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CLASSICALLY ARCHETYPAL RULES

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Abstract. A one-premiss rule is said to be archetypal for a consequence relation when not only is the conclusion of any application of the rule a consequence (according to that relation) of the premiss, but whenever one formula has another as a consequence, these formulas are respectively equivalent to a premiss and a conclusion of some application of the rule. We are concerned here with the consequence relation of classical propositional logic and with the task of extending the above notion of archetypality to rules with more than one premiss, and providing an informative characterization of the set of rules falling under the more general notion.

§1. Introduction and background. For present purposes languages—meaning sentential languages—can be identified with their sets of formulas, taken to be freely generated from a countably infinite set of sentence letters or propositional variables $p_1, p_2, \ldots$ (the first three of which we abbreviate to p, q, r), by means of connectives of varying arities. We take an n-premiss rule over such a language L to be a set of $n + 1$-tuples of formulas $\langle A_1, \ldots, A_{n+1}\rangle$ ($A_i \in L, i = 1, \ldots, n + 1$), thinking of each such tuple as an application of the rule, with premisses $A_1, \ldots, A_n$ and conclusion A_{n+1}. In what follows we are concerned only with n-premiss *sequential* rules, in the sense of [8], which is to say those rules R for which there is some $\langle A_1, \ldots, A_{n+1}\rangle \in R$ with every $\langle B_1, \ldots, B_{n+1}\rangle \in R$ being a substitution instance of $\langle A_1, \ldots, A_{n+1}\rangle \in R$, i.e., for every such $\langle B_1, \ldots, B_{n+1}\rangle$ there is a substitution s with $s(A_i) = B_i$ ($i = 1, \ldots, n+1$). These sequences $\langle A_1, \ldots, A_{n+1}\rangle \in R$ of which every other tuple in R is a substitution instance are called *skeletons* of R in [12]: they are unique to within relettering, so it is for most purposes safe to speak of *the* skeleton of a rule, and to allow transfer of terminology from applying to rules to applying to their skeletons, and vice versa.[1] If there are m distinct sentence letters (counting by type rather than token) appearing in the formulas $A_1, \ldots, A_{n+1}$ altogether, we call this an

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[1] In [3] such skeletons are referred to as forms of inference, and the terminology of being archetypal (introduced below) is applied to these rather than the induced rules; here we follow [10] in applying it to the rules themselves. This is an example of the transfer of terminology mentioned in the text to which this note is appended. Strictly speaking Rautenberg's skeletons are not of the form $\langle A_1, \ldots, A_{n+1}\rangle$ but are rather the corresponding pairs $\langle\{A_1, \ldots, A_n\}, A_{n+1}\rangle$, to emphasize the irrelevance of a particular ordering of the premiss formulas. For many purposes an intermediate version is desirable, in which the premisses are collected into a multiset rather than a set, so that while order is disregarded, the "n" is stabilized in talk of the applications of an n-premiss rule, but these details need not worry us here.

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m-variable rule. Rather than writing "$\langle A_1, \ldots, A_{n+1}\rangle$" we write "$A_1, \ldots, A_n/A_{n+1}$" to denote the skeleton in question, or alternatively, the unique rule with that skeleton. Given a substitution-invariant consequence relation $\vdash$,[2] we say that such a rule is derivable for, or (more often) is *correct* for, $\vdash$, when $A_1, \ldots, A_n \vdash A_{n+1}$. For ease of exposition and in order for convenient terminology to have a natural interpretation—see Remark 1.5 below—we restrict attention to consequence relations which are not only substitution-invariant but satisfy some additional conditions:

Convention 1.1. From this point on, by a *consequence relation* will be meant a finitary consequence relation which is both substitution-invariant and congruential. ($\vdash$ is *congruential* if for any formulas A, B in the language of $\vdash$, if $A \dashv\vdash B$ then for any context $C(\cdot)$ provided by the language, we have $C(A) \dashv\vdash C(B)$. This terminology is taken from Segerberg [13], adapting an earlier usage by Makinson. Alternative terms with some currency are "extensional" and "self-extensional"—though the first of these is dangerously misleading for the present concept.)

Humberstone [3] concentrates on the question of which rules are not only correct for a consequence relation but archetypally so, in the sense that they subsume, modulo equivalence (mutual consequence), every other correct rule. More precisely, where "$C \dashv\vdash s(A)$" means "$C \vdash s(A)$ and $s(A) \vdash C$":[3]

Definition 1.2. *A 1-premise rule A/B is archetypal for a consequence relation $\vdash$ when A/B is correct for $\vdash$ and for any formulas C, D, for which $C \vdash D$, there exists a substitution s with $C \dashv\vdash s(A)$, and $D \dashv\vdash s(B)$.*

It is easy to see, for example, that the two-variable rule $A \wedge B/A$—i.e., the rule with skeleton $\langle p \wedge q, p\rangle$—is archetypal for the consequence relation, $\vdash_{\mathsf{CL}}$, of classical propositional logic: any classically sanctioned inference from a formula C to a formula D can be rewritten as the inference from a conjunction to its first conjunct, where the conjunction is classically equivalent to C and the conjunct is classically equivalent to D (since we can take the conjunction as $D \wedge C$ and the inferred conjunct as D itself). On the other hand, it is equally evident that the rule $A \vee A/A$ is not archetypal for $\vdash_{\mathsf{CL}}$: it could at most subsume inferences from C to D in which C and D were (classically) equivalent. Such evident nonstarters for the status of archetypal rules we collect together in the following definition:

Definition 1.3. *A 1-premiss rule A/B which is correct for a consequence relation $\vdash$ is degenerate* (*w.r.t.* $\vdash$) *if one of the following three conditions obtains*:

(i) $A \vdash C$ for all C in the language of $\vdash$;
(ii) $\vdash B$ (i.e., $\varnothing \vdash B$);
(iii) $B \vdash A$.

Otherwise the rule A/B is *nondegenerate* w.r.t. $\vdash$.

2 A consequence relation $\vdash$ is *substitution-invariant* (sometimes called 'structural') if $A_1, \ldots, A_n, \ldots \vdash B$ implies $s(A_1), \ldots, s(A_n), \ldots \vdash s(B)$ for all substitutions s.

3 When the "$\vdash$" notation appears decorated with a sub- or superscript, etc., we give the decoration only in the forward direction, writing, for instance "$A \dashv\vdash_{\mathsf{CL}} B$" to mean that $A \vdash_{\mathsf{CL}} B$ and $B \vdash_{\mathsf{CL}} A$.

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The following appears as Theorem 2.1, with the parenthetical material included, in [3] which also notes that the result can be strengthened into an equivalence, for correct 2-variable 1-premiss rules for $\vdash_{\mathsf{CL}}$, between *archetypal* and *nondegenerate*.[4] [10] shows that the parenthetical qualification can be dropped: the result holds without the restriction to 2-variable rules—that is, that it held for n-premiss rules for any n. (A streamlined version of the proof appears in [11].)

THEOREM 1.4. *Every nondegenerate (2-variable) 1-premiss rule which is correct for* $\vdash_{\mathsf{CL}}$ *is archetypal for* $\vdash_{\mathsf{CL}}$.

In what follows, we shall be interested in relaxing the restriction in Theorem 1.4 (or in Połacik's generalization of it [10]) to 1-premiss rules. This will require coming up with an appropriate definition of a notion analogous to (non)degeneracy—though perhaps not deserving to be spoken of in precisely those terms, as we shall observe—since Definition 1.3 is tailored specifically to the 1-premiss case.

