p_n)$ and $\boldsymbol{q} = (q_1, q_2, \ldots, q_n)$, if $\sum_{j=i}^{n} q_j / \sum_{j=i}^{n} p_j$ is increasing in i, then $\boldsymbol{p}$ is said to be less than $\boldsymbol{q}$ in the hazard rate order (denoted by $\boldsymbol{p} \leq_{hr} \boldsymbol{q}$).

For more details on various notions of partial orderings and their applications, we refer to Shaked and Shanthikumar [33] and Lucia [19].

Definition 1.3. (Karlin [15].) A nonnegative function $h(x,y)$ is totally positive of order 2 (TP_2) if $h(x_1, y_1)h(x_2, y_2) - h(x_1, y_2)h(x_2, y_1) \geq 0$ whenever $x_1 < x_2$ and $y_1 < y_2$. The function $h(x, y)$ is said to be reverse regular of order 2 (RR_2) if $h(x_1, y_1)h(x_2, y_2) - h(x_1, y_2)h(x_2, y_1) \leq 0$ whenever $x_1 < x_2$ and $y_1 < y_2$.

2. The probability of the number of failed components of the system

In the present section, first we obtain the functional forms of the conditional probabilities $p_{k,n}^{t}(i)$ and $p_{k,n}^{t_1,t_2}(i)$. These conditional probabilities are useful in our derivations in Section 3 to establish the new optimal maintenance strategies on the coherent systems.

Single inspection

Consider a coherent system with lifetime T, as described in the introduction. The system begins to operate at time $t = 0$ and each component is subject to failure over time. Assume that the operator inspects the system at time t and he/she observes that at least k components have already failed before t, but the system is still working. As we mentioned in the previous section, the probability of the number of failed components up to time t is given by $p_{k,n}^{t}(i)$ in (2). Asadi and Berred [2] explored several properties of the above conditional probability $p_{k,n}^{t}(i)$ in the case that $k = 0$. Eryilmaz [10] considered the number of failed components for a coherent system whose component lifetimes are exchangeable. Ashrafi and Asadi [3] studied the number of failed components in three-state networks under different conditions and applied their results for age-replacement of three-state networks. The conditional probability $p_{k,n}^{t}(i)$ can be represented as follows:

$$\begin{aligned} p_{k,n}^{t}(i) &= \mathbb{P}(N_t = i \mid X_{k:n} \leq t < T) \\ &= \frac{\bar{S}_i \binom{n}{i} \phi^i(t)}{\sum_{j=k}^{n-1} \bar{S}_j \binom{n}{j} \phi^j(t)}, \quad i = k, \ldots, n-1, \end{aligned} \tag{4}$$

where $\bar{S}_i = \sum_{j=i+1}^{n} s_j$ and $\phi(t) = F(t)/\bar{F}(t)$, provided that $\bar{F}(t) = 1 - F(t) > 0$; see Appendix A for the proof. The expectation of $(N_t \mid X_{k:n} \leq t < T)$ can be calculated as follows:

$$\mathbb{E}(N_t \mid X_{k:n} \leq t < T) = \frac{n \sum_{i=k}^{n-1} \bar{S}_i \binom{n-1}{i-1} \phi^i(t)}{\sum_{j=k}^{n-1} \bar{S}_j \binom{n}{j} \phi^j(t)}, \quad k = 0, 1, \ldots, n-1. \tag{5}$$

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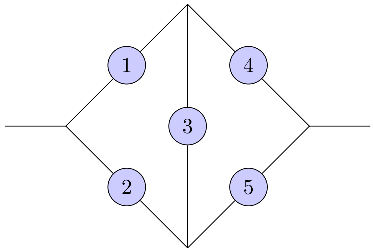

FIGURE 1: The bridge system.

One can easily verify that the common reliability function $\bar{F}(t)$ of the components can be recovered from two successive values $p^t_{k,n}(i)$ and $p^t_{k,n}(i+1)$ as follows:

$$\bar{F}(t) = \left(1 + \frac{i+1}{n-i}\left(\frac{s_{i+1}}{\bar{S}_{i+1}} + 1\right)\frac{p^t_{k,n}(i+1)}{p^t_{k,n}(i)}\right)^{-1}, \quad t > 0.$$

Remark 2.1. The probability $p^t_{k,n}(i)$ may be interpreted from a Bayesian viewpoint. Assume that the system designer is interested in the failure probability of the system components at time t, denoted by $\mathbb{P}(N_t = i)$, $i = 1, 2, \ldots, n$, as his/her prior belief (prior distribution) when the system has not yet been put into operation. Now suppose that the system starts to perform its mission at time $t = 0$, and the designer has information that at least k components have failed before t, while the system is functioning. Then the probability $p^t_{k,n}(i)$ can be viewed as the designer's posterior belief on N_t, given the information that is provided for the designer.

Remark 2.2. Asadi and Berred [2] investigated the time-dependent behavior of $p^t_{0,n}(i)$ for different values of i. The following results can be established regarding the time-dependent behavior of $p^t_{k,n}(i)$; these are similar to Theorem 2.3 of [2]. Let $i^* = \max\{i\colon s_i > 0\}$. Then (a) $p^t_{k,n}(k)$ is decreasing in t; (b) for $i = k+1, \ldots, i^* - 2$, $p^t_{k,n}(i)$ first increases with respect to t until it attains its maximum, then declines; and (c) $p^t_{k,n}(i^* - 1)$ is increasing in t. We omit the proof, which is analogous to that of Theorem 2.3 in Asadi and Berred [2]. The next example gives applications of these results.

Example 2.1. In Figure 1, a bridge system consisting of five identical components is pictured. We assume that the component lifetimes are i.i.d. random variables having Weibull distribution with CDF $F(t) = 1 - \exp\{-t^2\}$, $t \geq 0$. The system signature can be computed as $\boldsymbol{s} = (0, 0.2, 0.6, 0.2, 0)$. Then we have $\phi(t) = e^{t^2} - 1$ and

$$p^t_{k,5}(i) = \frac{\bar{S}_i\binom{5}{i}(e^{t^2} - 1)^i}{\sum_{j=k}^{4} \bar{S}_j\binom{5}{j}(e^{t^2} - 1)^j}, \quad 0 \leq k \leq i \leq 4. \tag{6}$$

As can be seen in Figure 2(a), $p^t_{1,5}(1)$ is decreasing in t, $p^t_{1,5}(2)$ increases for a period of time and then decreases, and $p^t_{1,5}(3)$ is increasing in t (see Remark 2.2). It should be mentioned that in this system, $p^t_{1,5}(4) = 0$, since for the component with lifetime $X_{5:5}$, $\mathbb{P}(T = X_{5:5}) = s_5 = 0$; i.e., the component with lifetime $X_{5:5}$ will never entail the failure of the system. Also, using Equation (5), we obtain

$$H^t_{k,n} := \mathbb{E}(N_t \mid X_{k:n} \leq t < T) = \frac{5\sum_{i=k}^{4} \bar{S}_i\binom{4}{i-1}(e^{t^2} - 1)^i}{\sum_{j=k}^{4} \bar{S}_j\binom{5}{j}(e^{t^2} - 1)^j}, \quad 0 \leq k \leq 4. \tag{7}$$

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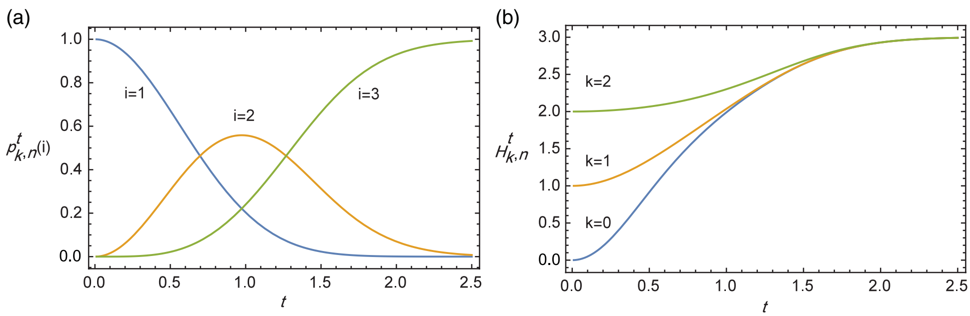

FIGURE 2: (a) The plots of $p^t_{k,n}(i)$ for $i=1, 2, 3$ and $k=1$ in Example 2.1. (b) The plots of $H^t_{k,n}$ for $k=0, 1, 2$ in Example 2.1.

Figure 2(b) shows the graph of $H^t_{k,n}$ as a function of t for $k=0, 1, 2$. It is seen that $H^t_{k,n}$ is increasing in k and t. We have proved the same result for a general system structure and an arbitrary baseline distribution in Corollary 2.1.

The following theorem gives some stochastic properties of $(N_t \mid X_{k:n} \leq t < T)$. Before giving the theorem, we mention the well-known result that, if X and Y are two nonnegative random variables with probability density functions f_1 and f_2, respectively, then the order $X \leq_{lr} Y$ is equivalent to saying that $f_m(t)$ is TP$_2$ in $(m, t) \in \{1, 2\} \times [0, \infty)$ (see Shaked and Shanthikumar [33]).

Theorem 2.1. *Assume that the distribution function F is absolutely continuous. Then the conditional random variable $(N_t \mid X_{k:n} \leq t < T)$ in (2) satisfies in the following orderings:*

(a) *for $0 \leq k \leq n-1$, $(N_t \mid X_{k:n} \leq t < T) \leq_{\mathrm{lr}} (N_t \mid X_{k+1:n} \leq t < T)$;*

(b) *for each $0 < t_1 \leq t_2$, $(N_{t_1} \mid X_{k:n} \leq t_1 < T) \leq_{\mathrm{lr}} (N_{t_2} \mid X_{k:n} \leq t_2 < T)$.*

Proof. See Appendix B. □

It is known that the likelihood ratio order between two random variables implies the usual stochastic order between them. Using this fact and Theorem 2.1, one can verify the behavior of $\mathbb{E}(N_t \mid X_{k:n} \leq t < T)$ in terms of k and t, as shown in the following corollary.

Corollary 2.1. *$\mathbb{E}(N_t \mid X_{k:n} \leq t < T)$ is increasing in terms of k and t.*

We now provide a comparison between two different coherent systems based on the number of failed components in each system.

Theorem 2.2. *Suppose that two coherent systems with, respectively, orders n and $n+1$ have signature vectors $\mathbf{s}^{(1)} = (s_1, s_2, \ldots, s_n)$ and $\mathbf{s}^{(2)} = (p_1, p_2, \ldots, p_{n+1})$. Let the component lifetimes of the first system be $X_1, X_2, \ldots, X_n$, and let those of the second system be $Y_1, Y_2, \ldots, Y_{n+1}$, where in the two systems the components are independent and have a common distribution function F. Denote by N_t and N^*_t the number of failed components of the first and second systems, respectively, at time t. If $\mathbf{s}^{(1)} \leq_{\mathrm{hr}} \mathbf{s}^{(2)}$, then*

$$(N^*_t \mid Y_{k:n+1} \leq t < T_2) \geq_{\mathrm{lr}} (N_t \mid X_{k:n} \leq t < T_1), \quad t \geq 0,$$

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where T_1 and T_2 denote the lifetimes of the system with order n and the system with order $n+1$, respectively.

