



COMPOSITIO MATHEMATICA

Connected components of affine Deligne–Lusztig varieties in mixed characteristic

Miaofen Chen, Mark Kisin and Eva Viehmann

Compositio Math. **151** (2015), 1697–1762.

[doi:10.1112/S0010437X15007253](https://doi.org/10.1112/S0010437X15007253)



FOUNDATION
COMPOSITIO
MATHEMATICA



LONDON
MATHEMATICAL
SOCIETY
150 YEARS



Connected components of affine Deligne–Lusztig varieties in mixed characteristic

Miaofen Chen, Mark Kisin and Eva Viehmann

ABSTRACT

We determine the set of connected components of minuscule affine Deligne–Lusztig varieties for hyperspecial maximal compact subgroups of unramified connected reductive groups. Partial results are also obtained for non-minuscule closed affine Deligne–Lusztig varieties. We consider both the function field case and its analog in mixed characteristic. In particular, we determine the set of connected components of unramified Rapoport–Zink spaces.

Contents

1	Introduction	1698
2	Affine Deligne–Lusztig varieties in mixed characteristic	1701
2.1	The Cartan decomposition in families	1701
2.2	The affine Grassmannian in mixed characteristic	1705
2.3	Affine Deligne–Lusztig varieties	1708
2.4	Reduction to adjoint groups	1710
2.5	Hodge–Newton indecomposability	1712
3	The superbasic case	1715
3.1	Superbasic σ -conjugacy classes	1715
3.2	The superbasic case for GL_h	1716
3.3	The superbasic case in general	1718
3.4	Reduction to the superbasic case	1719
4	Connecting points	1722
4.1	Main results: formulation and overview of the proofs	1722
4.2	Some maximal rank subgroups of G	1726
4.3	Proof of Proposition 4.1.6	1729
4.4	Immediate distance case	1736
4.5	Proof of Proposition 4.1.7	1739
4.6	Proof of Proposition 4.1.9	1743
4.7	Proof of Proposition 4.1.10	1745

Received 23 September 2013, accepted in final form 19 December 2014, published online 5 May 2015.

2010 Mathematics Subject Classification 14G35, 20G25 (primary).

Keywords: affine Deligne–Lusztig variety, affine Grassmannian, Rapoport–Zink space.

The second author was partially supported by NSF grant DMS-1001139. The first and third authors were partially supported by the SFB/TR45 ‘Periods, Moduli Spaces and Arithmetic of Algebraic Varieties’ of the DFG and by ERC starting grant 277889 ‘Moduli spaces of local G -shtukas’. The first author was also partially supported by NSFC grant no. 11301185 and SRFDP grant no. 20130076120002. The third author was also partially supported by a Heisenberg fellowship of the DFG.

This journal is © Foundation Compositio Mathematica 2015.

5 Application to Rapoport–Zink spaces	1757
Acknowledgements	1760
References	1761

1. Introduction

Let k be a finite field with $q = p^r$ elements and let \bar{k} be an algebraic closure of k . Let $F = k((t))$ or $F = W(k)[1/p]$. Accordingly, let $L = \bar{k}((t))$ or $L = W(\bar{k})[1/p]$. Let \mathcal{O}_F and \mathcal{O}_L be the valuation rings of F and L . We denote by ϵ the uniformizer t or p . We write $\sigma : x \mapsto x^q$ for the Frobenius of \bar{k} over k and also the induced Frobenius of L over F .

Let G be a connected reductive group over \mathcal{O}_F . We denote by G_F the generic fiber of G , and write $K = G(\mathcal{O}_L)$. Since k is finite, G is automatically quasi-split. Let $B \subset G$ be a Borel subgroup and $T \subset B$ the centralizer of a maximal split torus in B . We denote by $X_*(T)$ the set of cocharacters of T , defined over \mathcal{O}_L .

For $b \in G(L)$ and a dominant cocharacter $\mu \in X_*(T)$ the affine Deligne–Lusztig variety $X_\mu^G(b) = X_\mu(b)$ (which is in fact generally just a set of points) is defined as

$$X_\mu(b) = \{g \in G(L)/K \mid g^{-1}b\sigma(g) \in K\epsilon^\mu K\}.$$

Left multiplication by $g \in G(L)$ induces a bijection $X_\mu(b) \xrightarrow{\sim} X_\mu(gb\sigma(g)^{-1})$. Thus the isomorphism class of the affine Deligne–Lusztig variety only depends on the σ -conjugacy class $[b]$ of b , and not on b .

When F has mixed characteristic and μ is minuscule, the sets $X_\mu(b)$ are closely related to the \bar{k} -points on Shimura varieties which lie in a fixed isogeny class, and in special cases to \bar{k} -valued points of a moduli space of p -divisible groups as defined by Rapoport and Zink [RZ96].

If F is a function field, then $X_\mu(b)$ is the set of \bar{k} -valued points of a locally closed, locally of finite type subscheme of the affine Grassmannian LG/K where LG denotes the loop group of G (cf. [Rap00, GHKR06]). If F has mixed characteristic, there is, in general, no known scheme structure on the affine Deligne–Lusztig varieties.¹ Nevertheless, they admit some kind of geometric structure, and in particular a meaningful notion of a set of connected components $\pi_0(X_\mu(b))$ which is compatible with the corresponding notion for Rapoport–Zink spaces.

The aim of this paper is to compute the set of connected components of $X_\mu(b)$ for any b when μ is minuscule. To state our main results, we begin by recalling when $X_\mu(b) \neq \emptyset$. This condition is a relation between μ and the σ -conjugacy class of b . Let $B(G)$ denote the set of σ -conjugacy classes of all elements of $G(L)$. They are described by two invariants. Write $\pi_1(G)$ for the quotient of $X_*(T)$ by the coroot lattice of G . Recall that there is the Kottwitz homomorphism (cf. [RR96]) $w_G : G(L) \rightarrow \pi_1(G)$ which for $\mu \in X_*(T)$ sends an element $g \in K\epsilon^\mu K \subset G(L)$ to the class of μ . We denote by κ_G the composite of w_G with the projection $\pi_1(G) \rightarrow \pi_1(G)_\Gamma$, where $\Gamma = \text{Gal}(\bar{k}/k)$ acts in the natural way on L and hence on $\pi_1(G)$. Let $\nu_{\text{dom}} \in X_*(T)_\mathbb{Q}$ be the dominant cocharacter conjugate to the Newton cocharacter of b . Then ν_{dom} is Γ -invariant and together with $\kappa_G(b)$ determines the σ -conjugacy class.

Let $\bar{\mu} \in X_*(T)_\mathbb{Q}$ denote the average of the Γ -conjugates of μ . Then the set $X_\mu(b)$ is non-empty if and only if $\kappa_G(b) = [\mu]$ in $\pi_1(G)_\Gamma$, and $\bar{\mu} - \nu_{\text{dom}}$ is a linear combination of positive coroots with non-negative rational coefficients; see [KR03, Win05], [GHKR06, Proposition 5.6.1], and [Gas10].

¹ In fact in this case $X_\mu(b)$ is defined as a functor not on \bar{k} -algebras, but rather on certain p -adically complete $W(\bar{k})$ -algebras equipped with a lift of Frobenius. For this reason, what we have denoted $X_\mu(b)$ in the introduction is denoted $X_\mu(b)(W(\bar{k}))$ in the body of the paper.

We denote by $B(G, \mu)$ the set of σ -conjugacy classes $[b] \in B(G)$ satisfying these conditions, and we assume from now on that $[b] \in B(G, \mu)$.

Then $w_G(b) - \mu = (1 - \sigma)(c_{b,\mu})$ for an element $c_{b,\mu} \in \pi_1(G)$ whose $\pi_1(G)^\Gamma$ -coset is uniquely determined by this condition. The following is one of our main results.

THEOREM 1.1. *Assume that G^{ad} is simple and that μ is minuscule, and suppose that (μ, b) is Hodge–Newton indecomposable in G . Then w_G induces a bijection*

$$\pi_0(X_\mu(b)) \cong c_{b,\mu}\pi_1(G)^\Gamma$$

unless $[b] = [e^\mu]$ with μ central, in which case

$$X_\mu(b) \cong G(F)/G(\mathcal{O}_F)$$

is discrete.

Here G^{ad} denotes the adjoint group of G . The definition of Hodge–Newton indecomposability will be recalled below in § 2.5. In fact, without assuming that G^{ad} is simple, we show that w_G induces an isomorphism as in the first case of the theorem provided (μ, b) is *Hodge–Newton irreducible*, a condition slightly stronger than Hodge–Newton indecomposability, which is also recalled in § 2.5. When G^{ad} is simple, a Hodge–Newton indecomposable pair (μ, b) is Hodge–Newton irreducible unless $[b] = [e^\mu]$ with μ central. At the end of § 2.5 we also give the easy direct calculation showing the last assertion of the theorem.

The theorem describes $\pi_0(X_\mu(b))$ (for μ minuscule) when G^{ad} is simple and (μ, b) Hodge–Newton is indecomposable in G . The general case without these assumptions (but with μ still being minuscule) can be reduced to this one. Indeed, for any element $b \in G(L)$ there exists a $b' \in G(L)$ that is σ -conjugate to b , and a standard Levi subgroup $M \subset G$ such that $b' \in M(L)$ and (μ, b') is Hodge–Newton indecomposable in M , and such that the natural map $X_\mu^M(b') \rightarrow X_\mu^G(b')$ is a bijection.

To reduce to the case where G is adjoint and simple, we again denote by b and μ the images of b and μ in G^{ad} . Then the sets of connected components of $X_\mu^G(b)$ and $X_\mu^{G^{\text{ad}}}(b)$ are closely related. More precisely, we prove in § 2.4 that the diagram

$$\begin{array}{ccc} \pi_0(X_\mu^G(b)) & \longrightarrow & \pi_0(X_\mu^{G^{\text{ad}}}(b)) \\ w_G \downarrow & & w_{G^{\text{ad}}} \downarrow \\ c_{b,\mu}\pi_1(G)^\Gamma & \longrightarrow & c_{b,\mu}\pi_1(G^{\text{ad}})^\Gamma \end{array}$$

is Cartesian. Furthermore, affine Deligne–Lusztig varieties for products of groups are products of the affine Deligne–Lusztig varieties for the individual factors. This reduces the description of $\pi_0(X_\mu(b))$ from the general case to the case where G itself is simple.

In the course of the proof we obtain the following theorem (which is also a consequence of Theorem 1.1). It is less precise but has the advantage that it does not require any additional assumptions. Define an F -group J_b by setting

$$J_b(R) := \{g \in G(R \otimes_F L) : \sigma(g) = b^{-1}gb\}$$

for R an F -algebra. There is an inclusion $J_b \subset G$, defined over L , which is given on R -points (R an L -algebra) by the natural map $G(R \otimes_F L) \rightarrow G(R)$.

THEOREM 1.2. *If μ is minuscule then $J_b(F)$ acts transitively on $\pi_0(X_{\leq \mu}(b))$.*

In fact we will show in Theorem 4.1.12 that the action of a certain subgroup of $J_b(F)$ is transitive.

Our description of the connected components is used in an essential way in the work of one of us (Kisin) on the Langlands–Rapoport conjecture for mod p points on Shimura varieties. Our results also allow us to obtain a description of the set of connected components of (simple) unramified Rapoport–Zink spaces of PEL type.

More precisely, suppose that (G, b, μ) is a (simple) unramified Rapoport–Zink datum of EL type or unitary/symplectic PEL type (for the precise definition, see § 5). To this kind of datum we can associate a Rapoport–Zink space $\mathcal{M} = \mathcal{M}(G, b, \mu)$ which is a formal scheme locally formally of finite type over $\mathrm{Spf} \mathcal{O}_L$ (cf. [RZ96]). By the Dieudonné–Manin classification of isocrystals over \mathbb{F}_p , there exists a natural bijection $\theta : \mathcal{M}(G, b, \mu)(\mathbb{F}_p) \simeq X_\mu^G(b)$. Let $\check{\mathcal{M}}^{\mathrm{an}}$ be the generic fiber of $\check{\mathcal{M}}$ as a Berkovich analytic space. There exists a tower of finite étale covers $(\check{\mathcal{M}}_{\tilde{K}})_{\tilde{K} \subset G(\mathbb{Z}_p)}$ on $\check{\mathcal{M}}^{\mathrm{an}}$ parametrizing the \tilde{K} -level structures on the Tate module of the universal p -divisible group, where \tilde{K} runs through the open subgroups in $G(\mathbb{Z}_p)$. Let \mathbb{C}_p be the completion of an algebraic closure of \mathbb{Q}_p , and write $\pi_0(\check{\mathcal{M}}_{\tilde{K}} \hat{\otimes} \mathbb{C}_p)$ for the set of geometrically connected components of $\check{\mathcal{M}}_{\tilde{K}}$. The group $J_b(\mathbb{Q}_p) \times G(\mathbb{Q}_p) \times \mathrm{Gal}(\bar{L}/L)$ acts naturally on $\pi_0(\check{\mathcal{M}}_{\tilde{K}} \hat{\otimes} \mathbb{C}_p)$, where \bar{L} is the integral closure of L in \mathbb{C}_p . Moreover, there is a natural map

$$\delta = (\delta_{J_b}, \delta_G, \chi_{\delta_G, \mu}) : J_b(\mathbb{Q}_p) \times G(\mathbb{Q}_p) \times \mathrm{Gal}(\bar{L}/L) \rightarrow G^{\mathrm{ab}}(\mathbb{Q}_p),$$

where the maps δ_{J_b} and δ_G are the natural ones, and $\chi_{\delta_G, \mu}$ is given by the Artin reciprocity map and the reflex norm of μ . Then our main result implies the following theorem (see Theorem 5.1.10 below; cf. [Che14, Theorem 6.3.1]).

THEOREM 1.3. *If (b, μ) is Hodge–Newton irreducible, then the action of $J_b(\mathbb{Q}_p) \times G(\mathbb{Q}_p) \times \mathrm{Gal}(\bar{L}/L)$ on $\pi_0(\check{\mathcal{M}}_{\tilde{K}} \hat{\otimes} \mathbb{C}_p)$ factors through δ , and makes $\pi_0(\check{\mathcal{M}}_{\tilde{K}} \hat{\otimes} \mathbb{C}_p)$ into a $G^{\mathrm{ab}}(\mathbb{Q}_p)/\delta(\tilde{K})$ -torsor. In particular, there exist bijections*

$$\pi_0(\check{\mathcal{M}}_{\tilde{K}} \hat{\otimes} \mathbb{C}_p) \xrightarrow{\sim} G^{\mathrm{ab}}(\mathbb{Q}_p)/\delta(\tilde{K})$$

which are compatible when \tilde{K} varies.

For dominant elements $\mu, \mu' \in X_*(T)$ we say that $\mu' \preceq \mu$ if $\mu - \mu'$ is a non-negative integral linear combination of positive coroots. The closed affine Deligne–Lusztig variety is defined as

$$X_{\preceq \mu}(b) = \bigcup_{\mu' \preceq \mu} X_{\mu'}(b).$$

If μ is minuscule, $X_\mu(b) \cong X_{\preceq \mu}(b)$. We conjecture that Theorem 1.1 remains true without the assumption that μ is minuscule if we replace $X_\mu(b)$ by $X_{\preceq \mu}(b)$ in the statement. For split groups this is proved in [Vie08] in the function field case. For split groups in mixed characteristic it can be deduced by combining the arguments in [Vie08] with the theory of connected components of affine Deligne–Lusztig varieties in mixed characteristic developed in the present paper.

The proofs of the theorems are organized as follows. In § 2 we collect some foundational results including the behavior of the Cartan decomposition in a family, the definition of the affine Grassmannian and affine Deligne–Lusztig varieties in mixed characteristic. We also make the reductions discussed above, first to the case where (μ, b) is Hodge–Newton indecomposable, and then to the case where G is adjoint and simple. In § 3 we prove Theorem 1.1 for the case where b is superbasic, i.e. under the assumption that b is not σ -conjugate to an element of any

proper Levi subgroup of G . In Proposition 3.4.1 we show that each connected component contains an element of $J_b(F)X_{\preceq \mu'}^M(b)$ where M is a standard Levi subgroup such that b is superbasic in M and μ' is an M -dominant cocharacter with $\mu'_{\text{dom}} \preceq \mu$. Until this point we do not assume that μ is minuscule. Finally, in § 4 we assume that μ is minuscule and we connect suitable representatives of the connected components of all $X_{\preceq \mu'}^M(b)$ by one-dimensional subvarieties in $X_{\mu}^G(b)$. Here the reader may wish to first consider the case where G is a split group, as this substantially simplifies the arguments.

Apart from this introduction we only consider the arithmetic case. Proofs for the function field case are completely analogous, but simpler.

2. Affine Deligne–Lusztig varieties in mixed characteristic

2.1 The Cartan decomposition in families

2.1.1. Let $F = W(k)[1/p]$ with k a finite field with $q = p^r$ elements. Fix an algebraic closure \bar{k} of k , and let $L = W(\bar{k})[1/p]$. Write $\Gamma = \text{Gal}(\bar{k}/k)$. Then Γ has a canonical topological generator σ given by $x \mapsto x^q$, and acts in the natural way on L . Let G, B, T be as above, and write $W = W_G$ for the Weyl group of T in G .

We have the Cartan decomposition [BT72, 4.4.3]

$$G(L) = \coprod_{\mu} G(\mathcal{O}_L)p^{\mu}G(\mathcal{O}_L)$$

where μ runs over the dominant elements of $X_*(T)$. In particular, $\mu \mapsto p^{\mu}$ induces a bijection

$$X_*(T)/W \xrightarrow{\sim} G(\mathcal{O}_L) \backslash G(L) / G(\mathcal{O}_L). \tag{2.1.2}$$

We write $\mu_{G\text{-dom}}$ (or μ_{dom} if the group is clear) for the dominant element in the orbit of $\mu \in X_*(T)$ under W . For $\mu_1, \mu_2 \in X_*(T)$, we write $\mu_1 \preceq \mu_2$ if $\mu_2 - \mu_1$ is a linear combination of positive coroots with *integral*, non-negative coefficients. For $\nu_1, \nu_2 \in X_*(T)_{\mathbb{R}}$ we write $\nu_1 \leq \nu_2$ if $\nu_2 - \nu_1$ is a linear combination of positive coroots with *real*, non-negative coefficients.

2.1.3. Let R be a \bar{k} -algebra. A *frame* for R is a p -torsion free, p -adically complete and separated \mathcal{O}_L -algebra \mathcal{R} equipped with an isomorphism $\mathcal{R}/p\mathcal{R} \xrightarrow{\sim} R$, and a lift (again denoted σ) of the q -Frobenius σ on R to \mathcal{R} . When $q=p$, this is a special case of Zink’s definition [Zin01, Definition 1]. If $\theta : R \rightarrow R'$ is a map of \bar{k} -algebras, then a frame for θ is a morphism of \mathcal{O}_L -algebras $\tilde{\theta} : \mathcal{R} \rightarrow \mathcal{R}'$ from a frame of R to a frame of R' , which lifts θ and is compatible with σ .

Let κ be a perfect field. Any map $s : R \rightarrow \kappa$ admits a unique σ -equivariant map $\mathcal{R} \rightarrow W(\kappa)$, which we will often again denote by s .

LEMMA 2.1.4. *Let \mathcal{R} be a frame for R . Then any étale morphism $R \rightarrow R'$ admits a canonical frame $\mathcal{R} \rightarrow \mathcal{R}'$.*

Proof. Since the étale site is invariant under nilpotent thickenings, R' lifts canonically to an étale $\mathcal{R}/p^n\mathcal{R}$ algebra \mathcal{R}'_n , and we set $\mathcal{R}' = \varprojlim \mathcal{R}'_n$.

Similarly, the canonical isomorphism $R' \otimes_{R, \sigma} R \xrightarrow[\sigma \otimes 1]{\sim} R'$ lifts to a unique isomorphism $\mathcal{R}'_n \otimes_{\mathcal{R}, \sigma} \mathcal{R}'_n \xrightarrow{\sim} \mathcal{R}'_n$, and the composite

$$\mathcal{R}'_n \xrightarrow{a \mapsto a \otimes 1} \mathcal{R}'_n \otimes_{\mathcal{R}, \sigma} \mathcal{R}'_n \xrightarrow{\sim} \mathcal{R}'_n$$

lifts σ on \mathcal{R}'_n . Passing to the limit with n , we get a lift of σ on \mathcal{R}' . □

2.1.5. Fix a frame \mathcal{R} for R , and let $g \in G(\mathcal{R}_L)$. For a dominant $\mu \in X_*(T)$ let

$$S_\mu(g) = \{s \in \text{Spec } R : s(g) \in G(W(\bar{\kappa}(s)))p^\mu G(W(\bar{\kappa}(s)))\}$$

where $\bar{\kappa}(s)$ denotes an algebraic closure of $\kappa(s)$, and set

$$S_{\preceq\mu}(g) = \cup_{\mu' \preceq \mu} S_{\mu'}(g),$$

where μ' runs over dominant cocharacters $\preceq \mu$.

LEMMA 2.1.6. *Let R be a Noetherian, formally smooth \bar{k} -algebra, \mathcal{R} a frame for R , and $g \in G(\mathcal{R}_L)$.*

- (i) *The subset $S_{\preceq\mu}(g) \subset S = \text{Spec } R$ is Zariski closed.*
- (ii) *The subset $S_\mu(g)$ is locally closed and is closed if μ is minuscule.*
- (iii) *The function $s \mapsto [\mu_{s(g)}] \in \pi_1(G)$ is locally constant on $s \in \text{Spec } R$.*

Proof. We begin by checking that $S_{\leq\mu}(g) = \{s : \mu_{s(g)} \leq \mu\}$ is closed in S . By [RR96, 2.2(iv)] we have $\mu_{G,s(g)} \leq \mu$ if and only if for every representation $\rho : G_L \rightarrow \text{GL}(V)$ on an L -vector space V , we have $\rho \circ \mu_{G,s(g)} \leq \rho \circ \mu$.

Choosing a suitable \mathcal{O}_L -lattice $Q \subset V$, we may assume that ρ is induced by a map $G \rightarrow \text{GL}(Q)$ over \mathcal{O}_L (cf. the proof of [Kis13, 2.3.1]). Let $T' \subset \text{GL}(Q)$ be a maximal L -split torus containing the image of T . Then $\rho \circ \mu_{G,s(g)} = \mu_{\text{GL},s(\rho(g))}$ in $X_*(T')/W_{\text{GL}}$ where W_{GL} is the Weyl group of T' in $\text{GL}(Q)$. By [Kat79, Corollary 2.3.2] the set of points at which the Hodge polygon of a σ -isocrystal lies on or above a given polygon and has the same endpoints, is closed in S . Hence $S_{\leq\rho\circ\mu}(\rho(g)) \subset S$ is closed, and hence $S_{\leq\mu}(g) \subset S$ is closed.

It follows, in particular, that the function $s \mapsto [\mu_{G,s(g)}] \in \pi_1(G) \otimes_{\mathbb{Z}} \mathbb{Q}$ is locally constant on S , which proves (iii) when $\pi_1(G)$ has no torsion. To prove (iii) in general, let \tilde{G} be the universal cover of G^{der} and let $G' = \tilde{G} \times T$. The kernel of the natural map $G' \rightarrow G$ is a maximal torus $T' \subset \tilde{G}$. The obstruction to lifting g to a point of $G'_L(\mathcal{R}_L)$ lies in $H^1(\text{Spec } \mathcal{R}_L, T')$. Since T' is a split torus this obstruction corresponds to a finite collection of line bundles over $\text{Spec } \mathcal{R}_L$. Since \mathcal{R} is regular any line bundle on $\text{Spec } \mathcal{R}_L$ extends to a line bundle on $\text{Spec } \mathcal{R}$. Hence, after replacing S by a Zariski covering by affines, and \mathcal{R} by the corresponding frame (see Lemma 2.1.4), we may assume that g lifts to a point $g' \in G'_L(\mathcal{R}_L)$. By what we have already seen, the function $s \mapsto [\mu_{G',s(g')}] \in \pi_1(G')$ is locally constant, so $s \mapsto [\mu_{G,s(g)}] \in \pi_1(G)$ is locally constant.

To prove (i) and (ii) we may assume that S is connected. Then $[\mu_{s(g)}] \in \pi_1(G)$ does not depend on s , and $S_{\preceq\mu}(g)$ is empty unless $[\mu]$ is equal to this constant class. If this condition holds, then $\mu_{s(g)} \preceq \mu$ if and only if $\mu_{s(g)} \leq \mu$. Thus, $S_{\preceq\mu}(g) = \{s : \mu_{s(g)} \leq \mu\}$, which we have seen is closed. This proves (i) and that $S_\mu(g)$ is locally closed. If μ is minuscule and $\mu' \preceq \mu$ is dominant with $[\mu] = [\mu']$ in $\pi_1(G)$, then $\mu' = \mu$, so (ii) follows. \square

2.1.7. Suppose that $g \in G(\mathcal{R}_L)$ and $S_\mu(g) = S = \text{Spec } R$. Then a natural question is whether $G(\mathcal{R})p^\mu G(\mathcal{R}) \subset G(\mathcal{R}_L)$ contains g . We will show that this condition holds étale locally on R , when R is a reduced, Noetherian \bar{k} -algebra. This will be used in § 2.5 below. To do this we need some preparation.

By an *étale covering*, we mean a faithfully flat, étale morphism $R \rightarrow R'$. We begin with the following simple lemma which allows us to work with frames étale locally on R , and will allow us to often replace R by an étale covering in arguments.