Continuing with these recapitulations, a principal theme in [3] is the contrast between classical and intuitionistic (propositional) logic in respect of results like Theorem 1.4. Denoting the latter's consequence relation by $\vdash_{\mathsf{IL}}$, we find that not only does Theorem 1.4 not survive the replacement of $\vdash_{\mathsf{CL}}$ by $\vdash_{\mathsf{IL}}$, but the exceptions in the latter case are many and various. Because this theme is not directly pertinent to the developments below, we confine ourselves here to three examples from [3]: none of the following rules (identified here by their skeletons), all of which are correct and nondegenerate for $\vdash_{\mathsf{IL}}$, are archetypal for $\vdash_{\mathsf{IL}}$: $p/q \to p$; $p \leftrightarrow q/p \to q$; $p/\neg\neg p$. Further details may be found in [3].[5] Among the consistent extensions of $\vdash_{\mathsf{IL}}$ in the same language—one thing that might be meant by 'intermediate consequence relations'—the only one relative to which every correct nondegenerate rule is archetypal is $\vdash_{\mathsf{CL}}$; see Połacik [10, 11].

One can also consider extensions of $\vdash_{\mathsf{IL}}$, $\vdash_{\mathsf{CL}}$, etc., which *expand* the language, such as arise in modal logic, adding an additional primitive 1-ary connective $\Box$ (or instead $\Diamond$, or for the case of $\vdash_{IL}$, both). If we want to consider the smallest monotone modal logic—thought of as a consequence relation—with classical logic as the underlying nonmodal logic, or more explicitly the smallest modal logic in which $\Box$ is monotone, one adds the further conditional constraint that whenever $A \vdash B$, we have $\Box A \vdash \Box B$. Note that the corresponding constraint with "$\vdash$" replaced by "$\dashv\vdash$" is already built into the present discussion by the congruentiality condition in Convention 1.1. A typical application[6] of the notion of archetypal rule in this setting is the observation that the monoton(icit)y constraint here can be formulated in unconditional terms as by requiring that $\Box(A \wedge B) \vdash \Box A$ for all A, B, here making use of the fact that $p \wedge q \,/\, p$ is archetypal, and so one could equally use any other one-premiss rule which is archetypal for $\vdash_{\mathsf{CL}}$ instead. For example, using $p \,/\, q \to p$ in this capacity one could isolate the monotone modal logics as those satisfying the condition that $\Box A \vdash \Box(B \to A)$. On the other hand, since, as recalled above, $p \,/\, q \to p$ is not archetypal for $\vdash_{\mathsf{IL}}$, the smallest (congruential) intuitionistically based

[4] In [3], m-variable rules—or their skeletons—are called m-ary, but this is confusing since we are identifying rules with relations among formulas, so an m-ary rule should be an $(m-1)$-premiss rule, rather than a rule with an m-variable skeleton.

[5] In the case of the last of these rules the situation is especially simple: no rule of the form $\neg A/B$ or $A/\neg B$ could be archetypal for $\vdash_{\mathsf{IL}}$ because $\neg$ is not universally representative according to $\vdash_{\mathsf{IL}}$. This point is stressed in [3] and [10]; on universally representative connectives generally, see Chapter 9 of [5].

[6] See the Digression on p. 347 of [7].

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modal logic in which $\Box$ is monotone cannot be similarly captured as the least consequence relation in the present language satisfying $\Box A \vdash \Box(B \to A)$. For that consequence relation we do not have, for example, $\Box(r \wedge s) \vdash \Box r$, since $r \wedge s \,/\, r$ cannot be subsumed under $p \,/\, q \to p$ in intuitionistic logic. (This follows from Proposition 4.6 in [3] together with the fact that $(r \to (r \wedge s)) \to (r \wedge s) \nvdash_{\mathsf{IL}} r$.) The following remark, occasioned by this digression into the modal area, may be skipped by readers wanting to get straight to the agenda of the present article.

REMARK 1.5. A referee commented that in view of part (iii) of Definition 1.3 the modal rule $A \,/\, \Box A$, of Necessitation, counts as degenerate in (for example) $\mathsf{S4}$, making the term "degenerate" somewhat misleading. This has prompted the inclusion of Convention 1.1, to make explicit our tacit preference for how the concepts in play here apply to the modal case. Recall, sticking with the case of $\mathsf{S4}$ (though any normal modal logic extending KT would do for present purposes) that one distinguishes the local consequence relation associated with a modal logic (in the set-of-formulas sense of 'logic') from the global consequence relation associated with that logic: the former deems A to be a consequence of a set of formulas if it can be derived from that set of formulas together with theorems of the logic, with the aid of Modus Ponens, with the latter allowing also applications of Necessitation. (The terminology is based on fact that the former consequence relation preserves the property of being true at an arbitrarily selected point in a model for the logic, while the latter preserves the property of being true throughout such a model.) Let us call these relations, for the case of $\mathsf{S4}$, $\vdash^{loc}_{\mathsf{S4}}$ and $\vdash^{glo}_{\mathsf{S4}}$, respectively. Convention 1.1 with its condition of congruentiality directs us specifically to $\vdash^{loc}_{\mathsf{S4}}$ because according to $\vdash^{glo}_{\mathsf{S4}}$ each of $p, \Box p$ is a consequence of the other, while these two formulas are not interreplaceable in arbitrary contexts ($p \to \Box p$ and $\Box p \to \Box p$, being, for example, respectively, unprovable and provable in $\mathsf{S5}$). By contrast, for the (congruential) local consequence relations, one does not have $\Box A$ and A as consequences of each other. The referee's observation remains correct for the case in which degeneracy is understood as in Definition 1.3 and then applied to the discussion of noncongruential consequence relations: there is the misleading connotation that there is no significant distinction to be drawn between two formulas which are consequences of each other. (Otherwise, why is the transition from one to another of two such formulas being described as denegerate?) To reinforce the local perspective, conditions like congruentiality, monotony and closure under necessitation were not described using the vocabulary of rules, though of course one can retain this perspective while adopting that vocabulary—for the purposes of presenting a proof system, for example—provided one distinguishes between rules of proof and rules of inference. The latter distinction does more work than the simple local/global contrast in play here, a famous example being the case of uniform substitution, as a rule of proof which is not a rule of inference, not preserving either truth at a point or truth throughout a model but rather validity at a point (and therefore validity on a frame). See [4] for further discussion. ◀

The remainder of our discussion returns us specifically to the case of classical logic, as governing the connectives $\wedge$, $\vee$, $\to$, $\leftrightarrow$, $\to$, $\neg$ as well as, for good measure, the nullary connectives $\top$ and $\bot$, and the question raised above as to what an analogue of Theorem 1.4 might look like once the restriction to 1-premiss rules is relaxed. A rule which is archetypal for $\vdash_{\mathsf{CL}}$ will be called *classically* archetypal. So, although we begin with the definition of archetypality for an arbitrary consequence relation, without restriction to one-premiss rules, we will be aiming eventually for a characterization, specifically, of those rules which are classically archetypal. This will be appear as Theorem 3.9, the "if" direction of which provides the analogue of Theorem 1.4 for the general many-premiss case.

§2. Archetypality for multi-premiss rules. First, we need to generalize the notion of archetypality so as to apply to rules with more than one premiss.

