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Homotopic distance between maps

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Abstract

We show that well-known invariants like Lusternik–Schnirelmann category and topological complexity are particular cases of a more general notion, that we call *homotopic distance* between two maps. As a consequence, several properties of those invariants can be proved in a unified way and new results arise.

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1. *Introduction*

In this paper we prove that well-known homotopic invariants like the Lusternik–Schnirelmann category $\mathrm{cat}(X)$ of the topological space X ([**2**]) or the topological complexity $\mathrm{TC}(X)$ ([**3**]) can be seen as particular cases of a more general notion, that we call *homotopic distance* between two continuous maps f, g, denoted $\mathrm{D}(f, g)$. As a consequence, the proofs of several properties of those invariants can be unified in a systematic way, and new results arise.

It can be conjectured that this unifying approach will give a new insight about the relationship between $\mathrm{cat}(X)$ and $\mathrm{TC}(X)$; moreover, the inequalities we found could serve as new lower bounds for the difficult problem of computing the category and topological complexity in explicit examples.

The contents of the paper are as follows.

Section 2 is devoted to the basic definitions and examples. Given two continuous maps $f, g\colon X \to Y$ between topological spaces, we say that $\mathrm{D}(f, g) \leqslant n$ if there exists an open covering $\{U_0, \ldots, U_n\}$ of X such that the restrictions $f_{|U_j}, g_{|U_j}\colon U_j \to Y$ are homotopic maps, for all $j = 0, \ldots, n$. Then, by definition, $\mathrm{cat}(X)$ is the distance between id_X and a constant map. We show that $\mathrm{cat}(X)$ also equals the homotopic distance $\mathrm{D}(i_1, i_2)$ between the two axis inclusions $i_1, i_2\colon X \to X \times X$ (Proposition 2·5), while $\mathrm{TC}(X)$ equals the homotopic distance $\mathrm{D}(p_1, p_2)$ between the projections $p_1, p_2\colon X \times X \to X$ (Proposition 2·6).

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In Section 3 we prove several properties of the homotopic distance, namely its behaviour under compositions and products, and its homotopical invariance. They imply as particular cases well-known inequalities like $\operatorname{cat}(X) \leqslant \operatorname{TC}(X) \leqslant \operatorname{cat}(X \times X)$ (Corollary 3·6 and Corollary 3·9), or $\operatorname{TC}(X \times X') \leqslant \operatorname{TC}(X) + \operatorname{TC}(X')$ (Example 3·21).

In Section 4 we study H-spaces. For instance we prove that for any pair of maps $f, g\colon G \times G \to G$, where G is an H-space, we have $\mathrm{D}(f, g) \leqslant \operatorname{cat}(G)$ (Theorem 4·1), thus generalising the theorem of Farber, Lupton, and Scherer $\operatorname{TC}(G) = \operatorname{cat}(G)$ [**4**, **15**].

In Section 5 we give a lower cohomological bound for the homotopic distance, in terms of the length of the cup product, namely $D(f, g) \geqslant \text{l.c.p.}\ \mathcal{J}(f, g)$ (Theorem 5·2), where we denote by $\mathcal{J}(f, g)$ the image of $f^* - g^*$ in $H(X)$. Similar results are well known for the particular cases of $\operatorname{cat}(X)$ or $\operatorname{TC}(X)$.

A better result is obtained after defining the so-called *homotopy weight* $\operatorname{hw}_{f,g}(u) \geqslant 1$ of the non-zero cohomology class $u \in \mathcal{J}(f, g) \subset H(X)$. This generalises ideas from Fadell–Husseini and other authors, and we are able to prove (Theorem 5·9) that if $u_0 \smile \cdots \smile u_k \neq 0$, then

$$\mathrm{D}(f, g) \geqslant \sum_{j=0}^{k} \operatorname{hw}(u_j).$$

Section 6 is about fibrations. We generalise both Varadarajan's result [**25**] about the relationship between the LS-category of the total space E, the fiber F and the base B, and a similar result for the topological complexity, due to Farber and Grant [**6**, lemma 7]. Explicitly, we prove (Theorem 6·1) that, when the base B is path-connected,

$$\mathrm{D}(f, g) + 1 \leqslant \big(\mathrm{D}(f_0, g_0) + 1\big)\big(\operatorname{cat}(B) + 1\big)$$

for fibre preserving maps f, g that induce maps $f_0, g_0\colon F_0 \to F_0'$ between the fibers.

In Section 7 we show an example of a Lie group G and two maps $f, g\colon G \to G$ such that $\mathrm{D}(f, g) = 2 = \text{l.c.p.}\ H(G)$, while $\operatorname{TC}(G) = \operatorname{cat}(G) = 3$.

Finally, Section 8 contains an overview of possible generalisations, like a version in the simplicial setting or an analog of higher topological complexity.

2. *Basic notions*

All along the paper we work with unpointed spaces, unless otherwise stated.

2·1. *Homotopic distance*

Let $f, g\colon X \to Y$ be two continuous maps.

Definition 2·1. The *homotopic distance* $\mathrm{D}(f, g)$ between f and g is the least integer $n \geqslant 0$ such that there exists an open covering $\{U_0, \ldots, U_n\}$ of X with the property that $f_{|U_j} \simeq g_{|U_j}$, for all $j = 0, \ldots, n$. If there is no such covering, we define $\mathrm{D}(f, g) = \infty$.

Remark. In order to simplify several proofs, we shall denote by $\boldsymbol{U} = U_0 \sqcup \cdots \sqcup U_n$ the disjoint union, and by $\boldsymbol{u}\colon \boldsymbol{U} \to X$ the map induced by the inclusions, by the coproduct property. Then, $f \circ \boldsymbol{u} \simeq g \circ \boldsymbol{u}$, if and only if $f_{|U_j} \simeq g_{|U_j}$, for all $j = 0, \ldots, n$.

Notice that:

(i) $\mathrm{D}(f, g) = \mathrm{D}(g, f)$;
(ii) $\mathrm{D}(f, g) = 0$ if and only if the maps f, g are homotopic.

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In fact, the homotopic distance only depends on the homotopy class.

PROPOSITION 2·2. If $f \simeq f'$ and $g \simeq g'$ then $\mathrm{D}(f, g) = \mathrm{D}(f', g')$.

Later (Proposition 3·13) we shall show that $\mathrm{D}(-, -)$ is an invariant of the homotopy class of a pair of maps.

Example 2·3. Let $X = S^1$ be the Lie group of unit complex numbers. The distance between the identity z and the inversion $1/z$ is 1. Let $X = S^2$ be the unit sphere. The distance between the identity and the antipodal map is 1 (see Corollary 3·8).

Example 2·4. Let G be the unitary group U(2). The distance between the identity id_G and the inversion $I(A) = A^*$ is $\mathrm{D}(\mathrm{id}_G, I) = 2$ (see Equation (5·1)).

The two key examples of homotopic distance are Lusternik–Schnirelmann category ([**2**]) and Farber's topological complexity ([**3**]), as we shall show in the next paragraphs.

2·2. *Lusternik–Schnirelmann category*

Assume the space X to be path-connected. An open set $U \subset X$ is *categorical in* X if the inclusion is null-homotopic. The (normalised) LS-category $\mathrm{cat}(X)$ is the least integer $n \geqslant 0$ such that X can be covered by $n + 1$ categorical open sets. Then, $\mathrm{cat}(X)$ is the homotopic distance between the identity id_X and any constant map, that is, $\mathrm{cat}(X) = \mathrm{D}(\mathrm{id}_X, *)$.

More generally, the Lusternik–Schnirelmann category of the map $f : X \to Y$ ([**2**], exercise 1·16, p. 43]) is the distance between f and any constant map, $\mathrm{cat}(f) = \mathrm{D}(f, *)$, when Y is path-connected. For instance, the category of the diagonal $\Delta_X : X \to X \times X$ equals $\mathrm{cat}(X)$.

Given a base point $x_0 \in X$ we define the inclusion maps $i_1, i_2 : X \to X \times X$ as $i_1(x) = (x, x_0)$ and $i_2(x) = (x_0, x)$.

PROPOSITION 2·5. *The homotopic distance between* i_1 *and* i_2 *equals the LS-category of* X, *that is*, $\mathrm{D}(i_1, i_2) = \mathrm{cat}(X)$.

Proof. First, we show that $\mathrm{D}(i_1, i_2) \leqslant \mathrm{cat}(X)$.