LEMMA 2.1.8. *Let R be a reduced \bar{k} -algebra, and \mathcal{R} a frame for R . Suppose that $g \in G(\mathcal{R}_L)$ and $S_\mu(g) = S$. If $\kappa \supset \bar{k}$ is a perfect field of characteristic p , and $L'/W(\kappa)[1/p]$ a finite extension with ring of integers $\mathcal{O}_{L'}$, then for any map of \mathcal{O}_L -algebras $\xi : \mathcal{R} \rightarrow \mathcal{O}_{L'}$, we have*

$$\xi(g) \in G(\mathcal{O}_{L'})p^\mu G(\mathcal{O}_{L'}).$$

Proof. As in the proof of Lemma 2.1.6, it suffices to consider the case $G = \text{GL}(Q)$ for a finite free \mathcal{O}_L -module Q .

For ξ as in the lemma, let i_ξ denote the greatest number in $e(L')^{-1}\mathbb{Z}$ such that $\xi(g)(Q \otimes_{\mathcal{O}_L} \mathcal{O}_{L'}) \subset \pi_{L'}^{e(L')i_\xi} Q \otimes_{\mathcal{O}_L} \mathcal{O}_{L'}$, where $\pi_{L'}$ is a uniformizer for L' and $e(L')$ is the absolute ramification degree of L' . Our assumptions imply that if ξ is a map $s : \mathcal{R} \rightarrow \mathcal{O}_L = W(\bar{k})$ induced by a closed point $s : R \rightarrow \bar{k}$, then i_ξ has a value $i_0 \in \mathbb{Z}$ which does not depend on s .

We claim that $i_\xi = i_0$ for any ξ . To see this we may multiply μ by a central character and g by a scalar, and assume that $i_0 \geq 0$, and that g stabilizes $Q \otimes_{\mathcal{O}_L} \mathcal{R}$. If $i_0 > 0$, then g induces an endomorphism of $Q \otimes_{\mathcal{O}_L} R$ which vanishes at every closed point of R , and hence is identically 0 as R is reduced. Hence, $g(Q \otimes_{\mathcal{O}_L} \mathcal{R}) \subset p(Q \otimes_{\mathcal{O}_L} \mathcal{R})$. Thus, after again multiplying μ by a central character, we may assume that $i_0 = 0$ and g leaves $Q \otimes_{\mathcal{O}_L} \mathcal{R}$ stable. This implies that g induces an endomorphism of $Q \otimes_{\mathcal{O}_L} R$, which is non-zero at every closed point, and hence $i_\xi = 0$ for all ξ .

Now the lemma follows by applying the claim just proved to the exterior powers of Q . \square

2.1.9. Suppose that Q is a finite free \mathcal{O}_L -module equipped with an action of G . For $\mu \in X_*(G)$ we denote by μ^Q the $\text{GL}(Q)$ -valued cocharacter given by $z \mapsto z^i \mu(z)$, where i is the integer such that the eigenvalues of $p^i \mu(p)$ acting on Q are non-negative powers of p , and include 1. Let $P_\mu(Q) \subset G \times_{\mathcal{O}_F} G$ be the subgroup whose points are pairs (g_1, g_2) such that $g_1 \mu^Q(p) = \mu^Q(p) g_2$ in $\text{End } Q$. Note that this need not be a flat subgroup, in general.

Similarly, if α is a collection of finite free \mathcal{O}_F -modules equipped with an action of G , then we denote by $P_\mu(\alpha)$ the intersection of the $P_\mu(Q) \subset G \times_{\mathcal{O}_F} G$ for $Q \in \alpha$. Note that the generic fiber of $P_\mu(\alpha)$ may be identified with G via the embedding

$$G \rightarrow G \times G : g \mapsto (g, \mu(p)^{-1} g \mu(p)).$$

LEMMA 2.1.10. *Let $G \hookrightarrow \text{GL}(Q)$ be a faithful representation of G on a finite free module Q , let $\alpha = \{\wedge^i Q\}_{i \geq 1}$, and let $\mu \in X_*(G)$. Then $P_\mu(\alpha)$ is a smooth model of G , and may be identified with the closure of the embedding $G \rightarrow G \times G$ above.*

Proof. Let $P_\mu \subset G$ denote the parabolic defined by μ , so that $\text{Lie } P_\mu \subset \text{Lie } G$ is the submodule on which μ acts by non-negative weights. Similarly, let P_μ° denote the opposite parabolic and M_μ the common reductive quotient of P_μ and P_μ° . We will use a subscript of k to denote the special fiber of an \mathcal{O}_F -scheme.

Write $Q_k = \bigoplus Q_i$ where μ^Q acts on Q_i with weight n_i and $0 = n_0 < n_1 < \dots$, and for $i \geq 0$, let $d_i = \dim_k Q_i$ and $e_i = \sum_{j=0}^i d_j$. The condition $g_1 \mu^Q(p) = \mu^Q(p) g_2$ implies that if $(g_1, g_2) \in P_\mu(\alpha)$ then g_1 leaves Q_0 stable, g_2 leaves $\bigoplus_{i>0} Q_i$ stable and g_1, g_2 induce the same endomorphism of $Q_0 = Q_k / \bigoplus_{i>0} Q_i$.

Note that

$$(\wedge^{e_i+1} Q)_0 = \left(\bigotimes_{j=0}^i \wedge^{d_j} Q_j \right) \otimes Q_{i+1},$$

where $(\wedge^{e_i+1} Q)_0$ denotes the summand of $\wedge^{e_i+1} Q$ on which $\mu^{\wedge^{e_i+1} Q}$ acts with weight 0. Hence, for $i \geq 0$, g_1 leaves $\bigoplus_{j \leq i} Q_j$ stable, g_2 leaves $\bigoplus_{j > i} Q_j$ stable and g_1, g_2 induce the same endomorphism of Q_i .

It follows that $P_\mu(\alpha)_k$ is contained in $P_{\mu,k}^\circ \times_{M_{\mu,k}} P_{\mu,k}$. Thus, if P'_μ denotes the closure of $G \hookrightarrow G \times G$, under the embedding above, then we have

$$P'_{\mu,k} \subset P_\mu(\alpha)_k \subset P_{\mu,k}^\circ \times_{M_{\mu,k}} P_{\mu,k}.$$

Since $P_{\mu,k}^\circ \times_{M_{\mu,k}} P_{\mu,k}$ is a smooth connected group scheme with the same dimension as $P'_{\mu,k}$, the above inclusions must be equalities, which proves the lemma. \square

PROPOSITION 2.1.11. *Let R be a reduced, Noetherian \bar{k} -algebra, \mathcal{R} a frame for R , and $g \in G(\mathcal{R}_L)$. Suppose that $S_\mu(g) = S$. Then there exists an étale covering $R \rightarrow R'$ such that $g \in G(\mathcal{R}')p^\mu G(\mathcal{R}')$, where \mathcal{R}' is the canonical frame for R' produced in Lemma 2.1.4.*

Proof. Let $G \hookrightarrow \text{GL}(Q)$ and α be as in 2.1.10. Consider the map

$$G \times G \rightarrow \bigoplus_{i \geq 1} \text{End}_{\mathcal{O}_F} \wedge^i Q; \quad (g_1, g_2) \mapsto (g_1 \mu^{\wedge^i Q}(p)g_2)_{i \geq 1}. \tag{2.1.12}$$

Note that by Lemma 2.1.10 the non-empty fibers of (2.1.12) are torsors under the smooth group scheme $P_\mu(\alpha)$. More precisely, for any \mathcal{O}_F -scheme T the map on T -valued points induced by (2.1.12) has fibers which are either empty or torsors under $P_\mu(\alpha)(T)$. Hence, the pullback of (2.1.12) by the image of any point in $G \times G(T)$ is a $P_\mu(\alpha)$ -torsor.

Let $\gamma_i = \mu^{\wedge^i Q}(p)\mu(p)^{-1}g$ for each $i \geq 1$. Then $\gamma = (\gamma_i)_{i \geq 1}$ is an \mathcal{R} -point of $\bigoplus_{i \geq 1} \text{End}_{\mathcal{O}_F} \wedge^i Q$. By 2.1.8 for any perfect field $\kappa \supset \bar{k}$, any finite extension $L'/W(\kappa)[1/p]$, and any map of \mathcal{O}_L -algebras $\xi : \mathcal{R} \rightarrow \mathcal{O}_{L'}$, $\xi^*(\gamma)$ lifts to a point of $G \times G(\mathcal{O}_{L'})$, and hence for any such ξ the pullback of (2.1.12) by $\xi^*(\gamma)$ is a $P_\mu(\alpha)$ -torsor and, in particular, flat. It follows from Lemma 2.1.13 below that the pullback of (2.1.12) by γ is a (flat) $P_\mu(\alpha)$ -torsor.

Finally, the lemma follows, since the above torsor can be trivialized over some étale covering of R . \square

LEMMA 2.1.13. *Let \mathcal{R} be a p -adically complete and separated, p -torsion free \mathcal{O}_L -algebra, such that $\mathcal{R}/p\mathcal{R}$ is reduced and Noetherian, and X a finite type \mathcal{R} -scheme. Suppose that for any perfect field $\kappa \supset \bar{k}$, any finite extension $L'/W(\kappa)[1/p]$, and any map of \mathcal{O}_L -algebras $\xi : \mathcal{R} \rightarrow \mathcal{O}_{L'}$, the fiber X_ξ is flat over $\mathcal{O}_{L'}$. Then X is a flat \mathcal{R} -scheme.*

Proof. It suffices to check that X is flat at every closed point $x \in \text{Spec } R$. Let $\widehat{\mathcal{R}}_x$ denote the completion of \mathcal{R} at x . By [RG71, 4.2.8], $X \otimes_{\mathcal{R}} \widehat{\mathcal{R}}_x$ is flat, provided $\bigcap_\xi \ker \xi = 0$ where ξ runs over all maps $\widehat{\mathcal{R}}_x \rightarrow \mathcal{O}_{L'}$ with L' as in the lemma.

To see this, we first note that \mathcal{R} is reduced. Indeed, if $\alpha \in \mathcal{R}$ is a nilpotent element, then $\alpha^n = 0$ for some n , so that $\alpha \in p\mathcal{R}$, as $\mathcal{R}/p\mathcal{R}$ is reduced. Since \mathcal{R} is p -torsion free, we can apply the same argument to $p^{-1}\alpha$, and we find that α is infinitely divisible by p in \mathcal{R} . As \mathcal{R} is p -adically separated, this is a contradiction, unless $\alpha = 0$.

By [EGA, IV 10.5.8], $\widehat{\mathcal{R}}_x[1/p]$ is a Jacobson ring. Let $y \in \text{Spec } \widehat{\mathcal{R}}_x[1/p]$ be a closed point, and L_y the quotient of $\widehat{\mathcal{R}}_x[1/p]$ by the corresponding maximal ideal. Then L_y is equipped with a discrete valuation, and the corresponding valuation ring \mathcal{O}_{L_y} is a finite $\widehat{\mathcal{R}}_x$ -algebra (see [EGA, IV 10.5.10] and its proof). In particular, if $\bar{\kappa}(x)$ is an algebraic closure of $\kappa(x)$, then L_y admits an embedding into a finite extension $L'/W(\bar{\kappa}(x))[1/p]$. Since any map $\xi : \mathcal{R} \rightarrow L'$ factors through $\mathcal{O}_{L'}$, we see that $\bigcap_\xi \ker \xi = 0$ as required. \square

COROLLARY 2.1.14. *Let R be a Noetherian, formally smooth \bar{k} -algebra, \mathcal{R} a frame for R , and $g \in G(\mathcal{R}_L)$. Suppose that μ is minuscule and that $S_\mu(g)$ contains the generic points of $\text{Spec } R$.*

Then there exists an étale covering $R \rightarrow R'$ such that $g \in G(\mathcal{R}')p^\mu G(\mathcal{R}')$, where \mathcal{R}' is the canonical frame for R' produced in Lemma 2.1.4.

Proof. Since $S_\mu(g)$ contains the generic points of $\text{Spec } R$, and μ is minuscule, we have

$$S_\mu(g) = S_{\leq \mu}(g) = \text{Spec } R$$

by Lemma 2.1.6, and the corollary follows from Proposition 2.1.11. □

2.2 The affine Grassmannian in mixed characteristic

2.2.1. Let \mathcal{R} be a p -torsion free, p -adically complete and separated \mathcal{O}_L -algebra. Let $X(\mathcal{R}) = X_G(\mathcal{R})$ denote the set of isomorphism classes of pairs (\mathcal{T}, τ) where \mathcal{T} is a G -torsor over $\text{Spec } \mathcal{R}$, and τ is a trivialization of \mathcal{T} over $\text{Spec } \mathcal{R}_L$.

Let S be a flat p -adic formal scheme over \mathcal{O}_L , and let S_0 be the reduced subscheme of S . An étale morphism $U_0 \rightarrow S_0$ lifts canonically to a formally étale morphism of p -adic formal schemes $U \rightarrow S$. We call such a morphism a *formal étale neighborhood* of S . We call such a morphism a covering if U_0 is a covering of S_0 . We say that U is a *formal affine étale neighborhood* if in addition U is formal affine (or equivalently U_0 is affine).

In particular, $X_G : \text{Spf } \mathcal{R} \mapsto X(\mathcal{R})$ defines a functor on formal affine étale neighborhoods of S . Equivalently, we may view X_G as a functor on affine étale neighborhoods of S_0 .

Given a section (\mathcal{T}, τ) of $X_G(S)$ there is a formal étale covering $\text{Spf } \mathcal{R} \rightarrow S$ over which \mathcal{T} becomes trivial. To $g \in G(\mathcal{R}_L)$, we can associate the trivial G -torsor over $\text{Spec } \mathcal{R}$ given by G itself, equipped with the trivialization over $\text{Spec } \mathcal{R}_L$ corresponding to left multiplication by g . Two elements $g, g' \in G(\mathcal{R}_L)$ give rise to the same torsor with trivialization over $\text{Spec } \mathcal{R}_L$ if and only if they have the same image in $G(\mathcal{R}_L)/G(\mathcal{R})$. The set of elements of $X_G(\text{Spf } \mathcal{R})$ where the underlying G -torsor over $\text{Spec } \mathcal{R}$ is trivial is in natural bijection with $G(\mathcal{R}_L)/G(\mathcal{R})$. Thus, the functor X_G is an analog of the affine Grassmannian in mixed characteristic.²

We will often reduce questions about G -bundles to questions about vector bundles. For this we will need the following lemma.

LEMMA 2.2.2. *Let Y be a flat \mathcal{O}_L -scheme. Let \mathcal{F} denote the category of exact, faithful tensor functors from representations of G on finite free \mathcal{O}_L -modules to vector bundles on Y .*

If P is a G -bundle on Y , and V is a representation of G on a finite free \mathcal{O}_L module, write $F_P(V) = G \backslash (P \times V)$. Then $P \mapsto F_P$ is an equivalence between the category of G -bundles on Y , and the category \mathcal{F} .

Proof. See [Bro08, Theorem 2.1.5.5] (cf. also [Nor76]). □

LEMMA 2.2.3. *The functor X_G extends to a sheaf (again denoted X_G) on the étale topology of S_0 .*

Proof. We extend X_G to a presheaf X_G^- on the étale topology of S_0 by setting $X_G^-(U_0) = \varprojlim X_G(V_0)$ where V_0 runs over affine étale neighborhoods of $V_0 \rightarrow U_0$ (cf. [EGA, § 0, 3.2]), and we let X_G^+ denote the sheafification of X_G^- . Note that we do not claim that X_G^- is a sheaf, but only that its values agree with those of X_G^+ on affine étale neighborhoods.

² This definition works well for our purposes, but has the esthetic disadvantage that it depends on \mathcal{R} and not just on $R = \mathcal{R}/p\mathcal{R}$. Haboush [Hab05] (see also Kreidl [Kre14] and Lusztig [Lus12]) has proposed an approach to the affine Grassmannian in mixed characteristic which uses Witt vectors and the Greenberg functor, and does not depend on the choice of lifting. However, this works well only when R is perfect. Since perfect rings are typically not Noetherian many of our commutative algebra arguments would break down in this setting.

We have to show that for any formal affine étale neighborhoods $\mathrm{Spf} \mathcal{R} \rightarrow S$, $X_G(\mathcal{R}) = X_G^+(\mathcal{R})$. By definition, an element of $X_G^+(\mathcal{R})$ is defined by giving a collection $\{\mathrm{Spf} \mathcal{R}_i\}_i$ of formal affine étale neighborhoods of $\mathrm{Spf} \mathcal{R}$, whose union is a covering of $\mathrm{Spf} \mathcal{R}$, an element (\mathcal{T}_i, τ_i) of $X_G(\mathcal{R}_i)$ for each i , and isomorphisms $(\mathcal{T}_i, \tau_i) \xrightarrow{\sim} (\mathcal{T}_j, \tau_j)$ over $\mathrm{Spec} \mathcal{R}_{ij}$ satisfying the cocycle condition. Here $\mathrm{Spf} \mathcal{R}_{i,j} = \mathrm{Spf} \mathcal{R}_i \times_{\mathrm{Spf} \mathcal{R}} \mathrm{Spf} \mathcal{R}_j$. We have to show that any such collection of data arises from an element (\mathcal{T}, τ) in $X_G(\mathcal{R})$, which is unique up to a unique isomorphism.

By Lemma 2.2.2 it suffices to prove the analogous statement for vector bundles of some fixed rank d . Thus let $\{(V_i, \tau_i)\}_i$ be a collection consisting of vector bundles V_i of rank d , over $\mathrm{Spec} \mathcal{R}_i$ together with trivializations τ_i over $\mathrm{Spec} \mathcal{R}_{i,L}$. Suppose that we are given isomorphisms $\{(V_i, \tau_i) \xrightarrow{\sim} (V_j, \tau_j)\}_{i,j}$ over $\mathrm{Spec} \mathcal{R}_{i,j,L}$ for all i, j satisfying the cocycle condition. We have to show that this data arises from a vector bundle V over $\mathrm{Spec} \mathcal{R}$ together with a trivialization over $\mathrm{Spec} \mathcal{R}_L$, determined up to unique isomorphism.

By étale descent, for $n \geq 0$ this data gives rise to a uniquely determined vector bundle V_n on $\mathrm{Spec} \mathcal{R}/p^n \mathcal{R}$, and hence to a vector bundle on $\mathrm{Spec} \mathcal{R}$. To construct the trivialization τ , we may first assume that the above covering consists of finitely many formal affine étale neighborhoods, since $\mathrm{Spf} \mathcal{R}$ is quasi-compact. Now choose a sufficiently large integer n such that for each i , $p^n \tau_i$ and $p^n \tau_i^{-1}$ induce maps $V_i \rightarrow \mathcal{R}_i^d$ and $\mathcal{R}_i^d \rightarrow V_i$ whose composite is multiplication by p^{2n} . As above, by étale descent these maps give rise to maps $\mathcal{R}^d \rightarrow V$ and $V \rightarrow \mathcal{R}^d$ whose composite is multiplication by p^{2n} . Inverting p and dividing the resulting maps by p^n produces the required trivialization τ . □

2.2.4. Now suppose that $S = \mathrm{Spf} \mathcal{R}$ is formal affine, and locally Noetherian. We will give a description of $X_G(\mathcal{R})$ using the étale topology on $\mathrm{Spec} \mathcal{R}$, which will be useful for computations.

Let $j : \mathrm{Spec} \mathcal{R}_L \hookrightarrow \mathrm{Spec} \mathcal{R}$ and $i : \mathrm{Spec} \mathcal{R}/p\mathcal{R} \rightarrow \mathrm{Spec} \mathcal{R}$ denote the inclusions. We again write X_G for the étale sheaf $i_* X_G$ on $\mathrm{Spec} \mathcal{R}$.

Let U be an étale neighborhood of $\mathrm{Spec} \mathcal{R}$. Using the fact that a G -torsor over $\mathrm{Spec} \mathcal{R}$ is étale locally trivial, one sees that $(j_* G/G)(U)$ is in bijection with the set of isomorphism class of pairs consisting of a G -torsor over U , equipped with a trivialization over $U \otimes_{\mathcal{O}_L} L$. Thus, we have a natural map of étale sheaves $j_* G/G \rightarrow X_G$.

LEMMA 2.2.5. *The map $j_* G/G \rightarrow X_G$ is an isomorphism.*

Proof. We first consider the case $G = \mathrm{GL}_n$. Let $U = \mathrm{Spec} \mathcal{R}'$ be an étale neighborhood of S , $U_0 = U \otimes_{\mathbb{Z}} \mathbb{Z}/p\mathbb{Z}$, and $\hat{U} = \mathrm{Spf} \mathcal{R}'$ the p -adic completion of U . Let $\tilde{U} = \mathrm{Spec} \tilde{\mathcal{R}}'$ be the localization of U along U_0 , so $\tilde{\mathcal{R}}'$ is obtained from \mathcal{R}' by inverting all elements which map to a unit in $\mathcal{R}'/p\mathcal{R}'$. Note that any maximal ideal of $\tilde{\mathcal{R}}'$ contains (p) so that \mathcal{R}' , which is the p -adic completion of $\tilde{\mathcal{R}}'$, is a faithfully flat $\tilde{\mathcal{R}}'$ -algebra.

Since X_G is a sheaf, it suffices to show that for any \mathcal{R}' as above, the map $\mathrm{GL}_n(\tilde{\mathcal{R}}'_L)/\mathrm{GL}_n(\tilde{\mathcal{R}}') \rightarrow \mathrm{GL}_n(\mathcal{R}'_L)/\mathrm{GL}_n(\mathcal{R}')$ is a bijection. The injectivity follows from the fact that \mathcal{R}' is faithfully flat over $\tilde{\mathcal{R}}'$, which implies that $\tilde{\mathcal{R}}'_L/\tilde{\mathcal{R}}'$ injects into $\mathcal{R}'_L/\mathcal{R}'$. For the surjectivity, suppose that $g \in \mathrm{GL}_n(\mathcal{R}'_L)$. Let s be an integer such that $g, g^{-1} \in M_n(p^{-s}\mathcal{R}')$. For any $m > 0$ there exists $h \in M_n(p^{-s}\mathcal{R}')$ such that $g - h = p^m \delta$ for some $\delta \in M_n(\mathcal{R}')$. For m sufficiently large, $h \in \mathrm{GL}_n(\tilde{\mathcal{R}}')$ and $1 + p^m h^{-1} \delta \in \mathrm{GL}_n(\mathcal{R}')$. As $g = h(1 + p^m h^{-1} \delta)$, this proves the surjectivity.

Now suppose that G is arbitrary, and let P be a G -bundle over $\mathrm{Spec} \mathcal{R}'$ equipped with a trivialization over $\mathrm{Spec} \mathcal{R}'_L$. Then P gives rise to an exact, faithful tensor functor F_P which associates to each \mathcal{O}_L -representation V of G the vector bundle $F_P(V) = G \setminus V \times P$, together with an isomorphism $\tau_V : V \otimes \mathcal{R}'_L \xrightarrow{\sim} F_P(V) \otimes_{\mathcal{O}_L} L$. By the case of vector bundles proved above, $(F_P(V), \tau_V)$ arises from a pair $(\tilde{F}_P(V), \tilde{\tau}_V)$ consisting of a vector bundle $\tilde{F}_P(V)$ over $\mathrm{Spec} \mathcal{R}'$

together with an isomorphism $\tilde{\tau}_V : V \otimes \mathcal{R}'_L \xrightarrow{\sim} \tilde{F}_P(V) \otimes_{\mathcal{O}_L} L$, and this pair is unique up to canonical isomorphism. In particular, $\tilde{F}_P(V)$ is a faithful tensor functor. Moreover, $\tilde{F}_P(V)$ is exact: over (p) this follows from the fact that \mathcal{R}' is a faithful $\tilde{\mathcal{R}}'$ algebra, and after inverting p it is forced by the existence of the isomorphisms $\tilde{\tau}_V$. Using Lemma 2.2.2, we obtain the required G -bundle over $\text{Spec } \mathcal{R}'$ equipped with a trivialization over $\text{Spec } \mathcal{R}'_L$. \square

2.2.6. The following lemma, in the case where R is a Dedekind domain, shows that X_G satisfies an extension property which is analogous to of the valuative criterion for properness.

LEMMA 2.2.7. *Suppose that $R = \mathcal{R}/p\mathcal{R}$ is a Noetherian, formally smooth domain over \bar{k} . Let $f \in \mathcal{R} \setminus p\mathcal{R}$, and $\hat{\mathcal{R}}_f$ the p -adic completion of $\mathcal{R}_f = \mathcal{R}[f^{-1}]$. Denote by $r_{G,f}$ the natural functor from the category of G -torsors on $\text{Spec } \mathcal{R}$ equipped with a trivialization over $\text{Spec } \mathcal{R}_L$, to the category of G -torsors on $\text{Spec } \hat{\mathcal{R}}_f$ equipped with a trivialization over $\text{Spec } \hat{\mathcal{R}}_{f,L}$.*

Then:

- (i) $r_{G,f}$ is fully faithful, and an equivalence if R is a Dedekind domain. In particular, the natural map

$$X_G(\mathcal{R}) \rightarrow X_G(\hat{\mathcal{R}}_f)$$

is an injection, and a bijection if R is a Dedekind domain;

- (ii) if $M \subset G$ is a reductive, closed \mathcal{O}_F -subgroup, the diagram

$$\begin{array}{ccc} X_M(\mathcal{R}) & \longrightarrow & X_M(\hat{\mathcal{R}}_f) \\ \downarrow & & \downarrow \\ X_G(\mathcal{R}) & \longrightarrow & X_G(\hat{\mathcal{R}}_f) \end{array}$$

is Cartesian.