DEFINITION 2.1. *A rule $A_1, \ldots, A_n/B$ is archetypal for a consequence relation $\vdash$ when $A_1, \ldots, A_n/B$ is correct for $\vdash$ and for any formulas $C_1, \ldots, C_n, D$, for which $C_1, \ldots, C_n \vdash D$, there exist a permutation $\pi : \{1, \ldots, n\} \longrightarrow \{1, \ldots, n\}$ and a substitution s with $C_i \dashv\vdash s(A_{\pi(i)})$ for $i = 1, \ldots, n$, and $D \dashv\vdash s(B)$.*

Note that the $n = 1$ case of this definition coincides with archetypality for 1-premiss rules as understood as in §1. A simpler definition would discard the part about permutations here, and simply declare $A_1, \ldots, A_n/B$ to be archetypal for $\vdash$ when $C_1, \ldots, C_n \vdash D$ implies the existence of a substitution s with $C_i \dashv\vdash s(A_i)$ for $i = 1, \ldots, n$, and $D \dashv\vdash s(B)$. We shall have occasion to observe below (Remark 2.8) that this, for the case of $\vdash_{\mathsf{CL}}$, at least, is equivalent to that given in Definition 2.1; but it seems wiser not to preempt at this stage the possibility that the order of the A_i may have to be changed for $\langle C_1, \ldots, C_n, D\rangle$ to be subsumable under $\langle A_1, \ldots, A_n, B\rangle$.

In the interests of completeness, we should pause for a moment to note that the range of archetypal n-premiss rules, rather than just the $n = 1$ case, includes not only the case(s) of $n > 1$ but also that of $n = 0$. Here Definition 2.1 applies, with formulas $A_1, \ldots, A_n$ and $C_1, \ldots, C_n$ dropping out of the picture, leaving only B and D. Such a rule is, when correct for a consequence relation, essentially just a theorem schema (or axiom schema) whose applications are its instances, all of them consequences of $\varnothing$ by the consequence relation in question. So when B is (the skeleton of) such a rule and for D with $\vdash D$ (or $\varnothing \vdash D$ if you prefer), we have $B \dashv\vdash s(D)$ automatically, taking s as the identity substitution. Thus for 0-premiss rules, correctness and archetypality coincide.

For many purposes it is possible to avoid considering multi-premiss rules once single-premiss rules have been addressed, since one can transpose the one-premiss treatment to the general case by replacing the rule $A_1, \ldots, A_n \,/\, B$ with the rule $A_1 \wedge \ldots \wedge A_n \,/\, B$. That would certainly not be possible here. Consider for example the rules with skeletons $p, q \,/\, p$ on the one hand and $p \wedge q \,/\, p$ on the other. As already remarked in §1, the latter rule is classically archetypal; the former is certainly not, subsuming by substitution only those two-premiss inferences in which the conclusion is equivalent to one of the premisses.

EXAMPLE 2.2. (i) The rule with skeleton $p, q/(p \wedge q) \vee r$ is classically archetypal, since if $A, B \vdash_{\mathsf{S}} C$ we have s with $s(p) = A$, $s(q) = B$ and $s(r) = C$. One need only check that $s((p \wedge q) \vee r)$ is in this case CL-equivalent to C.
(ii) By contrast the rule with skeleton $p, q/p \wedge q$ is not classically archetypal, since for example, whenever $C_1, C_2 \vdash_{\mathsf{CL}} D$ while $D \nvdash_{\mathsf{CL}} C_1$ or $D \nvdash_{\mathsf{CL}} C_2$—e.g., recalling (i), taking C_1, C_2, D as, respectively, $p, q, (p \wedge q) \vee r$—we cannot have $s(p)$, $s(q)$ and $s(p \wedge q)$, alias $s(p) \wedge s(q)$ equivalent, respectively, to C_1, C_2, D (which would force $D \vdash_{\mathsf{CL}} C_1$ and $D \vdash_{\mathsf{CL}} C_2$). ◀

The nonarchetypal Example 2.2(ii) illustrates the same kind of phenomenon as those cited in [3]—features of the example not shared by every (two-premiss) inference correct for the consequence relation, such as having its premisses be consequences of its conclusion—but there is very little temptation to talk of the rule involved here,

$\wedge$-introduction as one might naturally call it, as degenerate. One of the best known two-premiss rules, Modus Ponens, with skeleton $p \rightarrow q, p \,/\, q$ would also not easily be called degenerate, but it exhibits also too much specificity to be classically archetypal, since one of the premisses is a classical consequence of the conclusion. (This feature of the rule shows incidentally that, as with that in Example 2.2(ii), this rule is not intuitionistically archetypal either. Another feature of the present case that could have been cited to show that it is not classically archetypal—namely that the disjunction of its premisses is a consequence of $\varnothing$—would not have carried across to the intuitionistic setting.)

The impressionistic verdict of nondegeneracy for Modus Ponens could be justified by the following extension of degeneracy as given in Definition 1.3: a rule $A_1, \ldots, A_n/B$ assumed correct for a consequence relation $\vdash$ is *degenerate* (w.r.t. $\vdash$) if one of the following three conditions obtains:

(i)′ $A_1, \ldots, A_n \vdash C$ for all C in the language of $\vdash$;
(ii)′ $\vdash B$;
(iii)′ $B \vdash A_i$ for all A_i $(i = 1, \ldots, n)$.

If conjunction behaves as expected according to $\vdash$ this amounts to the degeneracy (by the lights of Definition 1.3) of the one-premiss surrogate $A_1 \wedge \ldots \wedge A_n \,/\, B$. This would undermine the intuitive verdict of nondegeneracy of $\wedge$-Introduction, and an even stronger notion of nondegeneracy would still not provide a sufficient condition for archetypality in the multi-premiss case. This stronger notion is the negation of degeneracy as defined by (i)′, (ii)′ and the following weakening of (iii)′: (iii)″ $B \vdash A_i$ for some A_i $(i = 1, \ldots, n)$. Note that this still coincides with degeneracy *à la* Definition 1.3 but now rules not only $\wedge$-Introduction but also Modus Ponens to be degenerate. But the classically correct rule (with skeleton)

$$p, p \wedge q \,/\, (p \wedge q) \vee r$$

is not degenerate while still classically archetypal, since it can only subsume an inference from two premisses to a conclusion when one of the premisses has the other as a (classical) consequence. Evidently we need to replace these notions of degeneracy by something less specific, forbidding, essentially, the obtaining of any logical relations among the premisses and conclusion of a rule other than as required for the correctness of the rule. This is most conveniently done in semantic terms, as follows.

Call an assignment υ of one of the two truth-values, T, F, to every formula a *valuation*. (Thus all valuations are ‘bivalent’.) A valuation υ is *Boolean*, if υ respects the conventional association of truth-functions with connectives (thus $\upsilon(A \wedge B) = T$ iff $\upsilon(A) = \upsilon(B) = T$, for all A, B, etc.). Now for the reformulation: a classically correct rule A/B is nondegenerate when there are Boolean valuations $\upsilon_1, \upsilon_2, \upsilon_3$ with: $\upsilon_1(A) = F$ and $\upsilon_1(B) = F$, $\upsilon_2(A) = F$ and $\upsilon_2(B) = T$, and $\upsilon_3(A) = T$ and $\upsilon_3(B) = T$. (Of course there are no such valuations υ with $\upsilon(A) = T$ and $\upsilon(B) = F$, since *ex hypothesi*, A/B is correct.) This has an obvious generalization to the case of n-premiss rules for $n > 1$, and similarly generalizing from the Boolean case, via the notion of V-validity for any class V of (bivalent) valuations, rule being V-valid when for all $\upsilon \in V$ verifying its premisses verifies its conclusion. (For V as the class of Boolean valuations, the V-valid rules are just those rules that are correct for $\vdash_{\mathsf{CL}}$, then.) What we now define is the more restrictive notion of being “exactly” V-valid—the terminology, adapted from [1], being explained below—for which purpose we first adapt some other terminology, from [2], p. 265:

DEFINITION 2.3. (*i*) *Where* $x_1, \ldots, x_m \in \{T, F\}$ *we say that a class of valuations* V *imposes a ban on* $\langle x_1, \ldots, x_m \rangle$ *for* $\langle A_1, \ldots, A_m \rangle$ *just in case there is no* $v \in V$ *for which* $v(A_i) = x_i$ *for* $i = 1, \ldots, m$.
(*ii*) *We denote by* $\beta_V(A_1, \ldots, A_m)$ *the set of all* $\langle x_1, \ldots, x_m \rangle$ *such that* V *imposes a ban on* $\langle x_1, \ldots, x_m \rangle$ *for* $\langle A_1, \ldots, A_m \rangle$. *Think of this as the set of bans* V *imposes in respect of* $\langle A_1, \ldots, A_m \rangle$.
(*iii*) *When* V *is clear from the context, we write* $\beta_V(\cdot)$ *as* $\beta(\cdot)$, *and refer to any* $\langle x_1, \ldots, x_m \rangle \in \beta(A_1, \ldots, A_m)$ *as a ban obeyed by* $\langle A_1, \ldots, A_m \rangle$.

Using this apparatus there is the obvious definition of V-validity would have it that the rule $A_1, \ldots, A_n / B$ is V-valid when $\langle T^n, F \rangle \in \beta_V(A_1, \ldots, A_n, B)$, where $\langle T^n, F \rangle$ abbreviates $\langle \underbrace{T, \ldots, T}_{n \text{ times}}, F \rangle$. What we need for present purposes is the following strengthening of this notion of V-validity:

DEFINITION 2.4. *A rule* $A_1, \ldots, A_n/B$ *is exactly* V*-valid just in case*

$$\beta_V(A_1, \ldots, A_n, B) = \{\langle T^n, F \rangle\}.$$

The terminology here is adapted from de Jongh and Chagrova [1], in which the result of substituting $A_1, \ldots, A_n$ for $p_1, \ldots, p_n$ in a formula $F(p_1, \ldots, p_n)$ is denoted by $F(A_1, \ldots, A_n)$ and is said to be *exactly provable* in a theory when $F(A_1, \ldots, A_n)$ is provable in that theory and for any other (propositional) context G of n formulas for which $G(A_1, \ldots, A_n)$ is provable in the theory, we have $\vdash_{\mathsf{IL}} F(p_1, \ldots, p_n) \to G(p_1, \ldots, p_n)$. De Jongh and Chagrova consider the case in which the theory in question may itself be a logic, and in particular may be IL, as well as having in mind the case of theories—such as Heyting Arithmetic—with IL (or rather, intuitionistic predicate logic) as the underlying logic. If we take the theory and the logic to be classical (propositional) logic (i.e., in the latter case replace the reference to $\vdash_{\mathsf{IL}}$ with one to $\vdash_{\mathsf{CL}}$), then in the case where $F(p_1, \ldots, p_n)$ is $(p_1 \wedge \ldots \wedge p_{n-1}) \to p_n$ the exact provability of $F(A_1, \ldots, A_n)$ adds to its provability—i.e., in the present setting, to A_n's being a classical consequence of $A_1, \ldots, A_{n-1}$—the requirement that anything else $\vdash_{\mathsf{CL}}$ tells us about $A_1, \ldots, A_n$ follows from this: these formulas stand in no further logical relations according to classical logic.[7] This amounts to the class of Boolean valuations imposing no further ban in respect of $\langle A_1, \ldots, A_n \rangle$ beyond the ban on $\langle T^n, F \rangle$ which the fact that $A_1, \ldots, A_{n-1} \vdash_{\mathsf{CL}} A_n$ records. We now return, for conformity with the earlier discussion, to the case in which there are n (rather than $n - 1$) premisses, and "B" symbolizes the conclusion. What the various considerations in play here suggest is the following result, which will appear as Theorem 3.9: a rule $A_1, \ldots, A_n/B$ is classically archetypal if and only if it is exactly V-valid, where V is the class of Boolean valuations. Thus what correspond in the general case to the correct but degenerate rules (in the sense of Definition 1.3) as precisely the correct rules that are not classically archetypal, are the V-valid rules which fail to be exactly V-valid for this choice of V.

The $n = 1$ form of the envisaged result is essentially a reworking of Theorem 1.4. We can see the failure of $p, q/p \wedge q$ to be archetypal, remarked on in Example 2.2(ii) above, in the light of the present considerations in the following way. Understood relative

[7] No further *coercive* logical relations, as it is put in Humberstone [6] (which, however, restricts the discussion to *binary* logical relations), since here we want to exclude such 'permissive' relations as consistency, etc.

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to the class of Boolean valuations, $\langle p, q, p \wedge q\rangle$ obeys not only a ban on $\langle T, T, F\rangle$, reflecting the classical correctness of the rule $p, q / p \wedge q$, but also further bans on $\langle F, F, T\rangle$, $\langle F, T, T\rangle$, and $\langle T, F, T\rangle$. To recast in the present terms another example—the case of Modus Ponens—from our earlier discussion (after Example 2.2), $\beta(p \to q, p, q)$ contains not only $\langle T, T, F\rangle$, required for correctness but also the supernumerary $\langle F, F, T\rangle$, $\langle F, F, F\rangle$, and $\langle F, T, T\rangle$.

The harder part of the main result of the present article, showing that a rule is classically archetypal if and only if it is exactly V-valid for V the class of Boolean valuations, is the main business of §3. But we can settle the easy ('only if') half at once. Since there will be no further mention of nonclassical logics from this point on, to expedite discussion we make the following:

Convention 2.5. For the remainder of our discussion *archetypal* means "classically archetypal", *correct* means "correct for $\vdash_{\mathsf{CL}}$", *valuation* means "Boolean valuation", *exactly valid* means "exactly V-valid for V = the class of (Boolean) valuations", and *equivalent* means being assigned the same value by every valuation in this class.

PROPOSITION 2.6. *A rule is archetypal only if it is exactly valid.*

Proof. Suppose that $R = A_1, \ldots, A_n / B$ is (classically) archetypal. Thus R is correct for $\vdash_{\mathsf{CL}}$. To show that R is exactly valid, suppose otherwise. Then for some $\langle x_1, \ldots, x_n, y\rangle \neq \langle T^n, F\rangle$, the class of Boolean valuations imposes a ban on $\langle x_1, \ldots, x_n, y\rangle$ for $\langle A_1, \ldots, A_n, B\rangle$. Let C_i be $\top$ if x_i is T and let C_i be $\bot$ if x_i is F, for $i = 1 \ldots n$, and let D be $\top$ if y is T and $\bot$ if y is F. Note that $C_1, \ldots, C_n \vdash_{\mathsf{CL}} D$ because $\langle x_1, \ldots, x_n, y\rangle \neq \langle T^n, F\rangle$, but $C_1, \ldots, C_n / D$ cannot be subsumed under R because any ban obeyed by $\langle A_1, \ldots A_n, B\rangle$ will be obeyed by $\langle s(A_1), \ldots s(A_n), s(B)\rangle$ (s any substitution), so no such $(n+1)$-tuple can be coordinatewise equivalent to $\langle C_1, \ldots, C_n, D\rangle$, this sequence having been chosen to assume (on any Boolean valuation) the banned sequence $\langle x_1, \ldots, x_n, y\rangle$ of truth-values. □

We can press Proposition 2.6 into service to help explain one aspect of the discussion above. Although our definitions concerning multi-premiss rules are fully general, our concrete illustrations have involved specifically two-premiss rules. We mentioned in Example 2.2(i) a classically archetypal two-premiss rule, and in 2.2(ii) and the immediately following discussion two correct but nonarchetypal rules—an $\wedge$-introduction rule and an $\to$-elimination (or Modus Ponens) rule. Thus our sample of concrete two-premiss cases involve an archetypal 3-variable rule and two nonarchetypal 2-variable rules. This is not a misleading feature of the sample: There are no archetypal two-premiss two-variable rules. For the succinct formulation of an observation explaining this fact, we use the following:

Notation. Let R^n_m be the set of all correct n-premiss m-variable rules, and AR^n_m be the set of all such rules which are archetypal.