Proof. See Appendix C. □

The next result gives a likelihood ratio order comparison on the failed components between two coherent systems.

Theorem 2.3. *Assume that $\mathcal{S}_1$ and $\mathcal{S}_2$ are two coherent systems with i.i.d. component lifetimes $X_1, X_2, \ldots, X_n$ and $Y_1, Y_2, \ldots, Y_n$ whose CDFs are F and G, respectively. Let N_t and N_t^* be the numbers of failed components in $\mathcal{S}_1$ and $\mathcal{S}_2$, respectively, on $[0, t]$. Further, suppose that the lifetime of the system $\mathcal{S}_1(\mathcal{S}_2)$ is denoted by $T_1(T_2)$ and the corresponding signature vector is denoted by $\mathbf{s}^{(1)}$ $(\mathbf{s}^{(2)})$. If $X_1 \le_{\rm st} Y_1$ and $\mathbf{s}^{(1)} \ge_{\rm hr} \mathbf{s}^{(2)}$, then*

$$(N_t \mid X_{k:n} \le t < T_1) \ge_{\rm lr} (N_t^* \mid Y_{k:n} \le t < T_2), \quad t \ge 0.$$

Proof. See Appendix D. □

Double inspection

Consider again a coherent system with n i.i.d. components and suppose that the system starts working at time $t=0$. We assume that the system is monitored by the operator at two time instances t_1 and t_2 (with $t_1 < t_2$). This method of inspection is known in the literature as *double monitoring*. Some recent references in this regard are Zhang and Meeker [43], Parvardeh *et al.* [26], and Navarro and Calì [22]. Suppose that the number of failed components up to time t_1 is k, and at time t_2 the system is still operating. Under these circumstances, we intend to study the probability of the number of failed components in the system at time t_2. The probability mass function of this random variable, $p_{k,n}^{t_1,t_2}(i)$, can be written as

$$\begin{aligned} p_{k,n}^{t_1,t_2}(i) &= \mathbb{P}(N_{t_2} = i \mid X_{k:n} \le t_1 < X_{k+1:n}, T > t_2) \\ &= \frac{\bar{S}_i \binom{n}{i}\binom{i}{k}\left(\frac{\bar{F}(t_1)}{\bar{F}(t_2)} - 1\right)^{i-k}}{\sum_{j=k}^{n-1} \bar{S}_j \binom{n}{j}\binom{j}{k}\left(\frac{\bar{F}(t_1)}{\bar{F}(t_2)} - 1\right)^{j-k}}, \quad i = k, \ldots, n-1. \end{aligned} \tag{8}$$

For the proof, see Appendix E. From (8), the mean number of failed components up to time t_2 can be represented as

$$\begin{aligned} \varphi(t_1, t_2) &= \mathbb{E}(N_{t_2} \mid X_{k:n} \le t_1 < X_{k+1:n}, T > t_2) \\ &= \frac{n \sum_{i=k}^{n-1} \bar{S}_i \binom{n-1}{i-1}\binom{i}{k}\left(\frac{\bar{F}(t_1)}{\bar{F}(t_2)} - 1\right)^{i}}{\sum_{j=k}^{n-1} \bar{S}_j \binom{n}{j}\binom{j}{k}\left(\frac{\bar{F}(t_1)}{\bar{F}(t_2)} - 1\right)^{j}}. \end{aligned} \tag{9}$$

The next theorem reveals some stochastic properties of $(N_{t_2} \mid X_{k:n} \le t_1 < X_{k+1:n}, T > t_2)$ in terms of k, t_1, and t_2.

Theorem 2.4. *Assume that the common baseline CDF F is absolutely continuous. Then*

(a) *for $0 \le k \le n-2$,*

$$(N_{t_2} \mid X_{k:n} \le t_1 < X_{k+1:n}, T > t_2) \le_{\rm lr} (N_{t_2} \mid X_{k+1:n} \le t_1 < X_{k+2:n}, T > t_2);$$

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(b) *for each* $0 < t_1 \le t_1^* < t_2 \le t_2^*$,

$$(N_{t_2} \mid X_{k:n} \le t_1^* < X_{k+1:n}, T > t_2) \le_{\text{lr}} (N_{t_2} \mid X_{k:n} \le t_1 < X_{k+1:n}, T > t_2^*).$$

Proof. See Appendix F. □

As the likelihood ratio order is a subclass of the usual stochastic order, we get the following corollary from Theorem 2.4.

Corollary 2.2. $\mathbb{E}(N_{t_2} \mid X_{k:n} \le t_1 < X_{k+1:n}, T > t_2)$ *is a decreasing function of* t_1 *and an increasing function of* k *and* t_2.

The next theorem provides a comparison between the failed components in two coherent systems in terms of likelihood ratio order.

Theorem 2.5. *Let* $\mathcal{S}_1$ *and* $\mathcal{S}_2$ *denote two coherent systems with i.i.d. component lifetimes* $X_1, X_2, \ldots, X_n$ *and* $Y_1, Y_2, \ldots, Y_n$ *whose CDFs are F and G, respectively. Let* N_t *and* N_t^* *be the number of failed components of* $\mathcal{S}_1$ *and* $\mathcal{S}_2$*, respectively, on* $[0, t]$*. Further, suppose the lifetime of the system* $\mathcal{S}_1$ *(*$\mathcal{S}_2$*) is* T_1 *(*T_2*) with signature vector* $\boldsymbol{s}^{(1)}$ *(*$\boldsymbol{s}^{(2)}$*). If* $X_1 \le_{\text{hr}} Y_1$ *and* $\boldsymbol{s}^{(2)} \le_{\text{hr}} \boldsymbol{s}^{(1)}$*, then*

$$(N_{t_2} \mid X_{k:n} \le t_1 < X_{k+1:n}, T_1 > t_2) \ge_{\text{lr}} (N_{t_2}^* \mid Y_{k:n} \le t_1 < Y_{k+1:n}, T_2 > t_2), \quad 0 \le t_1 < t_2.$$

Proof. See Appendix G. □

3. Optimal corrective and preventive maintenance models

In this section, we develop two maintenance strategies for n-component coherent systems based on the conditional probabilities introduced in the previous section. The following notation is used in our strategies:

- c_{cm}: cost of CM for each component;
- c_{pm}: cost of PM for each component;
- c_{cms}: cost of CM for the whole system;
- c_{pms}: cost of PM for the whole system;
- c_{pms}^*: cost of rigid PM for the whole system;
- c_{min}: cost of minimal repair for each component;
- w_1: time to perform CM together with PM;
- w_2: time to perform CM on the system;
- w_3: time to perform PM on the system;
- w_4: time to perform rigid PM on the system.

Strategy I

Assume that a coherent system begins to operate at time 0. A minimal repair has been performed on each component of the system that fails in the interval $(0, \tau)$. Thus, we can

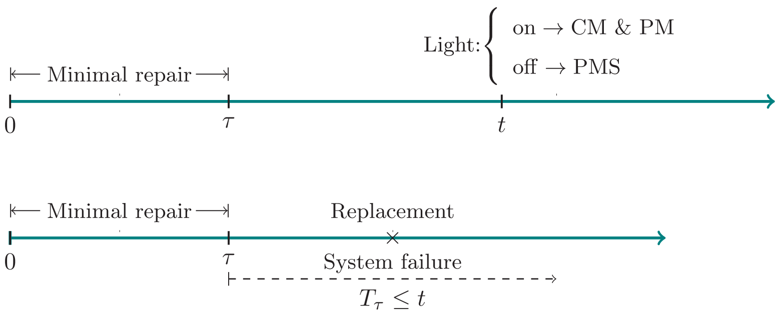

FIGURE 3: Maintenance Strategy **I**.

assume that the system, consisting of n unfailed components with age τ, is alive at τ. Here, τ is a predetermined constant which may be considered, for example, as a guarantee time of the system. We assume that, after τ, a warning lamp is installed on the system that turns on at the time of the kth component failure, where k is predetermined (for a realistic example where warning lamps are employed in a system, we refer to Shimizu and Kawai [35], where the authors consider a warning lamp in a vehicle's electronic power steering system (EPS) which will operate in case of failure of components of the EPS; see also Khaledi and Shaked [16]).

The operator decides to perform CM on the whole system at a cost of c_{cms} once the system fails in the interval (τ, t), or he/she decides to perform a PM action when the total operating time reaches t if the lamp turns on, whichever occurs first. More precisely, the operator decides to perform PM on all operating components together with CM on the failed ones at a cost of c_{pm} for PM and a cost of c_{cm} for CM, when the total operating time reaches t, provided that the warning light has lit up before or at time t. On the other hand, if the system is alive at t and the lamp does not turn on, he/she performs a perfect PM on the entire system with a cost of c_{pms}. It should be mentioned here that, in the above situation, the operator knows only whether the light is on or off. In other words, the warning light might have been turned on at a time before t; the operator does not know the exact time of the kth failure in the interval (τ, t). Two different cases of renewal cycle for Strategy **I** are shown in Figure 3. The upper axis in Figure 3 shows the case where system failure has not occurred up to time t; that is, the system age reaches t. The lower axis depicts the case where the system fails before the age reaches t.

It is evident that in the above policy, the inspection of the system is done in the interval $(0, \tau)$, where the components of the system have been monitored continuously, and at the single time instant t (see Pham and Wang [27]). As mentioned in [27], a justification for the first part is that in the interval $(0, \tau)$ the components are 'young' and hence a minor repair is adequate. Thus, before τ, only minimal repairs, which require little time and have low cost, are carried out. After the system reaches age τ, as the cost of continuous monitoring may be substantial and minimal repair may not be reasonable because of the increased failure rate of the components, the operator does not monitor the system; instead, an alarm is installed which operates at the time of the kth failure. Using this information we obtain the optimal time of PM under the following settings on the cost function.