Let $X = U_0 \cup \cdots \cup U_n$ be a categorical cover and let $\boldsymbol{u} : \boldsymbol{U} \to X$ as in the Remark after Definition 2·1; thus $\mathrm{id}_X \circ \boldsymbol{u} \simeq * \circ \boldsymbol{u}$. But then

$$i_1 \circ \boldsymbol{u} = (\mathrm{id}_X, *) \circ \boldsymbol{u} = (\mathrm{id}_X \circ \boldsymbol{u}, * \circ \boldsymbol{u}) \simeq (* \circ \boldsymbol{u}, \mathrm{id}_X \circ \boldsymbol{u}) = (*, \mathrm{id}_X) \circ \boldsymbol{u} = i_2 \circ \boldsymbol{u}.$$

Second, we show that $\mathrm{cat}(X) \leqslant \mathrm{D}(i_1, i_2)$. Assume that there is a homotopy $\mathcal{H} : U \times [0, 1] \to X \times X$ between $(i_1)_{|U}$ and $(i_2)_{|U}$, i.e. $\mathcal{H}(x, 0) = (x, x_0)$ and $\mathcal{H}(x, 1) = (x_0, x)$. Let $p_1 \circ \mathcal{F}$ be the first component of $\mathcal{F}$. Then $p_1 \circ \mathcal{F}$ is a homotopy between the inclusion $U \subset X$ and the constant map x_0.

2·3. *Topological complexity*

Let $PX = X^I$ be the path space of X. Let $\pi : PX \to X \times X$, with $\pi(\gamma) = \big(\gamma(0), \gamma(1)\big)$, be the path fibration sending each continuous path $\gamma : [0, 1] \to X$ onto its initial and final points. By definition, the (normalised) topological complexity $\mathrm{TC}(X)$ of X is the least integer n such that $X \times X$ can be covered by $n + 1$ open subsets U_j where the fibration π admits a continuous local section.

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PROPOSITION 2·6. *The topological complexity of X equals the homotopic distance between the two projections $p_1, p_2\colon X \times X \to X$, that is,* $\mathrm{TC}(X) = \mathrm{D}(p_1, p_2)$.

This result will be a consequence of Theorem 2·7 below.

2·4. *Švarc genus*

Both $\mathrm{cat}(X)$ and $\mathrm{TC}(X)$ are particular cases of the Švarc genus (also called sectional category) of some fibrations. Explicitly ([2]) the *Švarc genus* $\mathrm{secat}(\pi)$ of a fibration $\pi : E \to B$ is the minimum integer $n \geqslant 0$ such that the base B can be covered by open sets $V_0, \ldots, V_n$ with the property that over each V_j there exists a local section s of π. For instance, $\mathrm{cat}(X)$ is the Švarc genus of the fibration $\pi_0\colon P_0X \to X$ sending each path γ with initial point x_0 into the end point $\gamma(1)$.

What follows is an interpetation of the homotopic distance in terms of the Švarc genus.

THEOREM 2·7. *Let $f, g\colon X \to Y$ be two maps, and consider the pull-back $q\colon P \to X$ of the path fibration $\pi : PY \to Y \times Y$ by the map $(f, g)\colon X \to Y \times Y$:*

$$\begin{array}{ccc} P & \longrightarrow & PY \\ \downarrow q & & \downarrow \pi \\ X & \xrightarrow{(f,g)} & Y \times Y. \end{array}$$

Then $\mathrm{D}(f, g) = \mathrm{secat}(q)$.

Proof. The elements of P are the pairs (x, γ) where $x \in X$ and γ is a path on Y with $\gamma(0) = f(x)$ and $\gamma(1) = g(x)$. The map q is the projection onto the first factor. Then, if $U \subset X$ is an open set where there exists a homotopy $\mathcal{H}\colon U \times I \to Y$ between $f_{|U}$ and $g_{|U}$, we can define a section $s\colon X \to P$ as $s(x) = \big(x, \mathcal{H}(x, -)\big)$. Then $\mathrm{secat}(q) \leqslant D(f, g)$.

Conversely, if there is a map $s\colon U \to P$ such that $q \circ s$ is the inclusion $i_U\colon U \subset X$, we have $s(x) = (x, \gamma_x)$ for some path γ_x from $f(x)$ to $g(x)$. Then, the homotopy $\mathcal{H}(x, t) = \gamma_x(t)$ proves that $f \simeq g$ on U.

As a consequence, if we take $f = p_1$ and $g = p_2$ to be the projections from $X \times X \to X$ we have $(f, g) = \mathrm{id}_{X\times X}$ and $q = \pi$, thus proving Proposition 2·6, that is,

$$\mathrm{D}(p_1, p_2) = \mathrm{secat}(\pi) = \mathrm{TC}(X).$$

3. *Properties*

3·1. *Compositions*

We now prove several elementary properties, starting with the behaviour of the homotopic distance under compositions. Several known properties of cat and TC can be deduced from our general results.

PROPOSITION 3·1. *Suppose we have maps $f, g\colon X \to Y$ and $h\colon Y \to Z$. Then*

$$\mathrm{D}(h \circ f, h \circ g) \leqslant \mathrm{D}(f, g).$$

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Proof. Let $\mathrm{D}(f, g) \leqslant n$ and let $X = U_0 \cup \cdots \cup U_n$ be an open covering with $f_j = f_{|U_j}$ homotopic to $g_j = g_{|U_j}$. Then $\mathrm{D}(h \circ f, h \circ g) \leqslant n$ because

$$(h \circ f)_j = h \circ f_j \simeq h \circ g_j = (h \circ g)_j.$$

COROLLARY 3·2. *Let $f\colon X \to Y$ be a map with path-connected domain X and codomain Y. Then* $\mathrm{cat}(f) \leqslant \mathrm{cat}(X)$.

Proof. Take id_X and a constant map x_0. Then $\mathrm{D}(f \circ \mathrm{id}_X, f(x_0)) \leqslant \mathrm{D}(\mathrm{id}_X, x_0)$.

PROPOSITION 3·3. *Suppose we have maps $f, g\colon X \to Y$ and $h\colon Z \to X$. Then*

$$\mathrm{D}(f \circ h, g \circ h) \leqslant \mathrm{D}(f, g).$$

Proof. Let $\mathrm{D}(f, g) \leqslant n$ and let $X = U_0 \cup \cdots \cup U_n$ be an open covering with $f_j \simeq g_j\colon U_j \to Y$. Let $V_j = h^{-1}(U_j) \subset Z$. The restriction $h_j\colon V_j \to X$ can be written as the composition of a map $\bar{h}_j\colon V_j \to U_j$, where $\bar{h}_j(x) = h(x)$, and the inclusion $I_j\colon U_j \subset X$. Then we have that

$$(f \circ h)_j = f_j \circ \bar{h}_j \simeq g_j \circ \bar{h}_j = g \circ I_j \circ \bar{h}_j = g \circ h_j = (g \circ h)_j,$$

hence $D(fh, gh) \leqslant n$.

COROLLARY 3·4. *If $f\colon X \to Y$ is a continuous map with a path-connected codomain Y, then* $\mathrm{cat}(f) \leqslant \mathrm{cat}(Y)$.

Proof. Take id_Y and a constant map y_0. Then $D(\mathrm{id}_Y \circ f, y_0 \circ f) \leqslant D(\mathrm{id}_Y, y_0)$.

The latter result result can be extended.

COROLLARY 3·5. *Let $f, g : X \to Y$ be continuous maps with a path-connected codomain Y. Then*

$$\mathrm{D}(f, g) + 1 \leqslant (\mathrm{cat}(f) + 1)(\mathrm{cat}(g) + 1).$$

Proof. Denote by y_0 a constant map from X to Y. Assume that $\mathrm{cat}(f) = \mathrm{D}(f, y_0) \leqslant m$, $\mathrm{cat}(g) = \mathrm{D}(g, y_0) \leqslant n$ and let $\{U_i\}_{i=0}^m$, $\{V_j\}_{j=0}^n$ be the corresponding coverings of X. The open sets $W_{i,j} = U_i \cap V_j$ cover X. Moreover, $f \simeq y_0 \simeq g$ on $W_{i,j}$, so $D(f, g) \leqslant m \times n$.

The latter result will be greatly improved for normal spaces (see the remark after Proposition 3·16).

COROLLARY 3·6 ([3]). $\mathrm{cat}(X) \leqslant \mathrm{TC}(X)$.

Proof. In Proposition 3·3 consider the inclusion maps $i_1, i_2\colon X \to X \times X$, so

$$\mathrm{D}(*, \mathrm{id}_X) = \mathrm{D}(p_1 \circ i_2, p_2 \circ i_2) \leqslant \mathrm{D}(p_1, p_2).$$

In the next Proposition we shall prove a non-obvious inequality.

PROPOSITION 3·7. *Let $h, h'\colon Z \to X$ and $f, g\colon X \to Y$ be maps such that $f \circ h' \simeq g \circ h'$. Then*

$$\mathrm{D}(f \circ h, g \circ h) \leqslant D(h, h').$$

Proof. Assume $\mathrm{D}(h, h') \leqslant n$, and let $Z = U_0 \cup \cdots \cup U_n$ be a covering such that $h \circ \boldsymbol{u} \simeq h' \circ \boldsymbol{u}$, with the coproduct notation.