By Proposition 2.6, if $R \in \mathsf{AR}^n_m$, then all sequences $\langle x_1, \ldots, x_{n+1}\rangle \in \{T, F\}^{n+1}$, except $\langle T^n, F\rangle$ are realized as truth values for the formulas of the (skeleton of the) rule R. In particular, the subsequences $\langle x_1, \ldots, x_n\rangle \in \{T, F\}^n$ (with repetitions) must exhaust the whole set $\{T, F\}^n$. It follows that

$$2^m \geq 2^{n+1} - 1;$$

that is, we must have $m > n$. This proves the following:

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OBSERVATION 2.7. *There is no archetypal rule with m premisses and m or fewer variables; in particular, then,* $\mathsf{AR}^m_m = \emptyset$*, for all* $m \in \mathbb{N}$.

Thus our sample negative cases were those in which $m\ (= n) = 2$.

When we have not only Proposition 2.6 but also its converse, the latter to be established in the following section (Theorem 3.9), we shall have a simple decision procedure for archetypality, since the exact validity of a rule can be decided by a truth-table test.[8] Where $R = \langle A_1, \ldots, A_n, B\rangle$ is an m-variable rule draw up the 2^m-line truth-table noting the values given for the formulas $A_1, \ldots, A_n, B$ in these lines and check that every combination except for $\langle T^n, F\rangle$ appears, which means that to make up for the missing $\langle T^n, F\rangle$, some other combination appears exactly twice. (These various lines of the truth table correspond, in the reasoning of §3, to the atoms of the free Boolean algebra with m free generators.)

REMARK 2.8. The fact that archetypality and exact validity coincide also allows us to settle an issue raised after the formulation of Definition 2.1. The worry was that a rule—and for definiteness let us make it a three-premiss rule $A_1, A_2, A_3 \,/\, B$ might not be able to subsume a particular transition from formulas C_1, C_2, C_3, in that order, to D, in the sense of there being a substitution $\mathfrak{s}$ with $\mathfrak{s}(A_1)$ equivalent to C_1, $\mathfrak{s}(A_2)$ equivalent to C_2, $\mathfrak{s}(A_3)$ equivalent to C_3 (and also $\mathfrak{s}(B)$ equivalent to D), and instead have to do some permuting, so that in fact, say, there was only a substitution $\mathfrak{s}$ for which while $\mathfrak{s}(A_1)$ was equivalent to C_1 (and $\mathfrak{s}(B)$ equivalent to D), $\mathfrak{s}(A_2)$ and $\mathfrak{s}(A_3)$ were equivalent, respectively, to C_3 and C_2. At least in the case of classical archetypality, this cannot happen, since it is immediate from the definition of exact validity that if rules R and S differ only up to a permutation of their premisses, R is valid iff and only if S is. Thus for present purposes it would have made no difference if the reference to permutations had been omitted from Definition 2.1 (or equivalently, if we had insisted that the permutation π featuring there should be the identity permutation). ◀

Using the truth-table method sketched above, one can easily verify that a rule is exactly valid, when simply appealing to the definition of archetypality it would be far from obvious that the rule is archetypal. The following is an illustration of this with a rule in R^2_3, contrasting in respect of 'obvious archetypality' with the rule (also in R^2_3) of Example 2.2(i).

EXAMPLE 2.9. The truth-table method shows that the rule with skeleton

$$p \wedge (q \vee r),\ p \leftrightarrow (q \leftrightarrow r) \,/\, r$$

is exactly valid, and therefore, given the results of the following section, archetypal. Note that here the conclusion position is occupied by a sentence letter, while in the rule of Example 2.2(i), with skeleton $p, q \,/\, (p \wedge q) \vee r$ it was the premiss positions that were occupied by sentence letters. This makes it less obvious that the premisses are independent—capable of being assigned together all combinations of truth-values—than in the present case. That combined with validity is not, it should be noted in passing, sufficient for exact validity, as one sees from the case of the rule (with skeleton) $\langle p \leftrightarrow q, q \leftrightarrow r, p \leftrightarrow r\rangle$ which is not exactly valid (and therefore by Proposition 2.6, unlike the other rules mentioned here, lies outside AR^2_3) which obeys bans not only on the validity-required $\langle T, T, F\rangle$ but also on $\langle T, F, T\rangle$, $\langle F, T, F\rangle$, and $\langle F, F, F\rangle$. ◀

[8] Furthermore, our proof in §3 yields an effective procedure supplying all substitutions required for archetypality.

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Having sampled some of the convenience afforded to us by the fact that archetypal and exactly valid rules coincide, it remains only to get down to establishing that this is indeed a fact, as we shall at Theorem 3.9, supplementing Proposition 2.6 above with a proof of its converse. (We also return to the first rule mentioned under Example 2.9 in Example 3.10 to illustrate an important aspect of the proof.) Although Theorem 3.9 expressly concerns exact validity, we derive it from Proposition 3.7 and Corollary 3.8 thereto and these results pertain to arbitrary sets of bans on truth-values for sequences of formulas with no special privileging of the distinction between premisses and conclusions, as attention to the particular ban $\langle T^n, F\rangle$ suggests. It is for this reason that the second rule touched on in Example 2.9 used the notation $\langle p \leftrightarrow q, q \leftrightarrow r, p \leftrightarrow r\rangle$, rather than the notation $p \leftrightarrow q, q \leftrightarrow r \,/\, p \leftrightarrow r$. The sequence of formulas involved here does at least obey a ban on $\langle T^2, F\rangle$, which is of no special interest as far as Proposition 3.7 or Corollary 3.8 is concerned.[9] But we continue to refer to arbitrary $(n+1)$-tuples, or $(n+1)$-term sequences of formulas, as n-premiss rules, on the grounds that we have defined a rule to be any set of such sequences as are all substitution instances of one sequence in the set (the skeleton of the rule) and proceeded to identify the rule in question by citing one such representative sequence.

§3. The main result. This section is devoted to a proof of our main result, strengthening the "only if" in Proposition 2.6 to an "if and only if" in Theorem 3.9 and thereby showing that the appropriate replacement for nondegeneracy as we pass from the one-premiss case to characterizing archetypal rules with arbitrarily many premisses is indeed exact validity. The following notation will allow for more concise formulations at some points:

For any $k \in \mathbb{N}$, let $\mathbf{k}$ stand for the set $\{1, \ldots, k\}$ and let $\mathcal{P}(\mathbf{k})$ denote the power set of $\mathbf{k}$.