To evaluate the cost in interval $(0, \tau)$, we shall utilize a cost function used by Sheu [34]; see also Pham and Wang [27]. For each component, let the cost of the ith minimal repair depend on the deterministic part $a_1(t, i)$ and the age-dependent random part $a_2(t)$. Note that $a_1(t, i)$

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depends on the number i of minimal repairs performed on the component and the age t of that component. Two parts are now linked by a positive function h. In fact, the required cost of the ith minimal repair at age t for each component is $h(a_1(t, i), a_2(t))$, where h is a continuous nondecreasing function of t, and is a nondecreasing function of i. Thus, the expected cost of minimal repairs for the whole system in a renewal period is

$$c^*_{min} = n\mathbb{E}\left[\sum_{i=1}^{N(\tau)} h(a_1(S_i, i), a_2(S_i))\right],$$

where $N(\tau)$ denotes the total number of minimal repairs during the time interval $(0, \tau)$, and $S_1, S_2, \ldots$ are the successive failure times of each component at which minimal repairs have been performed. It is known that for the minimal repair the failure times follow a nonhomogeneous Poisson process with rate $r(t)$ (see Barlow and Hunter [6] or Gertsbakh [14]). Note that $r(t)$ is the failure rate of a component lifetime. It has been proved by Sheu [34] that

$$c^*_{min} = n\int_0^{\tau} \nu(y)r(y)dy, \tag{10}$$

where $\nu(y) = \mathbb{E}_{N(y)}\mathbb{E}_{a_2(y)}[h(a_1(y, N(y)+1), a_2(y))]$; see also Pham and Wang [27]. A special case considered in the literature is that in which $h(a_1(t, i), a_2(t))$ is assumed to be a constant c_{min} (see Barlow and Hunter [6] and Tahara and Nishida [37]). We assume that this cost includes the cost of monitoring and the cost of repair. Thus c^*_{min} reduces to

$$c^*_{\min} = nc_{\min}H(\tau),$$

where $H(\tau) = \int_0^{\tau} r(t)dt$ is known as the mean value function of the failure process.

In this strategy, we consider t as a decision variable and τ as a predetermined time instant. By using the renewal reward theorem (see, e.g., Ross [30], p. 52), the average cost of system maintenance per unit time is then defined as the ratio of the average cost of the system maintenance per renewal cycle to the expected duration of a renewal cycle. In other words,

$$\eta_I(t) = \frac{n\int_0^{\tau} \nu(y)r(y)dy + F_\tau(t-\tau)c_{cms} + c_{pms}P_{1,k}(\tau, t)}{\tau + \mathbb{E}(\min(t-\tau, T_\tau))} + \frac{P_{2,k}(\tau, t)\left[(c_{cm} - c_{pm})\mathbb{E}(N_{t,\tau} \mid (X_\tau)_{k:n} \le t < T_\tau) + nc_{pm}\right]}{\tau + \mathbb{E}(\min(t-\tau, T_\tau))}, \tag{11}$$

where $F_\tau(\cdot)$ denotes the CDF of the lifetime of an n-component system where each component's age is τ, $\mathbb{E}(N_{t,\tau} \mid (X_\tau)_{k:n} \le t < T_\tau)$ is the expectation of the number of failed components of the live system at time t with at least k failed components when all components are functioning at time τ (for $\tau < t$),

$$P_{1,k}(\tau, t) = \mathbb{P}(X_{k:n} > t, T > t \mid X_{1:n} > \tau),$$

and

$$P_{2,k}(\tau, t) = \mathbb{P}(X_{k:n} \le t < T \mid X_{1:n} > \tau).$$

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Note that $(X_\tau)_{k:n}$ is the kth order statistic from the CDF $F(t|\tau) = 1 - \bar{F}(t)/\bar{F}(\tau)$, $t > \tau$. We can easily show that

$$F_\tau(t-\tau) = 1 - \sum_{j=0}^{n-1} \bar{S}_j \binom{n}{j} \left(1 - \frac{\bar{F}(t)}{\bar{F}(\tau)}\right)^j \left(\frac{\bar{F}(t)}{\bar{F}(\tau)}\right)^{n-j}, \tag{12}$$

$$P_{1,k}(\tau, t) = \sum_{i=0}^{k-1} \bar{S}_i \binom{n}{i} \left(1 - \frac{\bar{F}(t)}{\bar{F}(\tau)}\right)^i \left(\frac{\bar{F}(t)}{\bar{F}(\tau)}\right)^{n-i},$$

and

$$P_{2,k}(\tau, t) = \sum_{i=k}^{n-1} \bar{S}_i \binom{n}{i} \left(1 - \frac{\bar{F}(t)}{\bar{F}(\tau)}\right)^i \left(\frac{\bar{F}(t)}{\bar{F}(\tau)}\right)^{n-i}.$$

Also, from Equation (5), we can obtain

$$\mathbb{E}(N_t \mid X_{k:n} \le t < T) = \frac{n \sum_{i=k}^{n-1} \bar{S}_i \binom{n-1}{i-1} \phi^i(t)}{\sum_{j=k}^{n-1} \bar{S}_j \binom{n}{j} \phi^j(t)}.$$

By replacing $\phi(t)$ with $\left(\frac{\bar{F}(\tau)}{\bar{F}(t)} - 1\right)$, we may obtain the corresponding formula for $\mathbb{E}(N_{t,\tau} \mid (X_\tau)_{k:n} \le t < T_\tau)$. Also, it can easily be shown that

$$\begin{aligned}\mathbb{E}(\min(t-\tau, T_\tau)) &= \int_0^{t-\tau} [1 - F_\tau(x)]dx \\ &= \frac{1}{\bar{F}^n(\tau)} \sum_{j=0}^{n-1} \bar{S}_j \binom{n}{j} \int_0^{t-\tau} (\bar{F}(\tau) - \bar{F}(x+\tau))^j (\bar{F}(x+\tau))^{n-j} dx.\end{aligned}$$

In the following proposition, in the case that the decision variable in $\eta_I(t)$ is t, we verify the existence of the optimal value t^* minimizing $\eta_I(t)$.

Proposition 3.1. *Let $r(t)$ be the failure rate of components and $\eta_I(t)$ be as given in Equation (11). If $\lim_{t\to\infty} r(t)$ is finite and*

$$\lim_{t\to\infty} r(t) > \frac{c_{cms} + n\int_0^\tau \nu(y) r(y) dy}{(n - i^* + 1)(c_{cms} - nc_{pm} - (i^* - 1)(c_{cm} - c_{pm}))(\tau + \mu_\tau)}, \tag{13}$$

then there exists a finite t^* which satisfies $\frac{d}{dt}\eta_I(t)\mid_{t=t^*} = 0$ and minimizes $\eta_I(t)$, where μ_τ is the expectation of the lifetime of an n-component system whose components have age τ, and i^* is defined in Remark 2.2.

Proof. See Appendix H. □

Remark 3.1. Although, in Strategy **I**, we have considered t as a decision variable and τ as a predetermined time instant, one can instead consider τ as a decision variable (or even consider both t and τ as decision variables). In this case, the cost function (11) has to be minimized with respect to τ (or with respect to both τ and t simultaneously). In Example 3.1, we have numerically and graphically illustrated Strategy **I** for the bridge system in the cases that either t or both τ and t are considered as the decision variables.

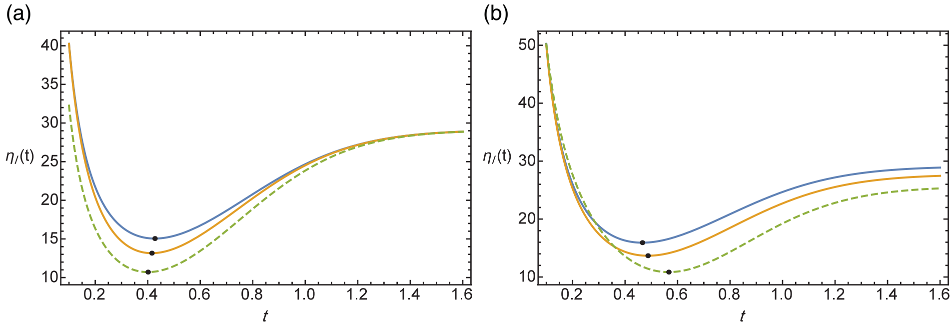

FIGURE 4: The expected cost of the system maintenance per unit time in Example 3.1: (a) $\tau = 0.1$, $k = 1, 2, 3$ from top to bottom; (b) $k = 0$, $\tau = 0.1, 0.3, 0.5$ from top to bottom.

It should be pointed out that, in the policy described above, we assume that all maintenance actions take negligible times. Now, suppose that minimal repair also takes negligible time, but PM combined with CM takes w_1 time units, CM on the whole system at time t takes w_2 time units, and PM on the whole system at time t takes w_3 time units. In the literature, a well-known criterion in the maintenance of systems is stationary availability. The stationary availability is defined as the ratio of the average time that the system is in a functioning state to the average length of a cycle. The stationary availability for Strategy **I** is then given by

$$A_I(t) = \frac{\tau + \mathbb{E}(\min(t-\tau, T_\tau))}{\tau + \mathbb{E}(\min(t-\tau, T_\tau)) + w_1 P_{2,k}(\tau, t) + w_2 F_\tau(t-\tau) + w_3 P_{1,k}(\tau, t)}.$$

Remark 3.2. We should mention here that the maintenance type depends not only on the action that is performed (replacement or repair of the failed component, a major overhaul of the system, and so on) but also on the complexity of the system structure. For example, the replacement of a failed component of a complex system does not generally improve the system's performance, and hence can be considered a minimal repair. By contrast, if the system is not complex, then the same replacement may produce a noticeable improvement and therefore cannot be considered a minimal repair; see Pulcini [28].

Now let us see the following example.

Example 3.1. Assume that the bridge system in Figure 1 has component lifetimes which are independent Weibull random variables with CDF $F(t) = 1 - \exp\{-t^2\}$, $t \geq 0$. It is known that the system signature is (0, 0.2, 0.6, 0.2, 0) (see, e.g., Samaniego [32]). Let $c_{min} = 0.5$, $c_{cms} = 25$, $c_{pms} = 4$, $c_{cm} = 2$, and $c_{pm} = 1$. In Figure 4, the graphs of $\eta_I(t)$ are presented for different values of k and τ.

In order to investigate the robustness of Strategy **I** with respect to the model parameters c_{pms}, c_{cms}, c_{min}, c_{cm}, and c_{pm}, we have provided some numerical results in Tables 1 and 2. As the tables show, when c_{cms} increases, t^* decreases and the operator should perform preventive action sooner. On the other hand, when c_{pms} gets larger, t^* increases too; i.e., the larger the cost of system PM, the later the time of performing system PM. When c_{min} gets larger, the optimal time of PM gets larger too, and when c_{cm} increases, as expected, t^* decreases. Hence, according to this example, it is evident that the model is robust in terms of c_{cms}, c_{pms}, c_{min}, and c_{cm}. In the same manner one can see that the model is robust in terms of the other parameters c_{pm} and τ.

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TABLE 1: Optimal maintenance time for Strategy **I**.