Then

$$f \circ h \circ \boldsymbol{u} \simeq f \circ h' \circ \boldsymbol{u} \simeq g \circ h' \circ \boldsymbol{u} \simeq g \circ h \circ \boldsymbol{u},$$

which implies that $\mathrm{D}(f \circ h, g \circ h) \leqslant n$, and the proof is complete.

3·2. *Domain and codomain*

Recall that the *geometric LS-category* of X, denoted by $\mathrm{gcat}(X)$, is the least integer $n \geqslant 0$ such that X can be covered by $n+1$ open sets which are *contractible in themselves*. This subtle difference with the LS-category —where the open sets are contractible in the ambient space— is important, because in general gcat is not a homotopy invariant. Since any map with a contractible domain is homotopic to a constant map, it is obvious that $\mathrm{D}(f, g) \leqslant \mathrm{gcat}(X)$ for any pair of continuous maps $f, g\colon X \to Y$.

The inequality $\mathrm{D}(f, g) \leqslant \mathrm{cat}(X)$ is less evident, but it follows directly from Proposition 3·7.

COROLLARY 3·8. *Let $f, g\colon X \to Y$ be two maps with path-connected domain X and codomain Y. Then*

$$\mathrm{D}(f, g) \leqslant \mathrm{cat}(X).$$

Proof. In Proposition 3·7, take $Z = X$, $h = \mathrm{id}_X$ and $h' = x_0$ a constant map, then the constant maps $f(x_0), g(x_0)\colon X \to Y$ are homotopic because Y is path-connected, so

$$\mathrm{D}(f, g) = \mathrm{D}(f \circ \mathrm{id}_X, g \circ \mathrm{id}_X) \leqslant D(\mathrm{id}_X, x_0) = \mathrm{cat}(X).$$

Another proof of this Corollary follows from Theorem 2·7: we have $\mathrm{D}(f, g) = \mathrm{secat}(q)$. Since q is a fibration, the homotopy lifting property implies that $\mathrm{secat}(q) \leqslant \mathrm{cat}(X)$.

COROLLARY 3·9 ([3]). $\mathrm{TC}(X) \leqslant \mathrm{cat}(X \times X)$.

Proof. In Corollary 3·8 take the maps $p_1, p_2\colon X \times X \to X$.

For the codomain, we have the following result.

PROPOSITION 3·10. *For maps $f, g\colon X \to Y$ we have* $\mathrm{D}(f, g) \leqslant \mathrm{TC}(Y)$.

Proof. This follows from Theorem 2·7, because if the fibration q is a pullback of the fibration π then $\mathrm{secat}(q) \leqslant \mathrm{secat}(\pi)$, which is exactly $\mathrm{D}(f, g) \leqslant \mathrm{TC}(Y)$.

Notice that in general it is not true that $\mathrm{D}(f, g) \leqslant \mathrm{cat}(Y)$. In fact, by taking the projections $p_1, p_2\colon Y \times Y \to Y$ this would imply that $\mathrm{TC}(Y) \leqslant \mathrm{cat}(Y)$, which is not true in general. However, this is true for H-spaces (see Section 4).

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3·3. *Invariance*

We prove the homotopy invariance of the homotopic distance.

PROPOSITION 3·11. *Let $f, g\colon X \to Y$ be maps and let $\alpha\colon Y \to Y'$ be a map with a left homotopy inverse. Then* $\mathrm{D}(\alpha \circ f, \alpha \circ g) = \mathrm{D}(f, g)$.

Proof. By Propositions 3·1 and 2·2, we have

$$\mathrm{D}(f, g) \geqslant \mathrm{D}(\alpha \circ f, \alpha \circ g) \geqslant \mathrm{D}(\beta \circ \alpha f, \beta \circ \alpha \circ g) = \mathrm{D}(f, g),$$

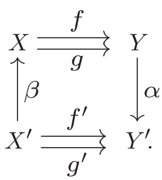
because $\beta \circ \alpha \simeq \mathrm{id}_Y$ implies $\beta \circ \alpha \circ f \simeq f$ and $\beta \circ \alpha \circ g \simeq g$.

Analogously:

PROPOSITION 3·12. *Let $f, g\colon X \to Y$ be maps and let $\beta\colon X' \to X$ be a map with a right homotopy inverse. Then* $\mathrm{D}(f \circ \beta, g \circ \beta) = \mathrm{D}(f, g)$.

As a consequence, D(,) is a homotopy invariant in the following sense:

PROPOSITION 3·13. *Assume that there exist homotopy equivalences $\beta : X' \simeq X$ and $\alpha : Y \simeq Y'$ such that $f\colon X \to Y$ (resp. g) and $f'\colon X' \to Y'$ (resp. g') verify $\alpha \circ f \circ \beta \simeq f'$ (resp. $\alpha \circ g \circ \beta \simeq g'$):*



Then $\mathrm{D}(f, g) = \mathrm{D}(f', g')$.

COROLLARY 3·14. *Both* cat( ) *and* TC( ) *are homotopy invariant, that is, if there exist homotopy equivalences $X \simeq X'$, then* $\mathrm{cat}(X) = \mathrm{cat}(X')$ *and* $\mathrm{TC}(X) = \mathrm{TC}(X')$.

3·4. *Normal spaces*

For normal spaces we shall use the following strikingly general Lemma, proved by Oprea and Strom [**17**, lemma 4·3].

LEMMA 3·15. *Let Z be a normal space with two open covers $\mathcal{U} = \{U_0, \ldots, U_m\}$ and $\mathcal{V} = \{V_0, \ldots, V_n\}$ such that each set of $\mathcal{U}$ satisfies Property (A) and each set of $\mathcal{V}$ satisfies Property (B). Assume that Properties (A) and (B) are inherited by open subsets and disjoint unions. Then Z has an open cover $\mathcal{W} = \{W_0, \ldots, W_{m+n}\}$ by open sets, each satisfying both Property (A) and Property (B).*

As a first consequence, we prove that the homotopic distance verifies the triangular inequality, thus being a true distance in the space of homotopy classes.

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PROPOSITION 3·16. *Let $f, g, h\colon X \to Y$ be maps defined on the normal space X. Then*

$$\mathrm{D}(f, h) \leqslant \mathrm{D}(f, g) + \mathrm{D}(g, h).$$

Proof. Let $\mathrm{D}(f, g) = m$ and $\mathrm{D}(g, h) = n$. Take coverings $\{U_0, \dots, U_m\}$ and $\{V_0, \dots, V_n\}$ of X such that $f_{|U_i} \simeq g_{|U_i}$ for all $i = 0, \dots m$ and $g_{|V_j} \simeq h_{|V_j}$ for all $j = 0, \dots, n$. Clearly these properties are closed for open subsets and disjoint unions. Then, by Lemma 3·15, there is a third covering $\{W_0, \dots, W_{m+n}\}$ such that $f_{|W_k} \simeq g_{|W_k} \simeq h_{|W_k}$, for all k, thus proving that $\mathrm{D}(f, h) \leqslant m + n$.

Remark. Proposition 3·16 does not hold in general for arbitrary topological spaces, as we shall show in Subsection 3·6.

Note that Corollary 3·5 could be improved (in normal spaces), because $\mathrm{D}(f, g) \leqslant \mathrm{D}(f, *) + \mathrm{D}(*, g)$ means that $\mathrm{D}(f, g) \leqslant \mathrm{cat}(f) + \mathrm{cat}(g)$.

Another result also follows from Lemma 3·15.

PROPOSITION 3·17. *Let X be a normal space. For maps $f, g\colon X \to Y$ and $f', g'\colon Y \to Z$ we have*

$$D(f' \circ f, g' \circ g) \leqslant D(f, g) + D(f', g').$$

Proof. If $D(f, g) = m$ there is a covering $U_0, \dots, U_m$ of X where $f \simeq g$. It follows that $g' \circ f \simeq g' \circ g$ for this covering. Clearly, the criteria of Lemma 3·15 are verified.

Now, if $D(f', g') = n$, there is a covering $V_0, \dots, V_n$ of Y where $f' \simeq g'$. But then, the covering $f^{-1}(V_0), \dots, f^{-1}(V_n)$ of X verifies $f' \circ f \simeq g' \circ f$. This property also fullfils the criteria of Lemma 3·15.

Hence there is a third covering $W_0, \dots, W_{m+n}$ of X where $f' \circ f \simeq g' \circ f \simeq g' \circ g$, which implies $D(f' \circ f, g' \circ g) \leqslant m + n$.

The latter result generalises Propositions 3·1 and 3·3, at least for normal spaces, because $D = 0$ for homotopic maps.

3·5. *Products*

We study the behaviour of the homotopic distance under products.

LEMMA 3·18. Let $f, g\colon X \to Y$ and $h\colon X' \to Y'$. Then the maps $f \times h, g \times h\colon X \times X' \to Y \times Y'$ verify $D(f \times h, g \times h) = D(f, g)$.