Proof. We first prove that the functor is fully faithful. By Lemma 2.2.2 it suffices to show this for vector bundles, and for this it is enough to check that $\hat{\mathcal{R}}_f \cap \mathcal{R}_L = \mathcal{R}$. Let $\tilde{\mathcal{R}}_f$ be the localization of \mathcal{R}_f along (p) . Then $\hat{\mathcal{R}}_f \cap \hat{\mathcal{R}}_{f,L} = \tilde{\mathcal{R}}_f$, since $\hat{\mathcal{R}}_f$ is a fully faithful $\tilde{\mathcal{R}}_f$ -algebra, and $\tilde{\mathcal{R}}_f \cap \mathcal{R}_L = \mathcal{R}$.

Now suppose that R is a Dedekind domain. By Lemma 2.2.5, to show essential surjectivity, it suffices to show that a G -bundle P over $\text{Spec } \mathcal{R}_f$ equipped with a trivialization over $\text{Spec } \mathcal{R}_{f,L}$ extends uniquely to a G -bundle over $\text{Spec } \mathcal{R}$. Using the trivialization, we may extend P to a G -bundle over the complement of a set of codimension 2 in $\text{Spec } \mathcal{R}$, equipped with a trivialization over $\text{Spec } \mathcal{R}_L$. By [CS79, Theorem 6.13], since G is reductive over \mathcal{O}_L , any such bundle extends to a G -bundle over $\text{Spec } \mathcal{R}$. This proves (i).

To prove (ii), it suffices, by (i), to show that if $(\mathcal{T}_M, \tau) \in X_M(\hat{\mathcal{R}}_f)$ lifts to an element of $X_G(\hat{\mathcal{R}})$ then it lifts to an element of $X_M(\hat{\mathcal{R}})$. Using the full faithfulness in (i) again, it suffices to prove this with R replaced by an étale covering. Thus we may assume that (\mathcal{T}_M, τ) is given by an element in $g \in G(\hat{\mathcal{R}}_L)$. By Lemma 2.2.5, \mathcal{T}_M arises from an M -bundle on $\text{Spec } \mathcal{R}_f$, and we extend it to an M -bundle \mathcal{T}'_M on $U := \text{Spec } \mathcal{R}_f \cup \text{Spec } \mathcal{R}_L$, equipped with a trivialization over $\text{Spec } \mathcal{R}_L$. Since \mathcal{T}'_M arises from g , the G -torsor induced by \mathcal{T}'_M is trivial. Thus it corresponds to a section in $G/M(U)$. The complement of U in $\text{Spec } \mathcal{R}$ has codimension 2 or greater. Since M is reductive, G/M is a smooth, affine scheme. It follows that any section in $G/M(U)$ extends to $\text{Spec } \mathcal{R}$. This shows that \mathcal{T}'_M extends to an M -bundle of $\text{Spec } \mathcal{R}$, and proves (ii). \square

2.2.8. Now suppose that \mathcal{R} has the structure of a frame for $R = \mathcal{R}/p\mathcal{R}$. If $\mathrm{Spf} \mathcal{R}' \rightarrow \mathrm{Spf} \mathcal{R}$ is a formal affine étale neighborhood, then as remarked in Lemma 2.1.4, $\mathrm{Spf} \mathcal{R}'$ has a canonical structure of frame for $R' = \mathcal{R}'/p\mathcal{R}'$. Thus given any $s \in \mathrm{Spec} R'$ and $g \in X_G(\mathcal{R}')$, we may consider the induced element $s(g) \in X_G(W(\bar{\kappa}(s))) = G(W(\bar{\kappa}(s))[1/p])/G(W(\bar{\kappa}(s)))$.

LEMMA 2.2.9. *Let R be a formally smooth, Noetherian \bar{k} -algebra, and \mathcal{R} a frame for R . We regard $\pi_1(G)$ as a constant étale sheaf on $\mathrm{Spec} R$ with value $\pi_1(G)$. Then there is a canonical map $w_G : X_G \rightarrow \pi_1(G)$ of étale sheaves on $\mathrm{Spec} R$, such that for any étale covering $\mathrm{Spec} R' \rightarrow \mathrm{Spec} R$, $s \in \mathrm{Spec} R'$, and $g \in X_G(\mathcal{R}')$, we have*

$$[\mu_{s(g)}] = w_G(g)_s \in \pi_1(G).$$

Proof. This follows immediately from Lemma 2.1.6. □

2.3 Affine Deligne–Lusztig varieties

2.3.1. Let \mathcal{R} be a p -torsion free, p -adically complete and separated \mathcal{O}_L -algebra. Recall that for $g \in G(\mathcal{R}_L)$ and $\mu \in X_*(T)$ dominant, we defined

$$S_\mu(g) = \{s \in \mathrm{Spec} R : s(g) \in G(W(\bar{\kappa}(s)))p^\mu G(W(\bar{\kappa}(s)))\}$$

where $\bar{\kappa}(s)$ denotes an algebraic closure of $\kappa(s)$. Note that the condition on g in the definition of $S_\mu(g)$ depends only on the image of g in $G(\mathcal{R}_L)/G(\mathcal{R})$. We may therefore define $S_\mu(g)$ and $S_{\leq \mu}(g)$ in the same way for any $g \in G(\mathcal{R}_L)/G(\mathcal{R})$.

Now let R be a \bar{k} -algebra, $S = \mathrm{Spec} R$ and \mathcal{R} a frame for R . For $b \in G(L)$ we set

$$X_{\leq \mu}(b)(\mathcal{R}) = \{g \in X_G(\mathcal{R}) : S_{\leq \mu}(g^{-1}b\sigma(g)) = S\},$$

and we define $X_\mu(b)(\mathcal{R})$ in an analogous way, replacing $S_{\leq \mu}$ by S_μ . If $\mathrm{Spf} \mathcal{R}' \rightarrow \mathrm{Spf} \mathcal{R}$ is a formal affine étale neighborhood, then as remarked above, $\mathrm{Spf} \mathcal{R}'$ has a canonical structure of frame for $R' = \mathcal{R}'/p\mathcal{R}'$. Thus we may consider $X_{\leq \mu}(b)(\mathcal{R}')$ (respectively $X_\mu(b)(\mathcal{R}')$). Note that the above definition probably needs to be refined if one wants to obtain a good notion of non-reduced structure on affine Deligne–Lusztig sets. However, for our study of connected components this is not relevant.

For $g_0 \in G(L)$ we have natural bijections $X_{\leq \mu}(b)(\mathcal{R}) \rightarrow X_{\leq \mu}(g_0^{-1}b\sigma(g_0))(\mathcal{R})$ with $g \mapsto g_0^{-1}g$. Therefore, all of the following notions for these sets and, in particular, the set of connected components of $X_{\leq \mu}(b)$ only depend on the σ -conjugacy class of b .

In the analogous situation, when F has characteristic p , any \bar{k} -algebra R admits the canonical frame $R[[t]]$. Thus $X_\mu(b)$ can be regarded as a functor on \bar{k} -algebras, by setting $X_\mu(b)(R)$ to be the set $X_\mu(b)(R[[t]])$ defined as above. In fact, in this setting, $X_\mu(b)$ is a scheme in characteristic p . Although one would like to have a similar interpretation in mixed characteristic there is no canonical frame, and we do not know of any way to formalize this heuristic.

We will sometimes write simply $X_{\leq \mu}(b)$ for $X_{\leq \mu}(b)(W(\bar{k}))$. When we want to make the group G explicit we will write $X_{\leq \mu}^G(b)$ for $X_{\leq \mu}(b)$.

LEMMA 2.3.2. *The functors $X_{\leq \mu}(b)$ and $X_\mu(b)$ are subsheaves of X_G in the étale topology of $\mathrm{Spec} R$.*

Proof. This follows from Lemma 2.2.3 together with the fact that the conditions defining $X_{\leq \mu}(b)$ and $X_\mu(b)$ are local for the étale topology on $\mathrm{Spec} R$. □

LEMMA 2.3.3. *Suppose that $R = \mathcal{R}/p\mathcal{R}$ is Noetherian and formally smooth over \bar{k} . Let $f \in \mathcal{R} \setminus p\mathcal{R}$, and $\widehat{\mathcal{R}}_f$ the p -adic completion of \mathcal{R}_f . Then the diagram*

$$\begin{array}{ccc} X_{\leq \mu}(b)(\mathcal{R}) & \longrightarrow & X_{\leq \mu}(b)(\widehat{\mathcal{R}}_f) \\ \downarrow & & \downarrow \\ X_G(\mathcal{R}) & \longrightarrow & X_G(\widehat{\mathcal{R}}_f) \end{array}$$

is Cartesian, and similarly with X_μ in place of $X_{\leq \mu}$ if μ is minuscule. In particular:

- (i) The natural map $X_{\leq \mu}(b)(\mathcal{R}) \rightarrow X_{\leq \mu}(b)(\widehat{\mathcal{R}}_f)$ is injective, and is bijective if R is a Dedekind domain.
- (ii) If μ is minuscule the natural map $X_\mu(b)(\mathcal{R}) \rightarrow X_\mu(b)(\widehat{\mathcal{R}}_f)$ is injective, and is bijective if R is a Dedekind domain.
- (iii) If $M \subset G$ is a closed, reductive \mathcal{O}_F -subgroup with $b \in M(L)$, then the diagram

$$\begin{array}{ccc} X_{\leq \mu}^M(b)(\mathcal{R}) & \longrightarrow & X_{\leq \mu}^M(b)(\widehat{\mathcal{R}}_f) \\ \downarrow & & \downarrow \\ X_{\leq \mu}^G(b)(\mathcal{R}) & \longrightarrow & X_{\leq \mu}^G(b)(\widehat{\mathcal{R}}_f) \end{array}$$

is Cartesian, and similarly with X_μ in place of $X_{\leq \mu}$ if μ is minuscule.

Proof. Let $g \in X_{\leq \mu}(\widehat{\mathcal{R}}_f)$, and suppose that g arises from an element $\tilde{g} \in X_G(\mathcal{R})$. By Lemma 2.1.6, the condition $S_{\leq \mu}(g^{-1}b\sigma(g)) = \text{Spec } R[1/f]$ implies $S_{\leq \mu}(\tilde{g}^{-1}b\sigma(\tilde{g})) = S$, so $\tilde{g} \in X_{\leq \mu}(\mathcal{R})$. Similarly, if μ is minuscule and $g \in X_\mu(\widehat{\mathcal{R}}_f)$, then $\tilde{g} \in X_\mu(\mathcal{R})$. It follows that the first diagram in the lemma is Cartesian. This implies the other claims in the lemma, using Lemma 2.2.7. \square

2.3.4. Let \mathbb{D} denote the pro-torus with character group \mathbb{Q} . Recall the Newton cocharacter

$$\nu = \nu_b : \mathbb{D} \rightarrow G$$

defined by Kottwitz [Kot85, 4.2]. If $G = GL(Q)$ for an F -vector space Q , then ν is the cocharacter which induces the slope decomposition of $b\sigma$ acting on Q . In general, ν is determined by requiring that it be functorial in the group G . We denote by $M_b \subset G$ the centralizer of ν_b . A σ -conjugacy class is called basic if the associated Newton cocharacter is central in G . Let $\nu_{\text{dom}} \in X_*(T)_{\mathbb{Q}}^{\Gamma}$ be the dominant cocharacter conjugate to the Newton cocharacter of b .

The group Γ acts on $X_*(T)$ through a finite quotient, and we denote by

$$\bar{\mu} = [\Gamma : \Gamma_\mu]^{-1} \sum_{\tau \in \Gamma/\Gamma_\mu} \tau(\mu) \in X_*(T)_{\mathbb{Q}}$$

the average of the Γ -conjugates of μ . As mentioned in the introduction, the set $X_\mu(b)$ is non-empty if and only if $[b] \in B(G, \mu)$. That is, $\kappa_G(b) = [\mu]$ in $\pi_1(G)_{\Gamma}$, and $\bar{\mu} - \nu_{\text{dom}}$ is a linear combination of positive coroots with non-negative rational coefficients. We assume from now on that this condition holds.

For any $\bar{b} \in G^{\text{ad}}(L)$, we define an F -group $J_{\bar{b}}$ by setting

$$J_{\bar{b}}(R) = J_{\bar{b}}^G(R) := \{g \in G(R \otimes_F L) : \sigma(g) = \bar{b}^{-1}g\bar{b}\},$$

for R an F -algebra. There is an inclusion $J_{\bar{b}} \subset G$, defined over L , which is given on R -points (R an L -algebra) by the natural map $G(R \otimes_F L) \rightarrow G(R)$, and which identifies $J_{\bar{b}}$ with the preimage of $M_{\bar{b}}$ in G . The group $J_{\bar{b}}$ is an inner form of $M_{\bar{b}}$ [Kot97, 3.3], [RZ96, 1.12].

If $b \in G(L)$ we write $J_b = J_{\bar{b}}$ where \bar{b} denotes the image of b in $G^{\text{ad}}(L)$. Then $J_b(F)$ acts naturally on $X_{\leq \mu}(b)$ and $X_{\mu}(b)$.

2.3.5. Let $g_0, g_1 \in X_{\leq \mu}(b)(W(\bar{k}))$, and R a smooth \bar{k} -algebra with connected spectrum, equipped with a frame \mathcal{R} . We say that g_0 is connected to g_1 via \mathcal{R} if there exist $g \in X_{\leq \mu}(b)(\mathcal{R})$ and $s_0, s_1 \in (\text{Spec } R)(\bar{k})$ such that $s_0(g) = g_0$ and $s_1(g) = g_1$. We denote by \sim the smallest equivalence relation on $X_{\leq \mu}(b)(W(\bar{k}))$ such that $g_0 \sim g_1$ if g_0 is connected to g_1 via some \mathcal{R} as above, and we write $\pi_0(X_{\leq \mu}(b))$ for the set of equivalence classes under \sim .

We could have defined a notion of connected components without assuming that R is smooth. However, the stronger notion of connectedness is useful in the applications in [Kis13] and, happily, this condition is also convenient in several of our arguments. On the other hand, we conjecture that the two definitions of connected components are equivalent. This follows *a posteriori* from our main result, when μ is minuscule, G^{ad} is simple, (μ, b) is Hodge–Newton indecomposable and G^{der} is simply connected (so that $\pi_1(G)$ has no torsion). To see this one uses the first part of the proof of Lemma 2.1.6, which shows (without assuming R formally smooth) that $s \mapsto [\mu_{s(g)}] \in \pi_1(G) \otimes_{\mathbb{Z}} \mathbb{Q}$ is locally constant on $\text{Spec } R$. We believe that all of Lemma 2.1.6 remains true without assuming R formally smooth, in which case the two notions of connected component would agree without assuming G^{der} simply connected.

The natural action of $J_b(F)$ on $X_{\leq \mu}(b)$ clearly induces an action on $\pi_0(X_{\leq \mu}(b))$. Note that we also have an action of $J_b(F)$ on $\pi_1(G)$ by left multiplication via $J_b(F) \xrightarrow{w_{J_b}} \pi_1(J_b) \rightarrow \pi_1(G)$.

LEMMA 2.3.6. (i) *The homomorphism $w_G : G(L) \rightarrow \pi_1(G)$ induces a well-defined map $w_G : \pi_0(X_{\leq \mu}(b)) \rightarrow \pi_1(G)$, which is compatible with the action of $J_b(F)$.*

(ii) *Let $c_{b,\mu}$ be as in Theorem 1.1. Then the image of the map defined above is contained in $c_{b,\mu}\pi_1(G)^{\Gamma}$.*

Proof. The first assertion of (i) follows from Lemma 2.1.6, where the claim regarding the action of $J_b(F)$ is clear.

For (ii) let $g \in X_{\leq \mu}(b)$. As K is in the kernel of w_G , this implies $w_G(g^{-1}b\sigma(g)) = [\mu] \in \pi_1(G)$. Hence, $-w_G(g) + \sigma(w_G(g)) = [\mu] - w_G(b)$. By definition of $c_{b,\mu}$, this implies the claim. \square

2.4 Reduction to adjoint groups

We continue to use the notation above. In particular, \mathcal{R} is a frame for $R = \mathcal{R}/p\mathcal{R}$, and we continue to assume that $[b] \in B(G, \mu)$.

LEMMA 2.4.1. *Let $G \rightarrow G'$ be a morphism of reductive groups over \mathcal{O}_L which takes Z_G to $Z_{G'}$ and induces an isomorphism on adjoint groups. Suppose that R is Noetherian and formally smooth over \bar{k} . Then the diagram of étale sheaves on $\text{Spec } R$*

$$\begin{array}{ccc} X_G & \longrightarrow & X_{G'} \\ w_G \downarrow & & w_{G'} \downarrow \\ \pi_1(G) & \longrightarrow & \pi_1(G') \end{array}$$

is Cartesian.

Proof. Using Lemma 2.2.5, we identify the top line of the diagram with the map $j_*G/G \rightarrow j_*G'/G'$. Let $Z = \ker(G \rightarrow G')$ and let G'' be the pushout of G by an embedding $Z \hookrightarrow T$ where T is a \mathcal{O}_L -torus. Then we have maps $G \rightarrow G'' \rightarrow G'$, where the first map is an embedding, and the second map has kernel a torus. Hence, it suffices to prove the lemma in the two cases where $G \rightarrow G'$ is faithfully flat with Z a torus, or an embedding.

For the first case, we begin by computing the fiber of this map over the identity. Let g be a local section in this fiber. Since any G -torsor is étale locally trivial, g admits a local lift to a section \tilde{g} of j_*G . Since the image of \tilde{g} is trivial in j_*G'/G' for any point $s \in \text{Spec } R$, we obtain that $\mu_{s(\tilde{g})}$ is in $X_*(Z)$. Hence, this cocharacter is a locally constant function on $\text{Spec } R$ by Lemma 2.1.6. It follows by 2.1.11 that \tilde{g} is étale locally of the form $p^{\mu_g}h$ with $\mu_g \in X_*(Z)$ and h a section of G . Hence, g is in the image of

$$X_*(Z) \rightarrow j_*G/G \quad \mu \mapsto p^\mu.$$

This map is injective (for example by the pointwise Cartan decomposition) and is equal to the fiber of $j_*G/G \rightarrow j_*G'/G'$ over the identity. In particular, we see that the non-empty fibers of both the horizontal maps in the diagram are $X_*(Z)$ -torsors.

Since Z is a torus the map $\pi_1(G) \rightarrow \pi_1(G')$ is surjective. Hence, it suffices to show that a local section of j_*G' lifts to j_*G . Note that $R^1j_*\mathbb{G}_m = 0$. Indeed, if $\mathfrak{p} \supset (p)$ is a prime of \mathcal{R} , then a line bundle \mathcal{L} on $\text{Spec } \mathcal{R}_{\mathfrak{p}}[1/p]$ extends to $\text{Spec } \mathcal{R}_{\mathfrak{p}}$: Our assumptions imply that $\mathcal{R}_{\mathfrak{p}}$ is a regular local ring. Thus, we may first extend \mathcal{L} as a coherent sheaf, and then take the determinant of the extension. Hence, $R^1j_*Z = 0$, which shows that $j_*G \rightarrow j_*G'$ is surjective.

For the case of an embedding, we have to show that if g is a local section of j_*G'/G' whose image in $\pi_1(G')$ is in $\pi_1(G)$, then g lifts locally to j_*G/G . We may assume that g lifts to a section \tilde{g} of j_*G' . Let $T' \subset G'$ be a maximal (necessarily split) torus, and $T \subset G$ its preimage. Using the fact that $R^1j_*\mathbb{G}_m = 0$, we have $j_*(G'/G) = j_*(T'/T) = j_*T'/j_*T$. Hence, after modifying \tilde{g} by an element of j_*G , we may assume that $\tilde{g} \in j_*T'$. Since the map $j_*T/T \rightarrow j_*T'/T'$ may be identified with $X_*(T) \rightarrow X_*(T')$, and the cokernel of the latter map is $X_*(G'/G)$, it follows that \tilde{g} lifts to an element of j_*T . □

COROLLARY 2.4.2. *Let $Z \subset Z_G$ be a closed \mathcal{O}_L -subgroup, and $G' = G/Z$. Write $T' = T/Z$, $b' \in G'(\mathcal{O}_L)$ and $\mu' \in X_*(T')$ for the elements induced by b and μ . Suppose that R is Noetherian and formally smooth over \bar{k} . Then the diagrams of étale sheaves on $\text{Spec } R$*

$$\begin{CD} X_\mu(b) @>>> X_{\mu'}(b') \\ @V w_G VV @VV w_{G'} V \\ c_{b,\mu}\pi_1(G)^\Gamma @>>> c_{b',\mu'}\pi_1(G')^\Gamma \end{CD}$$

and

$$\begin{CD} X_{\leq\mu}(b) @>>> X_{\leq\mu'}(b') \\ @V w_G VV @VV w_{G'} V \\ c_{b,\mu}\pi_1(G)^\Gamma @>>> c_{b',\mu'}\pi_1(G')^\Gamma \end{CD}$$

are Cartesian.

Proof. It follows from Lemma 2.4.1 that the non-empty fibers of all the horizontal maps in both diagrams are torsors under $X_*(Z)^\Gamma$. Hence, it suffices to show that a local section g of $X_{\mu'}(b')$

(respectively $X_{\leq \mu'}(b')$) whose image in $c_{b',\mu'}\pi_1(G')^\Gamma$ lifts to $c_{b,\mu}\pi_1(G)^\Gamma$, lifts étale locally to $X_\mu(b)$ (respectively $X_{\leq \mu}(b)$).

By Lemma 2.4.1, g lifts to a local section \tilde{g} of X_G . By assumption, there exists $\chi \in X_*(Z)$ such that $w_G(\tilde{g}) + \chi \in c_{b,\mu}\pi_1(G)^\Gamma$. Hence, after replacing \tilde{g} by $\tilde{g}p^\chi$, we may assume that $w_G(\tilde{g}) \in c_{b,\mu}\pi_1(G)^\Gamma$. To check that $\tilde{g} \in X_\mu(b)$ (respectively $X_{\leq \mu}(b)$), it suffices to pull back to geometric points, and consider the special case $\mathcal{R} = W(\bar{\kappa})[1/p]$ for an algebraically closed field $\bar{\kappa}$. In this case we have $\mu_{\tilde{g}^{-1}b\sigma(\tilde{g})} + \alpha = \mu$ (respectively $\mu_{\tilde{g}^{-1}b\sigma(\tilde{g})} + \alpha \leq \mu$) for some $\alpha \in X_*(Z)$. Since $w_G(\tilde{g}) \in c_{b,\mu}\pi_1(G)^\Gamma$, the image of α in $\pi_1(G)$ is trivial, and $\alpha = 0$. \square

COROLLARY 2.4.3. *With the notation above, the diagrams*

$$\begin{array}{ccc} \pi_0(X_\mu(b)) & \longrightarrow & \pi_0(X_{\mu'}(b')) \\ w_G \downarrow & & w_{G'} \downarrow \\ c_{b,\mu}\pi_1(G)^\Gamma & \longrightarrow & c_{b',\mu'}\pi_1(G')^\Gamma \end{array}$$

and

$$\begin{array}{ccc} \pi_0(X_{\leq \mu}(b)) & \longrightarrow & \pi_0(X_{\leq \mu'}(b')) \\ w_G \downarrow & & w_{G'} \downarrow \\ c_{b,\mu}\pi_1(G)^\Gamma & \longrightarrow & c_{b',\mu'}\pi_1(G')^\Gamma \end{array}$$

are Cartesian.

Proof. The vertical maps are given by Lemma 2.3.6, which also implies that $Z(F) \subset J_b(F)$ acts on the fibers of the top horizontal maps via $Z(F) \rightarrow X_*(Z)^\Gamma$. Thus the non-empty fibers of all the horizontal maps are $X_*(Z)^\Gamma$ -torsors. That the diagrams are Cartesian now follows from Corollary 2.4.2. \square

2.5 Hodge–Newton indecomposability

2.5.1. Let $b \in G(L)$, and $M_b \subset G$ the centralizer of ν_b , as above.

LEMMA 2.5.2. (i) *If $b' = gb\sigma(g)^{-1}$ for $g \in G(L)$, then $\nu_{b'} = g\nu_b g^{-1}$.*

(ii) *There exists a b' in the σ -conjugacy class of b such that $\nu_{b'} \in X_*(T) \otimes_{\mathbb{Z}} \mathbb{Q}$ is dominant and σ -invariant, and $b' \in M_{b'}$.*

Proof. (i) is clear from the definition of ν .

Applying this with $g = b^{-1}$, we find that $\sigma(\nu_b) = \nu_{\sigma(b)} = b^{-1}\nu_b b$ is conjugate to ν_b , so the $G(L)$ -conjugacy class of ν is stable by σ . Since G is quasi-split, this implies that ν is conjugate to a dominant σ -invariant cocharacter in $X_*(T) \otimes_{\mathbb{Z}} \mathbb{Q}$ [Kot84, 1.1.3(a)], which shows that there is a b' with $\nu_{b'} \in X_*(T) \otimes_{\mathbb{Z}} \mathbb{Q}$ and σ -invariant. Then $\nu_{b'} = \sigma(\nu_{b'}) = b'^{-1}\nu_{b'} b'$, so $b' \in M_{b'}$. \square

2.5.3. By the lemma, after replacing b by an element in its σ -conjugacy class, we may assume that $\nu = \nu_b \in X_*(T)$ is dominant, and thus defined over F (so that M_b is also defined over F), and that $b \in M_b(L)$. In particular, b is then basic as an element of $M_b(L)$. We assume that b has been chosen with these properties.