Let $\mathcal{FB}_m$ be the free m-generated Boolean algebra, i.e., the Lindenbaum algebra of classical propositional logic in the language with m propositional variables $p_1, \ldots, p_m$. Recall that the elements of $\mathcal{FB}_m$ are equivalence classes $[A]$ of formulas A with respect to the equivalence relation $\equiv$ defined as

$$A \equiv A' \text{ iff } A \dashv\vdash_{\mathsf{CL}} A'.$$

(We use the "$\equiv$" notation for its familiarity in this setting.) Let $\leq$ stand for the usual ordering of the algebra $\mathcal{FB}_m$, that is

$$[A] \leq [A'] \text{ iff } A \vdash_{\mathsf{CL}} A'.$$

As is well known, $\mathcal{FB}_m$ is a Boolean algebra whose meet and join are infimum and supremum with respect to $\leq$, respectively. The algebra $\mathcal{FB}_m$ is freely generated by the set of $[p_i]$ for $i \in \mathbf{m}$ and the atoms of $\mathcal{FB}_m$ are of the form

$$[p_1^{x_1} \wedge \ldots \wedge p_m{}^{x_m}],$$

9 For example, we may be interested in ordered triples $\langle A, B, C\rangle$ obeying a ban on $\langle T, T, T\rangle$, i.e., inconsistent triples of formulas, for which it would not be natural to think of A and B as premisses and C as a conclusion. Corollary 3.8 then implies that if $\langle A, B, C\rangle$ obeys this ban and no other bans then any three jointly inconsistent formulas (whatever other logical relations may obtain between them) are equivalent to substitution instances of A, B, C by some single substitution. Similarly if $\langle A, B, C\rangle$ obeys this ban and some additional ban then such a substitution can be found to formulas equivalent to any other three formulas obeying those two bans, regardless of what further bans they may obey.

where $p_i^0 = \neg p_i$ and $p_i^1 = p_i$, for any propositional variable p_i. The set of all the atoms of the algebra $\mathcal{FB}_m$ will be denoted by Atoms($\mathcal{FB}_m$).

DEFINITION 3.1. *We say that the sequence $\bar{R} = \langle [A_1], \ldots, [A_n], [A_{n+1}] \rangle$ of elements of $\mathcal{FB}_m$ is*
(i) *correct iff for every atom a of $\mathcal{FB}_m$,*

$$\text{if } a \leq \bigwedge_{1 \leq i \leq n} [A_i] \text{ then } a \leq [A_{n+1}];$$

(ii) *flexible iff $\bar{R}$ is correct and for every set $I \in \mathcal{P}(\mathbf{n+1}) \setminus \{\mathbf{n}\}$ there is an atom a_I such that*

$$a_I \leq [A_i] \text{ iff } i \in I.$$

There is a one-to-one correspondence between the atoms of $\mathcal{FB}_m$ and valuations $v : \{p_1, \ldots, p_m\} \to \{T, F\}$.[10] Namely, for every atom $a = [p_1^{x_1} \wedge \ldots \wedge p_m{}^{x_m}]$ we uniquely assign the valuation v_a such that $v_a(p_i) = x_i$. On the other hand for every valuation v we uniquely assign the atom a_v such that $a_v = [p_1^{v(p_1)} \wedge \ldots \wedge p_m^{v(p_m)}]$. Then a and v_a as well as a_v and v will be said to be comprise a *corresponding atom and valuation*.

PROPOSITION 3.2. *Let a and v be a pair of corresponding atoms of $\mathcal{FB}_m$ and valuation. Then, for every $[A] \in \mathcal{FB}_m$,*

$$a \leq [A] \text{ iff } v(A) = T. \tag{1}$$

Proof. Let $a = [p_1^{x_1} \wedge \ldots \wedge p_m{}^{x_m}]$. By the definition of $\leq$, the condition $a \leq [A]$ is equivalent to

$$p_1^{x_1} \wedge \ldots \wedge p_m{}^{x_m} \vdash_{\mathsf{CL}} A. \tag{2}$$

The fact that $v(A) = T$ implies (2) follows from the well-known Kalmár Lemma (for the Completeness Theorem for CL), as in [9, Lemma 1.13]. If (2) holds, then (by the Soundness Theorem for CL), since $v(p_1^{x_1} \wedge \ldots \wedge p_m{}^{x_m}) = T$, we have $v(A) = T$. □

PROPOSITION 3.3. *The rule $\langle A_1, \ldots, A_{n+1} \rangle \in \mathsf{R}_m^n$ obeys a ban on $\langle x_1, \ldots, x_{n+1} \rangle$ iff there is no atom a of $\mathcal{FB}_m$ such that*

$$a \leq [A_i] \text{ iff } x_i = T.$$

Proof. From the definition of bans (Definition 2.3(iii)) and Proposition 3.2. □

By Propositions 3.2 and 3.3 we get the following:

COROLLARY 3.4. *A rule $\langle A_1, \ldots, A_n, A_{n+1} \rangle \in \mathsf{R}_m^n$ is correct iff there is no atom a of $\mathcal{FB}_m$ such that $a \leq [A_i]$ iff $i \in \mathbf{n}$.*

This explains the restriction on the set I imposed in Definition 3.1(ii).

PROPOSITION 3.5. *A rule $R = \langle A_1, \ldots, A_n, A_{n+1} \rangle \in \mathsf{R}_m^n$ is exactly valid iff the corresponding sequence $\bar{R} = \langle [A_1], \ldots, [A_n], [A_{n+1}] \rangle$ of elements of $\mathcal{FB}_m$ is flexible.*

[10] As valuations were introduced in the discussion before Definition 2.3 they assigned a truth-value to every formula of the language, so the language for present purposes is the set of formulas constructed from sentence letters $p_1, \ldots, p_m$, and since Boolean valuations are uniquely determined by their behaviour on the sentence letters, here we are identifying valuations with assignments of truth-values to the sentence letters $p_1, \ldots, p_m$.

Proof. First we show that the rule R is correct iff the sequence $\bar{R}$ is correct. Assume that R is correct, i.e., $A_1, \ldots, A_n \vdash_{\mathsf{CL}} A_{n+1}$. Then in $\mathcal{FB}_m$, we have $\bigwedge_{1\le i\le n}[A_i] \le [A_{n+1}]$, and in particular, for every atom a, if $a \le \bigwedge_{1\le i\le n}[A_i]$ then $a \le [A_{n+1}]$ which amounts to the correctness of the sequence $\bar{R}$. Now suppose that the rule R is not correct. Then $A_1, \ldots, A_n \nvdash_{\mathsf{CL}} A_{n+1}$, so there is a valuation υ such that $\upsilon(A_i) = T$ for $1 \le i \le n$, and $\upsilon(A_{n+1}) = F$. Then it follows from Proposition 3.2 that for the corresponding atom a_υ we have $a_\upsilon \le \bigwedge_{1\le i\le n}[A_i]$ and $a_\upsilon \not\le [A_{n+1}]$, i.e., the sequence $\bar{R}$ is not correct.

Assume that $\bar{R}$ is exactly valid. Consider $I \subseteq (\mathbf{n}+\mathbf{1})$ such that $I \ne \mathbf{n}$. Then there is a valuation υ such that

$$\upsilon(A_i) = \begin{cases} T & \text{if } i \in I, \\ F & \text{if } i \in (\mathbf{n{+}1}) \smallsetminus I. \end{cases}$$

Hence $\upsilon(\bigwedge_{i\in I} A_i \wedge \bigwedge_{i\in(\mathbf{n+1})\smallsetminus I} \neg A_i) = T$. By Proposition 3.2, for the corresponding atom a_υ we have $a_\upsilon \le [A_i]$ iff $i \in I$.