$\tau = 0.1, k = 2, c_{min} = 0.5, c_{cm} = 2, c_{pm} = 1$					
$c_{pms} = 4$			$c_{cms} = 25$		
c_{cms}	t^*	$\eta_I(t^*)$	c_{pms}	t^*	$\eta_I(t^*)$
10	0.5910	10.4489	3	0.3714	10.9460
15	0.4883	11.6615	4	0.4153	13.1557
20	0.4436	12.4957	5	0.4566	15.0298
25	0.4153	13.1557	6	0.4962	16.6257
30	0.3947	13.7096	7	0.5341	17.9800
35	0.3787	14.1903	8	0.5703	19.1462

TABLE 2: Optimal maintenance time for Strategy **I**.

$\tau = 0.1, k = 2, c_{cms} = 25, c_{cm} = 2, c_{pms} = 4$					
$c_{cm} = 2$			$c_{min} = 0.5$		
c_{min}	t^*	$\eta_I(t^*)$	c_{cm}	t^*	$\eta_I(t^*)$
0.1	0.4147	13.1071	1	0.4310	12.5609
0.3	0.4150	13.1314	1.5	0.4231	12.8639
0.5	0.4153	13.1557	2	0.4153	13.1557
0.75	0.4157	13.1861	2.5	0.4077	13.4367
1	0.4159	13.2043	3	0.4005	13.7071

TABLE 3: Optimal maintenance times for Strategy **I** with $k = 2$, $c_{min} = 0.5$, $c_{cms} = 25$, $c_{pms} = 4$, $c_{cm} = 2$, and $c_{pm} = 1$.

τ	t^*	$\eta_I(t^*)$
1.00	1.1263	6.45966
1.05	1.06135	6.40032
1.10	1.11026	6.35702
1.20	1.20856	6.31107
1.25	**1.25787**	**6.30534**
1.30	1.30728	6.30943
1.35	1.35676	6.32228
1.40	2.40631	6.34300

The optimal times t^* that minimize the average cost per unit of time and $\eta_I(t^*)$ are presented in Table 3 for several time instants τ. Note that if we consider τ and t as two decision variables, then the optimal value for the pair (τ, t) is (1.25, 1.25787), which results in the minimum maintenance cost 6.30534. The optimal values of (τ, t) are also tabulated in Table 4 for different values of c_{cms} and c_{pms}.

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TABLE 4: Bivariate optimal maintenance times for Strategy **I**.

$k = 2$, $c_{min} = 0.5$, $c_{cm} = 2$, and $c_{pm} = 1$							
$c_{pms} = 4$				$c_{cms} = 25$			
c_{cms}	τ^*	t^*	$\eta_I(\tau^*, t^*)$	c_{pms}	τ^*	t^*	$\eta_I(\tau^*, t^*)$
10	1.241	1.257	6.287	3	1.084	1.092	5.458
15	1.247	1.259	6.296	4	1.253	1.261	6.305
20	1.251	1.260	6.302	5	1.402	1.410	7.051
25	1.253	1.261	6.305	6	1.536	1.545	7.724
30	1.255	1.262	6.308	7	1.658	1.628	8.341
35	1.256	1.262	6.310	8	1.770	1.782	8.912

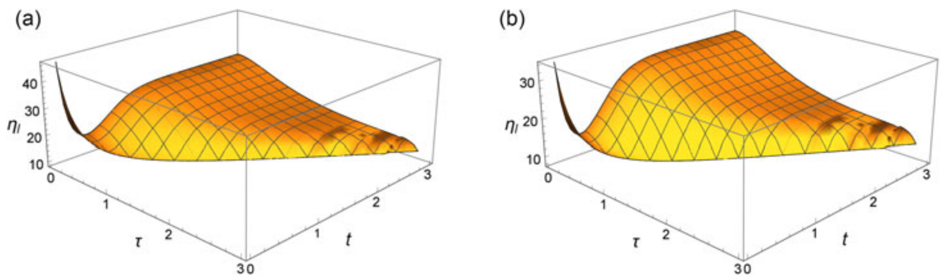

FIGURE 5: The three-dimensional plot of cost function in Example 3.1: (a) $k = 0$, (b) $k = 2$.

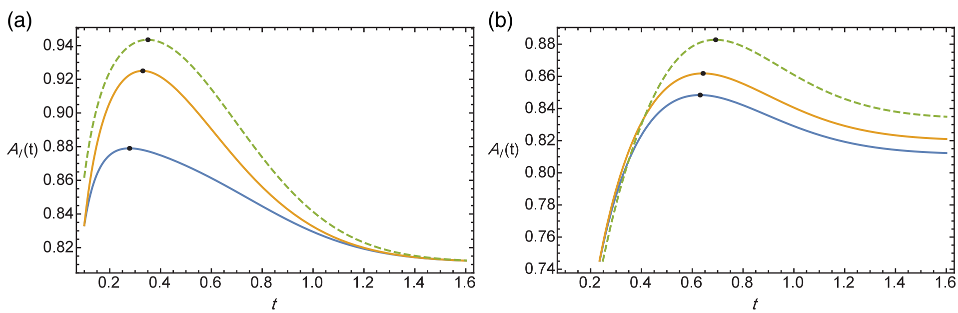

FIGURE 6: The stationary availability in Example 3.1: (a) $\tau = 0.1$, $k = 1, 2, 3$ from top to bottom; (b) $k = 0$, $\tau = 0.1, 0.3, 0.5$ from top to bottom.

Figure 5 depicts the three-dimensional plots of the cost function in terms of (τ, t) for $k = 0, 2$, $c_{min} = 0.5$, $c_{cms} = 25$, $c_{pms} = 4$, $c_{cm} = 2$, and $c_{pm} = 1$.

Figure 6 depicts the plots of $A_I(t)$ for $w_1 = 0.08$, $w_2 = 0.2$, and $w_3 = 0.02$, and for several values of k and τ. As the plots show, the system availability first increases to attain its maximum and then decreases.

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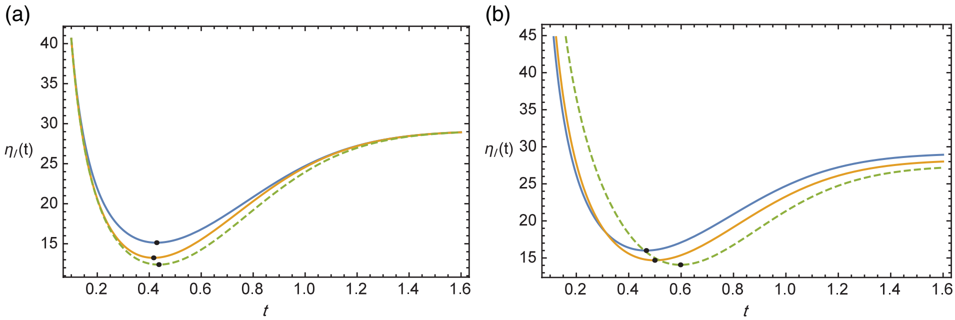

FIGURE 7: The expected cost of the system maintenance per unit time in Example 3.1 with $h(a_1(t, i), a_2(t)) = 3t + a_2(t)$ where $a_2(t)$ follows the normal distribution with mean 1: (a) $\tau = 0.1$, $k = 1, 2, 3$ from top to bottom; (b) $k = 0$, $\tau = 0.1, 0.3, 0.5$ from top to bottom.

Now, assume that $h(a_1(t, i), a_2(t)) = a_1(t) + a_2(t)$, where $a_1(t) = 3t$ and $a_2(t)$ follows the normal distribution with mean 1. Then, by (10), $c^*_{min} = n(2\tau^3 + \tau^2)$. The expected maintenance cost $\eta_I(t)$ is presented in Figure 7 for several values of k and τ.

The special case of this policy in which there is no minimal repair, i.e. $\tau = 0$, may be of interest. In this particular case, the average cost of the system maintenance per unit of time can be reduced to

$$\eta_I(t) = \frac{c_{cms}F_T(t) + c_{pms}\mathbb{P}(T > t, X_{k:n} > t)}{\mathbb{E}(\min(t, T))} + \frac{\mathbb{P}(X_{k:n} \le t < T)[(c_{cm} - c_{pm})\mathbb{E}(N_t \mid X_{k:n} \le t < T) + nc_{pm}]}{\mathbb{E}(\min(t, T))},$$

where

$$\mathbb{P}(X_{k:n} \le t < T) = \sum_{i=k}^{n-1} \bar{S}_i \binom{n}{i} F^i(t)\bar{F}^{n-i}(t),$$

$$\mathbb{P}(T > t, X_{k:n} > t) = \sum_{i=0}^{k-1} \bar{S}_i \binom{n}{i} F^i(t)\bar{F}^{n-i}(t),$$

and $\mathbb{E}(N_t \mid X_{k:n} \le t < T)$ is defined in (5).

Also, for this particular case where $\tau = 0$, the stationary availability may be written as

$$A_I(t) = \frac{\mathbb{E}(\min(t, T))}{\mathbb{E}(\min(t, T)) + w_1\mathbb{P}(X_{k:n} \le t < T) + w_2F_T(t) + w_3\mathbb{P}(T > t, X_{k:n} > t)}.$$

Now we consider the following example.

Example 3.2. Consider again the bridge system in Example 2.1 whose component lifetimes are independent Weibull random variables with CDF $F(t) = 1 - \exp\{-t^2\}$, $t \ge 0$. Figure 8(a) depicts the plots of $\eta_I(t)$ for $c_{cms} = 20$, $c_{pms} = 4$, $c_{cm} = 2$, and $c_{pm} = 1$ and the values $k = 1, 2, 3$. Figure 8(b) depicts the plots of $A_I(t)$ for $k = 2$, $w_1 = 0.08$, $w_2 = 0.2$, and $w_3 = 0.02$. As the plot shows, the system availability first increases to attain its maximum and then decreases. The availability attains its maximum at $t^* = 0.32645$ and $A_I(t^*) = 0.921662$.

An analysis of the results of Table 5 indicates that the maintenance strategy **I** with $\tau = 0$ is robust. It is seen that when c_{cms} increases, the optimal time of PM decreases, as expected.

TABLE 5: Optimal maintenance time for Strategy **I** with $\tau = 0$.