PROPOSITION 2.5.4. *Let $M \supset M_b$ be a standard Levi defined over F . Assume that $\kappa_M(b) = [\mu] \in \pi_1(M)_\Gamma$. Then the natural inclusion $X_\mu^M(b)(W(\bar{k})) \hookrightarrow X_\mu^G(b)(W(\bar{k}))$ is a bijection, and similarly for the closed affine Deligne–Lusztig varieties. Furthermore, it induces bijections between the corresponding sets of connected components.*

Proof. The bijection between the two Deligne–Lusztig sets is shown in [MV10, Theorem 6, i]. Note that that theorem has a slightly different assumption on M , which is incorrect. The present assertion is the correct statement and follows from the proof of [MV10], which in turn is nothing but a variant of the original proof of Kottwitz in [Kot03].

It remains to show that if $g \in X_{\leq \mu}^G(b)(\mathcal{R})$, where \mathcal{R} is a frame for a smooth connected \bar{k} -algebra R , and if $g_1 = g(s_1), g_2 = g(s_2)$ for two \bar{k} -valued points s_1, s_2 of R , then the corresponding elements of $X_{\leq \mu}^M(b)$ are in the same connected component. The strategy is to show that g is induced by a connecting family in $X_{\leq \mu}^M(b)$. We may replace $R = \mathcal{R}/p\mathcal{R}$ by an étale covering, and assume that g arises from an element $g \in G(\mathcal{R}_L)$.

Let \mathcal{R}_n denote \mathcal{R} regarded as an \mathcal{R} -algebra via $\mathcal{R} \xrightarrow{\sigma^n} \mathcal{R}$. Let η denote the generic point of $\text{Spec } R$, set $\mathcal{R}_{\eta, \infty} = \lim_n \mathcal{R}_{n, \eta}$, and let $\widehat{\mathcal{R}}_{\eta, \infty}$ be the p -adic completion of $\mathcal{R}_{\eta, \infty}$. Then $\widehat{\mathcal{R}}_{\eta, \infty}$ is a frame for a perfect closure $R_{\eta, \infty}$ of R_η .

By the Iwasawa decomposition, we have $g \in M(\widehat{\mathcal{R}}_{\eta, \infty, L})N(\widehat{\mathcal{R}}_{\eta, \infty, L})G(\widehat{\mathcal{R}}_{\eta, \infty})$. By the (pointwise) Hodge–Newton decomposition the factor in N may be assumed to be 1. Write $g = m_\eta h_\eta$ where $m_\eta \in M(\widehat{\mathcal{R}}_{\eta, \infty, L})$ and $h_\eta \in G(\widehat{\mathcal{R}}_{\eta, \infty})$. Using the Cartan decomposition and the formal smoothness of M , we may approximate m_η by an element of $\mathcal{R}_{n, \eta}$ for some n , and assume that $m_\eta \in M(\mathcal{R}_{n, \eta, L})$ and $h_\eta \in G(\mathcal{R}_{n, \eta})$.

It follows that there exists an $f \in \mathcal{R}_n \setminus p\mathcal{R}_n$ such that as a section of $X_{\leq \mu}^G(b)(\widehat{\mathcal{R}}_{n, f})$, g arises from an element $m_f \in X_{\leq \mu}^M(b)(\widehat{\mathcal{R}}_{n, f})$. Hence, g arises from an element $m \in X_{\leq \mu}^M(b)(\widehat{\mathcal{R}}_n)$ by Lemma 2.3.3. This shows that s_1 and s_2 are connected via \mathcal{R}_n . \square

2.5.5. We now suppose that $[b] \in B(G, \mu)$, and we continue to assume that $b \in M_b(L)$ and that ν_b is dominant.³ We say that the pair (μ, b) is indecomposable with respect to the Hodge–Newton decomposition if for all proper standard Levi subgroups $M \supset M_b$ that are defined over F , we have $\kappa_M(b) \neq \mu$ in $\pi_1(M)_\Gamma$. Given G, μ , and $[b]$, we may always pass to a Levi subgroup M of G defined over F in which (μ, b) is indecomposable. Proposition 2.5.4 shows that to describe the connected components of affine Deligne–Lusztig varieties it is sufficient to consider pairs (μ, b) which are indecomposable with respect to the Hodge–Newton decomposition. For a pair (μ, b) that is indecomposable with respect to the Hodge–Newton decomposition, we say that it is irreducible with respect to the Hodge–Newton decomposition (or HN-irreducible for short) if $\kappa_M(b) \neq \mu$ for every proper standard Levi M in G containing an element $b \in [b]$ such that the M -dominant Newton point of b is G -dominant.

The following theorem gives a stronger characterization of indecomposability that is used in §4.

THEOREM 2.5.6. *Let G, μ , and b be as above and assume that G^{ad} is simple. Then the following conditions are equivalent.*

- (i) *The pair (μ, b) is HN-irreducible.*
- (ii) *For any proper standard Levi subgroup M of G , we do not have $\nu_b \leq \bar{\mu}$ in the positive Weyl chamber of M in $X_*(A) \otimes \mathbb{Q}$, where $A \subset T$ is the maximal split torus.*
- (iii) *All the coefficients of simple coroots of G in $\bar{\mu} - \nu_b$ are strictly positive.*

If these conditions are not satisfied then either (μ, b) is already HN-decomposable or b is σ -conjugate to p^μ and μ is central.

³ We emphasize that one gets the correct notion of HN-indecomposability only if b is chosen so that ν_b is dominant.

Proof. Conditions (ii) and (iii) are clearly equivalent. For any standard proper Levi subgroup M with $b \in M(L)$, we have $\kappa_M(b) - \mu = \nu_b - \bar{\mu} \in \pi_1(M)_\Gamma \otimes \mathbb{Q}$. Therefore (iii) implies (i).

We now assume that (iii) is not satisfied, i.e. the coefficient of some simple coroot α_0^\vee vanishes.

Claim: (μ, b) is HN-decomposable or $\nu_b = \bar{\mu}$.

We first show that this claim implies the last assertion of the theorem. Suppose that (μ, b) is HN-indecomposable, so that $\nu_b = \bar{\mu}$.

Since $\mu - \kappa_{M_b}(b) = \bar{\mu} - \nu_b = 0$ in $\pi_1(M_b)_\Gamma \otimes \mathbb{Q}$, and $\mu = \kappa_G(b)$, it follows by Corollary 2.5.12 below that $\kappa_{M_b}(b) = \mu$. Hence, $M_b = G$, since we are assuming (μ, b) is HN-indecomposable. Thus $\langle \alpha, \bar{\mu} \rangle = n^{-1} \sum_{i=1}^n \langle \alpha, \sigma^i \mu \rangle = 0$ for every positive root α of G and some n with $\sigma^n(\mu) = \mu$. As B is defined over F and μ is dominant, each of the summands is non-negative. Hence, all of them are zero, and μ is central.

In particular, we see that $p^\mu \in [b] \cap T(L) \subsetneq G(L)$ with $\kappa_T(p^\mu) = \mu$, hence (μ, b) is not HN-irreducible.

It remains to prove the claim. Let us assume that (μ, b) is HN-indecomposable, because otherwise the claim holds. We want to use induction on the distance between a simple root α and the Galois orbit of α_0 in the Dynkin diagram of G to show that also the coefficient of α^\vee in $\bar{\mu} - \nu_b$ is 0. As $\bar{\mu} - \nu_b$ is Γ -invariant, our assumption on α_0 shows that the coefficients of all α^\vee for $\alpha \in \Gamma\alpha_0$ vanish. Assume that the statement is shown for some simple root α . Let $\Omega = \Gamma\alpha$ and let M_Ω be the standard Levi subgroup corresponding to the set of simple roots $\{\gamma : \text{simple root}, \gamma \notin \Omega\}$. If α is not a simple root in M_b then $M_\Omega \supset M_b \ni b$. As (μ, b) is HN-indecomposable, $\mu - \kappa_{M_\Omega}(b) = \lambda\alpha^\vee \in \pi_1(M_\Omega)_\Gamma$ with $\lambda > 0$ in contradiction to our assumption. Thus, α is a simple root in M_b . As μ is dominant, this implies

$$\langle \alpha, \bar{\mu} - \nu_b \rangle = \langle \alpha, \bar{\mu} \rangle + 0 \geq 0. \tag{2.5.7}$$

On the other hand,

$$\langle \alpha, \bar{\mu} - \nu_b \rangle = \left\langle \alpha, \sum_{\beta \text{ simple}} \lambda_\beta \beta^\vee \right\rangle = \sum_{\beta \text{ neighbor of } \alpha} \lambda_\beta \langle \alpha, \beta^\vee \rangle.$$

As all λ_β are non-negative, this can only be non-negative if $\lambda_\beta = 0$ for all neighbors β of α . This finishes the induction and shows that $\nu_b = \bar{\mu}$. \square

Remark 2.5.8. Using Corollary 2.5.12, as in the proof of the lemma, we obtain the following fact. Let $[b] \in B(G)$ and ν_b its Newton point. Let M be a standard Levi subgroup with $M(L) \cap [b] \neq \emptyset$. Then κ_M is constant on

$$\{x \in [b] \cap M(L) \mid \nu_x^M = \nu_b \in \pi_1(M) \otimes \mathbb{Q}\}.$$

Here ν^M denotes the Newton point for an element of M , an M -dominant element of $X_*(T)_\mathbb{Q}$.

Remark 2.5.9. In [Che14], we take the second condition in Theorem 2.5.6 as the definition of HN-irreducibility (cf. [Che14, Definition 5.0.4]).

Remark 2.5.10. In the particular case of the above theorem where b is σ -conjugate to p^μ and μ is central, we have

$$\begin{aligned} X_\mu(b) &= \{g \in G(L)/K \mid g^{-1}b\sigma(g) \in Kp^\mu K\} \\ &= \{g \in G(L)/K \mid g^{-1}\sigma(g) \in K\} \\ &= G(F)/G(\mathcal{O}_F) \end{aligned}$$

where the third equality follows from Lang’s lemma.

LEMMA 2.5.11. *Let G be a reductive group over \mathcal{O}_F , let T be the centralizer of a maximal split torus, and let $T^{\text{ad}} = T/Z_G$. Then the following diagram is Cartesian with surjective vertical maps.*

$$\begin{array}{ccc} X_*(T)^\Gamma & \longrightarrow & X_*(T^{\text{ad}})^\Gamma \\ w_G \downarrow & & w_{G^{\text{ad}}} \downarrow \\ \pi_1(G)^\Gamma & \longrightarrow & \pi_1(G^{\text{ad}})^\Gamma \end{array}$$

Proof. Let \tilde{G} denote the simply connected cover of G^{ad} , and \tilde{T} the preimage of T in \tilde{G} . The fibers of both horizontal maps are torsors under $X_*(Z_G)$, and the fibers of both vertical maps are torsors under $X_*(\tilde{T})^\Gamma$. Using this, one sees easily that it suffices to show that the vertical maps are surjective. Thus it remains to check that $H^1(\Gamma, X_*(\tilde{T})) = 0$.

Suppose that r is a non-negative integer, and consider any continuous action of Γ on \mathbb{Z}^r , which permutes the standard basis vectors. We claim that $H^1(\Gamma, \mathbb{Z}^r) = 0$. It suffices to consider the case where Γ permutes the basis vectors transitively. If Γ' is the stabilizer of one of the basis vectors, then \mathbb{Z}^r can be identified with $\text{Ind}_{\Gamma'}^\Gamma \mathbb{Z}$, and the claim follows since $H^1(\Gamma', \mathbb{Z}) = 0$.

Applying this to $X_*(\tilde{T})$ with its basis of simple coroots proves the lemma. □

COROLLARY 2.5.12. *Let $M \subset G$ be a standard Levi. Then:*

- (i) *the map $\pi_1(M)^\Gamma \rightarrow \pi_1(G)^\Gamma$ is surjective, and its kernel is spanned by the sum of Γ -orbits of coroots of G ;*
- (ii) *$\ker(\pi_1(M)^\Gamma \rightarrow \pi_1(G)^\Gamma)$ is torsion free.*

Proof. The first claim in (i) follows from Lemma 2.5.11, and (ii) then follows by the snake lemma. To see the second claim in (i), let \tilde{T} be as in Lemma 2.5.11, and let $\tilde{T}_M \subset \tilde{T}$ be the analogous torus for M in place of G . Then the kernel of the map in (i) is $(X_*(\tilde{T})/X_*(\tilde{T}_M))^\Gamma$. By what we saw in Lemma 2.5.11, $X_*(\tilde{T}_M)$ and $X_*(\tilde{T})$ are a sum of induced modules. It follows that $(X_*(\tilde{T})/X_*(\tilde{T}_M))^\Gamma = X_*(\tilde{T})^\Gamma/X_*(\tilde{T}_M)^\Gamma$, and that $X_*(\tilde{T})^\Gamma$ is spanned by the sum of Γ -orbits in $X_*(\tilde{T})^\Gamma$. □

3. The superbasic case

3.1 Superbasic σ -conjugacy classes

As recalled above, an element $b \in G(L)$ is called *basic* if ν_b factors through the center of G . This condition depends only on the σ -conjugacy class of b . We say that b is *superbasic* if no σ -conjugate of b is contained in a proper Levi subgroup of G defined over F . Since all maximal F -split tori of G are conjugate over F , this is equivalent to asking that no σ -conjugate of b is contained in a proper Levi subgroup of G defined over F , and containing T . If b is superbasic, then $M_b = G$, by Lemma 2.5.2(ii), and ν_b is central, so b is basic.

LEMMA 3.1.1. *If $b \in G(L)$ is superbasic, then J_b is anisotropic modulo center, and in particular the simple factors of G^{ad} are of the form $\text{Res}_{E_i/F} \text{PGL}_{h_i}$ for some unramified extension E_i/F and $h_i \geq 2$.*

This is analogous to [GHKR06, 5.9.1]. We are grateful to Kottwitz for explaining how to adapt the proof of [GHKR06] to the quasi-split setting.

Proof. A cocharacter $\psi \in X_*(J_b)^\Gamma$ may be regarded as a cocharacter of G such that $\sigma(\psi) = b^{-1}\psi b$. Then, as above, ψ is conjugate by a $g \in G(L)$ to a dominant cocharacter $\psi' \in X_*(T)$ defined

over F . That is, $\sigma(g^{-1}\psi'g) = b^{-1}g^{-1}\psi'gb$, which implies that $gb\sigma(g^{-1})$ commutes with ψ' . Since $gb\sigma(g^{-1})$ is not contained in a proper Levi subgroup of G containing T , ψ must be central.

The fact that J_b is anisotropic modulo center implies that all the factors of J_b^{ad} are isomorphic to the group of units of a division algebra over an extension of F modulo its center [Tit79, § 4]. Since $G^{\text{ad}} = M_b^{\text{ad}}$ is an inner form of J_b^{ad} , which is quasi-split, its simple factors have the form $\text{Res}_{E_i/F} \text{PGL}_{h_i}$ for some finite extensions E_i/F . As G is unramified, E_i must be an unramified extension of F . □

3.1.2. For every $[b] \in B(G)$ there exists a standard parabolic subgroup P of G defined over F with Levi factor M containing T , unipotent radical N and the following properties. There exists $b' \in [b] \cap M(L)$ such that b' is superbasic in M , i.e. no σ -conjugate of b lies in a proper Levi subgroup of M . Thus, we may assume that $b \in M(L)$ is superbasic.

3.2 The superbasic case for GL_h

Let E/F be a finite unramified extension and suppose that $G = \text{Res}_{\mathcal{O}_E/\mathcal{O}_F} \text{GL}_h$, with T the standard diagonal torus and B the Borel subgroup of upper triangular matrices. In this subsection we will prove Theorem 1.1 for this G when b is superbasic. For the rest of this subsection, we suppose that b is a superbasic element of $G(L)$.

Let $n = [E : F]$. The F -algebra embeddings $E \hookrightarrow L$ are permuted cyclically by Frobenius, so over \mathcal{O}_E we may identify G with $(\text{GL}_h)^n$, such that σ acts on $G(L) = \text{GL}_h(L)^n$, by

$$\sigma(g_1, \dots, g_n) = (\sigma(g_n), \sigma(g_1), \dots, \sigma(g_{n-1})).$$

We get an analogous decomposition of $X_*(T)$, and for $r = 1, \dots, n$ we denote by μ_r the projection of μ onto the r th factor of $X_*(T)$. Let $\mu_{r,\text{min}} \in X_*(T)$ denote the unique dominant minuscule cocharacter with $\mu_{r,\text{min}} \preceq \mu_r$ (i.e. with $\det(\mu_{r,\text{min}}(p)) = \det(\mu_r(p))$; cf. (4.1.2) below) and set $\mu_{\text{min}} = (\mu_{r,\text{min}})_r$.

Let $h \geq 1$ be an integer and e_1, \dots, e_h the standard basis of L^h . We define e_i for $i \in \mathbb{Z}$ so that $e_{i+h} = pe_i$. Let $s \in \text{GL}_h(F)$ be defined by $s(e_i) = e_{i+1}$ for all i .

Note that for $i \in \mathbb{Z}$, $s^i = {}^i\mu_{\text{min}}(p)w^i$ where w is the Weyl group element given by $w(e_i) = e_{i+1}$ for $i = 1, \dots, h-1$ and $w(e_h) = e_1$, and ${}^i\mu_{\text{min}}$ is the unique dominant minuscule cocharacter of GL_h such that $\det({}^i\mu_{\text{min}}(p)) = p^i$.

LEMMA 3.2.1. *If $X_{\preceq\mu}(b) \neq \emptyset$, b is σ -conjugate to $b_{\text{min}} = (s^{m_r}) \in G(L)$, where $m_r \in \mathbb{Z}$ satisfies ${}^{m_r}\mu_{\text{min}} = \mu_{r,\text{min}}$. Moreover, we have $(\sum_r m_r, h) = 1$.*

Proof. Recall from [Kot85, Proposition 5.6] that κ_G induces a bijection between the set of basic σ -conjugacy classes in $G(L)$ and $\pi_1(G)_\Gamma$. The Newton cocharacter of (s^{m_r}) is the central cocharacter of $\text{GL}_h \subset G$ corresponding to the rational number $n^{-1}h^{-1} \sum m_r$. In particular, (s^{m_r}) is basic. As $X_{\preceq\mu}(b) \neq \emptyset$, we have $\kappa_G(b) = \mu$ in $\pi_1(G)_\Gamma$. Furthermore, μ and (s^{m_r}) both have image $\sum_r m_r$ in $\pi_1(G)_\Gamma \xrightarrow{\sim} \mathbb{Z}$. Thus, b and (s^{m_r}) are σ -conjugate.

If $(\sum_r m_r, h) \neq 1$, then there exist integers m'_r with $\sum_r m'_r = \sum_r m_r$, and such that $\text{gcd}(m'_1, \dots, m'_r, h) > 1$. Then the same argument as above shows that b is σ -conjugate to $(s^{m'_r})$. The latter element is contained in a proper Levi subgroup of G , defined over F , which contradicts the fact that b is superbasic. □

3.2.2. Let $i, \delta \in \mathbb{Z}$. If $\delta \neq 0$, set $\mathcal{R}_\delta = \mathcal{O}_L\langle x \rangle$, the p -adic completion of $\mathcal{O}_L[x]$. Similarly, if $\delta = 0$, we set \mathcal{R}_δ equal to the p -adic completion of $\mathcal{O}_L[x, (1+x)^{-1}]$. Let $a_{i,\delta} \in \text{GL}_h(\mathcal{R}_\delta)$, which sends e_j to $e_j + xe_{j+\delta}$ if $h|(j-i)$ and fixes e_j otherwise.

LEMMA 3.2.3. *Let $g \in \mathrm{GL}_h(L)$ and let $\delta_g \in \mathbb{Z}$ be minimal such that $a_{i,\delta}(x) \circ g \in g \mathrm{GL}_h(\mathcal{R}_\delta)$ for all $\delta > \delta_g$ and i . Then:*

- (i) *either $\delta_g \geq 1$ or $\delta_g = -1$;*
- (ii) *if $\delta_g = -1$ then $g \mathrm{GL}_h(\mathcal{O}_L)$ contains an element of the form s^j for some $j \in \mathbb{Z}$;*
- (iii) *if $\delta_g \geq 1$ then there exists a unique $i_g \in \{1, \dots, h\}$ with $a_{i_g, \delta_g}(x) \circ g \notin g \mathrm{GL}_h(\mathcal{R}_{\delta_g})$;*
- (iv) *if $i, i' \in \{1, \dots, h\}$, and $\delta \geq \delta' > 0$, then $a_{i,\delta}(x)a_{i,\delta}(x')a_{i,\delta}(-x-x')$ and the commutator $[a_{i,\delta}(x), a_{i',\delta'}(x')]$ can be written as a (possibly infinite, p -adically convergent) product of terms of the form $a_{i_j, \delta_j}(x_j)$ with $\delta_j > \delta$.*

Proof. This is a translation of [Vie08, Lemma 2]. The proof given in [Vie08] goes over verbatim, except that the elements $\beta_j \in \bar{k}$ which appear in it should be replaced by Teichmüller representatives in $W(\bar{k})$. Note that in [Vie08] the definition of δ_g and condition (iii) are formulated by asking that $a_{i,\delta}(x) \circ g$ is contained (respectively not contained) in $g \mathrm{GL}_h(\mathcal{O}_L)$ for every specialization of x at a point of \bar{k} . This is equivalent to the formulation here, for example using Lemma 2.1.11. □

LEMMA 3.2.4. *Let $s \in \mathrm{GL}_h(F) \subset G(F)$ be as above, and suppose that $b = b_{\min}$. Then $\langle s \rangle \subset J_b(F)$ acts transitively on $\pi_0(X_{\leq \mu}(b))$.*

Proof. For $r = 1, \dots, n$, let δ_{g_r} be the integer obtained by applying Lemma 3.2.3 to g_r , and if $\delta_{g_r} \geq 1$, let i_{g_r} be the integer produced by (iii) of that lemma. Suppose that $g, g' \in X_{\leq \mu}(b)$, and that $\delta_{g_r} = \delta_{g'_r} = -1$ for all r . We claim that g and g' are in the same $\langle s \rangle$ -orbit. By Lemma 3.2.3(ii) we may assume that for $r = 1, 2, \dots, n$ we have $g_r = s^{j_r}$ and $g'_r = s^{j'_r}$ for some $j_r, j'_r \in \mathbb{Z}$. Note that $\sigma(s) = s \in J_b(F)$, so that

$$s^{j_{r-1}-j_r} b_r = s^{-j_r} b_r s^{j_{r-1}} \in \mathrm{GL}_h(\mathcal{O}_L) p^{\mu'_r} \mathrm{GL}_h(\mathcal{O}_L)$$

for some $\mu'_r \leq \mu_r$. Here we set $j_{-1} = j_n$, and we have again written b_r for the image of b under the r th projection $G(L) \rightarrow \mathrm{GL}_h(L)$. Hence, $v_p(\det(s^{j_{r-1}-j_r} b_r)) = v_p(\det(b_r)) + j_r - j_{r-1}$ depends only on μ_r and not on g . It follows that $j = j_r - j'_r$ is independent of r , so that $g = s^j g'$.

Note that if $h = 1$, then $\delta_{g_r} = -1$ for all r for any g , so we are done in this case (which can of course be easily checked directly). If $h > 1$, it remains to show that, given $g \in X_{\leq \mu}(b)$ with $\delta_{g_r} > 0$ for some r , there exists $g' \in X_{\leq \mu}(b)$ in the same connected component as g , with $\delta_{g'_r} \leq \delta_{g_r}$ for $r = 1, \dots, n$ and such that this inequality is strict for some r .

Let $\mathcal{R} = \mathcal{O}_L \langle x \rangle$ equipped with the lift of Frobenius given by $x \mapsto x^q$. Choose r_0 such that $\delta_{g_{r_0}}$ is maximal among the δ_{g_r} and set $\delta = \delta_{g_{r_0}} > 0$. (In the following it will be convenient to view the indices r in $\mathbb{Z}/n\mathbb{Z}$.) Define $a = (a_r) \in (\mathrm{GL}_h)^n(\mathcal{R})$ as follows. If not all the δ_{g_r} are equal δ , let $r_1 < r_0$ be an integer with $\delta_{g_{r_1}} < \delta$. Then for $r = r_1, \dots, r_1 + h - 1$ we set $a_r = \sigma^{r-r_1}(a_{j_r, \delta}(x))$, where $j_{r_1} = i_{g_{r_0}} - m_{r_0} - \dots - m_{r_1+1}$ and $j_r = j_{r_1} + m_{r_1+1} + \dots + m_r$ for $r = r_1 + 1, \dots, r_1 + n - 1$. If all the $\delta_{g_r} = \delta$ we choose $r_1 = r_0$ so that $h \nmid m_{r_0}$ and set $a_r = \sigma^{r-r_0}(a_{j_r, \delta}(x))$, where $j_{r_0} = i_{g_{r_0}}$ and $j_r = i_{g_{r_0}} + m_{r_0+1} + \dots + m_r$ for $r = r_0 + 1, \dots, r_0 + n - 1$.