Assume that the sequence $\bar{R} = \langle [A_1], \ldots, [A_n], [A_{n+1}]\rangle$ is flexible. Let

$$\vec{x} = \langle x_1, \ldots, x_{n+1}\rangle \in \{T, F\}^{n+1}$$

such that $\vec{x} \ne \langle T^n, F\rangle$. Let $I = \{i \in (\mathbf{n}+\mathbf{1}) : x_i = T\}$. Then there is an atom a such that $a \le [A_i]$ iff $i \in I$. Then, by Proposition 3.2, for the valuation υ_a corresponding to a, we have $\upsilon_a(A_i) = T$ iff $i \in I$. Hence it follows that $\upsilon_a(A_i) = x_i$ for $1 \le i \le n+1$. □

DEFINITION 3.6. *Let $\bar{R} = \langle [A_1], \ldots, [A_n], [A_{n+1}]\rangle$ be a sequence of elements of the algebra $\mathcal{FB}_m$. For every set $I \subseteq (\mathbf{n}+\mathbf{1})$ we define the following sets of atoms:*

$$\mathrm{At}(\bar{R}, I) := \{a \in \mathrm{Atoms}(\mathcal{FB}_m) : a \le [A_i] \text{ iff } i \in I\},$$

and let

$$\bar{\beta}(\bar{R}) = \{I \subseteq (\mathbf{n}+\mathbf{1}) : \mathrm{At}(\bar{R}, I) = \emptyset\}.$$

PROPOSITION 3.7. *Let $\bar{R} = \langle [A_1], \ldots, [A_{n+1}]\rangle$ and $\bar{S} = \langle [B_1], \ldots, [B_{n+1}]\rangle$ be sequences of elements of the free Boolean algebra $\mathcal{FB}_m$ such that $\bar{\beta}(\bar{R}) \subseteq \bar{\beta}(\bar{S})$. Then there is an endomorphism $\varepsilon : \mathcal{FB}_m \to \mathcal{FB}_m$ such that $\varepsilon([A_i]) = [B_i]$, for all $i \in (\mathbf{n}+\mathbf{1})$.*

Proof. Notice that

$$\bigcup\{\mathrm{At}(\bar{R}, I) : I \subseteq (\mathbf{n}+\mathbf{1})\} = \bigcup\{\mathrm{At}(\bar{S}, I) : I \subseteq (\mathbf{n}+\mathbf{1})\} = \mathrm{Atoms}(\mathcal{FB}_m).$$

Indeed, for any atom a, we have $a \in \mathrm{At}(\bar{R}, I_a)$, where $I_a = \{i \in (\mathbf{n}+\mathbf{1}) : a \le [A_i]\}$. Similarly in the other case.

In terms of the families $\{\mathrm{At}(R, I) : I \subseteq (\mathbf{n}+\mathbf{1})\}$ and $\{\mathrm{At}(S, I) : I \subseteq (\mathbf{n}+\mathbf{1})\}$ we will define an endomorphism $\varepsilon : \mathcal{FB}_m \to \mathcal{FB}_m$.

First, we define a function $\bar{\varepsilon} : \mathrm{Atoms}(\mathcal{FB}_m) \to \mathcal{FB}_m$ in the following way. For every nonempty set $\mathrm{At}(\bar{R}, I)$ we choose an atom $a_I \in \mathrm{At}(\bar{R}, I)$ and put

$$\bar{\varepsilon}(a_I) = \bigvee\{b : b \in \mathrm{At}(\bar{S}, I)\}$$

and $\bar{\varepsilon}(a) = 0$ for other atoms in $\mathrm{At}(\bar{R}, I)$ and when $\mathrm{At}(\bar{S}, I) = \emptyset$.

Obviously, every nonzero element of the algebra $\mathcal{FB}_m$ can be uniquely represented by a join of the some atoms of $\mathcal{FB}_m$. In particular, for each $i \in \mathbf{m}$ we have

$$[p_i] = \bigvee\{[p_1^{x_1} \wedge \ldots \wedge p_m^{x_m}] : x_i = T\}.$$

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Now we can define the required endomorphism by setting its values on the set of generators of the algebra $\mathcal{FB}_m$:

$$\varepsilon([p_i]) = \bigvee\{\bar{\varepsilon}([p_1^{x_1} \wedge \ldots \wedge p_m^{x_m}]) : x_i = T\}. \tag{3}$$

Notice that for any elements $[A_i]$ and $[B_i]$ of $\mathcal{FB}_m$ where $i \in (\mathbf{n}+\mathbf{1})$, we have

$$\begin{aligned}[A_i] &= \bigvee\{a : a \in \mathrm{At}(\bar{R}, I) \text{ and } i \in I\}\\ [B_i] &= \bigvee\{b : b \in \mathrm{At}(\bar{S}, I) \text{ and } i \in I\}.\end{aligned}$$

Recall that if $a \in \mathrm{At}(\bar{R}, I)$ for some set I, then $\varepsilon(a)$ is a join of atoms of $\mathrm{At}(\bar{S}, I)$ or $\varepsilon(a)$ is equal to 0 when $\mathrm{At}(\bar{S}, I)$ is the empty set. Moreover, by the assumption

$$\bar{\beta}(\bar{R}) \subseteq \bar{\beta}(\bar{S}),$$

each nonempty set $\mathrm{At}(\bar{S}, I)$ of atoms is represented as $\bar{\varepsilon}(a)$ for some a. So, in a sense, all the atoms of $\mathcal{FB}_m$ are in the range of $\bar{\varepsilon}$. Consequently, for every $i \in (\mathbf{n+1})$, we get

$$\begin{aligned}\varepsilon([A_i]) &= \bigvee\{\varepsilon(a) : a \in \mathrm{At}(\bar{R}, I) \text{ and } i \in I\}\\ &= \bigvee\{b : b \in \mathrm{At}(\bar{S}, I) \text{ and } i \in I\}\\ &= [B_i].\end{aligned}$$

Hence ε is the desired endomorphism of the algebra $\mathcal{FB}_m$. □

Notice that from Proposition 3.7 it follows that if $\bar{R} = \langle [A_1], \ldots, [A_{n+1}]\rangle$ a flexible sequence then for every correct sequence $\bar{S} = \langle [B_1], \ldots, [B_{n+1}]\rangle$ of elements of $\mathcal{FB}_m$, there is an endomorphism $\varepsilon : \mathcal{FB}_m \to \mathcal{FB}_m$ such that $\varepsilon([A_i]) = [B_i]$, for all $i \in (\mathbf{n}+\mathbf{1})$.

Recall that $\beta(A_1, \ldots, A_{n+1})$ is the set of bans to $\{T, F\}^{n+1}$ imposed in respect of the rule $\langle A_1, \ldots, A_{n+1}\rangle$.