Proof. The inequality $D(f \times h, g \times h) \leqslant D(f, g)$ follows from a simple argument, because if $f \simeq g$ on the open set $U \subset X$, then $f \times h \simeq g \times h$ on $U \times X'$.

On the other hand, for a fixed point $x_0 \in X$, let $i_1\colon X \to X \times X$ be the map $i_1(x) = (x, x_0)$, and let $p_1\colon X \times X \to X$ be the projection onto the first factor. Then $f = p_1 \circ (f \times h) \circ i_1$, an analogously for g. By the composition properties (Propositions 3·1 and 3·3), it follows

$$D(f, g) = D\left(p_1 \circ (f \times h) \circ i_1, p_1 \circ (g \times h) \circ i_1\right) \leqslant D(f \times h, g \times h).$$

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THEOREM 3·19. *Given $f, g\colon X \to Y$ and $f', g'\colon X' \to Y'$, assume that the space $X \times X'$ is normal. Then*

$$\mathrm{D}(f \times f', g \times g') \leqslant \mathrm{D}(f, g) + \mathrm{D}(f', g').$$

It is possible to give a proof identical to that given in [2, section 1·5] for the particular case of LS-category, just by replacing the notion of categorical sequence by a similar notion of *homotopical sequence*. However, a much simpler proof follows from Proposition 3·15 and Lemma 3·18.

Proof of Theorem 3·19. First note that $f \times f' = (f \times \mathrm{id}_{Y'}) \circ (\mathrm{id}_X \circ f')$, and similarly for $g \times g'$. We can just compute

$$\begin{aligned} D(f \times f', g \times g') &= D((f \times \mathrm{id}) \circ (\mathrm{id} \times f'), (g \times \mathrm{id}) \circ (\mathrm{id} \times g')) \\ &= D((f \times \mathrm{id}) \circ (\mathrm{id} \times f'), (g \times \mathrm{id}) \circ (\mathrm{id} \times f')) \\ &\quad + D((g \times \mathrm{id}) \circ (\mathrm{id} \times f'), (g \times \mathrm{id}) \circ (\mathrm{id} \times g')) \\ &\leqslant D(f \times \mathrm{id}, g \times \mathrm{id}) + D(\mathrm{id} \times f', \mathrm{id} \times g') \\ &= D(f, g) + D(f', g'). \end{aligned}$$

Example 3·20. Set $f\colon X \to X$ and $f'\colon X' \to X'$ to be the identity maps and $g\colon X \to X$ and $g'\colon X' \to Y'$ to be constant maps. Then

$$\mathrm{cat}(X \times X') \leqslant \mathrm{cat}(X) + \mathrm{cat}(X').$$

Example 3·21. Set $f\colon X \times X \to X$ and $f'\colon X' \times X' \to X'$ to be the projection maps onto the first factor and $g\colon X \times X \to X$ and $g'\colon X' \times X' \to X'$ to be the projection maps onto the second factor. Then

$$\mathrm{TC}(X \times X') \leqslant \mathrm{TC}(X) + \mathrm{TC}(X').$$

3·6. *Finite topological spaces*

It has been shown (Proposition 3·16) that the homotopic distance satisfies the triangular inequality under the assumption that the domains of the maps involved are normal spaces. This subsection is devoted to show that this does not hold for T_0 finite topological spaces.

We recall some basic facts about finite topological spaces; for a detailed exposition we refer the reader to [1]. Finite posets and finite T_0-spaces are in bijective correspondence. If $(X, \leqslant)$ is a poset, a basis for a topology on X is given by the sets

$$U_x = \{y \in X : y \leqslant x\}, \quad x \in X.$$

Conversely, if X is a finite T_0-space, define, for each $x \in X$, the *minimal open set* U_x as the intersection of all open sets containing x. Then X may be given a poset structure by defining $y \leqslant x$ if and only if $U_y \subset U_x$. Given two finite spaces X and Y, the product topology is given by the basic open sets

$$U_{(x,y)} = U_x \times U_y, \quad (x, y) \in X \times Y.$$

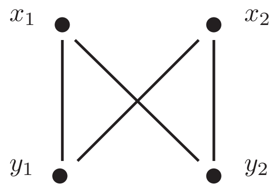

Fig. 1. The finite topological space S.

Example 3·22. Let S be the finite space corresponding to the poset depicted in Figure 1. Consider the finite space $X = S \times S$ and the continuous maps $f, g, h\colon X \to X$ given by $f = \mathrm{id}_X$, $g = \mathrm{id}_S \times c$ and $h = c \times c$ where $c\colon S \to S$ is a constant map. Recall from [1] that for any finite space Y and $y \in Y$, the subspace U_y is contractible. Therefore $\{S \times U_{x_1}, S \times U_{x_2}\}$ is an open cover of X such that the restrictions of f and g to each of the members of the cover are homotopic. This proves that $\mathrm{D}(f, g) \leqslant 1$. A symmetrical argument shows that $\mathrm{D}(g, h) \leqslant 1$. However $\mathrm{D}(f, h) = \mathrm{cat}(X) \geqslant 3$ [22, example 3·5]. Therefore, the maps f, g and h do not satisfy the triangular inequality.

4. *H-spaces*

A well-known result from Farber [4, lemma 8·2] states that for a Lie group G the topological complexity $\mathrm{TC}(G)$ equals the LS-category $\mathrm{cat}(G)$. This result was later extended to all H-spaces by Lupton and Scherer [15].

Here, an H-space is a topological space G endowed with a *multiplication* $\mu : G \times G \to G$, a *division* $\delta\colon G \times G \to G$ and an identity element $x_0 \in G$ such that $\mu(p_1, \delta) \simeq p_2$ and $\mu(-, x_0) \simeq \mathrm{id}_G$. Note that we do not ask the multiplication to be associative.

This definition is inspired by the discussion in [15, proof of theorem 1] of the results of James [13]. As an example, let G be a Lie group, with multiplication $\mu(x, y) = xy$ and division $\delta(x, y) = x^{-1}y$.

Farber and Lupton–Scherer results are particular cases of the following theorem.

THEOREM 4·1. *Let G be a path-connected H-space and let $f, g\colon G \times G \to G$ be two maps. Then* $\mathrm{D}(f, g) \leqslant \mathrm{cat}(G)$.

In fact we know that $\mathrm{D}(f, g) \leqslant \mathrm{TC}(G)$ (Proposition 3·10), so Theorem 4·1 is equivalent to the result of Lupton and Scherer. For the sake of completeness we shall give a direct proof.

Proof. Let $U \subset G$ be a categorical open set, that is, $i_U \simeq x_0$, and consider the preimage $\Omega \subset G \times G$ of U by the map $\delta \circ (f, g)\colon G \times G \to G$. Then

$$p_2 \circ (f, g) \circ i_\Omega \simeq \mu \circ (p_1, \delta) \circ (f, g) \circ i_\Omega,$$

that is,

$$g_{|\Omega} \simeq \mu \circ (f_{|\Omega}, \delta \circ (f, g) \circ i_\Omega).$$

But $\delta \circ (f, g) \circ i_\Omega$ factors through i_U, by the definition of Ω, so it is homotopic to the constant map $(x_0)_{|\Omega}\colon \Omega \to G$. Then

$$g_{|\Omega} \simeq \mu \circ (f_{|\Omega}, x_0) \simeq f_{|\Omega}.$$

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COROLLARY 4·2 ([15]). *For a path-connected H-space G we have* $\mathrm{TC}(G) = \mathrm{cat}(G)$.

Proof. Take $f = p_1$ and $g = p_2$ and apply Theorem 2·6. Then $\mathrm{TC}(G) \leqslant \mathrm{cat}(G)$. The other inequality was proven in Corollary 3·6.

In the next proposition we shall use the multiplication $f \cdot h$ of maps $f, h\colon X \to G$ into an H-space, defined as usual by the composition

$$X \xrightarrow{\Delta} X \times X \xrightarrow{f \times h} G \times G \xrightarrow{\mu} G.$$

PROPOSITION 4·3. *If G is an H-space and $f, g, h\colon X \to G$, then*

$$D(f \cdot h, g \cdot h) \leqslant D(f, g).$$

Proof. The proposition follows by applying the composition rules and Lemma 3·18. Namely, since

$$f \cdot h = \mu \circ (\mathrm{id} \times h) \circ (f \times \mathrm{id}) \circ \Delta,$$

and analogously for $g \cdot h$, we have

$$D(f \cdot h, g \cdot h) \leqslant D(f \times \mathrm{id}, g \times \mathrm{id}) = D(f, g).$$

COROLLARY 4·4. *In a Lie group, the distance between the multiplication μ and the division δ equals the distance between the identity* id_G *and the inversion map $I\colon G \to G$, $I(x) = x^{-1}$, that is,* $\mathrm{D}(\mu, \delta) = \mathrm{D}(\mathrm{id}_G, I)$.