$k=2,\ c_{min}=0.5,\ c_{cm}=2,\ c_{pm}=1$					
$c_{pms}=4$			$c_{cms}=20$		
c_{cms}	t^*	$\eta_I(t^*)$	c_{pms}	t^*	$\eta_I(t^*)$
10	0.5941	10.5716	3	0.3894	10.7757
15	0.4873	11.8695	4	0.4413	12.7725
20	0.4414	12.7725	5	0.4914	14.4032
25	0.4123	13.4943	6	0.5398	15.7332
30	0.3913	14.1054	7	0.5863	16.8148
35	0.3749	14.6403	8	0.6302	17.6933

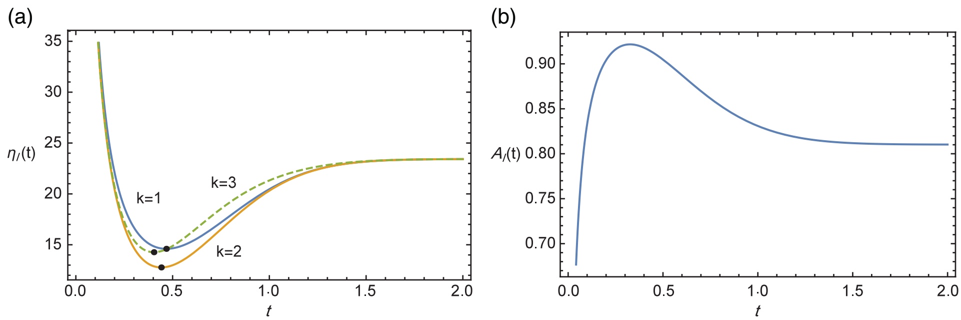

FIGURE 8: (a) The average maintenance cost per unit time in Example 3.2; (b) the stationary availability in Example 3.2.

Also, when c_{pms} gets larger, the optimal time of PM gets larger, too. One can easily see that, based on this example, the model is robust in terms of the costs c_{pm} and c_{cm}.

We should mention here that the only difference between the maintenance schedules given in this example and those of Example 3.1 is that, in Example 3.1, the operator performs minimal repair in a time interval at the beginning of the system operation, while in the present example there is no minimal repair. By considering the same initial values for these two maintenance policies, one can make a comparison based on the expected cost or the availability criterion. For example, let $c_{min} = 0.5$, $c_{cms} = 25$, $c_{pms} = 4$, $c_{cm} = 2$, $c_{pm} = 1$, $w_1 = 0.08$, $w_2 = 0.2$, and $w_3 = 0.02$. If $k = 1$ and $\tau = 0.1$, the optimal values of η_I and A_I are 15.0488 and 0.8790, respectively, whereas if we do not perform minimal repair, the optimal values of η_I and A_I are 15.4200 and 0.8727, respectively. Therefore, based on these observations, the operator prefers to perform minimal repair at the starting point of the maintenance. One can show that the situation is reversed if $k = 3$, in which case no minimal repair is preferred.

Strategy II

Assume that a new coherent system begins to operate at time 0. Suppose that the system has been inspected at two times t_1 and t_2, with $t_1 < t_2$. If the system fails before t_1, then the operator performs CM on the entire system with a cost of c_{cms} at the time of the system failure. He/she

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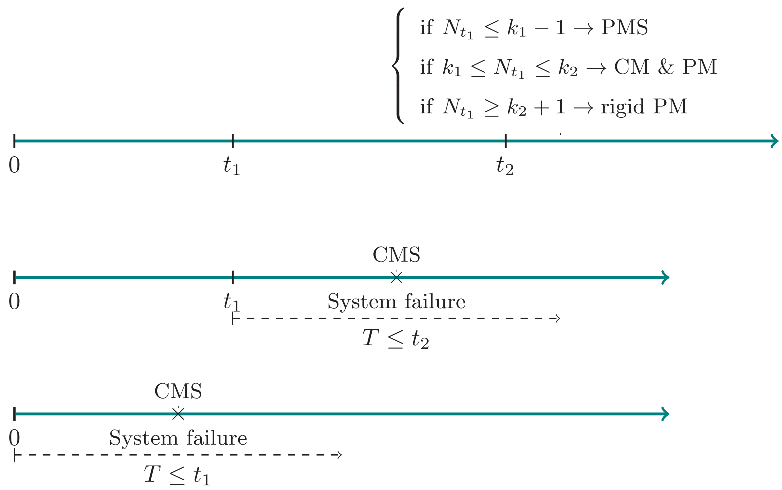

FIGURE 9: Maintenance Strategy **II**.

performs the same action if the system fails during the time interval (t_1, t_2). On the other hand, if the system is functioning at t_2, the operator decides among three different actions:

(a) If the number of components that have failed by t_1, namely N_{t_1}, is at most $(k_1 - 1)$, the operator performs PM on the whole system with a cost of c_{pms}.

(b) If $k_1 \leq N_{t_1} \leq k_2$, then he/she decides to perform PM on all operating components of the system together with CM on all failed ones, at a cost of c_{pm} for PM and a cost of c_{cm} for CM.

(c) If N_{t_1} is at least $(k_2 + 1)$, then the operator decides to perform a more rigid PM on the system (than in Case (a)) at a cost of c^*_{pms}.

In this strategy, we assume that t_2 is the decision variable, while t_1, k_1, and k_2 are fixed constants. Three different cases of renewal cycle for Strategy **II** are shown in Figure 9. The upper axis in Figure 9 depicts the case in which system failure has not occurred up to time t_2; that is, the age of the system reaches t_2. The middle axis shows the case in which the system is alive at the inspection time t_1 and fails before attaining the age t_2. The lower axis depicts the case where the system fails before its age reaches the time of inspection t_1.

The average cost of the system maintenance per unit of time is

$$\eta_{II}(t_2) = \frac{D(t_2)}{\mathbb{E}(\min(t_2, T))}, \tag{14}$$

where

$$\begin{aligned} D(t_2) = {} & c_{cms}\mathbb{P}(T \leq t_2) + c_{pms}\mathbb{P}(T > t_2, N_{t_1} \leq k_1 - 1) \\ & + [(c_{cm} - c_{pm})\mathbb{E}(N_{t_2} \mid k_1 \leq N_{t_1} \leq k_2, T > t_2) + nc_{pm}] \\ & \times \mathbb{P}(T > t_2, k_1 \leq N_{t_1} \leq k_2) + c^*_{pms}\mathbb{P}(T > t_2, N_{t_1} \geq k_2 + 1), \end{aligned}$$

and

$$\mathbb{E}(\min(t_2, T)) = \int_0^{t_2} \sum_{j=0}^{n-1} \bar{S}_j \binom{n}{j} F^j(t)\bar{F}^{n-j}(t)dt.$$

In the special case where $k_1 = k_2 = k$, $D(t_2)$ may be reduced to

$$D(t_2) = c_{cms}\mathbb{P}(T \le t_2) + c_{pms}\mathbb{P}(T > t_2, N_{t_1} \le k-1) + [(c_{cm} - c_{pm})\varphi(t_1, t_2) + nc_{pm}]\mathbb{P}(T > t_2, N_{t_1} = k) + c^*_{pms}\mathbb{P}(T > t_2, N_{t_1} \ge k+1),$$

where, from (9),

$$\varphi(t_1, t_2) = \frac{n\sum_{i=k}^{n-1}\bar{S}_i\binom{n-1}{i-1}\binom{i}{k}\left(\frac{\bar{F}(t_1)}{\bar{F}(t_2)} - 1\right)^i}{\sum_{j=k}^{n-1}\bar{S}_j\binom{n}{j}\binom{j}{k}\left(\frac{\bar{F}(t_1)}{\bar{F}(t_2)} - 1\right)^j}.$$

Also,

$$\mathbb{P}(T > t_2, N_{t_1} \le k-1) = \sum_{i=1}^{k} s_i \sum_{j=0}^{i-1}\binom{n}{j}F^j(t_2)\bar{F}^{n-j}(t_2) + \sum_{i=k+1}^{n} s_i \sum_{m=n-i+1}^{n}\sum_{l=\max(m,n-k+1)}^{n}\binom{n}{l}\binom{l}{m}F^{n-l}(t_1)(\bar{F}(t_1) - \bar{F}(t_2))^{l-m}\bar{F}^m(t_2)$$

and

$$\mathbb{P}(T > t_2, N_{t_1} \ge k+1) = \sum_{i=k+2}^{n} s_i \sum_{j=k+1}^{i-1}\sum_{m=n-i+1}^{n-k-1}\binom{n}{m}\binom{n-m}{j}F^j(t_1) \times (\bar{F}(t_1) - \bar{F}(t_2))^{n-j-m}\bar{F}^m(t_2).$$

On the other hand,

$$\mathbb{P}(T > t_2, N_{t_1} = k) = \sum_{i=k+1}^{n} s_i \sum_{j=k}^{i-1}\binom{n}{j}\binom{j}{k}F^k(t_1)\bar{F}^{n-j}(t_2)(\bar{F}(t_1) - \bar{F}(t_2))^{j-k} = \sum_{j=k}^{n-1}\bar{S}_j\binom{n}{j}\binom{j}{k}F^k(t_1)\bar{F}^{n-j}(t_2)(\bar{F}(t_1) - \bar{F}(t_2))^{j-k}.$$

Also, we may obtain

$$\mathbb{P}(T < t_2) = 1 - \sum_{j=0}^{n-1}\bar{S}_j\binom{n}{j}F^j(t_2)\bar{F}^{n-j}(t_2).$$

The aim here is to minimize $\eta_{II}(t_2)$ with respect to the decision variable t_2; that is, we should find the possible value t_2, if it exists, such that

$$\eta_{II}(t_2^*) = \min_{t_2 > t_1}\eta_{II}(t_2).$$

Now, let us assume that PM combined with CM takes w_1 time units, CM on the whole system takes w_2 time units, PM on the whole system takes w_3 time units, and the rigid PM on the system takes w_4 time units. The stationary availability for Strategy **II** is given by

$$A_{II}(t_2) = \frac{\mathbb{E}(\min(t_2, T))}{\mathbb{E}(\min(t_2, T)) + B(t_2)},$$

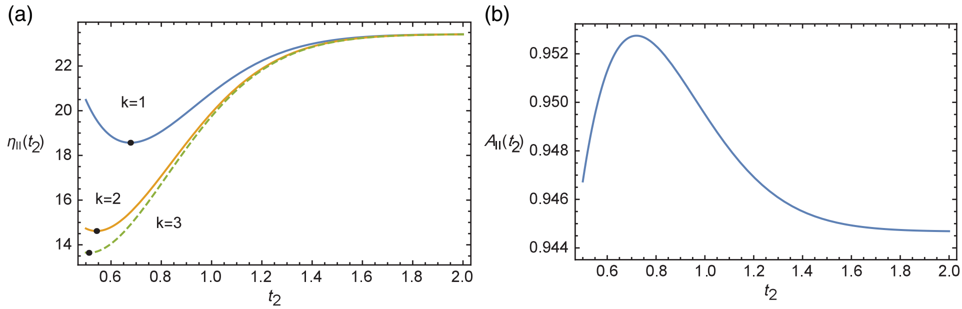

FIGURE 10: (a) The average maintenance cost per unit time in Example 3.3; (b) the stationary availability for Strategy **II** in Example 3.3.

where

$$B(t_2) = w_1\mathbb{P}(T > t_2, k_1 \le N_{t_1} \le k_2) + w_2\mathbb{P}(T \le t_2) \\ + w_3\mathbb{P}(T > t_2, N_{t_1} \le k_1 - 1) + w_4\mathbb{P}(T > t_2, N_{t_1} \ge k_2 + 1).$$

Remark 3.3. In Strategy **II**, we assumed that t_2 is the only decision variable and t_1 is a fixed constant. However, in this strategy, one can instead assume that t_1 is the decision variable (or even that both t_1 and t_2 are decision variables) and minimize the cost function (14) based on that. The other point we should mention here is that of the conditions under which there exists an optimum value which minimizes the cost function (14). Since the functional form of $\eta_{II}(t_2)$ is rather complicated in the general setting, we can verify the existence of a possible optimum value for a given lifetime distribution of the components numerically (or graphically) using mathematical software. In the next example, we illustrate this in more detail.