COROLLARY 3.8. *Let $R = \langle A_1, \ldots, A_{n+1}\rangle$ and $S = \langle B_1, \ldots, B_{n+1}\rangle$ be any rules of R_m^n for which $\beta(R) \subseteq \beta(S)$. Then there is a substitution $\mathfrak{s}$ for which we have $\mathfrak{s}(A_i) \dashv\vdash_{\mathsf{CL}} B_i$, for $i \in (\mathbf{n}+\mathbf{1})$.*

Proof. Consider the sequences $\bar{R}$ and $\bar{S}$ corresponding to the rules R and S, respectively. It is easy to see that

$$\bar{\beta}(\bar{R}) \subseteq \bar{\beta}(\bar{S}) \text{ iff } \beta(R) \subseteq \beta(S).$$

Then, by Proposition 3.7, there is an endomorphism ε of the algebra $\mathcal{FB}_m$ such that $\varepsilon([A_i]) = [B_i]$, for all $i \in (\mathbf{n}+\mathbf{1})$. The required substitution $\mathfrak{s}$ can be defined, up to equivalence in CL, by means of ε in the natural way by putting $\mathfrak{s}(p_i)$ to be the disjunction of the representatives of the equivalence classes of atoms as in (3). □

THEOREM 3.9. *A rule is archetypal if and only if it is exactly valid.*

Proof. The 'only if' half was Proposition 2.6. The 'if' half is a special case of Corollary 3.8, since that tells us that if a rule R is exactly valid, and hence $\beta(R) \subseteq \beta(S)$ for any valid rule S with the same number of premisses, then there is a substitution subsuming S under R, showing R to be archetypal. □

EXAMPLE 3.10. Identifying rules with their skeletons, recall the rules

$$R = p_1, p_2 \,/\, (p_1 \wedge p_2) \vee p_3 \textit{ and } S = p_1 \wedge (p_2 \vee p_3), p_1 \leftrightarrow (p_2 \leftrightarrow p_3) \,/\, p_3$$

noted in Examples 2.2(i) and 2.9 to be, respectively, archetypal and exactly valid (at a stage before we had proved these two notions to be coextensive; for continuity with our more recent discussion, here write p, q, r explicitly p_1, p_2, p_3). To illustrate the procedure described in the proof of Proposition 3.7 we follow that procedure to find a substitution $\mathfrak{s}$ such that $\mathfrak{s}(S) = R$.

We consider the Lindenbaum algebra $\mathcal{FB}_3$ and the atoms a_i of $\mathcal{FB}_3$. Let $x_1x_2x_3$ be the binary representation of the number i, then, identifying 0 and 1 with F and T, respectively,

$$a_i = [p_1^{x_1} \wedge p_2{}^{x_2} \wedge p_3{}^{x_3}],$$

where, as at the start of this section, $p_i^0 = \neg p_i$ and $p_i^1 = p_i$, for the propositional variables p_i.

One can check that

$$\begin{array}{rl}
\mathrm{At}(\bar{S}, \emptyset) = \{a_0\} & \mathrm{At}(\bar{R}, \emptyset) = \{a_0\} \\
\mathrm{At}(\bar{S}, \{1\}) = \{a_6\} & \mathrm{At}(\bar{R}, \{1\}) = \{a_4\} \\
\mathrm{At}(\bar{S}, \{2\}) = \{a_2, a_4\} & \mathrm{At}(\bar{R}, \{2\}) = \{a_2\} \\
\mathrm{At}(\bar{S}, \{3\}) = \{a_3\} & \mathrm{At}(\bar{R}, \{3\}) = \{a_1\} \\
\mathrm{At}(\bar{S}, \{1, 2\}) = \emptyset & \mathrm{At}(\bar{R}, \{1, 2\}) = \emptyset \\
\mathrm{At}(\bar{R}, \{1, 3\}) = \{a_5\} & \mathrm{At}(\bar{S}, \{1, 3\}) = \{a_5\} \\
\mathrm{At}(\bar{R}, \{2, 3\}) = \{a_1\} & \mathrm{At}(\bar{S}, \{2, 3\}) = \{a_3\} \\
\mathrm{At}(\bar{R}, \{1, 2, 3\}) = \{a_7\} & \mathrm{At}(\bar{S}, \{1, 2, 3\}) = \{a_6, a_7\}.
\end{array}$$

Since $\mathrm{At}(\bar{S}, \{2\})$ contains two elements, we have to consider two cases according to the choice of the element of this set. So, we have the functions ε_1 and ε_2 defined on the set of atoms.

$$\begin{array}{rl}
\varepsilon_1(a_0) = a_0 & \varepsilon_2(a_0) = a_0 \\
\varepsilon_1(a_1) = a_3 & \varepsilon_2(a_1) = a_3 \\
\varepsilon_1(a_2) = a_2 & \varepsilon_2(a_2) = 0 \\
\varepsilon_1(a_3) = a_1 & \varepsilon_2(a_3) = a_1 \\
\varepsilon_1(a_4) = 0 & \varepsilon_2(a_4) = a_2 \\
\varepsilon_1(a_5) = a_5 & \varepsilon_2(a_5) = a_5 \\
\varepsilon_1(a_6) = a_4 & \varepsilon_2(a_6) = a_4 \\
\varepsilon_1(a_7) = a_6 \vee a_7 & \varepsilon_2(a_7) = a_6 \vee a_7.
\end{array}$$

So we have

$$\begin{array}{rcl}
\varepsilon_1([p_1]) & = & \varepsilon_1(a_4 \vee a_5 \vee a_6 \vee a_7) = a_4 \vee a_5 \vee a_6 \vee a_7 \\
\varepsilon_1([p_2]) & = & \varepsilon_1(a_2 \vee a_3 \vee a_6 \vee a_7) = a_1 \vee a_2 \vee a_4 \vee a_6 \vee a_7 \\
\varepsilon_1([p_3]) & = & \varepsilon_1(a_1 \vee a_3 \vee a_5 \vee a_7) = a_1 \vee a_3 \vee a_5 \vee a_6 \vee a_7.
\end{array}$$

Now we can compute the required substitution:

$$\begin{array}{rcl}
\mathfrak{s}_1(p_1) & = & p_1 \\
\mathfrak{s}_1(p_2) & = & (p_1 \to (p_3 \to p_2) \wedge ((p_2 \leftrightarrow p_3) \to p_1) \\
\mathfrak{s}_1(p_3) & = & (p_1 \wedge p_2) \vee p_3.
\end{array}$$

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In a similar way we can compute the other substitution

$$\begin{aligned} s_2(p_1) &= (p_2 \to p_3) \to p_1 \\ s_2(p_2) &= (p_3 \to p_2) \leftrightarrow p_1 \\ s_2(p_3) &= (p_1 \wedge p_2) \vee p_3. \end{aligned}$$ ◀

We illustrate the same procedure but now considering a rule which is correct though not archetypal, with a view to subsuming it under the rule S of Example 3.10.

EXAMPLE 3.11. The rule in question is the Modus Ponens rule, $MP = \langle p_1 \to p_2, p_1, p_2 \rangle$, observed in the discussion immediately following Example 2.2 not to be archetypal (something later seen in Observation 2.7 to follow immediately from the fact that $MP \in \mathsf{R}^2_2$. We wish to find s with $s(S) = MP$, S being as in Example 3.10. Proceeding as before we can find two required substitutions either of which will serve as such an s:

$$\begin{aligned} s_3(p_1) &= p_1 \to p_2 \\ s_3(p_2) &= p_2 \to p_1 \\ s_3(p_3) &= p_2 \end{aligned}$$

and

$$\begin{aligned} s_4(p_1) &= \top \\ s_4(p_2) &= p_1 \leftrightarrow p_2 \\ s_4(p_3) &= p_2. \end{aligned}$$ ◀

REMARK 3.12. Recall (from Observation 2.7) that if $R = \langle A_1, \ldots, A_{n+1} \rangle \in \mathsf{R}^n_m$ is exactly valid then $m \geq n + 1$. One can show that for such a rule R there is a substitution s for which the rule $s(R) = \langle s(A_1), \ldots, s(A_{n+1}) \rangle \in \mathsf{R}^n_{n+1}$ is an exactly valid $(n + 1)$-ary rule. The proof is similar to that of Theorem 2 of [11]. (This fact, in case of $n = 1$, was used in [10] to prove that every nondegenerate 1-premiss rule is archetypal by reducing the problem for the rules in more than two variables to the 2-variable rules, the latter having been settled in [3], as was recalled in Theorem 1.4 and the surrounding discussion.) ◀

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