Proof. Let $x_0 = e$ be the identity element, and consider the map $i_1(x) = (x, x_0)$. Then $\mu \circ i_1 = \mathrm{id}_G$ and $\delta \circ i_1 = I$. From Proposition 3·3 it follows that $\mathrm{D}(\mathrm{id}_G, I) \leqslant \mathrm{D}(\mu, \delta)$.

On the other hand, we have $\mu = (\mathrm{id}_G \circ p_1) \cdot p_2$, while $\delta = (I \circ p_1) \cdot p_2$. Then

$$D(\mu, \delta) \leqslant D(\mathrm{id}_G \circ p_1, I \circ p_1) \leqslant D(\mathrm{id}_G, I).$$

Note that Corollary 4·2 and Proposition 3·10 imply that $D(\mu, \delta) \leqslant \mathrm{cat}(G)$.

COROLLARY 4·5. *In any Lie group, the distance* $\mathrm{D}(\mu_a, \mu_b)$ *between two power maps $\mu_a, \mu_b\colon G \to G$, where $\mu_c(x) = x^c$, equals* $\mathrm{D}(\mu_{a-b}, e)$.

Proof. The result follows from Proposition 4·3 applied to $x^a = x^{a-b} \cdot x^b$ and $x^b = e \cdot x^b$.

5. *Cohomology*

5·1. *Cup length*

For the LS category it is well known ([2]) that

$$\text{l.c.p. } H(X; R) \leqslant \mathrm{cat}(X),$$

where l.c.p. denotes the length of the cup product of the cohomology (with coefficients in any commutative ring R with unit).

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Analogously, Farber [3] proved that l.c.p. $\ker \Delta^* \leqslant \mathrm{TC}(X)$. When the coefficients are a field K, $\ker \Delta^*$ is isomorphic to the kernel of the cup product

$$H(X; K) \otimes H(X; K) \xrightarrow{\smile} H(X; K).$$

We shall give a general cohomological lower bound for the homotopic distance between two maps.

Let $f, g\colon X \to Y$ be two maps and let $f^*, g^*\colon H(Y; R) \to H(X; R)$ be the induced morphisms in cohomology (for an arbitrary unitary commutative ring of coefficients). We denote by $\mathcal{J}(f, g) \subset H(X; R)$ the image of the linear morphism $f^* - g^*\colon H(Y; R) \to H(X; R)$.

Definition 5·1. We denote by l.c.p. $\mathcal{J}(f, g)$ the least integer k such that any product $u_0 \smile \cdots \smile u_k$ of elements of $\mathcal{J}(f, g)$ is null in $H(X)$.

Note that we do not ask $\mathcal{J}(f, g)$ to be a ring. Also note that

$$\text{l.c.p. } \mathcal{J}(f, g) \leqslant \text{l.c.p. } H(X; R).$$

THEOREM 5·2. *Let* $\mathcal{J}(f, g) \subset H(X; R)$ *be the image of the morphism* $f^* - g^*\colon H(Y; R) \to H(X; R)$. *Then* l.c.p.$\mathcal{J}(f, g) \leqslant \mathrm{D}(f, g)$.

Proof. Assume $\mathrm{D}(f, g) \leqslant n$ and let $X \times X = U_0 \cup \cdots \cup U_n$ be a covering such that the restrictions of p_1 and p_2 to each open set U_k, $k = 0, \ldots, n$, are homotopic. For $U = U_k$ let us consider the long exact sequence of the pair (X, U) (from now on we shall not make explicit the ring R):

$$\begin{array}{ccccccccc} & & & & H^m(Y) & & & & \\ & & & & \big\downarrow{\scriptstyle f^*-g^*} & \searrow{\scriptstyle (f_{|U})^*-(g_{|U})^*=0} & & & \\ \cdots & \longrightarrow & H^m(X, U) & \xrightarrow{j_U} & H^m(X) & \xrightarrow{(i_U)^*} & H^m(U) & \longrightarrow & \cdots \end{array}$$

Then $(f_{|U})^* = (g_{|U})^*\colon H(Y) \to H(U)$, which implies that every element ω in $\mathcal{J}$ belongs to $\ker(i_U)^* = \operatorname{im} j_U$, then $\omega = j_U(\tilde{\omega})$ for some $\tilde{\omega} \in H(X, U)$.

Now, let us remember the relative cup product ([12, p. 209])

$$\smile\colon H^m(X, U) \otimes H^n(X, V) \to H^{m+n}(X, U \cup V),$$

where U, V are open subsets of X. From [23, p. 251] it follows that the following diagram is commutative:

$$\begin{array}{ccc} H(X, U_0) \otimes \cdots \otimes H(X, U_n) & \xrightarrow{\smile} & H^*(X, \bigcup_{k=0}^n U_k) \\ \big\downarrow{\scriptstyle j_{U_0} \otimes \cdots \otimes j_{U_n}} & & \big\downarrow{\scriptstyle j_{U_0 \cup \ldots \cup U_n}} \\ H(X) \otimes \cdots \otimes H(X) & \xrightarrow{\smile} & H(X). \end{array}$$

If $\omega_0 \smile \cdots \smile \omega_n$ is a product of length $n+1$ of elements $\omega_k \in \mathcal{J}$, there exist elements $\widetilde{\omega_k} \in H(X, U_k)$ such that $\omega_k = j_k(\widetilde{\omega_k})$, hence

$$\omega_0 \smile \cdots \smile \omega_n = j_0(\widetilde{\omega_0}) \smile \cdots \smile j_n(\widetilde{\omega_n}) = j_{0\ldots n}(\widetilde{\omega_0} \smile \cdots \smile \widetilde{\omega_n}) = 0$$

because $\widetilde{\omega_0} \smile \cdots \smile \widetilde{\omega_n} \in H(X, X) = 0$. Then l.c.p. $\mathcal{J} \leqslant n$.

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Example 5·3. Consider the inclusion maps $i_1, i_2 \colon X \to X \times X$ as in Proposition 2·5. Then $\mathcal{J}(i_1, i_2)$ is isomorphic to $H(X)$. Therefore, we recover the classical cohomological lower bound for the LS category.

Example 5·4. Let $G = \mathrm{U}(2)$ be the Lie group of 2×2 complex matrices A such that $A^{-1} = A^*$. It is known ([21]) that $\mathrm{cat}(G) = 2$. In fact, topologically G is the product $S^1 \times S^3$, so its real cohomology is $H(G) = H(S^1) \otimes H(S^3)$, the exterior algebra $\bigwedge(x_1, x_3)$. Consider the maps $f = \mathrm{id}_G$ the identity and $g = I$ the inversion $I(A) = A^*$. Then $I^*(x_1) = -x_1$ and $I^*(x_3) = -x_3$, so $\mathcal{J}(f, g) = H(G)$. Then

$$2 = \mathrm{l.c.p.}\, \mathcal{J} \leqslant \mathrm{D}(f, g) \leqslant \mathrm{cat}(G) = 2.$$

Hence, the distance between the identity and the inversion is 2, and the distance between the multiplication and the division is 2 too (Proposition 4·4). The same argument applies to the groups $U(n)$, $n \geqslant 2$, that is, $\mathrm{D}(\mathrm{id}_G, I) = n = \mathrm{cat}(G)$.

5·2. *Homotopy weight*

Following the ideas of Fadell–Husseini for the LS-category and Farber–Grant for the Topological Complexity ([6]), we can define a notion of *homotopy weight* that serves to improve inequality (5·2). Our proofs follow the lines of those in [5, section 6] for the TC-weight, which is a particular case.

Let $f, g \colon X \to Y$ be two maps, and let $u \in H(X; R)$ be a cohomology class.

Definition 5·5. We say that u has homotopy weight $\mathrm{hw}(u) = k + 1$ (with respect to f, g) if k is the greatest integer such that the following condition is satisfied: given any continuous map $\phi \colon A \to X$ with $D(f \circ \phi, g \circ \phi) \leqslant k$, then $\phi^* u = 0 \in H(A; R)$. We put $\mathrm{hw}(0) = \infty$.

In other words, $\mathrm{hw}(u) \geqslant k + 1$ means that $\phi^* u = 0 \in H(A; R)$ for all maps $\phi \colon A \to X$ with $D(f \circ \phi, g \circ \phi) \leqslant k$.

We first prove the homotopy invariance of the homotopy weight.

PROPOSITION 5·6. *If α, β in the following commutative diagram are homotopy equivalences,*

$$\begin{array}{ccc} X' & \overset{f'}{\underset{g'}{\rightrightarrows}} & Y' \\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} \\ X & \overset{f}{\underset{g}{\rightrightarrows}} & Y \end{array}$$

then $\mathrm{hw}_{f',g'}(\alpha^* u) = \mathrm{hw}_{f,g}(u)$ *for* $u \in H(X)$.