Example 3.3. Let us look at again the bridge system in Example 2.1, where the component lifetimes are i.i.d. with Weibull distribution, with reliability function $\bar{F}(t) = \exp\{-t^2\}$, $t \ge 0$. Figure 10(a) shows the plot of $\eta_{II}(t_2)$ for $t_1 = 0.5$, $c^*_{pms} = 20$, $c_{cms} = 20$, $c_{pms} = 5$, $c_{cm} = 2$, and $c_{pm} = 1$, and different values $k = 1, 2, 3$. Figure 10(b) depicts the plots of $A_{II}(t)$ for $k = 2$, $t_1 = 0.5$, $w_1 = 0.04$, $w_2 = 0.05$, $w_3 = 0.02$, and $w_4 = 0.06$. It can be seen that the system availability first increases to arrives at its maximum and then decreases. The availability attains its maximum at $t^*_2 = 0.719917$ and $A_{II}(t^*_2) = 0.952746$.

An analysis of the results in Table 6 indicates that the maintenance strategy **II** is robust. It is seen that when c_{cms} increases (or c_{pm} decreases), the optimal time of PM decreases, as expected. This robustness is true also in terms of c^*_{pms}. That is, when c^*_{pms} gets larger, the optimal time of PM gets larger, too. One can easily see that, based on this example, the model is robust in terms of the costs c_{pms} and c_{cm}.

Considering t_1 and t_2 as two decision variables, the three-dimensional plot of the cost function is depicted in Figure 11. The optimal values of (t_1, t_2) are also tabulated in Table 7 for different values of c_{cms} and c^*_{pms}.

Recalling Example 3.2, it is interesting to compare Strategies **I** and **II** for the bridge system. To do so, let $\tau = 0.1$, $t_1 = 0.5$, $c_{min} = 0.5$, $c_{cms} = 25$, $c_{pms} = 4$, $c_{cm} = 2$, $c_{pm} = 1$, $c^*_{pms} = 20$, $w_1 = 0.08$, $w_2 = 0.2$, $w_3 = 0.02$, and $w_4 = 0.06$. If $k = 2$ in Strategy **I** and $k_1 = k_2 = 1$ in Strategy **II**, then the optimal values of η_I and η_{II} are 13.1557 and 20.6165, and the optimal

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TABLE 6: Optimal maintenance time for Strategy **II** with $t_1 = 0.5, k = 1$.

$c_{cm} = 2, c_{pms} = 5$								
$c_{pm} = 1, c^*_{pms} = 20$			$c_{cms} = 20, c^*_{pms} = 20$			$c_{pm} = 1, c_{cms} = 20$		
c_{cms}	t^*_2	$\eta_{II}(t^*_2)$	c_{pm}	t^*_2	$\eta_{II}(t^*_2)$	c^*_{pms}	t^*_2	$\eta_{II}(t^*_2)$
10	1.3000	11.6949	0.6	0.6465	17.8146	8	0.5477	15.4899
15	0.8010	15.8536	0.75	0.6581	18.1069	10	0.5725	16.1256
20	0.6774	18.5673	1.0	0.6774	18.5673	12	0.5957	16.7025
25	0.6101	20.6165	1.25	0.6966	18.9959	15	0.6281	17.4757
30	0.5656	22.2806	1.5	0.7159	19.3944	17	0.6483	17.9389
35	0.5330	23.6913	1.75	0.7352	19.7645	20	0.6774	18.5673
40	0.5077	24.4218	2.0	0.7545	20.1078	22	0.6959	18.9474

TABLE 7: Bivariate optimal maintenance times for Strategy **II**.

$k = 2, c_{pms} = 8, c_{cm} = 2$ and $c_{pm} = 1$							
$c^*_{pms} = 12$				$c_{cms} = 10$			
c_{cms}	t^*_1	t^*_2	$\eta_{II}(t^*_1, t^*_2)$	c^*_{pms}	t^*_1	t^*_2	$\eta_{II}(t^*_1, t^*_2)$
10	0.3017	1.4270	11.7054	9	0.7051	1.3849	11.7043
11	0.4780	1.0876	12.7653	10	0.4417	1.4205	11.7052
12	0.5712	0.9344	13.6611	11	0.3520	1.4530	11.7053
13	0.6303	0.8425	14.4204	12	0.3017	1.4270	11.7054
14	0.6718	0.7794	15.0765	13	0.2683	1.4277	11.7054

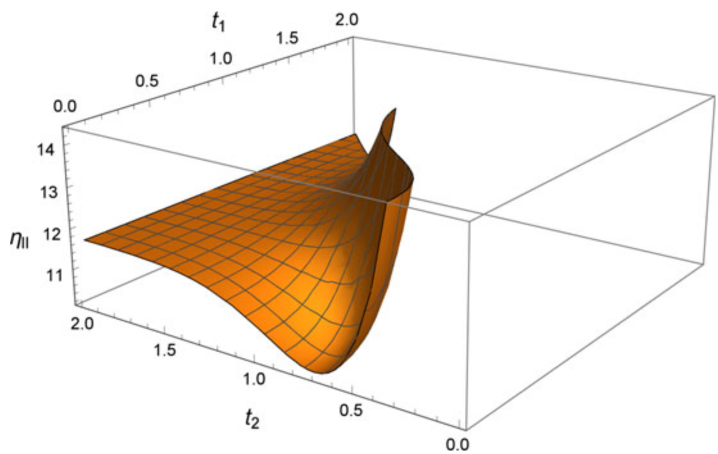

FIGURE 11: The three-dimensional plot of the cost function in Example 3.3.

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values of A_I and A_{II} are 0.9250 and 0.8728, respectively. This means that based on both the expected maintenance cost and the stationary availability criteria, Strategy **I** is preferred.

4. Conclusions

In reliability engineering, although an extensive number of research works have been devoted to optimal maintenance of one-unit systems, only a small portion of the literature has considered maintenance of multi-component systems. This paper aimed to propose some optimal strategies for the maintenance of a multi-component coherent system using some partial information on the number of destroyed components in the system. For this purpose, first, we proposed two criteria for evaluating the conditional probability function of the number of destroyed components in the system. In the computation of the proposed measures, we utilized some partial information on the status of the components of the system based on some inspection strategies. The derivations of both criteria rely on the notion of the signature associated with a coherent system. Using these criteria, we then introduced two different approaches to the optimal maintenance of the system. In the first approach, before a predetermined time τ, the system undergoes minimal repair, and after τ, the system is equipped with a warning light that turns on at the time of the kth component failure. Then the system is inspected at time t, where $t > \tau$, and the operator decides to perform CM on the entire system once the system fails or to perform a PM action when the total operating time reaches t if the warning light turns on, whichever occurs soonest. In the second approach, the system is inspected at two times t_1 and t_2, where $t_1 < t_2$, and depending on the information obtained at t_1, the operator performs different maintenance actions at t_2. In the proposed approaches, optimality criteria were defined based on the long-run expected costs of maintenance and availability of the system. The results of the paper were applied to the bridge system, for which several illustrative plots were presented. In this paper, we assumed that the component lifetimes were i.i.d. One may also consider other scenarios for component failure to propose optimal strategies for maintaining complex systems. The extension of the results of this paper to systems with dependent and/or non-identical components would be an interesting area for future research.

Appendix A. Proof of Equation (4)

The probability mass function of $(N_t \mid X_{k:n} \le t < T)$ can be computed as

$$\begin{aligned} p^t_{k,n}(i) &= \mathbb{P}(N_t = i \mid X_{k:n} \le t < T) \\ &= \frac{\sum_{m=i+1}^{n} \mathbb{P}(X_{i:n} \le t < X_{i+1:n}, X_{k:n} \le t < T, T = X_{m:n})}{\mathbb{P}(X_{k:n} \le t < T)} \\ &= \frac{\sum_{m=i+1}^{n} \mathbb{P}(T = X_{m:n})\mathbb{P}(X_{i:n} \le t < X_{i+1:n}, X_{k:n} \le t < X_{m:n})}{\mathbb{P}(X_{k:n} \le t < T)} \\ &= \frac{\bar{S}_i \binom{n}{i} F^i(t) \bar{F}^{n-i}(t)}{\mathbb{P}(X_{k:n} \le t < T)}, \qquad i = k, \ldots, n-1, \end{aligned} \tag{15}$$

where the second equality follows from the law of total probability; the third and fourth equalities follow from the facts that the order statistics are independent from their ranks (see

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Kochar *et al.* [17]), and that $[X_{i:n} \le t < X_{i+1:n}] \subseteq [X_{k:n} \le t < X_{m:n}]$ for $k \le i < m$, respectively; and $\bar{S}_i = \sum_{j=i+1}^{n} s_j$. On the other hand, using the law of total probability,

$$\begin{aligned}\mathbb{P}(X_{k:n} \le t < T) &= \sum_{m=k+1}^{n} \mathbb{P}(X_{k:n} \le t < T \mid T = X_{m:n})\mathbb{P}(T = X_{m:n}) \\ &= \sum_{m=k+1}^{n} s_m \sum_{j=k}^{m-1} \binom{n}{j} F^j(t)\bar{F}^{n-j}(t) \\ &= \sum_{j=k}^{n-1} \bar{S}_j \binom{n}{j} F^j(t)\bar{F}^{n-j}(t). \qquad (16)\end{aligned}$$

Therefore

$$p^t_{k,n}(i) = \frac{\bar{S}_i \binom{n}{i} \phi^i(t)}{\sum_{j=k}^{n-1} \bar{S}_j \binom{n}{j} \phi^j(t)}, \qquad i = k, \ldots, n-1.$$

Appendix B. Proof of Theorem 2.1

To prove Part (a), first note that

$$p^t_{k,n}(i) = \frac{\bar{S}_i \binom{n}{i} \phi^i(t)}{\sum_{j=k}^{n-1} \bar{S}_j \binom{n}{j} \phi^j(t)} I_{\{k,k+1,\ldots,n-1\}}(i),$$

where $I_{\{k,k+1,\ldots,n-1\}}$ denotes the indicator function on the set $\{k, k+1, \ldots, n-1\}$. It is easy to show that $I_{\{k,k+1,\ldots,n-1\}}(i)$ is TP_2 in $(i, k) \in \{k, \ldots, n-1\} \times \{0, \ldots, n-1\}$. This, in turn, implies that $p^t_{k,n}(i)$ is TP_2 in $(i, k) \in \{k, \ldots, n-1\} \times \{0, \ldots, n-1\}$.