Then, as in [Vie08, p. 322], for $r \neq r_1$, we have, using Lemma 3.2.3,

$$\begin{aligned} & \mathrm{GL}_h(\mathcal{R}) g_r^{-1} a_r^{-1} b_r \sigma(a_{r-1}) \sigma(g_{r-1}) \mathrm{GL}_h(\mathcal{R}) \\ &= \mathrm{GL}_h(\mathcal{R}) g_r^{-1} \sigma^{r-r_1}(a_{j_r, \delta}(x))^{-1} b_r \sigma^{r-r_1}(a_{j_{r-1}, \delta}(x)) \sigma(g_{r-1}) \mathrm{GL}_h(\mathcal{R}) \\ &= \mathrm{GL}_h(\mathcal{R}) g_r^{-1} \sigma^{r-r_1}(a_{j_r, \delta}(x))^{-1} a_{j_{r-1}+m_r, \delta}(x) b_r \sigma(g_{r-1}) \mathrm{GL}_h(\mathcal{R}) \\ &= \mathrm{GL}_h(\mathcal{R}) g_r^{-1} \sigma^{r-r_1}(a_{j_r, \delta}(-x)) a_{j_{r-1}+m_r, \delta}(x) b_r \sigma(g_{r-1}) \mathrm{GL}_h(\mathcal{R}). \end{aligned}$$

From the definition of the a_r and j_r , it follows that this is equal to $\mathrm{GL}_h(\mathcal{R})g_r^{-1}b\sigma(g_{r-1})\mathrm{GL}_h(\mathcal{R})$. For $r = r_1$ a similar calculation shows that

$$\begin{aligned} & \mathrm{GL}_h(\mathcal{R})g_{r_1}^{-1}a_{r_1}^{-1}b_{r_1}\sigma(a_{r_1-1})\sigma(g_{r_1-1})\mathrm{GL}_h(\mathcal{R}) \\ &= \mathrm{GL}_h(\mathcal{R})g_{r_1}^{-1}a_{j_{r_1},\delta}(-x)\sigma^n(a_{j_{r_1-1}+m_{r_1},\delta}(x))b_{r_1}\sigma(g_{r_1-1})\mathrm{GL}_h(\mathcal{R}). \end{aligned} \tag{3.2.5}$$

We claim that this is again equal to $\mathrm{GL}_h(\mathcal{R})g_{r_1}^{-1}b_{r_1}\sigma(g_{r_1-1})\mathrm{GL}_h(\mathcal{R})$. If not all the δ_{g_r} are equal to δ , this follows from $\delta > \delta_{g_{r_1}}$. If all the δ_{g_r} are equal to δ , then, using Lemma 3.2.3(iv), expression (3.2.5) is equal to

$$\begin{aligned} & \mathrm{GL}_h(\mathcal{R})g_{r_1}^{-1}\sigma^n(a_{j_{r_1-1}+m_{r_1},\delta}(x))a_{j_{r_1},\delta}(-x)b_{r_1}\sigma(g_{r_1-1})\mathrm{GL}_h(\mathcal{R}) \\ &= \mathrm{GL}_h(\mathcal{R})g_{r_1}^{-1}\sigma^n(a_{j_{r_1-1}+m_{r_1},\delta}(x))b_{r_1}a_{j_{r_1}-m_{r_1},\delta}(-x)\sigma(g_{r_1-1})\mathrm{GL}_h(\mathcal{R}). \end{aligned}$$

Now $j_{r_1} - m_{r_1} = i_{g_{r_1}} - m_{r_1} \neq i_{g_{r_1}}$ in $\mathbb{Z}/h\mathbb{Z}$ as $h \nmid m_{r_1}$, while $j_{r_1-1} + m_{r_1} = i_{g_{r_1}} + \sum_r m_r \neq i_{g_{r_1}}$. Hence, the uniqueness of $i_{g_{r_1}}$ in Lemma 3.2.3(iii) implies the claim in this case also. It follows that $ag \in X_{\leq \mu}(b)(\mathcal{R})$.

Let \mathcal{R}' and \mathcal{R}'' denote the p -adic completions of $\mathcal{O}_L[y]$ and $\mathcal{O}_L[x, x^{-1}]$ respectively, equipped with the lifts of Frobenius σ given by $y \mapsto y^q$ and $x \mapsto x^q$. We consider \mathcal{R}' as a subring of \mathcal{R}'' via $y \mapsto x^{-1}$. We may consider $ag \in X_{\leq \mu}(b)(\mathcal{R}'')$. Then by Lemma 2.3.3, ag is induced by an element $\gamma \in X_{\leq \mu}(b)(\mathcal{R}')$.

Now $(a \circ g)|_{x=0} = g$, and a computation as in [Vie08, proof of Proposition 1] for superbasic b , using Lemma 3.2.3(iv), shows that $g' = \gamma|_{y=0}$ satisfies $\delta_{g'_{r_0}} < \delta_{g_{r_0}}$ and $\delta_{g'_r} \leq \delta_{g_r}$ for $r \neq r_0$. Since g and g' are in the same connected component of $X_{\leq \mu}(b)(W(\bar{k}))$, the lemma follows. \square

3.2.6. It will be convenient to formulate a slight variant of Lemma 3.2.4. Recall the element w defined at the beginning of this subsection, which permutes the chosen basis e_1, \dots, e_h cyclically. Then $\det(w) = (-1)^{h-1}$. Let $w' = tw$ where $t(e_1) = (-1)^{h-1}(e_1)$ and $t(e_i) = e_i$ for $i > 1$. Then $w' \in \mathrm{SL}_h(F)$. We set $s' = ts = {}^1\mu_{\min}(p)w'$, and $b'_{\min} = ((s')^{m_r})_r \in G(L)$.

COROLLARY 3.2.7. *If $b = b'_{\min}$ then b is superbasic in G , and $\langle s' \rangle \subset J_b(F)$ acts transitively on $\pi_0(X_{\leq \mu}(b))$.*

Proof. The same argument as in Lemma 3.2.1 shows that b'_{\min} is superbasic in $G(L)$ and σ -conjugate to b_{\min} . By Lemma 3.2.4, $\pi_0(X_{\leq \mu}(b))$ maps isomorphically to $\pi_1(G)_\Gamma = \mathbb{Z}$. Since s' maps to a generator of $\pi_1(G)_\Gamma$, $\langle s' \rangle$ acts transitively on $\pi_0(X_{\leq \mu}(b))$. \square

3.3 The superbasic case in general

We return to the notation and assumptions introduced in §3.1.

PROPOSITION 3.3.1. *Suppose that $b \in G(L)$ is superbasic. Then*

$$\pi_0(X_{\leq \mu}^G(b)) \xrightarrow{\sim} c_{b,\mu}\pi_1(G)^\Gamma,$$

and $J_b(F)$ acts transitively on $\pi_0(X_{\leq \mu}^G(b))$.

Proof. By Lemma 3.1.1, G^{ad} is isomorphic to $\prod_{i \in I} \mathrm{Res}_{E_i/F} \mathrm{PGL}_{h_i}$ with E_i/F some finite unramified extension of degree n_i , and $h_i > 1$. Fix such an isomorphism. Let $\mu_{\min} \in X_*(T)$ denote the unique dominant minuscule cocharacter whose image in $\pi_1(G)$ is equal to that of μ . The induced cocharacter of $\prod_{i \in I} \mathrm{Res}_{E_i/F} \mathrm{PGL}_{h_i}$ has the form $({}^{m_{i,r}}\mu_{\min})_{i,r}$ where i runs over elements of I , $1 \leq r \leq n_i$, and $m_{i,r} \in \mathbb{Z}$.

Write w'_{h_i} and s'_{h_i} for the elements introduced in 3.2.6 above, when $h = h_i$. Since $w'_{h_i} \in \mathrm{SL}_{h_i}(F)$, we may regard $((w'_{h_i})^{m_{i,r}})_{i,r} \in \tilde{G}^{\mathrm{ad}}$ where \tilde{G}^{ad} denotes the simply connected cover of G^{ad} . In particular, we may regard $((w'_{h_i})^{m_{i,r}})_{i,r}$ and hence $b'_{\min} := \mu_{\min}(p)((w'_{h_i})^{m_{i,r}})_{i,r}$ as elements of $G(L)$. The image of b'_{\min} in G^{ad} is $((s'_{h_i})^{m_{i,r}})_{i,r}$. Hence, b'_{\min} is basic, and the same argument as in the proof of Lemma 3.2.1 shows that b is σ -conjugate to b'_{\min} . Thus we may assume that $b = b'_{\min}$.

Let b^{ad} be the image of b in G^{ad} and $b_{\mathrm{GL}} = ((s')^{m_{i,r}})_{i,r} \in \prod_{i \in I} \mathrm{Res}_{E_i/F} \mathrm{GL}_{h_i}$. Similarly, let μ^{ad} be the cocharacter of G^{ad} induced by μ . Let μ_{GL} be the cocharacter of $\prod_{i \in I} \mathrm{Res}_{E_i/F} \mathrm{GL}_{h_i}$ lifting μ^{ad} whose image in $\pi_1(\prod_{i \in I} \mathrm{Res}_{E_i/F} \mathrm{GL}_{h_i})$ is equal to $(^{m_{i,r}}\mu_{\min})_{i,r}$.

By Corollary 3.2.7, $\prod_{i \in I} \langle s'_{h_i} \rangle$ acts transitively on $\pi_0(X_{\preceq \mu_{\mathrm{GL}}}(b_{\mathrm{GL}}))$, and, in particular, the first claim of the proposition holds for $(\mu_{\mathrm{GL}}, b_{\mathrm{GL}})$. It follows from Corollary 2.4.3 that $\prod_{i \in I} \langle s'_{h_i} \rangle$ acts transitively on $\pi_0(X_{\preceq \mu^{\mathrm{ad}}}(b^{\mathrm{ad}}))$, and that the first claim of the proposition holds for $(\mu^{\mathrm{ad}}, b^{\mathrm{ad}})$.

Using Corollary 2.4.3 again, we see that the first claim of the proposition holds, and that, since $Z_G(F) \subset J_b(F)$, to prove the second claim it suffices to show that, if the image of $((s'_{h_i})^{j_i})_{i \in I}$ in $\pi_1(G^{\mathrm{ad}})^{\Gamma}$ lifts to $\pi_1(G)^{\Gamma}$ for some integers j_i , then $((s'_{h_i})^{j_i})_{i \in I} \in J_{b^{\mathrm{ad}}}(F) \cap G^{\mathrm{ad}}(F)$ lifts to an element of $G(F)$. But $((s'_{h_i})^{j_i})_{i \in I} = (^{j_i}\mu_{\min}(p)(w'_{h_i})^{j_i})_{i \in I}$, so it suffices to show that (the image of) $(^j\mu_{\min}(p))_{i \in I}$ lifts to $G(F)$. This follows, for example, from Lemma 2.5.11. \square

3.4 Reduction to the superbasic case

Let $[b] \in B(G, \mu)$ and $M \subset G$ a smallest standard Levi subgroup of G , defined over F and containing T , and which contains an element of $[b]$. Fix a representative $b \in M(L)$ of $[b]$, so that b is superbasic in $M(L)$. Let $P \supset B$ be the parabolic with reductive quotient M , and $N \subset P$ its unipotent radical.

Let $\bar{I}_{\mu,b}$ be the set of M -conjugacy classes of cocharacters $\mu' : \mathbb{G}_m \rightarrow M$ (defined over some finite extension of F) such that $\mu' : \mathbb{G}_m \rightarrow G$ satisfies $\mu' \preceq \mu$ and such that $b \in B(M, \mu')$. We identify an element of $\bar{I}_{\mu,b}$ with its M -dominant representative in $X_*(T)$. Note that in general (even for minuscule μ) this set is non-empty and finite, but may have more than one element. For each $\mu' \in \bar{I}_{\mu,b}$ we have a canonical inclusion $X_{\preceq \mu'}^M(b) \rightarrow X_{\preceq \mu}^G(b)$. The following proposition is the main goal of this subsection.

PROPOSITION 3.4.1. *Each connected component of $X_{\preceq \mu}^G(b)$ contains an element jk where $j \in J_b(F) \cap N(L)$ and $k \in X_{\preceq \mu'}^M(b)$ for some $\mu' \in \bar{I}_{\mu,b}$.*

The proof of this is very similar to [Vie08, proof of Proposition 1].

3.4.2. For any $l \geq 0$, let $b^{(l)} = b\sigma(b) \cdots \sigma^l(b)$. By [Kot85, 4.3], after replacing b by a σ -conjugate in M , we may assume that for some $l_0 > 0$, $b^{(l_0)} = p^{l_0\nu}$, where $\nu = \nu_b$ is defined over F , as before.

Let $\{\alpha_i\}_{i=1}^r$ denote the roots of T in N . We denote by $U_{\alpha_i} \subset N$ the corresponding root subgroup. It will be convenient to identify U_{α_i} with \mathbb{G}_a . Then, for an F -algebra R and $\beta \in R$, we can regard β as a point $U_{\alpha_i}(\beta) \in U_{\alpha_i}(R)$.

For $j \geq 1$ let $N[j] \subset N$ denote the subgroup generated by those U_{α_i} for which the sum of the coefficients of α_i , expressed as a linear combination of simple roots of A in N , is greater than or equal to j . Then for $j, j' \geq 1$, $[N[j], N[j']] \subset N[j + j']$. The filtration $N \supset N[1] \supset N[2] \cdots$ may be refined into a filtration $N \supset N_1 \supset N_2 \cdots$ such that N_i/N_{i+1} is one-dimensional. After reordering the α_i we may assume that N_i is generated by $U_{\alpha_{i'}}$ for $i' \geq i$.

Now suppose that R is a \bar{k} -algebra, \mathcal{R} a frame for R , and $y \in N(\mathcal{R}_L)$. We set

$$f_b(y) = y^{-1}b\sigma(y)b^{-1}.$$

Then $f_b(y) \in N(\mathcal{R}_L)$.

LEMMA 3.4.3. *Let R be a smooth \bar{k} -algebra, \mathcal{R} a frame for R , and $\beta \in \mathcal{R}_L$. Assume that there is an element $x \in (\text{Spec } R)(\bar{k})$ with $\beta(x) = 0$. If $i \geq 1$, and j is maximal such that $N[j] \supset N_i$ then for $n \geq 1$ a positive integer, there exists a finite étale covering $R \rightarrow R'$, with frame $\mathcal{R} \rightarrow \mathcal{R}'$, and $z \in N[j](\mathcal{R}'_L)$ such that:*

- (i) $f_b(z) \in U_{\alpha_i}(\beta + \varepsilon)N_{i+1}(\mathcal{R}'_L)$ for some $\varepsilon \in p^n\mathcal{R}'$;
- (ii) there exists $x' \in (\text{Spec } R')(\bar{k})$ mapping to x such that $z(x') = 1$.

Proof. This is analogous to the argument of [Vie08, pp. 324–325].

Suppose first that $\langle \alpha_i, \nu \rangle > 0$, and set

$$z(l) = b^{(l)}\sigma^l(U_{\alpha_i}(-\beta))(b^{(l)})^{-1} \dots b\sigma(U_{\alpha_i}(-\beta))b^{-1}U_{\alpha_i}(-\beta).$$

Note that conjugation by $b^{(l_0)}$ acts on U_{α_i} by $p^{\langle \alpha_i, \nu \rangle}$. Using this, one sees as in [Vie08] that the sequence $z(l)$ converges to an element $z \in U_{\alpha_i}(\mathcal{R}_L)$ such that $f_b(z) = U_{\alpha_i}(\beta)$. Thus we may take $R' = R$.

Suppose that $\langle \alpha_i, \nu \rangle = 0$. Let R' be finite étale over R , and $z_0 \in \mathcal{R}'_L$. Set

$$z = b^{(l_0-1)}\sigma^{l_0-1}(U_{\alpha_i}(z_0))(b^{(l_0-1)})^{-1} \dots b\sigma(U_{\alpha_i}(z_0))b^{-1}U_{\alpha_i}(z_0).$$

Then we have

$$\begin{aligned} f_b(z) &= z^{-1}b\sigma(z)b^{-1} = z^{-1}b^{(l_0)}\sigma^{l_0}(U_{\alpha_i}(z_0))(b^{(l_0)})^{-1}zU_{\alpha_i}(-z_0) \\ &= z^{-1}\sigma^{l_0}(U_{\alpha_i}(z_0))zU_{\alpha_i}(-z_0). \end{aligned}$$

Since all the terms in the product defining z are in $N[j]$, we have $z \in N[j]$. Assume that l_0 is such that σ^{l_0} acts trivially on $X^*(T)$. Then the final term is equal to $U_{\alpha_i}(\sigma^{l_0}(z_0) - z_0) \bmod N[j + 1]$, and z will have the desired property if z_0 satisfies

$$\sigma^{l_0}(z_0) - z_0 = \beta \bmod p^n\mathcal{R}'.$$

To show this equation has a solution for some \mathcal{R}'/\mathcal{R} finite étale we may replace β and n by $p^m\beta$ and $n + m$ respectively and assume that $\beta \in \mathcal{R}$. Then one sees by induction on n , that the above equation has a solution over a finite étale covering of \mathcal{R} . \square

LEMMA 3.4.4. *Let R be a smooth \bar{k} -algebra with frame \mathcal{R} , and $x_1 \in (\text{Spec } R)(\bar{k})$. Suppose that $y \in N(\mathcal{R}_L)$ and $z_1 \in N(L)$ satisfy $f_b(z_1) = y(x_1)$. Then for any bounded open subgroup $K' \subset N(L)$ there exists a finite étale covering $R \rightarrow R'$, with canonical frame $\mathcal{R} \rightarrow \mathcal{R}'$, and $z \in N(\mathcal{R}'_L)$ such that:*

- (i) for every \bar{k} -valued point x of R' ,

$$f_b(z(x))K' = y(x)K';$$

- (ii) there exists a point $x'_1 \in (\text{Spec } R')(\bar{k})$ over x_1 such that $z(x'_1) = z_1$.

Proof. We remind the reader that in the statement of the lemma and below, a map $R \rightarrow \bar{k}$ and the induced map $\mathcal{R} \rightarrow W(\bar{k})$ are denoted by the same symbol.

We will construct a finite étale covering $R \rightarrow R_i$ with canonical frame $\mathcal{R} \rightarrow \mathcal{R}_i$, together with a point $x_{1,i} \in (\text{Spec } R_i)(\bar{k})$ over x_1 and elements $z_i \in N(\mathcal{R}_L)$ and $\delta_i \in N_i(\mathcal{R}_L)$, such that for every $x \in (\text{Spec } R_i)(\bar{k})$,

$$f_b(z_i(x))\delta_i(x)K' = y(x)K',$$

$z_i(x_{1,i}) = z_1$, and $\delta_i(x_{1,i}) = 1$.

When $i = 1$, we have $N_1 = N$, and the element $z_1 \in N(L) \subset N(\mathcal{R}_L)$ satisfies these conditions, with $\delta_1 = f_b(z_1)^{-1}y$. Suppose that z_i, δ_i and $x_{1,i}$ with these properties have already been constructed. Let j be maximal such that $N_i \subset N[j]$. Then $\delta_i \in U_{\alpha_i}(\beta)N_{i+1}(\mathcal{R}_{i,L})$ for some $\beta \in \mathcal{R}_{i,L}$. By Lemma 3.4.3, for any $n \geq 0$, there exist a finite étale faithful R_i -algebra R_{i+1} and elements $\tilde{z} \in N[j](\mathcal{R}_{i+1,L})$ and $\varepsilon_i \in p^n \mathcal{R}_{i+1}$ such that

$$f_b(\tilde{z}) \in U_{\alpha_i}(\beta + \varepsilon_i)N_{i+1}(\mathcal{R}_{i+1,L}).$$

Note that $\delta_i(x_{1,i}) = 1$ implies $\beta(x_{1,i}) = 0$, so by Lemma 3.4.3(ii) we may assume that there is a point $x_{1,i+1} \in (\text{Spec } R_{i+1})(\bar{k})$ over $x_{1,i}$ such that $\tilde{z}(x_{1,i+1}) = 1$.

Let $z_{i+1} = z_i \tilde{z}$. Since $\tilde{z}, b\sigma(\tilde{z})b^{-1} \in N[j](\mathcal{R}_{i+1,L})$, and $[N[j], f_b(z_i)] \subset N[j+1]$, we have

$$f_b(z_{i+1}) = \tilde{z}^{-1}f_b(z_i)(b\sigma(\tilde{z})b^{-1}) = f_b(z_i)f_b(\tilde{z})\gamma_{i+1}$$

for some $\gamma_{i+1} \in N_{i+1}(\mathcal{R}_{i+1,L})$. Hence,

$$f_b(z_{i+1}) = f_b(z_i)f_b(\tilde{z})\gamma_{i+1} = f_b(z_i)\delta_i[U_{\alpha_i}(\varepsilon)\delta_{i+1}^{-1}]$$

for some $\delta_{i+1} \in N_{i+1}(\mathcal{R}_{i+1,L})$. Now choose n so that $U_{\alpha_i}(p^n \mathcal{O}_L) \subset K'$. Then for every $x \in (\text{Spec } R_{i+1})(\bar{k})$ we have

$$f_b(z_{i+1})(x)\delta_{i+1}(x)K' = f_b(z_i)\delta_i(x)U_{\alpha_i}(\varepsilon(x))K' = y(x)K'.$$

Moreover, since $\tilde{z}(x_{1,i+1}) = 1$, we have $\varepsilon(x_{1,i+1}) = 0$ and $z_{i+1}(x_{1,i+1}) = z_i(x_{1,i+1})$, which implies that $\gamma_{i+1}(x_{1,i+1}) = \delta_{i+1}(x_{1,i+1}) = 1$.

This completes the induction step. Taking i large enough that $N_i = 0$, the lemma follows. \square

LEMMA 3.4.5. *Let $m \in M(L)$. Then there exists a compact open subgroup $K' \subset N(L)$ such that*

$$K' \subset f_b(N(L) \cap mKm^{-1})$$

Proof. This can be shown using the methods of [GHKR06, 5.3.1, 5.3.2]. In our present situation, when $\text{char } L = 0$, there is a simpler argument which we now sketch.

Let $\mathfrak{n} = \text{Lie } N$ regarded as an L -scheme. The map f_b induces the map

$$df_b : \mathfrak{n} \rightarrow \mathfrak{n} : n \mapsto \text{ad}(b)(\sigma(n)) - n.$$

Since $N(L) \cap mKm^{-1}$ is a bounded open subgroup of $N(L)$, an argument using the exponential shows that it suffices to show that df_b maps a bounded open subset of $\mathfrak{n}(L)$ to a bounded open subset of $\mathfrak{n}(L)$.

Now for any L -vector space V equipped with a σ -semi-linear map σ_V , the map $\sigma_V - 1$ maps bounded open subsets onto bounded open subsets. This may be checked as in [GHKR06, 4.3.1] using the classification of σ -isocrystals (V, σ_V) . \square

Proof of Proposition 3.4.1. Let $g_1 \in X_{\leq \mu}(b)$. By the Iwasawa decomposition, g_1 has a representative in $G(L)$ of the form nm with $n \in N(L)$ and $m \in M(L)$. Let $\chi \in X_*(Z_M)$ be such that $\langle \chi, \alpha \rangle > 0$ for every root α of T in N .

Let $\mathcal{O}_L\langle s, s^{-1} \rangle$ and $\mathcal{O}_L\langle s \rangle$ denote the p -adic completions of $\mathcal{O}_L[s, s^{-1}]$ and $\mathcal{O}_L[s]$, respectively. We equip these rings with the Frobenius lifts given by $s \mapsto s^q$, and consider them as frames of their mod p reductions. Define $y = \chi(s)f_b(n)\chi(s)^{-1} \in N(\mathcal{O}_L\langle s, s^{-1} \rangle_L)$. For any root α , conjugation by $\chi(s)$ maps $U_\alpha(\beta)$ to $U_\alpha(s^{\langle \chi, \alpha \rangle} \beta)$. Hence, $y \in N(\mathcal{O}_L\langle s \rangle_L)$. Note also that $y(0) = 1$, while $y(1) = f_b(n)$.

Using Lemma 3.4.5, we choose a bounded open subgroup $K' \subset N(L)$ such that $K' \subset f_b(N(L) \cap mKm^{-1})$. We may also assume that $K' \subset (b\sigma(m))K(b\sigma(m))^{-1}$. Applying Lemma 3.4.4, we find a finite étale covering $\bar{k}[s] \rightarrow R$, with canonical frame $\mathcal{O}_L\langle s \rangle \rightarrow \mathcal{R}$, an element $z \in N(\mathcal{R}_L)$, and a point $x_1 \in (\text{Spec } R)(\bar{k})$ over 1, such that $f_b(z(x))K' = y(x)K'$ for every x in $(\text{Spec } R)(\bar{k})$, and $z(x_1) = n$. The first condition implies that

$$f_b(z(x))b\sigma(m)K = y(x)b\sigma(m)K.$$

We may replace $\text{Spec } R$ with the connected component containing x_1 and assume that this scheme is connected.

Let $g = zm \in G(\mathcal{R}_L)$. For $x \in (\text{Spec } R)(\bar{k})$ such that $s(x) \in \bar{k}^\times$, we have

$$\begin{aligned} g(x)^{-1}b\sigma(g(x))K &= m^{-1}f_b(z(x))b\sigma(m)K \\ &= m^{-1}y(x)b\sigma(m)K \\ &= \chi(s(x))^{-1}m^{-1}f_b(n)b\sigma(m)\chi(s(x))K \\ &= \chi(s(x))^{-1}g_1^{-1}b\sigma(g_1)\chi(s(x))K \\ &\subset Kp^\mu K. \end{aligned}$$

Hence, $g \in X_{\leq \mu}(b)(\mathcal{R})$ by Lemma 2.1.6.