Proof. Let $\mathrm{hw}'(\alpha^* u) \geqslant k + 1$, and consider $\phi \colon A \to X$ such that $\mathrm{D}(f \circ \phi, g \circ \phi) \leqslant k$. If $\bar{\alpha} \colon X \to X'$ is the homotopy inverse of α, then (Corollaries 3·11 and 3·12),

$$\begin{aligned} \mathrm{D}(f' \circ \bar{\alpha} \circ \phi, g' \circ \bar{\alpha} \circ \phi) &= \mathrm{D}(\beta \circ f' \circ \bar{\alpha} \circ \phi, \beta \circ g' \circ \bar{\alpha} \circ \phi) \\ &= D(f \circ \alpha \circ \bar{\alpha} \circ \phi, g \circ \alpha \circ \bar{\alpha} \circ \phi) = \mathrm{D}(f, g) \leqslant k \end{aligned}$$

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because $\alpha\circ\bar\alpha\circ\phi\simeq\phi$, so $0=(\bar\alpha\circ\phi)^*(\alpha^*u)=\phi^*u$. This proves that $\mathrm{hw}(u)\geqslant k+1$. The other implication is analogous.

Our invariant can be seen as a generalization of those introduced by several authors, including Rudyak [18] and Strom [24].

From now on we shall assume that our cohomology classes are in $\mathcal{J}(f,g)=\mathrm{im}\,(f^*-g^*)$ because it is there where hw is defined, as the following Lemma proves.

LEMMA 5·7. *If* $u\in\mathcal{J}(f,g)$ *then* $\mathrm{hw}(u)\geqslant 1$.

Proof. If $\mathrm{D}(f\circ\phi,g\circ\phi)=0$ then $f\circ\phi\simeq g\circ\phi$, so $\phi^*(f^*-g^*)=(f\circ\phi)^*-(g\circ\phi)^*=0$. Since $u=(f^*-g^*)v$ for some $v\in H(Y;R)$, we have $\phi^*u=0$, and the result follows.

LEMMA 5·8. *For any non-zero class* $u\in\mathcal{J}(f,g)\subset H(X)$ *we have* $\mathrm{hw}(u)\leqslant\mathrm{D}(f,g)$.

Proof. If $\mathrm{hw}(u)\geqslant\mathrm{D}(f,g)+1=k+1$, then $\mathrm{D}(f\circ\mathrm{id}_X,g\circ\mathrm{id}_X)\leqslant k$, so $u=\mathrm{id}_X^*u=0$.

THEOREM 5·9. *Let* $u=u_0\smile\cdots\smile u_k$ *be a cup product of cohomology classes in* $\mathcal{J}(f,g)$. *Then*

$$\mathrm{hw}(u)\geqslant\sum_{j=0}^{k}\mathrm{hw}(u_j).$$

Proof. It is enough to prove the result when $k=1$. Let $\mathrm{hw}(u_0)=m+1$ and $\mathrm{hw}(u_1)=n+1$. We want to prove that $\mathrm{hw}(u_0\smile u_1)\geqslant m+n+2$. Let $\phi\colon A\to X$ such that $\mathrm{D}(f\circ\phi,g\circ\phi)\leqslant m+n+1$, then there exists an open covering $\{U_0,\dots,U_{m+n+1}\}$ of A such that $f\circ\phi_{|U_j}\simeq g\circ\phi_{|U_j}$ for all j. Define $V_0=U_0\cup\cdots\cup U_m$ and $V_1=U_{m+1}\cup\cdots\cup U_{m+n+1}$. Then $\mathrm{D}(f\circ\phi_{|V_0},g\circ\phi_{|V_0})=m$, so $\phi^*_{|V_0}u_0=0$. Analogously, $\mathrm{D}(f\circ\phi_{|V_0},g\circ\phi_{|V_0})=n$ implies $\phi^*_{|V_1}u_1=0$.

Now, we consider the long exact sequence of the pair (A,V_0),

$$\begin{array}{ccccccccc} & & & & H^m(X) & & & & \\ & & & & \Big\downarrow{\scriptstyle\phi^*} & & & & \\ \cdots & \longrightarrow & H^m(A,V_0) & \xrightarrow{j_0} & H^m(A) & \xrightarrow{i_0^*} & H^m(V_0) & \longrightarrow & \cdots \end{array}$$

Since $i_0^*\phi^*u_0=0$, there exists $\xi_0\in H(A)$ such that $j_0(\xi)=\phi^*(u_0)$. Analogously, $\phi^*(u_1)=j_1(\xi_1)$. Then, as in the proof of Theorem 5·2, we have

$$\begin{aligned}\phi^*(u_0\smile u_1)&=\phi^*(u_0)\smile\phi^*(u_1)=j_0(\xi_0)\smile j_1(\xi_1)\\&=j_{01}(\xi_0\smile\xi_1)=0\in H(A;V_0\cup V_1).\end{aligned}$$

Theorem 5·2 can be read as follows: if $u=u_1\smile\cdots\smile u_k\neq 0$ is a non-zero product of k cohomology classes in $\mathcal{J}(f,g)$, then $D(f,g)>k$. Combining the latter Lemmas and Proposition we have proved:

THEOREM 5·10. *If* $u_0\smile\cdots\smile u_k\neq 0$ *is a non-zero product of* $k+1$ *cohomology classes in* $\mathcal{J}(f,g)$ *then* $\mathrm{D}(f,g)\geqslant\sum_{j=0}^{k}\mathrm{hw}(u_j)$.

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The interest of this result is that it is possible to find elements of high category weight. For instance, we can mimic [6, theorem 6], originally stated only for the TC weight. For simplicity we only consider Steenrod squares, but it is possible to state it in a much larger context for other cohomology operations.

Let $\theta = \mathrm{Sq}^i\colon H^p(X;\mathbb{Z}_2) \to H^{p+i}(X;\mathbb{Z}_2)$, for $0 \leqslant i \leqslant p$, be a Steenrod square. Each one of these squaring operations is a morphism of abelian groups that is natural and commutes with the connecting morphisms in the Mayer–Vietoris sequence. Its *excess* equals i, so $\theta(u) = 0$ if $u \in H^{n-1}(X;\mathbb{Z}_2)$ with $n \leqslant i$ [16, theorem 1].

THEOREM 5·11. *If* $n \leqslant i$ *and* $u \in \mathcal{J}(f, g) \subset H^n(X;\mathbb{Z}_2)$, *then* $\mathrm{hw}(\theta(u)) \geqslant 2$.

Proof. Let $\phi\colon A \to X$ with $D(f \circ \phi, g \circ \phi) \leqslant 1$, so $A = U_0 \cup U_1$ with $f \circ \phi_{|U_j} \simeq g \circ \phi_{|U_j}$, for $j = 0, 1$. Consider the Mayer–Vietoris sequence

$$\cdots \longrightarrow H^{n-1}(U_0 \cap U_1) \xrightarrow{\delta} H^n(A) \longrightarrow H^n(U_0) \oplus H^n(U_1) \longrightarrow \cdots$$

Since $\phi^* u = \phi^*(f^* - g^*)v = \big((f \circ \phi)^* - (g \circ \phi)^*\big)v$ is zero on each U_j, there exists $w \in H^{n-1}(U_0 \cap U_1)$ such that $\delta\omega = \phi^* u$. But then

$$\phi^*(\theta u) = \theta(\phi^* u) = \theta(\delta\omega) = \delta(\theta\omega) = 0,$$

where the nullity holds because ω has degree $n - 1 < i$.

6. *Fibrations*

6·1. *Statement of results*

A well-known result from Varadarajan [25] states that if $\pi\colon E \to B$ is a (Hurewicz) fibration with generic fiber F and path-connected base B, then

$$\mathrm{cat}(E) + 1 \leqslant \big(\mathrm{cat}(B) + 1\big)\big(\mathrm{cat}(F) + 1\big). \qquad (6·1)$$

On the other hand, Farber and Grant [6] proved that

$$\mathrm{TC}(E) + 1 \leqslant \big(TC(F) + 1\big) \times \mathrm{cat}\big(B \times B\big). \qquad (6·2)$$

We shall see that both results are particular cases of a much more general situation. Let $\pi'\colon E' \to B'$ be another fibration with path-connected base B' and generic fibre F', and take two fiber-preserving maps $f, g\colon E \to E'$, with induced maps $\bar{f}, \bar{g}\colon B \to B'$. That is, we have $\pi' \circ f = \bar{f} \circ \pi$ and $\pi' \circ g = \bar{g} \circ \pi$, as in the commutative diagram below:

$$\begin{array}{ccc} E & \overset{f}{\underset{g}{\rightrightarrows}} & E' \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi'} \\ B & \overset{\bar{f}}{\underset{\bar{g}}{\rightrightarrows}} & B'. \end{array}$$

Our aim is to prove the following result (both B and B' are assumed to be path-connected):

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THEOREM 6·1. *Let $b_0 \in B$ with $\bar{f}(b_0) = b_0' = \bar{g}(b_0)$. If $f_0, g_0 \colon F_0 \to F_0'$ are the induced maps between the fibers of b_0 and b_0', then*

$$\mathrm{D}(f, g) + 1 \leqslant \big(\mathrm{D}(f_0, g_0) + 1\big) \times \big(\mathrm{cat}(B) + 1\big).$$

It is well known that all the fibers $F_b = \pi^{-1}(b)$ of the fibration π have the same homotopy type ([**12**, proposition 4·61]). Also it is known that if the base B is contractible then the fibration is fiber homotopy equivalent to a product fibration ([**12**, corollary 4·63]). We need a similar statement that will allow us to establish our notations.