Part (b) follows from the result that was stated right before the theorem and the fact that $\phi^i(t)$ and hence $p^t_{k,n}(i)$ are TP_2 in $(i, t) \in \{k, \ldots, n-1\} \times [0, \infty)$.

Appendix C. Proof of Theorem 2.2

The probability mass function of $(N_t \mid X_{k:n} \le t < T_1)$ is given in (4). Similarly, the probability mass function of $(N^*_t \mid Y_{k:n+1} \le t < T_2)$ can be expressed as

$$q^t_{k,n+1}(i) = \frac{\bar{S}^{(2)}_i \binom{n+1}{i} \phi^i(t)}{\sum_{j=k}^{n} \bar{S}^{(2)}_j \binom{n+1}{j} \phi^j(t)}, \quad i = k, \ldots, n,$$

where $\bar{S}^{(2)}_i = \sum_{j=i+1}^{n+1} p_j$. Therefore, the fraction $q^t_{k,n+1}(i)/p^t_{k,n}(i)$ is proportional to

$$\frac{\bar{S}^{(2)}_i}{\bar{S}^{(1)}_i} \frac{\binom{n+1}{i}}{\binom{n}{i}},$$

which is increasing in $i = 0, 1, \ldots, n$. This completes the proof.

Appendix D. Proof of Theorem 2.3

Denote by $p^{t,r}_{k,n}(i)$ the conditional probability in (4) for the system $\mathcal{S}_r$, $r = 1, 2$; that is

$$p^{t,r}_{k,n}(i) = \frac{\bar{S}^{(r)}_i \binom{n}{i} \phi^i_r(t)}{\sum_{j=k}^{n-1} \bar{S}^{(r)}_j \binom{n}{j} \phi^j_r(t)}, \qquad i = k, \ldots, n-1, \qquad (17)$$

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where $\phi_1(t) = F(t)/\bar{F}(t)$, $\phi_2(t) = G(t)/\bar{G}(t)$, and $\bar{S}_i^{(r)} = \sum_{j=i+1}^n s_j^{(r)}$, $r = 1, 2$. It follows from the assumptions of the theorem that both $\bar{S}_i^{(r)}$ and $\phi_r^i(t)$ are RR_2 in $(i, r) \in \{k, ..., n-1\} \times \{1, 2\}$. Since the product of two RR_2 functions is again an RR_2 function, we conclude that $p_{k,n}^{t,r}(i)$ is RR_2 in $(i, r) \in \{k, ..., n-1\} \times \{1, 2\}$. This completes the proof.

Appendix E. Proof of Equation (8)

The probability mass function of $(N_{t_2} \mid X_{k:n} \le t_1 < X_{k+1:n}, T > t_2)$ can be computed as

$$\begin{aligned} p_{k,n}^{t_1,t_2}(i) &= \mathbb{P}(N_{t_2} = i \mid X_{k:n} \le t_1 < X_{k+1:n}, T > t_2) \\ &= \frac{\sum_{m=i+1}^n \mathbb{P}(X_{i:n} \le t_2 < X_{i+1:n}, X_{k:n} \le t_1 < X_{k+1:n}, T > t_2, T = X_{m:n})}{\mathbb{P}(X_{k:n} \le t_1 < X_{k+1:n}, T > t_2)} \\ &= \frac{\sum_{m=i+1}^n \mathbb{P}(T = X_{m:n})\mathbb{P}(X_{i:n} \le t_2 < X_{i+1:n}, X_{k:n} \le t_1 < X_{k+1:n}, X_{m:n} > t_2)}{\mathbb{P}(X_{k:n} \le t_1 < X_{k+1:n}, T > t_2)} \\ &= \frac{\sum_{m=i+1}^n s_m \binom{n}{i}\binom{i}{k} F^k(t_1)(\bar{F}(t_1) - \bar{F}(t_2))^{i-k}\bar{F}^{n-i}(t_2)}{\mathbb{P}(X_{k:n} \le t_1 < X_{k+1:n}, T > t_2)}. \end{aligned} \tag{18}$$

But

$$\begin{aligned} &\mathbb{P}(X_{k:n} \le t_1 < X_{k+1:n}, T > t_2) \\ &\quad = \sum_{m=k+1}^n \mathbb{P}(X_{k:n} \le t_1 < X_{k+1:n}, T > t_2 \mid T = X_{m:n})\mathbb{P}(T = X_{m:n}) \\ &\quad = \sum_{m=k+1}^n \mathbb{P}(X_{k:n} \le t_1 < X_{k+1:n}, X_{m:n} > t_2)\mathbb{P}(T = X_{m:n}) \\ &\quad = \sum_{m=k+1}^n s_m \sum_{j=k}^{m-1} \binom{n}{j}\binom{j}{k} F^k(t_1)(\bar{F}(t_1) - \bar{F}(t_2))^{j-k}\bar{F}^{n-j}(t_2) \\ &\quad = \sum_{j=k}^{n-1} \bar{S}_j \binom{n}{j}\binom{j}{k} F^k(t_1)(\bar{F}(t_1) - \bar{F}(t_2))^{j-k}\bar{F}^{n-j}(t_2). \end{aligned} \tag{19}$$

Therefore

$$p_{k,n}^{t_1,t_2}(i) = \frac{\bar{S}_i\binom{n}{i}\binom{i}{k}\left(\frac{\bar{F}(t_1)}{\bar{F}(t_2)} - 1\right)^{i-k}}{\sum_{j=k}^{n-1} \bar{S}_j\binom{n}{j}\binom{j}{k}\left(\frac{\bar{F}(t_1)}{\bar{F}(t_2)} - 1\right)^{j-k}}, \quad i = k, ..., n-1. \tag{20}$$

Appendix F. Proof of Theorem 2.4

The probability mass function in (8) may be written as

$$p_{k,n}^{t_1,t_2}(i) = \frac{\bar{S}_i\binom{n}{i}\binom{i}{k}\left(\frac{\bar{F}(t_1)}{\bar{F}(t_2)} - 1\right)^{i-k}}{\sum_{j=k}^{n-1} \bar{S}_j\binom{n}{j}\binom{j}{k}\left(\frac{\bar{F}(t_1)}{\bar{F}(t_2)} - 1\right)^{j-k}} I_{\{k,...,n-1\}}(i).$$

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To prove Part (a), first note that $\binom{i}{k}$ and $I_{\{k,\ldots,n-1\}}(i)$ are TP$_2$ in $(i, k) \in \{k, \ldots, n-1\} \times \{0, \ldots, n-1\}$. Since the product of two TP$_2$ functions is a TP$_2$ function, we conclude that $p_{k,n}^{t_1,t_2}(i)$ is TP$_2$ in $(i, k) \in \{k, \ldots, n-1\} \times \{0, \ldots, n-1\}$, which implies that $p_{k+1,n}^{t_1,t_2}(i)/p_{k,n}^{t_1,t_2}(i)$ is increasing in i. Hence the proof of Part (a) is complete.

To prove Part (b), it can be shown that $p_{k,n}^{t_1,t_2}(i)$ is RR$_2$ in $(i, t_1) \in \{k, \ldots, n-1\} \times [0, t_2)$ and TP$_2$ in $(i, t_2) \in \{k, \ldots, n-1\} \times (t_1, \infty)$, and hence the result follows.

Appendix G. Proof of Theorem 2.5

Denote by $p_{k,n}^{t_1,t_2,r}(i)$ the conditional probability in (8) for the system $\mathcal{S}_r$, $r = 1, 2$; that is,

$$p_{k,n}^{t_1,t_2,r}(i) = \frac{\bar{S}_i^{(r)}\binom{n}{i}\binom{i}{k}(\psi_r(t_1, t_2) - 1)^{i-k}}{\sum_{j=k}^{n-1} \bar{S}_j^{(r)}\binom{n}{j}\binom{j}{k}(\psi_r(t_1, t_2) - 1)^{j-k}}, \qquad i = k, \ldots, n-1, \tag{21}$$

where $\bar{S}_i^{(r)} = \sum_{j=i+1}^{n} s_j^{(r)}$, $r = 1, 2$, and

$$\psi_r(t_1, t_2) = \begin{cases} \frac{\bar{F}(t_1)}{\bar{F}(t_2)}, r = 1; \\ \frac{\bar{G}(t_1)}{\bar{G}(t_2)}, r = 2. \end{cases} \tag{22}$$

By the assumption $\boldsymbol{s}^{(1)} \geq_{\text{hr}} \boldsymbol{s}^{(2)}$, $\bar{S}_i^{(2)}/\bar{S}_i^{(1)}$ is non-increasing in i, and hence $\bar{S}_i^{(r)}$ is RR$_2$ in $(i, r) \in \{k, \ldots, n-1\} \times \{1, 2\}$. On the other hand, it can be concluded from $X_1 \leq_{\text{hr}} Y_1$ that $(\psi_r(t_1, t_2) - 1)^i$ is also RR$_2$ in $(i, r) \in \{k, \ldots, n-1\} \times \{1, 2\}$. From the fact that product of two RR$_2$ functions is an RR$_2$ function, we conclude that $p_{k,n}^{t_1,t_2,r}(i)$ is RR$_2$ in $(i, r) \in \{k, \ldots, n-1\} \times \{1, 2\}$. This completes the proof of the theorem.