Let $x_0 \in (\text{Spec } R)(\bar{k})$ be a point mapping to 0 in $\text{Spec } \bar{k}[s]$. Then $f_b(z(x_0)) \in K'$, so there exists $k \in N(L) \cap mKm^{-1}$ such that $f_b(z(x_0)) = f_b(k^{-1})$. This implies that $z(x_0)k \in J_b(F) \cap N(L)$. Hence,

$$g(x_0) = z(x_0)m = [z(x_0)k] \cdot k^{-1}m \in (J_b(F) \cap N(L))M(L)K.$$

Since $g(x_1) = nm = g_1$, we see that $g_1 \sim jm$ for some $j \in J_b(F) \cap N(L)$ and $m \in M(L)$. □

4. Connecting points

4.1 Main results: formulation and overview of the proofs

In this subsection we reduce the proofs of our main results Theorems 1.1 and 1.2 to four technical propositions whose proof will be the subject of the remainder of this section. At the end of the subsection we also explain how the arguments simplify if one is only interested in the case where G is split.

We let $G \supset B \supset T$ be as above, $\mu \in X_*(T)$ a dominant, minuscule cocharacter, and $b \in B(G, \mu)$.

4.1.1. For every standard Levi subgroup M of G , the projection $X_*(T) \rightarrow \pi_1(M)$ induces a bijection

$$\left\{ \begin{array}{l} M\text{-minuscule, } M\text{-dominant} \\ \text{cocharacters in } X_*(T) \end{array} \right\} \xrightarrow{\sim} \pi_1(M). \tag{4.1.2}$$

For $x \in \pi_1(M)$, denote by μ_x the preimage of x via (4.1.2). For any $b \in M(L)$ and G -minuscule $\mu \in X_*(T)$, let

$$\begin{aligned} \bar{I}_{\mu,b}^{M,G} &= \{x \in \pi_1(M) \mid (\mu_x)_{G\text{-dom}} = (\mu)_{G\text{-dom}}, x = \kappa_M(b) \text{ in } \pi_1(M)_\Gamma\} \\ &= \{x \in \pi_1(M) \mid x = \mu \text{ in } \pi_1(G), x = \kappa_M(b) \text{ in } \pi_1(M)_\Gamma, \mu_x \text{ } G\text{-minuscule}\}, \end{aligned}$$

where $\Gamma = \text{Gal}(\bar{k}|k)$.

For every k -algebra R with frame \mathcal{R} and every $\mu' \in X_*(T)_{M\text{-dom}}$ we have the natural inclusion $X_{\mu'}^M(b)(\mathcal{R}) \hookrightarrow X_{\mu'_{\text{dom}}}^G(b)(\mathcal{R})$. Note that if $\mu'_{\text{dom}} = \mu$ then μ' is M -minuscule, hence of the form $\mu' = \mu_x$ for some $x \in \pi_1(M)$. Furthermore, μ' has the same image in $\pi_1(G)$ as μ . Finally, $X_{\mu'}^M(b)(\mathcal{R}) = \emptyset$ unless $\kappa_M(b) = x$ as elements of $\pi_1(M)_\Gamma$. Hence, $X_{\mu'}^M(b)(\mathcal{R})$ is a non-empty subset of $X_\mu^G(b)(\mathcal{R})$ if and only if the image of μ' via the natural projection $X_*(T) \rightarrow \pi_1(M)$ is in $\bar{I}_{\mu,b}^{M,G}$.

4.1.3. Recall that N is the unipotent radical of the standard parabolic subgroup of G corresponding to M . Let Φ_N be the set of roots in N , and let $\Phi_{N,\Gamma}$ be the set of Galois orbits of roots in N .

DEFINITION 4.1.4. (i) For any root $\alpha \in \Phi_N$, we say that α is *adapted* if α^\vee is M -antidominant, and we have $\langle \beta, \alpha^\vee \rangle \in \{-1, 0, 1\}$ for every root β in M .

(ii) For any $\Omega \in \Phi_{N,\Gamma}$, we say that Ω is *adapted* if some $\alpha \in \Omega$ is adapted.

As B and M are stable under the action of Γ , if Ω is adapted, then so is any element in Ω .

4.1.5. From now on, we assume that G^{ad} is simple, as in Theorem 1.1, although this assumption will be dropped toward the end of the subsection. We also suppose that $M \subseteq G$ is a standard Levi subgroup defined over F such that b is superbasic in M . Recall that this implies that $M^{\text{ad}} \cong \prod_i \text{Res}_{F_i/F} \text{PGL}_{n_i}$ with F_i/F unramified (Lemma 3.1.1). Using (4.1.2), we have an identification of sets $\bar{I}_{\mu,b}^{M,G} = \bar{I}_{\mu,b}$, where $\bar{I}_{\mu,b}$ is defined in § 3.4. If G is split, this set consists of a single element.

The proofs of the two main theorems are based on the following propositions.

PROPOSITION 4.1.6 (Convexity of $\bar{I}_{\mu,b}^{M,G}$). *Let $x, x' \in \bar{I}_{\mu,b}^{M,G}$. Then there are elements $x_i \in \bar{I}_{\mu,b}^{M,G}$ for $i = 1, \dots, m$ for some m , such that $x = x_1, x' = x_m$ and such that for each i ,*

$$x_{i+1} - x_i = \alpha^\vee - \alpha'^\vee \quad \text{in } \pi_1(M)$$

for some roots $\alpha, \alpha' \in \Omega$ with $\Omega \in \Phi_{N,\Gamma}$ (depending on i).

PROPOSITION 4.1.7. *Suppose that $x, x' \in \bar{I}_{\mu,b}^{M,G}$ with $x - x' = \alpha^\vee - \alpha'^\vee$ for some $\alpha, \alpha' \in \Omega$ with $\Omega \in \Phi_{N,\Gamma}$. Then for any $g \in X_{\mu_x}^M(b)$, there is a $g' \in X_{\mu_{x'}}^M(b)$ such that the images of g and g' in $X_\mu^G(b)$ are in the same connected component.*

4.1.8. Let $x \in \pi_1(M)$ and let P_x be the parabolic subgroup of M defined by μ_x , M_x its Levi subgroup containing T and N_x its unipotent radical. Let $w_x = w_{0,x}w_{0,M}$ where $w_{0,x}$ is the longest Weyl group element in M_x and where $w_{0,M}$ is the longest Weyl group element in M .

Let N_M be the normalizer of T in M . Recall that $W_M = N_M(L)/T(L)$ is the Weyl group of M_L . The natural map $N_M(L) \cap K \rightarrow W_M$ is surjective (see, for example, [HR08, Proposition 13]). In particular, w_x has a representative \dot{w}_x in K . Let $b_x = \mu_x(p)\dot{w}_x$ with $\dot{w}_x \in K$. Note that the representatives of superbasic σ -conjugacy classes chosen in § 3.3 are also of this form.

The elements b and b_x are in the same σ -conjugacy class for the group M (i.e., $[b] = [b_x]$ in $B(M)$). Indeed, as $\kappa_M(b_x) = x = \kappa_M(b)$, in order to show that the σ -conjugacy classes of b and b_x agree, it suffices to show that b_x is basic in M . This is shown in [VW13, proof of Proposition 9.17].

For the next two propositions, we assume that $b = b_{x_0}$ for some fixed $x_0 \in \bar{I}_{\mu,b}^{M,G}$.

PROPOSITION 4.1.9. *Suppose that (μ, b) is HN-irreducible. Let*

$$C := \{\alpha^\vee \in X_*(T) \mid \alpha \in \Phi_N \text{ is adapted, and } \langle \alpha, \mu_{x_0} \rangle < 0\}.$$

Then the sum of the coroot lattice of M and the \mathbb{Z} -lattice generated by the Galois orbit of the set C is the coroot lattice of G .

PROPOSITION 4.1.10. *Let $\Omega \in \Phi_{N,\Gamma}$ be adapted. Suppose that there exists $\alpha \in \Omega$ such that $\langle \alpha, \mu_{x_0} \rangle < 0$. Then there exist an $x \in \bar{I}_{\mu,b}^{M,G}$ and $g_1, g_2 \in X_{\mu_x}^M(b)$ such that:*

- g_1 and g_2 are in the same connected component of $X_\mu^G(b)$;
- $w_M(g_2) - w_M(g_1) = \sum_{\beta \in \Omega} \beta^\vee$ in $\pi_1(M)^\Gamma$,

where $w_M : M(L) \rightarrow \pi_1(M)$ is the Kottwitz homomorphism.

4.1.11. Before proving these propositions let us show how they can be used to prove the main theorems. We first show the following stronger version of Theorem 1.2 (assuming G^{ad} is simple) which we then use in the proof of Theorem 1.1. We continue to assume that $b \in M(L)$ is superbasic, and we let $P = NM$ be the parabolic subgroup corresponding to M . As usual, we write J_b^M for the group defined by $b \in M(L)$, so that $J_b^M(F) = J_b(F) \cap M(L)$.

THEOREM 4.1.12. *The image of*

$$\pi_0(X_{\mu_x}^M(b)) \rightarrow \pi_0(X_\mu^G(b))$$

does not depend on the choice of $x \in \bar{I}_{\mu,b}^{M,G}$. In particular, for any such x , the map

$$(J_b(F) \cap N(L)) \times \pi_0(X_{\mu_x}^M(b)) \rightarrow \pi_0(X_\mu^G(b)) \tag{4.1.13}$$

is surjective, and the group $J_b(F) \cap P(L)$ acts transitively on $\pi_0(X_\mu^G(b))$.

Proof. Let x_1, \dots, x_n be as in Proposition 4.1.6 for a pair $x, x' \in \bar{I}_{\mu,b}^{M,G}$. To prove the first claim of the theorem, it is enough to show that for every $g \in X_{\mu_{x_i}}^M(b)$ there is an element $g' \in X_{\mu_{x_{i-1}}}^M(b)$ such that g, g' are in the same connected component in $X_\mu^G(b)$. This follows by applying Proposition 4.1.7 to each successive pair (x_{i-1}, x_i) .

For the second claim note that, by Proposition 3.4.1, each connected component of $X_\mu^G(b)$ contains the image of some element of $(J_b(F) \cap N(L)) \times \bigsqcup_{x \in \pi_1(M)} X_{\mu_x}^M(b)$. We thus obtain a surjective map

$$(J_b(F) \cap N(L)) \times \bigsqcup_{x \in \bar{I}_{\mu,b}^{M,G}} \pi_0(X_{\mu_x}^M(b)) \rightarrow \pi_0(X_\mu^G(b)). \tag{4.1.14}$$

Hence, the first claim implies that (4.1.13) is surjective. Now the final claim follows as $J_b^M(F)$ acts transitively on $\pi_0(X_{\mu_x}^M(b))$ by Proposition 3.3.1. \square

Proof of Theorem 1.1. We fix some $x \in \bar{I}_{\mu,b}^{M,G}$ and $g \in X_{\mu_x}^M(b)$. Then left multiplication by g^{-1} induces a bijection $X_{\mu_x}^M(b)(\mathcal{R}) \cong X_{\mu_x}^M(g^{-1}b\sigma(g))(\mathcal{R})$ for every k -algebra R with frame \mathcal{R} , and similarly for G . In particular, the sets of connected components of the affine Deligne–Lusztig sets for b and $g^{-1}b\sigma(g)$ coincide. Thus we may assume that $b = b_x$. In particular, $1 \in X_{\mu_x}^M(b)$ and therefore $c_{b,\mu_x}^{(M)} = c_{b,\mu} = 1$.

By Proposition 3.3.1 we have $J_b^M(F)$ -equivariant morphisms

$$\pi_1(M)^\Gamma \cong \pi_0(X_{\mu_x}^M(b)) \rightarrow \pi_0(X_\mu^G(b)) \rightarrow \pi_1(G)^\Gamma$$

where the composite of all morphisms is induced by the natural projection $\pi_1(M) \rightarrow \pi_1(G)$. By Lemma 2.5.11 and Proposition 4.1.9, the kernel of the composition $\pi_1(M)^\Gamma \rightarrow \pi_1(G)^\Gamma$ is generated by the elements $\sum_{\beta \in \Omega} \beta^\vee$ where $\Omega \in \Phi_{N,\Gamma}$, satisfies $\Omega \cap C \neq \emptyset$ (C defined as in Proposition 4.1.9).

We claim that each of the elements $\sum_{\beta \in \Omega} \beta^\vee$ with $\Omega \cap C \neq \emptyset$ is mapped to 1 by the composite $\pi_1(M)^\Gamma \cong \pi_0(X_{\mu_x}^M(b)) \rightarrow \pi_0(X_\mu^G(b))$. Then the transitivity of the $J_b^M(F)$ -action on $\pi_0(X_{\mu_x}^M(b))$ implies that this composite factors through $\pi_1(G)^\Gamma$. Again, by the transitivity of the $J_b^M(F)$ -action on $\pi_0(X_{\mu_x}^M(b))$, our claim follows if we can show that there are elements $g_1, g_2 \in X_{\mu_x}^M(b)$ with $w_M(g_2) - w_M(g_1) = \sum_{\beta \in \Omega} \beta^\vee$ and such that g_1, g_2 are in the same connected component of $X_\mu^G(b)$.

To prove this, we apply Proposition 4.1.10 to $\alpha \in \Omega \cap C$. Let $x' \in \bar{I}_{\mu_x, b}^{M,G}$ and $g'_1, g'_2 \in X_{\mu_x'}^M(b)$ be the elements produced there. As $J_b^M(F)$ acts transitively on $\pi_0(X_{\mu_x'}^M(b))$, we can choose a $j_\Omega \in J_b^M(F)$ such that $j_\Omega g'_1$ is in the connected component of g'_2 in $X_{\mu_x'}^M(b)$. Then the image of j_Ω in $\pi_1(M)$ is equal to $\sum_{\beta \in \Omega} \beta^\vee$. By Theorem 4.1.12, we see that there is a $g_1 \in X_{\mu_x}^M(b)$ such that g_1, g'_1 are in the same connected component of $X_\mu^G(b)$. Hence, also $j_\Omega g'_1$ and $j_\Omega g_1$ are in the same connected component of $X_\mu^G(b)$. Altogether we obtain that in $X_\mu^G(b)$ the elements $j_\Omega g_1, j_\Omega g'_1, g'_1, g_1$ are all in the same connected component. As $j_\Omega \in M(L)$ we have $j_\Omega g_1, g_1 \in X_{\mu_x}^M(b)$, and $w_M(j_\Omega g_1) - w_M(g_1) = \sum_{\beta \in \Omega} \beta^\vee$. This shows our claim.

We have shown the existence of the following diagram.

$$\begin{array}{ccccc} \pi_1(M)^\Gamma \cong \pi_0(X_{\mu_x}^M(b)) & \longrightarrow & \pi_0(X_\mu^G(b)) & \longrightarrow & \pi_1(G)^\Gamma \\ \downarrow & \nearrow & \searrow & \text{=} & \nearrow \\ & \pi_1(G)^\Gamma & & & \end{array}$$

It remains to show that $\pi_0(X_{\mu_x}^M(b)) \rightarrow \pi_0(X_\mu^G(b))$ (or equivalently $\pi_1(G)^\Gamma \rightarrow \pi_0(X_\mu^G(b))$) is surjective. By the second claim in Theorem 4.1.12, it suffices to show that for each $j \in J_b(F) \cap N(L)$ and for each $z \in \pi_0(X_{\mu_x}^M(b))$, the two elements jz and z have the same image in $\pi_0(X_\mu^G(b))$. As $J_b^M(F)$ acts on $\pi_0(X_{\mu_x}^M(b))$, it is enough to show the same statement for mjz and mz for some $m \in J_b^M(F)$. We choose m such that mjm^{-1} is contained in the stabilizer in $G(L)/G(\mathcal{O}_L)$ of a chosen representative of z in $G(L)$ and such that the image of m in $\pi_1(G)$ is equal to 1. For example, we can choose m to be a sufficiently dominant element in $Z_M(F)$, in the image of $\tilde{G}(F)$, where \tilde{G} denotes the simply connected cover of G^{der} . Then, by what we saw above, the second property of m implies that mz and z are in the same connected component of $X_\mu^G(b)$. Hence, the same holds for $mjm^{-1}z$ and $mjm^{-1}mz = mjz$. Finally, the first property of m implies that $mjm^{-1}z$ and z are the same element. Altogether, we see that mjz and mz have the same image in $\pi_0(X_\mu^G(b))$. \square

4.1.15. We now drop the assumption that G^{ad} is simple. We have the following corollary and generalization of Theorem 1.1.

COROLLARY 4.1.16. *Suppose that (μ, b) is Hodge–Newton irreducible in G . Then w_G induces a bijection*

$$\pi_0(X_\mu(b)) \cong c_{b,\mu} \pi_1(G)^\Gamma.$$

Proof. Let $\mu^{\text{ad}} \in X_*(T/Z_G)$ and $b^{\text{ad}} \in G^{\text{ad}}(L)$ be the images of μ and b , and let $M \subset G$ be a Levi subgroup. Since $\ker(\pi_1(M)_\Gamma \rightarrow \pi_1(G)_\Gamma)$ is torsion free by Lemma 2.5.12, it has trivial intersection with the image of $X_*(Z_G)_\Gamma$. Using this, one sees that (μ, b) is HN-irreducible if and only if $(\mu^{\text{ad}}, b^{\text{ad}})$ is.

For $(\mu^{\text{ad}}, b^{\text{ad}})$ the corollary follows from Theorem 1.1 as the set of connected components of affine Deligne–Lusztig varieties for products of groups is the product of the corresponding sets for the individual factors. And this implies the result for (μ, b) by Corollary 2.4.3. \square

Proof of Theorem 1.2. Note that we have already proved Theorem 1.2 in Theorem 4.1.12 above when G^{ad} is simple. We now deduce the general case from Theorem 4.1.16.

By Proposition 2.5.4 we may assume that (μ, b) is HN-indecomposable in G . Let $(\mu^{\text{ad}}, b^{\text{ad}})$ be as in the proof of Theorem 4.1.16. Consider a decomposition $G^{\text{ad}} = G_1 \times G_2$, and let (μ_1, b_1) and (μ_2, b_2) denote the images of $(\mu^{\text{ad}}, b^{\text{ad}})$ in G_1 and G_2 , respectively. By Theorem 2.5.6, we may choose G_1 and G_2 so that (μ_1, b_1) is HN-irreducible, and b_2 is σ -conjugate to $p^{\mu_2} \in X_*(Z_{G_2})$.

Now suppose that $M \subset G$ is a Levi subgroup and $b \in M(L) \subset G(L)$ is superbasic. As in the proof of Theorem 4.1.12, it suffices to show that the image of $\pi_0(X_{\mu_x}^M(b)) \rightarrow \pi_0(X_{\mu}^G(b))$ is independent of $x \in \bar{I}_{\mu, b}^{M, G}$. We may assume that $c_{b, \mu_x} = 1$. Using Proposition 2.4.3, one sees that it suffices to show that image of $\pi_0(X_{\mu_x}^M(b)) \rightarrow \pi_0(X_{\mu^{\text{ad}}}^G(b^{\text{ad}}))$ is independent of x .

By Theorem 4.1.16 and Remark 2.5.10, the map $M(L) \rightarrow G^{\text{ad}}(L)$ induces a well-defined map $\pi_1(M)^\Gamma \rightarrow \pi_1(G_1)^\Gamma \times G_2(F)/G_2(\mathcal{O}_F)$ whose image may be identified with that of $\pi_0(X_{\mu_x}^M(b)) \rightarrow \pi_0(X_{\mu^{\text{ad}}}^G(b^{\text{ad}}))$. \square

4.1.17. Let us consider the case where G is split. Then $\bar{I}_{\mu, b}^{M, G}$ consists of a single element, so Propositions 4.1.6 and 4.1.7 are no longer needed. In the proof of Proposition 4.1.10 we have to distinguish essentially between all different Dynkin diagrams equipped with the Galois action, and a fixed Galois orbit of simple roots (subject to some restrictions). This case-by-case study is shortened drastically when assuming that G is split (i.e. that the Galois action is trivial). The reader only interested in this case is referred to [Vie08, 2.5], where the completely parallel proof for split groups in the function field case is given in less than five pages.

The remainder of this section will be devoted to the proof of the propositions above.

4.2 Some maximal rank subgroups of G

In this subsection we will introduce some subgroups of maximal rank of G . They will be needed in the proofs of Propositions 4.1.7 and 4.1.10 to distinguish several cases. From now on we again assume that G^{ad} is simple, and we denote by $T \subset M \subset G$ a standard Levi subgroup over F .

We begin with a (probably well-known) fact on root systems with an endomorphism.

LEMMA 4.2.1. *Let Φ be a root system with an action by a finite cyclic group Γ such that there exists a basis Δ that is stable under this action. Furthermore, we assume that Γ acts transitively on the set of connected components of the Dynkin diagram. Let $\alpha \in \Phi$ and $\alpha' \in \Gamma\alpha \setminus \{\alpha\}$. Then $\langle \alpha, (\alpha')^\vee \rangle \in \{0, -1\}$. Moreover:*

- if $\langle \alpha, (\alpha')^\vee \rangle = -1$ then the root system is a disjoint union of finitely many copies of root systems of type A_n for some even n ;
- $\Gamma\alpha$ has at most three elements in each connected component of the Dynkin diagram. If $\Gamma\alpha$ has three elements in each connected component of the Dynkin diagram, then the root system is a disjoint union of finitely many copies of root systems of type D_4 .

Proof. The first assertion can be found, for example, in [Spr06, Lemma 1]. The second and third assertions follow from the classification of Dynkin diagrams. \square

Example 4.2.2. Let τ be the non-trivial automorphism of the Dynkin diagram of type A_{2n} , and of the corresponding root system. Using the standard notation for this root system, we have $\tau(e_i) = e_{2n+2-i}$. Then a root $\alpha = e_i - e_j$ (for $i < j$) satisfies $\langle \alpha, \tau\alpha^\vee \rangle = -1$ if and only if i or j is equal to $n + 1$.

4.2.3. Let $\Phi = \Phi(G, T)$ be the root system of G , and $\Delta \subset \Phi$ a Γ -stable basis of simple roots for Φ corresponding to a Borel subgroup $B \subset G$. If $\phi = \sum_{\alpha \in \Delta} n_\alpha \alpha \in X^*(T)$ is an integral sum of roots ($n_\alpha \in \mathbb{Z}$), we define $|\phi| = \sum_{\alpha \in \Delta} |n_\alpha|$. For $\phi \in X_*(T)$ we define $|\phi|$ analogously, using the basis of coroots Δ^\vee . We will make repeated use of the following two simple lemmas.

LEMMA 4.2.4. *Let $\gamma, \gamma' \in \Phi$ with $\gamma \neq -\gamma'$.*

- (i) *If $\langle \gamma, (\gamma')^\vee \rangle < 0$ then $\gamma + \gamma'$ is a root.*
- (ii) *If $\langle \gamma, (\gamma')^\vee \rangle > 0$ then $\gamma - \gamma'$ is a root.*
- (iii) *If $\langle \gamma, (\gamma')^\vee \rangle > 0$ and γ, γ' are positive, then*

$$\|\gamma| - |\gamma'|\| = |\gamma - \gamma'| \neq 0.$$

Proof. Indeed, $\gamma \neq -\gamma'$ implies that $\langle \gamma, (\gamma')^\vee \rangle = -1$ or $\langle \gamma', (\gamma)^\vee \rangle = -1$. By symmetry, we may assume that the second is true. Then $s_\gamma(\gamma') = \gamma + \gamma'$ is a root. This proves (i), and (ii) follows immediately. To see (iii), write $\gamma - \gamma' = \sum_{\alpha \in \Delta} n_\alpha \alpha$. By (ii) all the non-zero n_α have the same sign, and (iii) follows easily. \square

LEMMA 4.2.5. *Let $\alpha \in X^*(T)$ be an integral sum of roots. Then α may be written as a sum of roots $\alpha = \sum_{i \in I} \gamma_i$ such that $\langle \gamma_i, \gamma_j^\vee \rangle \geq 0$ for $i, j \in I$.*

Moreover, if $\alpha = \sum_{j \in J} \alpha_j \in X^(T)$ with $\alpha_j \in \Phi$, then we may take each γ_i to be a sum of a subset of $\{\alpha_j\}_{j \in J}$. In particular, if α is positive, then the γ_i may be chosen to be positive.*

Proof. Write $\alpha = \sum_{i \in I} \gamma_i$ such that each γ_i is a root and $|I|$ is as small as possible. If $i, j \in I$ with $\langle \gamma_i, \gamma_j^\vee \rangle < 0$, then $\gamma_i \neq -\gamma_j$ by the minimality of I . Hence, $\gamma_i + \gamma_j$ is a root by Lemma 4.2.4, which contradicts the minimality of I .

If $\alpha = \sum_{j \in J} \alpha_j$ then write $\alpha = \sum_{i \in I} \gamma_i$ such that each γ_i is a root which is a sum of a subset of $\{\alpha_j\}_{j \in J}$, and $|I|$ is as small as possible. The same argument proves the second claim. If α is positive, we may take the α_j to be positive simple roots, which proves the final claim. \square

DEFINITION 4.2.6. Let Φ_1 be a subset of Φ .

- Φ_1 is said to be *symmetric* if $\Phi_1 = -\Phi_1$ where $-\Phi_1 = \{-\alpha \mid \alpha \in \Phi_1\}$.
- Φ_1 is said to be *closed* if $\alpha, \beta \in \Phi_1$ with $\alpha + \beta \in \Phi$ implies $\alpha + \beta \in \Phi_1$.

Remark 4.2.7. If $\Phi_1 \subset \Phi$ is a closed symmetric subset, then Φ_1 is a root system in the \mathbb{R} -vector space generated by Φ_1 [Bou02, ch. VI, no. 1.8, Proposition 23]. In this case we also say that Φ_1 is a root system if there is no confusion.