LEMMA 6·2. *If U is a categorical open set in B, which contracts to the point b_0, then the fibration $\pi^{-1}(U) \to U$ is fiber homotopy equivalent to the trivial fibration $F_0 \times U$. Moreover, on each fiber F_b, $b \in U$, the restriction of the homotopy equivalence is a homotopy equivalence $F_b \simeq F_0$.*

Proof. There is a homotopy $\mathcal{C} \colon U \times I \to B$ with $\mathcal{C}_0$ the inclusion $U \subset B$ and $\mathcal{C}_1$ the constant map $b_0 \colon U \to B$. Then, the homotopy lifting property in the following diagram

$$\begin{array}{ccccc} \pi^{-1}(U) \times \{0\} & & \hookrightarrow & & E \\ {\scriptstyle i_0}\downarrow & & \overset{\widetilde{\mathcal{C}}}{\dashrightarrow} & & \downarrow{\scriptstyle \pi} \\ \pi^{-1}(U) \times I & \xrightarrow[\pi\times\mathrm{id}]{} & U \times I & \xrightarrow{\mathcal{C}} & B \end{array} \qquad (6·3)$$

gives us a map $\widetilde{\mathcal{C}} \colon \pi^{-1}(U) \times I \to E$ such that $\widetilde{\mathcal{C}}_0$ is the inclusion $\pi^{-1}(U) \subset E$ and $(\pi \circ \widetilde{\mathcal{C}})(x, t) = \mathcal{C}(\pi(x), t)$. As a consequence, we have a map (we use the same name with a slight abuse of notation)

$$\widetilde{\mathcal{C}}_1 \colon \pi^{-1}(U) \longrightarrow F_0 = \pi^{-1}(b_0). \qquad (6·4)$$

For each $b \in U$, the path $\mathcal{C}_t(b)$ in B connects the points b and b_0, and it lifts, for each $x \in F_b$, to the path $\widetilde{\mathcal{C}}_t(x)$, so the map $(\widetilde{\mathcal{C}}_1)_{|F_b} \colon F_b \to F_0$ is the usual one giving the homotopy equivalence between the fibers.

The rest of the proof is similar to the usual one.

6·2. *Proof of Theorem 6·1*

Proof. We now start the proof of Theorem 6·1. Assume that $\mathrm{cat}(B) \leqslant m$ and let $B = U_0 \cup \cdots \cup U_m$ be a covering by categorical open sets.

For each $U = U_i$ take the homotopy $\widetilde{\mathcal{C}}$ in (6·3) and the map $\widetilde{\mathcal{C}}_1$ in (6·4). We shall use the "coproduct notation" introduced after Definition 2·1. Then $i_0 \circ \widetilde{\mathcal{C}}_1 \colon \boldsymbol{P} \to E$ is homotopic to the map $\boldsymbol{p} \colon \boldsymbol{P} \to E$ induced by the inclusion, for the disjoint union $\boldsymbol{P} = \pi^{-1}(U_0) \sqcup \cdots \sqcup \pi^{-1}(U_m)$ and the inclusion $i_0 \colon F_0 \subset E$.

If $\mathrm{D}(f_0, g_0) \leqslant n$, let $F_0 = V_0 \cup \cdots \cup V_n$ with $f_0 \circ \boldsymbol{v} \simeq g_0 \circ \boldsymbol{v}$, for the coproduct map $\boldsymbol{v} \colon \boldsymbol{V} = V_0 \sqcup \cdots \sqcup V_n \to F_0$.

For each $V = V_j \subset F_0$ take the open set

$$\Omega(U, V) = \pi^{-1}(U) \cap (\widetilde{\mathcal{C}}_1)^{-1}(V) \subset E.$$

It is clear that $\{\Omega(U_i, V_j)\}$ is an open covering of E.

We claim that f and g are homotopic in each $\Omega(U, V)$.

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To see it, we have that the map $\boldsymbol{\omega}\colon \boldsymbol{\Omega} \to E$, induced by the inclusions in the disjoint union $\boldsymbol{\Omega} = \sqcup_{ij}\Omega(U_i, V_j)$, is homotopic to the map $i_0 \circ \widetilde{\mathcal{C}}_{1|\boldsymbol{\Omega}}$, because $i_0 \circ \widetilde{\mathcal{C}}_1 \simeq \boldsymbol{p}$. Hence $f \circ \boldsymbol{\omega} \simeq (f \circ \widetilde{\mathcal{C}}_1)_{|\boldsymbol{\Omega}}$ and $g \circ \boldsymbol{\omega} \simeq (g \circ \widetilde{\mathcal{C}}_1)_{|\boldsymbol{\Omega}}$.

But $\widetilde{\mathcal{C}}_1(\boldsymbol{\Omega})$ is contained in $\boldsymbol{V}$, so in fact $\widetilde{\mathcal{C}}_{1|\boldsymbol{\Omega}}$ can be written as $\boldsymbol{v} \circ \widetilde{\mathcal{C}}_{1|\boldsymbol{\Omega}}$.

Moreover, there is a homotopy between $f_0 \circ \boldsymbol{v}$ and $g_0 \circ \boldsymbol{v}$, hence $i'_0 \circ f_0 \circ \boldsymbol{v} \simeq i'_0 \circ g_0 \circ \boldsymbol{v}$, where $i'_0\colon F'_0 \subset E'$ is the inclusion.

Finally we have

$$\begin{aligned} f \circ \omega &\simeq f \circ i_0 \circ \boldsymbol{v} \circ \widetilde{\mathcal{C}}_{1|\boldsymbol{\Omega}} = i'_0 \circ f_0 \circ \boldsymbol{v} \circ \widetilde{\mathcal{C}}_{1|\boldsymbol{\Omega}} \\ &\simeq i'_0 \circ g_0 \circ \boldsymbol{v} \circ \widetilde{\mathcal{C}}_{1|\boldsymbol{\Omega}} = g \circ i_0 \circ \boldsymbol{v} \circ \widetilde{\mathcal{C}}_{1|\boldsymbol{\Omega}} \\ &\simeq g \circ \omega. \end{aligned}$$

So the result follows.

Example 6·3. By taking $E = E'$, $B = B'$, $f = \mathrm{id}_E$, $\bar{f} = \mathrm{id}_B$, $g = e_0$ a constant map and $\bar{g}$ the constant map $b_0 = \pi(e_0)$ one obtains

$$\mathrm{D}(\mathrm{id}_E, e_0) + 1 \leqslant \big(D(\mathrm{id}_0, e_0) + 1\big) \times (\mathrm{cat}(B) + 1),$$

and we recover (6·1).

Example 6·4. In Theorem 6·1, take the projections p_1, p_2 as in the following diagram:

$$\begin{array}{ccc} E \times E & \overset{p_1}{\underset{p_2}{\rightrightarrows}} & E \\ {\scriptstyle \pi\times\pi}\downarrow & & \downarrow{\scriptstyle \pi} \\ B \times B & \overset{p_1}{\underset{p_2}{\rightrightarrows}} & B \end{array}$$

and use Theorem 2·6. We have $p_1(b_0, b_0) = b_0 = p_2(b_0, b_0)$. Since $(p_1)_0, (p_2)_0\colon F_0 \times F_0 \to F_0$ are the projections, (6·2) follows.

7. *Example*

In this section we show an example of a Lie group G and two maps $f, g\colon G \to G$ such that $\mathrm{D}(f, g) = 2 = \mathrm{l.c.p.}\,\mathrm{H}(G)$, while $\mathrm{TC}(G) = \mathrm{cat}(G) = 3$.

7·1. *Description of the example*

Let $G = \mathrm{Sp}(2)$ be the Lie group of 2×2 quaternionic matrices such that $AA^* = \mathrm{I}$ (where A^* denotes the conjugate transpose and $\mathrm{I} = \mathrm{I}_2$ is the identity matrix). Its dimension is 10. Its cohomology is $\mathrm{H}(G) = \Lambda(x^3, x^7)$, the exterior algebra with two generators, so the length of the cup product is $\mathrm{l.c.p.}\,\mathrm{H}(G) = 2$. It is also known that $\mathrm{TC}(G) = \mathrm{cat}(G) = 3$ [20].