Appendix H. Proof of Proposition 3.1

Let f_τ denote the density function corresponding to $\bar{F}_\tau$. On differentiating the cost function (11) with respect to t, we obtain

$$\begin{aligned} \frac{d}{dt}\eta_I(t) \overset{sgn}{=} \Bigg\{ & \left(c_{cms}\frac{f_\tau(t-\tau)}{\bar{F}_\tau(t-\tau)} + \frac{A_1(t)}{\bar{F}_\tau(t-\tau)} + \frac{A_2(t)}{\bar{F}_\tau(t-\tau)} + \frac{A_3(t)}{\bar{F}_\tau(t-\tau)} \right) \left(\tau + \int_0^{t-\tau} \bar{F}_\tau(x)dx \right) \\ & - P_{2,k}(\tau, t) \left[(c_{cm} - c_{pm})\mathbb{E}(N_{t,\tau} \mid (X_\tau)_{k:n} \leq t < T_\tau) + nc_{pm} \right] \\ & - n\int_0^\tau \nu(y)r(y)dy - F_\tau(t-\tau)c_{cms} - c_{pms}P_{1,k}(\tau, t) \Bigg\}, \end{aligned} \tag{23}$$

where '$\overset{sgn}{=}$' means to have the same sign,

$$A_1(t) = c_{pms}\sum_{i=0}^{k-1} \bar{S}_i \binom{n}{i} \left(1 - \frac{\bar{F}(t)}{\bar{F}(\tau)}\right)^{i-1} \left(\frac{\bar{F}(t)}{\bar{F}(\tau)}\right)^{n-i-1} \frac{f(t)}{\bar{F}(\tau)} \left\{ i\frac{\bar{F}(t)}{\bar{F}(\tau)} - (n-i)\left(1 - \frac{\bar{F}(t)}{\bar{F}(\tau)}\right) \right\},$$

$$\begin{aligned} A_2(t) = & n\left(c_{cm} - c_{pm}\right) \sum_{i=k}^{n-1} \bar{S}_i \binom{n-1}{i-1} \left(1 - \frac{\bar{F}(t)}{\bar{F}(\tau)}\right)^{i-1} \left(\frac{\bar{F}(t)}{\bar{F}(\tau)}\right)^{n-i-1} \frac{f(t)}{\bar{F}(\tau)} \\ & \times \left\{ i\frac{\bar{F}(t)}{\bar{F}(\tau)} - (n-i)\left(1 - \frac{\bar{F}(t)}{\bar{F}(\tau)}\right) \right\}, \end{aligned}$$

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and

$$A_3(t) = nc_{pm} \sum_{i=k}^{n-1} \bar{S}_i \binom{n}{i} \left(1 - \frac{\bar{F}(t)}{\bar{F}(\tau)}\right)^{i-1} \left(\frac{\bar{F}(t)}{\bar{F}(\tau)}\right)^{n-i-1} \frac{f(t)}{\bar{F}(\tau)} \left\{ i \frac{\bar{F}(t)}{\bar{F}(\tau)} - (n-i)\left(1 - \frac{\bar{F}(t)}{\bar{F}(\tau)}\right) \right\}.$$

By using Equation (12), it can be observed that

$$\lim_{t\to\infty} \frac{f_\tau(t-\tau)}{\bar{F}_\tau(t-\tau)} = \lim_{t\to\infty} r(t),$$

$$\lim_{t\to\infty} \frac{A_1(t)}{\bar{F}_\tau(t-\tau)} = 0,$$

$$\lim_{t\to\infty} \frac{A_2(t)}{\bar{F}_\tau(t-\tau)} = -(i^*-1)(n-i^*+1)(c_{cm}-c_{pm}) \lim_{t\to\infty} r(t),$$

$$\lim_{t\to\infty} \frac{A_3(t)}{\bar{F}_\tau(t-\tau)} = -n(n-i^*+1)c_{pm} \lim_{t\to\infty} r(t).$$

One can show that if

$$\lim_{t\to\infty} r(t) > \frac{c_{cms} + n\int_0^\tau \nu(y)r(y)dy}{(c_{cms} - nc_{pm} - (i^*-1)(c_{cm}-c_{pm}))(n-i^*+1)(\tau+\mu_\tau)},$$

then the right-hand side of Equation (23) is positive. This means that, under the condition (13), $\eta_I(t)$ is eventually strictly increasing. On the other hand, one can obtain

$$\lim_{t\to\tau} \frac{d}{dt}\eta_I(t) = -\frac{n\int_0^\tau \nu(y)r(y)dy + c_{pms}}{\tau^2}.$$

Therefore, $\eta_I(t)$ is initially decreasing. We conclude that $\eta_I(t)$ has at least a finite minimum.

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References

[1] Arriaza, A., Navarro, J. and Suárez-Llorens, A. (2018). Stochastic comparisons of replacement policies in coherent systems under minimal repair. *Naval Res. Logistics* **65**, 550–565.

[2] Asadi, M. and Berred, A. (2012). On the number of failed components in a coherent operating system. *Statist. Prob. Lett.* **82**, 2156–2163.

[3] Ashrafi, S. and Asadi, M. (2017). The failure probability of components in three-state networks with applications to age replacement policy. *J. Appl. Prob.* **54**, 1051–1070.

[4] Bada, F. G., Berrade, M. D., Cha, J. H. and Lee, H. (2018). Optimal replacement policy under a general failure and repair model: minimal versus worse than old repair. *Reliab. Eng. System Safety* **180**, 362–372.

[5] Balakrishnan, N., Navarro, J. and Samaniego, F. J. (2012). Signature representation and preservation results for engineered systems and applications to statistical inference. In *Recent Advances in System Reliability: Signatures, Multi-state Systems and Statistical Inference*, eds. A. Lisnianski and I. Frenkel, Springer, London, pp. 1–22.

[6] Barlow, R. E. and Hunter, L. C. (1960). Optimum preventive maintenance policies. *Operat. Res.* **8**, 90–100.
[7] Barlow, R. E. and Proschan, F. (1975). *Statistical Theory of Reliability and Life Testing*. Holt, Rinehart and Winston, New York.
[8] Belzunce, F., Martnez-Riquelme, C. and Ruiz, J. M. (2018). Allocation of a relevation in redundancy problems. To appear in *Appl. Stoch. Models Business Industry*.
[9] Cha, J. H., Finkelstein, M. and Levitin, G. (2017). On preventive maintenence of systems with lifetimes dependent on a random shock process. *Reliab. Eng. System Safety* **168**, 90–97.
[10] Eryilmaz, S. (2012). The number of failed components in a coherent system with exchangeable components. *IEEE Trans. Reliab.* **61**, 203–207.
[11] Eryilmaz, S. (2018). The number of failed components in a k-out-of-n system consisting of multiple types of components. *Reliab. Eng. System Safety* **175**, 246–250.
[12] Finkelstein, M. and Gertsbakh, I. (2015). 'Time-free' preventive maintenance of systems with structures described by signatures. *Appl. Stoch. Models Business Industry* **31**, 836–845.
[13] Finkelstein, M. and Gertsbakh, I. (2016). On preventive maintenance of systems subject to shocks. *P. I. Mech. Eng. O-J. Ris.* **230**, 220–227.
[14] Gertsbakh, I. (2000). *Reliability Theory with Applications to Preventive Maintenance*. Springer, London.
[15] Karlin, S. (1968). *Total Positivity*, Vol. I. Stanford University Press.
[16] Khaledi, B-E. and Shaked, M. (2007). Ordering conditional lifetimes of coherent systems. *J. Statist. Planning Infer.* **137**, 1173–1184.
[17] Kochar, S. C., Mukerjee, H., and Samaniego, F. J. (1999). The 'signature' of a coherent system and its application to comparisons among systems. *Naval Res. Logistics* **46**, 507–523.
[18] Levitin, G., Finkelstein, M. and Dai, Y. (2018). Optimizing availability of heterogeneous standby systems exposed to shocks. *Reliab. Eng. System Safety* **170**, 137–145.
[19] Lucia, U. (2013). Thermodynamic paths and stochastic order in open systems. *Physica A* **392**, 3912–3919.
[20] Mahmoudi, M. and Asadi, M. (2011). The dynamic signature of coherent systems. *IEEE Trans. Reliab.* **60**, 817–822.
[21] Morse, P. M. (1958). *Queues, Inventories, and Maintenance*. Wiley, New York.
[22] Navarro, J. and Calì, C. (2018). Inactivity times of coherent systems with dependent components under periodical inspection. To appear in *Appl. Stoch. Models Business Industry*.
[23] Navarro, J. and Rychlik, T. (2007). Reliability and expectation bounds for coherent systems with exchangeable components. *J. Multivariate Anal.* **98**, 102–113.
[24] Navarro, J., Samaniego, F. J. and Balakrishnan, N. (2011). Signature-based representations for the reliability of systems with heterogeneous components. *J. Appl. Prob.* **48**, 856–867.
[25] Navarro, J., Samaniego, F. J. and Balakrishnan, N. (2013). Mixture representations for the joint distribution of lifetimes of two coherent systems with shared components. *Adv. Appl. Prob.* **45**, 1011–1027.
[26] Parvardeh, A., Balakrishnan, N. and Arshadipour, A. (2018). A note on the conditional residual lifetime of a coherent system under double monitoring. *Commun. Statist. Theory Meth.* **47**, 2373–2378.
[27] Pham, H. and Wang, H. (2000). Optimal (τ, T) opportunistic maintenance of a k-out-of-$n : G$ system with imperfect PM and partial failure. *Naval Res. Logistics* **47**, 223–239.
[28] Pulcini, G. (2003). Mechanical reliability and maintenance models. In *Handbook of reliability engineering*, ed. H. Pham, Springer, London, pp. 317–348.
[29] Reed, S. (2017). An efficient algorithm for exact computation of system and survival signatures using binary decision diagrams. *Reliab. Eng. System Safety* **165**, 257–267.
[30] Ross, S. M. (1970). *Applied Probability Models with Optimisation Applications*. Holden-Day, San Francisco.
[31] Samaniego, F. J. (1985). On closure of the IFR class under formation of coherent systems. *IEEE Trans. Reliab.* **34**, 69–72.
[32] Samaniego, F. J. (2007). *System Signatures and their Applications in Engineering Reliability*. Springer, New York.
[33] Shaked, M. and Shanthikumar, J. G (2007). *Stochastic Orders*. Springer, New York.
[34] Sheu, S. (1991). Generalized block replacement policy with minimal repair and general random repair costs for a multi-unit system. *J. Operat. Res. Soc.* **42**, 331–341.
[35] Shimizu, Y. and Kawai, T. (1991). Development of electric power steering. SAE Technical Paper 910014. Available at https://doi.org/10.4271/910014.
[36] Spizzichino, F. (2018). Reliability, signature, and relative quality functions of systems under time-homogeneous load-sharing models. *Appl. Stoch. Models Business Industry* **35**, 158–176.
[37] Tahara, A. and Nishida, T. (1975). Optimal replacement policy for minimal repair model. *J. Operat. Res. Soc. Japan* **18**, 113–124.
[38] Tavangar, M. (2016). Conditional inactivity time of components in a coherent operating system. *IEEE Trans. Reliab.* **65**, 359–369.

[39] WANG, H. AND PHAM, H. (2006). *Reliability and Optimal Maintenence*. Springer, London.

[40] ZAREZADEH, S. AND ASADI, M. (2019). Coherent systems subject to multiple shocks with applications to preventative maintenance. To appear in *Reliab. Eng. System Safety*.

[41] ZHANG, Z. (2009). Ordering conditional general coherent systems with exchangeable components. *J. Statist. Planning Infer.* **140**, 454–460.

[42] ZHANG, N., FOULADIRAD, M. AND BARROS, A. (2017). Maintenance analysis of a two-component load-sharing system. *Reliab. Eng. System Safety* **177**, 24–34.

[43] ZHANG, Z. AND MEEKER, W. Q. (2013). Mixture representations of reliability in coherent systems and preservation results under double monitoring. *Commun. Statist. Theory Meth.* **42**, 385–397.