4.2.8. Now we will define some subgroups of maximal rank of G which will be used in the proof of the main results. For the general theory of these subgroups, we refer to [Hum95, § 2.1] or [SGA3, Exposé XXII].

Let $\Delta_M \subset \Delta$ (respectively $\Phi_M \subset \Phi$) denote the roots (respectively simple roots) contained in Lie M . The action of $\Gamma = \text{Gal}(\bar{k}|k)$ on Φ factors through some finite cyclic quotient of Γ .

Sometimes we also write Γ for that finite cyclic quotient if no confusion can arise. The Frobenius automorphism σ is a generator of Γ . Let Φ_N and $\Phi_{N,\Gamma}$ be as in §4.1. For any $\Omega \in \Phi_{N,\Gamma}$, let Φ_Ω be the smallest symmetric closed subset of Φ containing Φ_M and Ω . As M and Ω are stable under the Galois action, so is Φ_Ω . We let G_Ω be the subgroup of G_L generated by T and U_α for all $\alpha \in \Phi_\Omega$.

PROPOSITION 4.2.9. *For any $\Omega \in \Phi_{N,\Gamma}$, the group G_Ω is defined over F . Moreover, it is a reductive subgroup of G with root system Φ_Ω with respect to the maximal torus T .*

Proof. [BT65, Theorem 3.13] (cf. also [SGA3, Exposé 22, Theorem 5.4.7 and Proposition 5.10.1]). □

Remark 4.2.10. Note that, in general, G_Ω is not a Levi subgroup of G . For example, let G have Dynkin diagram of type C_2 . Then it may happen that G_Ω is generated by T and the root subgroups for all long roots, hence it is of type $A_1 \times A_1$. However, for $\mu \in X_*(T)$, $b \in M(L)$ and for any G_Ω -dominant $\mu' \in X_*(T)$ with $(\mu')_{G\text{-dom}} = \mu$, we always have a map $X_{\mu'}^{G_\Omega}(b) \rightarrow X_\mu^G(b)$ given by the natural inclusion and inducing a map between the sets of connected components.

PROPOSITION 4.2.11. *Suppose that Ω is adapted, and that all the roots in G_Ω have the same length. Then $B \cap G_\Omega$ is a Borel subgroup of G_Ω with basis $\Delta_M \cup \Omega$.*

Proof. Let Φ_Ω^+ be the set of roots in G_Ω which are positive as roots in G with respect to B . Then $\Phi_\Omega = \Phi_\Omega^+ \amalg -\Phi_\Omega^+$ and Φ_Ω^+ is the set of roots in $B \cap G_\Omega$. It is clear that $B \cap G_\Omega$ is a Borel subgroup of G_Ω (as the set of roots in a Borel subgroup is determined by a regular hyperplane in the corresponding root system). By the definition of Φ_Ω , all elements in Φ_Ω can be written as linear combinations of roots in $\Delta_\Omega := \Delta_M \cup \Omega$. It suffices to show that all elements in Ω are indecomposable. Moreover, since Φ_Ω^+ is stable under the action of Γ , we only need to show that some $\alpha \in \Omega$ is indecomposable.

Suppose that $\alpha \in \Omega$ is adapted and decomposable. Then there exists a root $\alpha_1 \in \Phi_\Omega$ such that $\alpha_1, \alpha - \alpha_1 \in \Phi_\Omega^+$. Write

$$\begin{aligned} \alpha_1 &= \sum_{\beta \in \Delta_\Omega} n_\beta \beta = \sum_{\beta \in \Delta_{\alpha_1}^+} n_\beta \beta + \sum_{\beta \in \Delta_{\alpha_1}^-} n_\beta \beta, \\ \alpha - \alpha_1 &= \sum_{\beta \in \Delta_\Omega} \tilde{n}_\beta \beta = \sum_{\beta \in \Delta_{\alpha-\alpha_1}^+} \tilde{n}_\beta \beta + \sum_{\beta \in \Delta_{\alpha-\alpha_1}^-} \tilde{n}_\beta \beta \end{aligned}$$

where $\Delta_{\alpha_1}^+ = \{\beta \in \Delta_\Omega | n_\beta > 0\}$, $\Delta_{\alpha_1}^- = \{\beta \in \Delta_\Omega | n_\beta < 0\}$ and $\Delta_{\alpha-\alpha_1}^+, \Delta_{\alpha-\alpha_1}^-$ are defined in the same way.

By Lemma 4.2.5 we may write $\sum_{\beta \in \Delta_{\alpha_1}^+} n_\beta \beta = \sum_{i \in I} \gamma_i^+$ and $\sum_{\beta \in \Delta_{\alpha_1}^-} n_\beta \beta = \sum_{j \in J} \gamma_j^-$ as sums of roots such that $\gamma_i^+, \gamma_j^- \in \Phi$ and for $i, i' \in I$ and $j, j' \in J$,

- $\langle \gamma_i^+, \gamma_{i'}^{+\vee} \rangle \geq 0$ and $\langle \gamma_j^-, \gamma_{j'}^{-\vee} \rangle \geq 0$;
- γ_i^+ (respectively γ_j^-) is a linear combination of roots in $\Delta_{\alpha_1}^+$ (respectively $\Delta_{\alpha_1}^-$) with non-negative (respectively non-positive) coefficients.

By Lemma 4.2.1 and the fact that α is M -antidominant, for distinct roots $\beta, \beta' \in \Delta_\Omega$, we have $\langle \beta, \beta'^\vee \rangle \leq 0$. Therefore $\langle \gamma_i^+, \gamma_j^{-\vee} \rangle \geq 0$ for any $i \in I$ and $j \in J$. We show that one of the two sets I and J is empty (or equivalently, that one of the two sets $\Delta_{\alpha_1}^+$ and $\Delta_{\alpha_1}^-$ is empty). Suppose

that I is non-empty, the other case being analogous. For $i_0 \in I$, the inequality

$$\langle \alpha_1, \gamma_{i_0}^{+\vee} \rangle = \left\langle \sum_{i \in I} \gamma_i^+ + \sum_{j \in J} \gamma_j^-, \gamma_{i_0}^{+\vee} \right\rangle \geq 2$$

implies that $\alpha_1 = \gamma_{i_0}^+$. Hence, J is empty and $\alpha - \alpha_1 = \alpha - \gamma_{i_0}^+$. Moreover, the sets $\Delta_{\alpha - \alpha_1}^+ = \{\alpha\}$ and $\Delta_{\alpha - \alpha_1}^- = \Delta_{\alpha_1}^+$ are both non-empty, which is impossible according to the same discussion as above, but applied to $\alpha - \alpha_1$. \square

Remark 4.2.12. If not all roots in G_Ω have the same length, then in general Proposition 4.2.11 does not hold. In fact, in this case, the root system generated by the root system of M and the roots in Ω is not necessarily the root system of G_Ω . Here is an example. Consider the split group $G = \mathrm{GSp}_4$. The Dynkin diagram is of type C_2 with simple roots $\beta_1 = (1, -1)$ and $\beta_2 = (0, 2)$. Let M be the standard Levi subgroup corresponding to β_1 . And let $\alpha = \beta_1 + \beta_2 = (1, 1)$. Then the sub-root system generated by β_1 and α is of type $A_1 \times A_1$, while $G_\Omega = G$ as the commutator $[U_\alpha(x), U_{\beta_1}(y)]$ is a non-trivial element of the root subgroup $U_{\alpha + \beta_1}$.

PROPOSITION 4.2.13. *Let $\Omega \in \Phi_{N,\Gamma}$ be adapted. Then M is a standard Levi subgroup of G_Ω .*

Proof. By the proof of Proposition 4.2.11, the basis of G_Ω corresponding to the Borel subgroup $B \cap G_\Omega$ is the set of indecomposable elements of Φ_Ω^+ . Therefore M is a standard Levi subgroup of G_Ω as any $\beta \in \Delta_M$ is indecomposable in Φ_Ω^+ . \square

4.3 Proof of Proposition 4.1.6

From now on let Γ be the image of the absolute Galois group of F in the group of automorphisms of the Dynkin diagram of G . It is thus a finite and cyclic group, generated by Frobenius. As G^{ad} is assumed to be simple, Γ acts transitively on the set of connected components of the Dynkin diagram. All assertions involving the Galois action on $X_*(T)$ can then be studied using the induced Γ -action.

The proof of Proposition 4.1.6 is divided into two steps: we first reduce the general statement to the special case where $M = T$ is a maximal torus of G . More precisely, we want to show the following proposition.

PROPOSITION 4.3.1. *Let $x, x' \in \bar{I}_{\mu,b}^{M,G}$. Then there exists a $w \in W_M$ (the Weyl group of M) such that $\mu_x = w\mu_{x'}$ in $X_*(T)_\Gamma$.*

In particular, $\mu_x, w\mu_{x'} \in X_*(T) = \pi_1(T)$ then satisfy that $w\mu_{x'} \in \bar{I}_{\mu,\mu_x(p)}^{T,G}$. Furthermore, under the canonical projection $X_*(T) \rightarrow \pi_1(M)$, the set $\bar{I}_{\mu,\mu_x(p)}^{T,G}$ is mapped to a subset of $\bar{I}_{\mu,b}^{M,G}$, and μ_x and $w\mu_{x'}$ have the same image. Proposition 4.1.6 is then implied by the following proposition.

PROPOSITION 4.3.2. *Let $x, x' \in \bar{I}_{\mu,b}^{T,G}$ for some $\mu \in X_*(T)$ and $b \in T(L)$. Then there are elements $x_i \in \bar{I}_{\mu,b}^{T,G} \subset X_*(T)$ for $i = 0, \dots, m$ for some m such that $x = x_0, x' = x_m$ and such that for each i ,*

$$x_{i+1} - x_i = \alpha^\vee - \alpha'^\vee$$

for some roots $\alpha, \alpha' \in \Omega$ with $\Omega \in \Phi_{N,\Gamma}$ (depending on i).

It remains to show these two propositions.

DEFINITION 4.3.3. (i) Let $\phi = \sum_{\alpha \in \Delta} n_\alpha \alpha^\vee \in X_*(T)$ be an integral sum of coroots. We write $|\phi|_\Gamma = \sum_{\Gamma\alpha \in \Gamma\Delta} |\sum_{\beta \in \Delta \cap \Gamma\alpha} n_\beta|$.
 (ii) For all $\mu_1, \mu_2 \in X_*(T)$ having the same image in $\pi_1(G)$ we define

$$d(\mu_1, \mu_2) = |\mu_1 - \mu_2|,$$

$$d_\Gamma(\mu_1, \mu_2) = |\mu_1 - \mu_2|_\Gamma.$$

(iii) For $x, x' \in \bar{I}_{\mu,b}^{M,G}$ let $d(x, x') = d(\mu_x, \mu_{x'})$, and similarly for d_Γ .

Note that $|x|_\Gamma \leq |x|$ (where the latter expression is as in 4.2.3) with equality if and only if for each Galois orbit $\Gamma\alpha$ all n_β for $\beta \in \Gamma\alpha$ have the same sign.

As a preparation for the proofs of the propositions we provide several smaller lemmas. For these we consider a root datum $(V, \Phi, V^\vee, \Phi^\vee)$ equipped with an action of Γ , together with a Γ -stable basis of simple roots Δ . We assume that Γ acts transitively on the set of connected components of the Dynkin diagram of $(V, \Phi, V^\vee, \Phi^\vee)$.

LEMMA 4.3.4. (i) Let $\sum_{i \in I} \gamma_i^\vee = \sum_{j \in J} \lambda_j^\vee \neq 0$ be two equal sums of coroots. Then there are an $i \in I$ and $j \in J$ with $\langle \gamma_i, \lambda_j^\vee \rangle > 0$.

(ii) Let $\gamma_i^\vee, \lambda_j^\vee$ (for $i \in I, j \in J$) be coroots with $\sum_{i \in I} \gamma_i^\vee = \sum_{j \in J} \lambda_j^\vee \neq 0$ as elements of V_Γ^\vee . Then there are $i \in I, j \in J$ and $\tau \in \Gamma$ with $\langle \gamma_i, \tau \lambda_j^\vee \rangle > 0$.

Proof. By Lemma 4.2.5, applied to $\alpha = \sum_{i \in I} \gamma_i$, we may assume that $\langle \gamma_{i_1}, \gamma_{i_2}^\vee \rangle \geq 0$ for all $i_1, i_2 \in I$. Then for all $i_0 \in I$, we have

$$0 < \left\langle \gamma_{i_0}, \sum_{i \in I} \gamma_i^\vee \right\rangle = \left\langle \gamma_{i_0}, \sum_{j \in J} \lambda_j^\vee \right\rangle.$$

Hence, there is a $j \in J$ with $\langle \gamma_{i_0}, \lambda_j^\vee \rangle > 0$.

Now let γ_i, λ_j be as in the second assertion. Then the first assertion holds for

$$\sum_{i \in I} \sum_{\tau \in \Gamma} \tau \gamma_i^\vee = \sum_{j \in J} \sum_{\tau \in \Gamma} \tau \lambda_j^\vee.$$

Indeed, V^\vee is a sum of induced Γ -modules (cf. the proof of Lemma 2.5.11), so V_Γ^\vee is a free abelian group and thus these sums are non-zero in V_Γ^\vee . This implies the second assertion. \square

LEMMA 4.3.5. Let $\sum_{i \in I} \gamma_i^\vee \in V^\vee$ be a sum of coroots which maps to 0 in V_Γ^\vee . Then there exist $\tau_i \in \Gamma$ for all $i \in I$ such that $\sum_{i \in I} \tau_i(\gamma_i^\vee) = 0 \in V^\vee$ and such that all $\tau_i(\gamma_i^\vee)$ are in the same connected component of the Dynkin diagram.

Proof. We use induction on $|I|$. Let I^+ be the set of $i \in I$ such that γ_i^\vee is positive and $I^- = I \setminus I^+$. Then

$$\sum_{i \in I} \gamma_i^\vee = \sum_{i \in I^+} \gamma_i^{+\vee} - \sum_{i \in I^-} \gamma_i^{-\vee}$$

where $\gamma_i^+ = \gamma_i$ and $\gamma_i^- = -\gamma_i$ are all positive. Assume that one of the sums on the right-hand side is zero. Then the left-hand side lies in the positive (respectively the negative) cone. As Γ fixes the set of simple roots and as $\sum_{i \in I} \gamma_i^\vee = 0$ in V_Γ^\vee , this implies that the other sum is also equal to 0 (first in V_Γ^\vee but then also in V^\vee). Furthermore, this only occurs if none of the sums contains any non-zero summand. Thus in this case the assertion of the lemma is trivial. From now on we exclude this case.

Then by Lemma 4.3.4(ii) there are a $j_+ \in I^+$, a $j_- \in I^-$ and a $\tau \in \Gamma$ such that $\langle \gamma_{j_+}^+, \tau \gamma_{j_-}^{-\vee} \rangle > 0$. If $\gamma_{j_+}^+ = \tau \gamma_{j_-}^-$ we have that $\sum_{i \in I} \gamma_i^\vee = \sum_{i \in I \setminus \{j_+, j_-\}} \gamma_i^\vee = 0$ in V_Γ^\vee . Then the statement follows by induction. Thus we may assume that $\gamma_{j_+}^+ \neq \tau \gamma_{j_-}^-$. Then by Lemma 4.2.4 (applied to $-\gamma_{j_+}^+, \tau \gamma_{j_-}^-$) we obtain that $\alpha^\vee = \tau \gamma_{j_-}^{-\vee} - \gamma_{j_+}^{+\vee}$ is a coroot. Then

$$\sum_{i \in I} \gamma_i^\vee = \sum_{i \in I \setminus \{j_+, j_-\}} \gamma_i^\vee - \alpha^\vee = 0$$

as elements of V_Γ^\vee . The assertion follows again by induction. □

LEMMA 4.3.6. *Let $v = \sum_{\beta \in \Delta} n_\beta \beta^\vee \in V^\vee \setminus \{0\}$ with $|v| = |v|_\Gamma$. Then there is a coroot α^\vee such that $|v| = |\alpha^\vee| + |v - \alpha^\vee|$ and $\langle \sum_{\tau \in \Gamma} \tau \alpha, v \rangle > 0$.*

Proof. We first consider the case where the Γ -action on the Dynkin diagram is trivial. Using Lemma 4.2.5, we may write v as a sum of coroots $v = \sum_{i \in I} \gamma_i^\vee$ in such a way that $|v| = \sum_i |\gamma_i^\vee|$ and $\langle \gamma_i, \gamma_j^\vee \rangle \geq 0$ for all $i, j \in I$. Then for all i , we have $\langle \gamma_i, v \rangle > 0$. Thus each $\alpha = \gamma_i$ is as claimed.

We now assume that Γ acts non-trivially on the (connected) Dynkin diagram. This implies that the Dynkin diagram is of type A, D or E_6 , and in particular all roots have equal length. Let β_1, \dots, β_n be representatives of the Γ -orbits on Δ . Note that $|v| = |v|_\Gamma$ implies that $n_{\beta_i}, n_{\tau \beta_i}$ have the same sign for all $\tau \in \Gamma$. For $1 \leq i \leq n$ let $m_i = |\sum_{\tau \in \Gamma} n_{\tau \beta_i}| = \sum_{\tau \in \Gamma} |n_{\tau \beta_i}|$. By possibly changing the representatives β_i we may assume that $n_{\beta_i} \neq 0$ whenever $m_i \neq 0$. We have

$$\left\langle \frac{n_{\beta_i}}{|n_{\beta_i}|} \beta_i, v \right\rangle = 2|n_{\beta_i}| - \sum_{\alpha \in \Delta, \langle \beta_i, \alpha^\vee \rangle = -1} \frac{n_{\beta_i}}{|n_{\beta_i}|} n_\alpha.$$

For $\alpha \in \Gamma \beta_j$, let $m_\alpha = m_j$. Then we obtain

$$\begin{aligned} \left\langle \frac{n_{\beta_i}}{|n_{\beta_i}|} \sum_{\tau \in \Gamma} \tau \beta_i, v \right\rangle &= 2m_i + \sum_{\alpha \in \Delta, \langle \beta_i, \alpha^\vee \rangle = -1} \frac{n_{\beta_i}}{|n_{\beta_i}|} \frac{n_\alpha}{|n_\alpha|} m_\alpha \\ &\geq 2m_i - \sum_{\alpha \in \Delta, \langle \beta_i, \alpha^\vee \rangle = -1} m_\alpha. \end{aligned}$$

If $2m_i - \sum_{\alpha \in \Delta, \langle \beta_i, \alpha^\vee \rangle = -1} m_\alpha > 0$ for some $i \leq n$ the claim is shown. Thus it suffices to show that

$$\left\{ m_i \in \mathbb{N}^n \mid 2m_i - \sum_{\alpha \in \Delta, \langle \beta_i, \alpha^\vee \rangle = -1} m_\alpha \leq 0 \right\} = \{(0, \dots, 0)\}.$$

This can be done by an easy case-by-case computation considering the different possible types of Dynkin diagrams. □

LEMMA 4.3.7. *Let $\mu', \mu'' \in X_*(T)$ be minuscule and such that $(\mu')_{G\text{-dom}} = (\mu'')_{G\text{-dom}}$. Then we have a decomposition $\mu' - \mu'' = \sum_{i \in I} \gamma_i^\vee$ as a sum of coroots such that:*

- $\langle \gamma_i, \gamma_j^\vee \rangle = 0$ for $i \neq j$;
- $d(\mu', \mu'') = \sum_{i \in I} |\gamma_i^\vee|$;
- $\langle \gamma_i, \mu' \rangle = 1, \langle \gamma_i, \mu'' \rangle = -1$ for all $i \in I$;
- $\mu'' = (\prod_{i \in I} s_{\gamma_i}) \mu'$ where the product does not depend on the order.

Proof. Applying Lemma 4.2.5 to $\mu' - \mu''$ written as an integral sum of simple coroots, we see that $\mu' - \mu'' = \sum_{i \in I} \gamma_i^\vee$ where the γ_i are roots such that $\gamma_i \neq -\gamma_j$ and $\langle \gamma_i, \gamma_j^\vee \rangle \geq 0$ for all $i, j \in I$, and $d(\mu', \mu'') = \sum_{i \in I} |\gamma_i^\vee|$. Then for all $i_0 \in I$,

$$2 \leq \left\langle \gamma_{i_0}, \sum_{i \in I} \gamma_i^\vee \right\rangle = \langle \gamma_{i_0}, \mu' - \mu'' \rangle \leq 2$$

where the last inequality follows from μ', μ'' minuscule. Thus both inequalities are equalities. We obtain $\langle \gamma_{i_0}, \mu' \rangle = 1$, $\langle \gamma_{i_0}, \mu'' \rangle = -1$ and $\langle \gamma_{i_0}, \gamma_j^\vee \rangle = 0$ for all $j \neq i_0$. \square

Proof of Proposition 4.3.1. Let $x_1, x_2 \in \bar{I}_{\mu, b}^{M, G}$. If $\mu_{x_1} = \mu_{x_2}$ in $X_*(T)_\Gamma$, then we are done. So we may assume that $\mu_{x_1} \neq \mu_{x_2}$ in $X_*(T)_\Gamma$. We use induction on $d_\Gamma(\mu_{x_1}, \mu_{x_2})$. Write $\mu_{x_2} - \mu_{x_1} = \sum_{i=1}^r \gamma_i^\vee$ as in Lemma 4.3.7.

Recall that Γ acts transitively on the set of connected components of the Dynkin diagram of G as G^{ad} is simple. As $\mu_{x_2} = \mu_{x_1}$ in $\pi_1(M)_\Gamma$, there exist roots $(\beta_j)_j$ in M such that $\sum_i \gamma_i^\vee = \sum_j \beta_j^\vee \neq 0$ as elements of $X_*(T)_\Gamma$ and $|\sum_i \gamma_i^\vee|_\Gamma = \sum_j |\beta_j^\vee|$. Then $d_\Gamma(\mu_{x_1}, \mu_{x_2}) = \sum_j |\beta_j^\vee|$. By Lemma 4.3.5 (applied to $\sum_i \gamma_i^\vee - \sum_j \beta_j^\vee$), and after replacing β_j by some representative in $\Gamma\beta_j$, there exist $(\tau_i)_{1 \leq i \leq r} \in \Gamma^r$ such that $\sum_i \tau_i \gamma_i^\vee = \sum_j \beta_j^\vee$. As $|\sum_j \beta_j^\vee|_\Gamma = \sum_j |\beta_j^\vee|$, we have $|\sum_j \beta_j^\vee| = |\sum_j \beta_j^\vee|_\Gamma$. By applying Lemma 4.3.6 to $\sum_j \beta_j^\vee$ in the root datum of M , there is a coroot α^\vee in M such that $|\sum_j \beta_j^\vee| = |\alpha^\vee| + |\sum_j \beta_j^\vee - \alpha^\vee|$ and $\langle \sum_{\tau \in \Gamma} \tau \alpha, \sum_j \beta_j^\vee \rangle > 0$. Thus

$$\begin{aligned} \left\langle \sum_{\tau \in \Gamma} \tau \alpha, \mu_{x_2} - \mu_{x_1} \right\rangle &= \left\langle \sum_{\tau \in \Gamma} \tau \alpha, \sum_i \gamma_i^\vee \right\rangle \\ &= \left\langle \sum_{\tau \in \Gamma} \tau \alpha, \sum_i \tau_i \gamma_i^\vee \right\rangle \\ &= \left\langle \sum_{\tau \in \Gamma} \tau \alpha, \sum_j \beta_j^\vee \right\rangle \\ &> 0. \end{aligned}$$

Thus there is a $\tau_0 \in \Gamma$ with $\langle \tau_0 \alpha, \mu_{x_2} - \mu_{x_1} \rangle > 0$. Hence, $\langle \tau_0 \alpha, \mu_{x_2} \rangle = 1$ or $\langle \tau_0 \alpha, \mu_{x_1} \rangle = -1$. In the first case,

$$\begin{aligned} d_\Gamma(s_{\tau_0 \alpha} \mu_{x_2}, \mu_{x_1}) &= \left| \sum_i \beta_i^\vee - \tau_0(\alpha) \right|_\Gamma < \left| \sum_i \beta_i^\vee \right| \\ &= d_\Gamma(\mu_{x_2}, \mu_{x_1}), \end{aligned}$$

and the statement is shown by induction. In the second case we proceed analogously using

$$d_\Gamma(\mu_{x_2}, s_{\tau_0 \alpha} \mu_{x_1}) < d_\Gamma(\mu_{x_2}, \mu_{x_1}). \quad \square$$

Proof of Proposition 4.3.2. By assumption Γ permutes the connected components of the Dynkin diagram of G transitively and each element $\tau \neq 1$ acts non-trivially.

Let $\mu', \mu'' \in \bar{I}_{\mu, b}^{T, G}$. We prove the proposition by induction on $d(\mu', \mu'')$. We assume that $\mu' \neq \mu''$. We write $\mu' - \mu'' = \sum_i \gamma_i^\vee$ as in Lemma 4.3.7. Gathering the positive (respectively the negative) γ_i , we obtain

$$\mu' - \mu'' = \sum_{i \in I} \gamma_i^{+\vee} - \sum_{j \in J} \gamma_j^{-\vee}$$

where now all γ_i^+, γ_j^- are positive. By Lemma 4.3.4, there are a $\gamma_{i_0}^+$, a $\gamma_{j_0}^-$ and a $\tau \in \Gamma$ such that $\langle \tau\gamma_{i_0}^+, \gamma_{j_0}^{-\vee} \rangle > 0$. By orthogonality of the γ_i , we have $\tau \neq 1$. Let $\gamma^+ = \gamma_{i_0}^+$ and $\gamma^- = \gamma_{j_0}^-$. Note that $s_{\gamma^-} s_{\gamma^+} \mu' = \mu' - \gamma^{+\vee} + \gamma^{-\vee}$. If $\gamma^+ = \tau\gamma^-$ then $d(s_{\gamma^-} s_{\tau\gamma^-} \mu', \mu'') < d(\mu', \mu'')$ and the induction hypothesis applies. So we may assume that $\gamma^+ \neq \tau\gamma^-$. Then $\langle \tau\gamma^+, \gamma^{-\vee} \rangle = 1$ or $\langle \gamma^-, \tau\gamma^{+\vee} \rangle = 1$, and by symmetry we may assume that the second equation holds. Let

$$\alpha^\vee = s_{\gamma^-}(\tau\gamma^{+\vee}) = \tau\gamma^{+\vee} - \gamma^{-\vee}.$$

We need to distinguish several cases.