Let $f = \mu_2\colon G \to G$ be the map $\mu_2(A) = A^2$ and let $g = \mathrm{I}$ be the constant map. We have $f^*\omega = 2\omega$ for any bi-invariant form, while $g^* = 0$. Then

$$\mathrm{l.c.p.}\,\mathcal{J}(f, g) = 2 \leqslant \mathrm{D}(f, g) \leqslant 3 = \mathrm{cat}(G).$$

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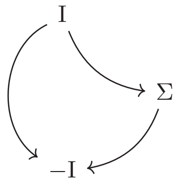

Fig. 2. The flow of the negative gradient of h.

We want to find a covering of G by three open sets where f, g are homotopic, by means of the gradient flow of the function $h\colon G \to \mathbb{R}$ given by the real part of the trace,

$$h(A) = \Re\,\mathrm{Tr}(A).$$

The reader can find more information about this map in [10]. It is a Morse–Bott function, whose critical set is

$$\mathrm{Crit}\, h = \{B \in G\colon B^2 = I\}.$$

That set has three connected components: the two points $\{\pm \mathrm{I}\}$ and the the orbit $\Sigma = \{UPU^*\colon U \in \mathrm{Sp}(2)\}$ of the matrix

$$P = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{7·1}$$

under the action of the group onto itself by conjugation. Hence $\Sigma \cong \mathrm{Sp}(2)/(\mathrm{Sp}(1) \times \mathrm{Sp}(1))$ is a compact Grassmannian manifold with $\dim \Sigma = 4$. Note that $-P \in \Sigma$.

To end this preliminaries, we describe the foliated local structure of the gradient flow near the critical set (see Figure 2). The stable manifold $W^+(\Sigma)$ of the critical submanifold Σ (that is, the points of G whose flow line ends at Σ) fibers over Σ, and this fiber bundle $W^+(\Sigma) \to \Sigma$ is isomorphic to the positive normal bundle $p^+\colon \nu^+(\Sigma) \to \Sigma$ defined as follows: for the critical point $B = UPU^* \in \Sigma$, the Hessian $H_B\colon T_BG \to T_BG$ is given by [11]:

$$H_B(X) = -\frac{1}{2}(XB + BX).$$

Its kernel is the tangent space $T_B\Sigma$ to the critical orbit, and the normal space $\nu_B\Sigma$ decomposes as $\nu_B^+ \oplus \nu_B^-$ depending on the sign of the eigenvalues of H_B. All these constructions are invariant by conjugation, so a simple computation for the particular case $B = P$ shows that $\nu_B^+ = U\nu_P^+U^*$, where $X \in \nu_P^+$ if and only if $X = \begin{bmatrix} 0 & 0 \\ 0 & b \end{bmatrix}$, with $\Re(b) = 0$, so $\dim \nu_B^+ = 3$. Hence, the index of the critical manifold Σ equals 3. Analogously, $W^s(-I)$ is a cell of dimension 10.

7·2. *Contractible open sets*

We also need to describe the explicit categorical covering of Sp(2) given by the first author in [11]. For that, for each of the matrices $B = \pm\mathrm{I}, \pm P$ (see (7·1)), let us consider the open neighbourhood

$$\Omega_G(B) = \{A \in G\colon B + A \text{ invertible}\}.$$

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There are global diffeomorphisms (called Cayley transforms)

$$c_B\colon \Omega_G(B) \longrightarrow T_{B^*}G,$$

given by

$$c_B(A) = (I - B^*A)(B + A)^{-1},$$

which prove that each $\Omega_G(B)$ is a contractible open set. The inverse map of c_B is c_{B^*}. Moreover

$$\Omega(I) \cup \Omega(-I) = G \setminus \Sigma \tag{7·2}$$

and

$$\Sigma \subset \Omega(P) \cup \Omega(-P).$$

7·3. *Homotopies*

We shall describe the following three open sets, covering G, where the maps $f = \mu_2$ and $g = \mathrm{I}$ are homotopic.

(i) Let $U_0 = G \setminus (\Sigma \cup \{I\})$. Since h is a Morse–Bott function, there is a well defined map sending each point $A \in U_0$ to the final point $\phi_A(+\infty) \in \Sigma \cup \{-I\}$ of the flow line $\phi_A(t)$ passing through it. This map is not continuous, but we shall use the fact that $\mu_2(\mathrm{Crit}\, h) = \{\mathrm{I}\}$.

We define

$$\mathcal{H}\colon U_0 \times [0, +\infty] \longrightarrow G$$

to be the map

$$\mathcal{H}(A, t) = \begin{cases} \phi_A(t) & \text{if } 0 \leqslant t < +\infty, \\ \lim\limits_{t\to\infty} \phi_A(t) & \text{if } t = +\infty. \end{cases}$$

Then $\mu_2 \circ \mathcal{H}_0 = (\mu_2)_{|U_0}$ while $\mu_2 \circ \mathcal{H}_\infty$ is the constant map I. The explicit formulas for the flow $\phi_A(t)$ given in [**11**] allow to prove that the map $\mu_2 \circ \mathcal{H}$ is continuous.

(ii) The second open set is the Cayley domain $U_1 = \Omega_G(\mathrm{I})$ (see 7·2), which is contractible, hence $\mu_2 \simeq \mathrm{I}$ on it.

(iii) Finally, the third one will be a tubular open neighbourhood $U_2 = N(\Sigma)$ of the critical Grassmannian manifold Σ, in such a way that Σ is a deformation retract of $N(\Sigma)$, by a retraction

$$\mathcal{R}\colon N(\Sigma) \times [0, 1] \longrightarrow N(\Sigma) \subset G.$$

That is, $\mathcal{R}_0$ is the inclusion $N(\Sigma) \subset G$; and the image of $\mathcal{R}_1$ is contained in Σ. Since the square of any critical point is I, we have that $\mu_2 \circ \mathcal{R}\colon N(\Sigma) \times [0, 1] \to G$ is a homotopy between $\mu_2 \circ \mathcal{R}_0 = (\mu_2)_{|N(\Sigma)}$ and the constant map $\mu_2 \circ \mathcal{R}_1 = \mathrm{I}$.

It is clear that $G = U_0 \cup U_1 \cup U_2$, so $\mathrm{D}(I, \mu_2) \leqslant 2$, as stated.

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8. *Further ideas*

8·1. *Contiguity distance between simplicial maps*

It is easy to adapt our definitions to the simplicial setting. For instance, in [7, 8, 9] simplicial versions of LS-category and topological complexity were given by one of the authors. With the classical notion of contiguous simplicial maps replacing that of homotopical continuous maps, one can define a notion of distance between simplicial maps.

Definition 8·1. The *contiguity distance* $\mathrm{SD}(\varphi, \psi)$ between two simplicial maps $\varphi, \psi\colon K \to K'$ is the least integer $n \geqslant 0$ such that there is a covering of K by subcomplexes $K_0, \ldots, K_n$ such that the restrictions $\varphi_{|K_j}, \psi_{|K_j}\colon K_j \to K'$ are in the same contiguity class, for all $j = 0, \ldots, n$. If there is no such covering, we define $\mathrm{SD}(f, g) = \infty$.

As expected, this notion of contiguity distance generalizes those of simplicial LS category $\mathrm{scat}(K)$ and discrete topological complexity $\mathrm{TC}(K)$:

Example 8·2. Given two simplicial complexes K and L, denote by $K \prod L$ their categorical product ([14]). The contiguity distance between the projections $p_1, p_2\colon K \prod K \to K$ equals $\mathrm{TC}(K)$, as follows from [7, theorem 3·4].

8·2. *Higher homotopic distance*

The notion of topological complexity has been extended to higher analogs ([19]). The same can be done for the homotopy distance.

Definition 8·3. Given m continuous maps $f_1, \ldots, f_m\colon X \to Y$, their mth *homotopy distance* $\mathrm{D}(f_1, \ldots, f_m)$ is the least integer $n \geqslant 0$ such that there exists a covering of X by open subspaces $\{U_0, \ldots, U_n\}$, such that the restrictions $f_{1_{|U_j}} \simeq \ldots \simeq f_{m_{|U_j}}\colon U_j \to Y$, for all $j = 0, \ldots, n$.

We denote the mth topological complexity of the space X by $\mathrm{TC}_m(X)$. As expected, the notion of mth homotopic distance generalizes the notion of higher topological complexity:

THEOREM 8·4. *Given a path-connected topological space X, consider the projections $p_1, \ldots, p_m\colon X \times \overset{m)}{\cdots} \times X \to X$. Then $\mathrm{D}(p_1, \ldots, p_m) = \mathrm{TC}_m(X)$.*

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