Case 1: $\langle \tau\gamma^+, \gamma^{-\vee} \rangle > 1$. In this case, the root system has roots of different lengths; in particular; the connected components do not have non-trivial automorphisms, and $\langle \tau\gamma^+, \gamma^+ \rangle = 0$, as $\tau \neq 1$.

We have $\alpha = s_{\gamma^-}(\tau\gamma^+) = \tau\gamma^+ - \langle \tau\gamma^+, \gamma^{-\vee} \rangle \gamma^-$. Thus

$$-1 \leq \langle \tau\gamma^+, \mu' \rangle = \langle \alpha + \langle \tau\gamma^+, \gamma^{-\vee} \rangle \gamma^-, \mu' \rangle \leq -1.$$

Here the first inequality follows from the fact that μ' minuscule. For the second we use μ' minuscule, $\langle \tau\gamma^+, \gamma^{-\vee} \rangle \geq 2$ and $\langle \gamma^-, \mu' \rangle = -1$ (the last equation following from our choice of the γ_i). Let $\tilde{\mu} = s_{\tau\gamma^+} s_{\gamma^-} \mu'$. Then $\tilde{\mu} \in \bar{I}_{\mu, b}^{T, G}$. Since $\langle \tau\gamma^+, \gamma^- \rangle > 0$, we have $|\tau\gamma^{+\vee} - \gamma^{-\vee}| < |\gamma^{+\vee}| + |\gamma^{-\vee}|$ by Lemma 4.2.4, which implies that $d(\tilde{\mu}, \mu'') < d(\mu', \mu'')$, so the induction hypothesis applies.

Case 2: $\langle \tau\gamma^+, \gamma^{-\vee} \rangle = 1$. By Lemma 4.2.1 we have $\langle \gamma^+, \tau\gamma^{+\vee} \rangle, \langle \gamma^-, \tau^{-1}\gamma^{-\vee} \rangle \in \{0, -1\}$. Since $\langle \tau\gamma^+, \gamma^- \rangle > 0$, $\tau\gamma^+$ and γ^- are in the same connected component of the Dynkin diagram. Using Lemma 4.2.1 again, we see that if one of the products above is equal to -1 , then the Dynkin diagram is of type A_n with n even. The explicit description of Example 4.2.2 then shows that $\langle \gamma^+, \gamma^{-\vee} \rangle = 0$ implies that at most one of the two products can in fact be equal to -1 . Hence, we have $\langle \gamma^+, \tau\gamma^{+\vee} \rangle = 0$ or $\langle \gamma^-, \tau^{-1}\gamma^{-\vee} \rangle = 0$.

Case 2.1: Assume that one of the following conditions is satisfied:

- $\langle \gamma^-, \tau^{-1}\gamma^{-\vee} \rangle = 0$ and $\langle \tau^{-1}\gamma^-, \mu' \rangle > 0$;
- $\langle \gamma^+, \tau\gamma^{+\vee} \rangle = 0$ and $\langle \tau\gamma^+, \mu' \rangle < 0$;
- $\langle \gamma^-, \tau^{-1}\gamma^{-\vee} \rangle = 0$ and $\langle \tau^{-1}\gamma^-, \mu'' \rangle < 0$;
- $\langle \gamma^+, \tau\gamma^{+\vee} \rangle = 0$ and $\langle \tau\gamma^+, \mu'' \rangle > 0$.

If the first assumption holds let $\tilde{\mu} = s_{\tau^{-1}\gamma^-} s_{\gamma^-} \mu'$. Then $\langle \gamma^+, \tau^{-1}\gamma^- \rangle > 0$ implies $d(\tilde{\mu}, \mu'') < d(\mu', \mu'')$, as above, and the induction hypothesis applies. The arguments for the other three assumptions are analogous.

Case 2.2: Assume that none of the four possible conditions of case 2.1 are satisfied, and that there is a $\tilde{\tau} \in \Gamma$ such that $\tilde{\tau}\alpha$ is not in the same connected component as γ^+ or γ^- and that one of the following conditions hold:

- $\langle \tilde{\tau}\alpha, \mu' \rangle = -1$;
- $\langle \tilde{\tau}\alpha, \mu'' \rangle = 1$.

Note that by the last assertion of Lemma 4.2.4 and the assumption of case 2, $|\gamma^{-\vee}| \neq |\gamma^{+\vee}|$. We show that statement for the first of the two alternative assumptions, the other one

being analogous, exchanging μ' and μ'' (and suitable signs). Furthermore, we assume that $\langle \gamma^-, \tau^{-1}\gamma^{-\vee} \rangle = 0$, which implies $\langle \tau^{-1}\gamma^-, \mu' \rangle \leq 0$, as we are excluding case 2.1. The alternative case for $\langle \gamma^+, \tau\gamma^{+\vee} \rangle = 0$ can be shown by the same argument exchanging γ^-, γ^+ (and suitable signs).

As $\langle \tau^{-1}\gamma^-, \mu' \rangle \leq 0$ we obtain

$$\langle \tau^{-1}\alpha, \mu' \rangle = \langle \gamma^+ - \langle \tau\gamma^+, \gamma^{-\vee} \rangle \tau^{-1}\gamma^-, \mu' \rangle \geq 1 - 1 \cdot 0 = 1. \tag{4.3.8}$$

Let $\tilde{\mu} = s_{\gamma^-} s_{\gamma^+} s_{\tilde{\tau}\alpha} \mu'$ and $\tilde{\mu}'' = s_{\tilde{\tau}\alpha} s_{\tau^{-1}\alpha} \mu''$. As $\alpha^\vee = \tau\gamma^{+\vee} - \gamma^{-\vee}$, these two coweights (in particular, the first) are still in $\bar{I}_{\mu,b}^{T,G}$. Notice that

$$\begin{aligned} \tilde{\mu} - \tilde{\mu}'' &= (\mu' + \gamma^{-\vee} - \gamma^{+\vee} + \tilde{\tau}\alpha^\vee) - (\mu'' + \tilde{\tau}\alpha^\vee - \tau^{-1}\alpha^\vee) \\ &= \gamma^{-\vee} - \tau^{-1}\gamma^{-\vee}. \end{aligned}$$

Here we have used that $\tau^{-1}\alpha$ is in the same component as γ^+ , so that $\langle \tilde{\tau}\alpha, \tau^{-1}\alpha \rangle = 0$. Therefore, in order to use induction, it is enough to show that $d(\tilde{\mu}, \mu'') < d(\mu', \mu'')$. We have

$$|\tilde{\tau}\alpha^\vee| = |\alpha^\vee| = |\tau\gamma^{+\vee} - \gamma^{-\vee}| = ||\tau\gamma^{+\vee}| - |\gamma^{-\vee}|| = ||\gamma^{+\vee}| - |\gamma^{-\vee}||.$$

Here the second equality follows from Lemma 4.2.4 as $\tau\gamma^+$ and γ^- are both positive roots. Thus

$$d(\tilde{\mu}, \mu'') \leq d(\mu', \mu'') - |\gamma^{-\vee}| - |\gamma^{+\vee}| + |\alpha^\vee| < d(\mu', \mu'').$$

This implies the assertion for this case.

Case 2.3: $\langle \gamma^+, \tau\gamma^{+\vee} \rangle = -1$ or $\langle \gamma^-, \tau^{-1}\gamma^{-\vee} \rangle = -1$, but none of the cases considered in 2.1 and 2.2 applies. We will show that this case is impossible. We have seen above that then the Dynkin diagram is a union of Dynkin diagrams of type A_n for even n . We assume that $\langle \gamma^+, \tau\gamma^{+\vee} \rangle = -1$, the other case being similar. Then $\langle \gamma^-, \tau^{-1}\gamma^{-\vee} \rangle = 0$. The roots $\gamma^+, \tau\gamma^+, \gamma^-, \tau\gamma^-$ all lie within one connected component of the Dynkin diagram.

The inequality (4.3.8) still holds, and

$$\langle \alpha, \mu' \rangle = \langle \tau\gamma^+ - \gamma^-, \mu' \rangle \geq 0.$$

Furthermore, excluding case 2.2 implies that for all $\tilde{\tau} \neq \tau, 1$ in Γ , we have $\langle \tilde{\tau}\alpha, \mu' \rangle \geq 0$. A similar argument applies to μ'' , and yields $\langle \tilde{\tau}\alpha, \mu'' \rangle \leq 0$. Recall that $\mu' = \mu''$ in $X_*(T)_\Gamma$. Altogether we obtain

$$0 < \left\langle \sum_{\tilde{\tau} \in \Gamma} \tilde{\tau}\alpha, \mu' \right\rangle = \left\langle \sum_{\tilde{\tau} \in \Gamma} \tilde{\tau}\alpha, \mu'' \right\rangle < 0,$$

a contradiction.

Case 2.4: $\langle \gamma^+, \tau\gamma^{+\vee} \rangle = 0 = \langle \gamma^-, \tau^{-1}\gamma^{-\vee} \rangle$, but none of the cases in 2.1 and 2.2 apply. As before, we have that $\langle \tau\gamma^+, \mu' \rangle \geq 0$ which implies $\langle \alpha, \mu' \rangle = 1$, and that $\langle \tau^{-1}\gamma^-, \mu' \rangle \leq 0$ which implies $\langle \tau^{-1}\alpha, \mu' \rangle = 1$. Similarly, we obtain $\langle \alpha, \mu'' \rangle = -1$ and $\langle \tau^{-1}\alpha, \mu'' \rangle = -1$. Notice again that

$$\left\langle \sum_{\tilde{\tau} \in \Gamma} \tilde{\tau}\alpha, \mu' \right\rangle = \left\langle \sum_{\tilde{\tau} \in \Gamma} \tilde{\tau}\alpha, \mu'' \right\rangle. \tag{4.3.9}$$

This equality implies that $\Gamma\alpha$ has at least two elements in each connected component of the Dynkin diagram. Indeed, otherwise we would have $\langle \tilde{\tau}\alpha, \mu' \rangle \geq 0 \geq \langle \tilde{\tau}\alpha, \mu'' \rangle$ for $\tilde{\tau} \neq 1$, as we are

excluding case 2.2, and $\langle \alpha, \mu' \rangle \geq 1 > -1 \geq \langle \alpha, \mu'' \rangle$, as we are excluding case 2.1. In particular, all roots have equal length. Therefore $\alpha \neq \tau^{-1}\alpha$ since $|\tau^{-1}\gamma^-| \neq |\tau\gamma^+|$, as we saw above, and

$$\langle \tau^{-1}\alpha, \alpha^\vee \rangle = \langle \gamma^+ - \tau^{-1}\gamma^-, \tau\gamma^{+\vee} - \gamma^{-\vee} \rangle = -\langle \tau^{-1}\gamma^-, \tau\gamma^{+\vee} \rangle \neq 2.$$

As we have excluded case 2.2, using (4.3.9) again, we obtain $\tau_1, \tau_2 \in \Gamma$ such that $\tau_1\alpha \neq \tau_2\alpha$, and $\tau_1\alpha, \tau_2\alpha$ are each in the connected component of α or $\tau^{-1}\alpha$ with one of the following two conditions satisfied:

- $\langle \tau_1\alpha, \mu' \rangle = -1$ and $\langle \tau_2\alpha, \mu' \rangle = -1$;
- $\langle \tau_1\alpha, \mu'' \rangle = 1$ and $\langle \tau_2\alpha, \mu'' \rangle = 1$.

Assume that the first of the above two alternative conditions holds, the other one being analogous. From our calculation of the products with μ', μ'' above, we see that $\tau_i\alpha \neq \alpha, \tau^{-1}\alpha$ for $i = 1, 2$. Moreover, α and $\tau^{-1}\alpha$ cannot be in the same connected component of the Dynkin diagram, otherwise the four roots $\alpha, \tau^{-1}\alpha, \alpha_1 := \tau_1\alpha$ and $\alpha_2 := \tau_2\alpha$ are in the same connected component, which is impossible according to Lemma 4.2.1.

Case 2.4.1: $\Gamma\alpha$ has two elements in each connected component. We assume that α_1 is in the same connected component as α (and thus as γ^-), the other case being analogous. Then $\alpha_2 = \tau^{-1}\alpha_1$. We want to show that $\langle \alpha_1, \gamma^{-\vee} \rangle \geq 0$. As $\langle \alpha_1, \mu' \rangle = -1$ and $\langle \gamma^-, \mu' \rangle = -1$, we have $\alpha_1 \neq -\gamma^-$. Hence, if $\langle \alpha_1, \gamma^{-\vee} \rangle < 0$ then $\gamma^- + \alpha_1$ is a root by Lemma 4.2.4. Since $\langle \gamma^- + \alpha_1, \mu' \rangle = -2$, this contradicts the condition that μ' is minuscule.

In the same way one shows that $\langle \alpha_2, \gamma^{+\vee} \rangle \leq 0$. On the other hand,

$$0 \geq \langle \alpha_2, \gamma^{+\vee} \rangle = \langle \alpha_1, \tau\gamma^{+\vee} \rangle = \langle \alpha_1, \alpha^\vee \rangle + \langle \alpha_1, \gamma^{-\vee} \rangle$$

and by Lemma 4.2.1 the first of the summands on the right-hand side is 0 or -1 . Thus $\langle \alpha_2, \gamma^{+\vee} \rangle = 0$ or $\langle \alpha_1, \gamma^{-\vee} \rangle = 0$. We consider the second case, the other being analogous. Let $\tilde{\mu} = s_{\alpha_1}s_{\gamma^-}s_{\gamma^+}\mu'$ and $\tilde{\mu} = s_{\alpha_1}s_{\tau^{-1}\alpha}\mu'$. Then $\tilde{\mu}, \tilde{\mu} \in \bar{I}_{\mu,b}^{T,G}$. Moreover, since we are excluding case 2.1, $\langle \tau^{-1}\alpha, \mu' \rangle = 1$, so

$$\tilde{\mu} - \tilde{\mu} = -\gamma^{+\vee} + \gamma^{-\vee} + \tau^{-1}\alpha^\vee = \gamma^{-\vee} - \tau^{-1}\gamma^{-\vee},$$

and $d(\tilde{\mu}, \mu'') < d(\mu', \mu'')$, as in case 2.2. Thus the assertion follows by induction.

Case 2.4.2: $\Gamma\alpha$ has three elements in each connected component. In this case, the Dynkin diagram is of type D_4 by Lemma 4.2.1. Suppose that $\alpha_1 := \tau_1\alpha$ is in the same connected component as α . Then $\langle \tau_1 \rangle \subset \Gamma$ is the stabilizer of each connected component of the Dynkin diagram. Let $\{\beta_i\}_{0 \leq i \leq 3}$ be the basis of the connected component of the root system containing α such that $\tau_1\beta_0 = \beta_0$ and τ_1 acts transitively on $\{\beta_i\}_{1 \leq i \leq 3}$. We may suppose that α is positive. Then α is of the form β_i or $\beta_i + \beta_0$ or $\beta_i + \beta_0 + \beta_j$ with $1 \leq i \neq j \leq 3$, and therefore $\alpha_1 - \alpha = \beta_{i_0} - \beta_{j_0}$ for some $1 \leq i \neq j \leq 3$. As $\langle \alpha_1 - \alpha, \mu' \rangle = -2$, we have $\langle \beta_{i_0}, \mu' \rangle = -1$, $\langle \beta_{j_0}, \mu' \rangle = 1$, and for $0 \leq k \leq 3$, $k \neq i_0, j_0$, $\langle \beta_k, \mu' \rangle = 0$ since μ' is minuscule. Thus $\langle \gamma^-, \mu' \rangle = -1$ implies that $\beta_{i_0}^\vee \preceq \gamma^{-\vee}$.

On the other hand, notice that

$$\langle \tau_1^2(\alpha) - \tau_1(\alpha), \mu' \rangle = \langle \tau_1(\beta_{i_0}) - \tau_1(\beta_{j_0}), \mu' \rangle \in \{\pm 1\}.$$

This implies that $\langle \tau_1^2(\alpha), \mu' \rangle = 0$ and $\tau_2(\alpha)$ is not in the same connected component as α , so it is in the same connected component as $\tau^{-1}\alpha$. By applying the same method as above to the connected component of the Dynkin diagram of $\tau^{-1}\alpha$, we can find $1 \leq j'_0 \leq 3$ such that $\langle \tau^{-1}\beta_{j'_0}, \mu' \rangle = 1$ and $\tau^{-1}\beta_{j'_0}^\vee \preceq \gamma^{+\vee}$. Let $\tilde{\mu} := s_{\tau^{-1}\beta_{j'_0}}s_{\beta_{i_0}}\mu'$; then $d(\tilde{\mu}, \mu'') < d(\mu', \mu'')$ and the induction hypothesis applies. □

4.4 Immediate distance case

In Proposition 4.1.6, for any two elements $x, x' \in \bar{I}_{\mu,b}^{M,G}$, we have found a series of elements $x_1, \dots, x_r \in \bar{I}_{\mu,b}^{M,G}$ with $x = x_1, x' = x_r$ such that the difference of each two successive elements in the series is of the form $\alpha^\vee - \sigma^m(\alpha^\vee)$ in $\pi_1(M)$, where α is a root in N . In this subsection we want to add some elements to that series such the each pair of successive elements in the enlarged series has ‘minimal distance’ in a sense that we will define below. Such pairs will be called in immediate distance (cf. Definition 4.4.8).

We now return to the assumptions of 4.1.5, so that G^{ad} is simple, $M \subset G$ is a standard Levi, and $b \in M(L)$ is superbasic. For any $\Omega \in \Phi_{N,\Gamma}$, we recall that the subgroup G_Ω of G is defined in 4.2.8. We first provide several useful lemmas that will be used in the sequel.

LEMMA 4.4.1. *Let $\alpha \in \Phi_N$ be a (positive) root, and let $\Omega = \Gamma\alpha$. There exists an adapted root α' in G_Ω such that $\alpha^\vee = \alpha'^\vee$ in $\pi_1(M)$.*

Proof. Let α_1 be the M -antidominant representative in $W_M\alpha$. If α_1 is adapted, then let $\alpha' = \alpha_1$ and we are done. If α_1 is not adapted, then there is a root β in M with $\langle \beta, \alpha_1^\vee \rangle < -1$. This means that the irreducible sub-root system (corresponding to a connected component of the Dynkin diagram) of G_Ω which contains α_1 and β has roots of different length, and β is a long root while α_1 is a short one. Let α' be the M -antidominant representative in $W_M(\alpha_1^\vee + \beta^\vee)^\vee$. By definition, α' is a long root and thus it is adapted. □

DEFINITION 4.4.2. Let $\Omega \in \Phi_{N,\Gamma}$. Then Ω is of type I (respectively II, III) if any irreducible sub-root system (corresponding to a connected component of the Dynkin diagram) of G_Ω which contains some element of Ω has 1 (respectively 2, 3) root(s) in Ω .

Remark 4.4.3. Suppose that Ω is adapted and that Γ acts transitively on the connected components of the Dynkin diagram of G_Ω . If Ω is of type II or III, then all roots in G_Ω have the same length and Proposition 4.2.11 applies. In particular, all the roots in Ω are simple roots in G_Ω for the Borel subgroup $B \cap G_\Omega$. Moreover, the fact that the stabilizer in Γ of each connected component of the Dynkin diagram of M acts trivially on that component (cf. Lemma 3.1.1) implies the following additional conditions on Ω . If Ω is of type III, then the Dynkin diagram of G_Ω is of type D_4 . If Ω is of type II, then only the following cases may occur. For type A_n with n even, Ω consists of the two middle simple roots in each connected component of the Dynkin diagram. For type A_n with n is odd, it consists of the two neighbors of the middle simple root in each connected component. For type D_n the intersection of Ω with any connected component consists of two of the roots with only one neighbor, which are exchanged by some element of Γ . For type E_6 , Ω consists of the two simple roots having two neighbors in each connected component.

4.4.4. Recall that for $x \in \pi_1(M)$, μ_x denotes the unique M -dominant, M -minuscule cocharacter with image x . As in 4.1.8, we write $M_x \subset M$ for the centralizer of μ_x , and we set $w_x = w_{0,x}w_{0,M}$ where $w_{0,x}$ is the longest Weyl group element in M_x and where $w_{0,M}$ is the longest Weyl group element in M .

LEMMA 4.4.5. *Suppose that $\Omega \in \Phi_{N,\Gamma}$ is adapted, and $x \in \pi_1(M)$. Then:*

- (i) $w_x^{-1}(\mu_x) = w_{0,M}(\mu_x)$;
- (ii) $(\mu_x + \gamma^\vee)_{M\text{-dom}} = \mu_{x+\gamma^\vee}$ for $\gamma \in \Omega$;
- (iii) $\mu_x - w_x\gamma^\vee = \mu_{x-\gamma^\vee}$ for $\gamma \in \Omega$.

Proof. (i) This follows as μ_x is by definition invariant under conjugation by $w_{0,x}$ and as $w_{0,M} = w_{0,M}^{-1}$.

(ii) It suffices to show that $\mu_x + \gamma^\vee$ is M -minuscule. For positive roots β in M , we have $\langle \beta, \mu_x \rangle \in \{0, 1\}$. As γ is adapted, we have $\langle \beta, \gamma^\vee \rangle \in \{0, -1\}$. Therefore $\langle \beta, \mu_x + \gamma^\vee \rangle \in \{-1, 0, 1\}$, thus $\mu_x + \gamma^\vee$ is M -minuscule.

(iii) It is enough to show that the element on the left-hand side is M -dominant and M -minuscule. To compute the pairing with all simple roots β of M , recall that, by definition,

$$\langle \beta, \mu_x \rangle = \begin{cases} 1 & \text{if } \beta \text{ is a simple root in } N_x, \\ 0 & \text{otherwise.} \end{cases}$$

On the other hand, $\langle \beta, w_x \gamma^\vee \rangle \in \{-1, 0, 1\}$ as γ is adapted. Notice that $w_x^{-1} \beta = w_{0,M} w_{0,x} \beta$. If β is a simple root in M_x , then $w_x^{-1} \beta$ is a simple root of $w_x^{-1} M_x w_x$ with respect to the Borel $B \cap w_x^{-1} M_x w_x$. In particular, it is a simple root of M . If β is a simple root in N_x , then $-w_x^{-1} \beta$ is a highest root in M . Therefore, $\langle \beta, w_x \gamma^\vee \rangle = \langle w_x^{-1} \beta, \gamma^\vee \rangle = 1$ if and only if β is a simple root in N_x and γ^\vee is not central on the connected component of the Dynkin diagram of M containing β . Moreover, $\langle \beta, w_x \gamma^\vee \rangle = -1$ occurs for at most one β in each connected component of the Dynkin diagram of M . This follows from the fact that $\langle \beta, \gamma^\vee \rangle = -1$ for at most one simple root in each connected component of the Dynkin diagram of M . \square

LEMMA 4.4.6. *Suppose that $x, x' \in \bar{I}_{\mu,b}^{M,G}$ such that $x' - x = \alpha^\vee - \tau(\alpha)^\vee$ with α an adapted root in N , such that $\alpha \neq \tau(\alpha)$. Then we have $(\mu_{x+\alpha^\vee})_{G\text{-dom}} = \mu$ and $(\mu_{x-\tau(\alpha)^\vee})_{G\text{-dom}} = \mu$. Moreover, $(\mu_x + \alpha^\vee)_{G\text{-dom}} = \mu$, $(\mu_x - w_x \tau(\alpha)^\vee)_{G\text{-dom}} = \mu$ and $(\mu_x + \alpha^\vee - w_x \tau(\alpha)^\vee)_{G\text{-dom}} = \mu$.*

Proof. Write $\mu_{x'} - \mu_x = \alpha^\vee - \tau(\alpha)^\vee + \sum_{\beta} n_{\beta} \beta^\vee$ where β runs over simple coroots of M , and $n_{\beta} \in \mathbb{Z}$. Let Δ^+ (respectively Δ^-) denote the set of β with $n_{\beta} > 0$ (respectively $n_{\beta} < 0$). Note that $\langle \alpha, \tau(\alpha)^\vee \rangle \leq 0$ by Lemma 4.2.1, and $\langle \beta, \alpha^\vee \rangle, \langle \beta, \tau(\alpha)^\vee \rangle \leq 0$ for any β since α is adapted. Hence, if $\gamma_1^\vee, \gamma_2^\vee$ are coroots of the form $\gamma_1^\vee = \alpha^\vee + \sum_{\beta \in \Delta^+} m_{\beta} \beta^\vee$, $\gamma_2^\vee = \tau(\alpha)^\vee + \sum_{\beta \in \Delta^-} m_{\beta} \beta^\vee$ with m_{β} positive integers, then $\langle \gamma_1, \gamma_2^\vee \rangle \leq 0$. It follows by the proof of Lemma 4.3.7, that we can write:

- $\mu_{x'} - \mu_x = \sum_{i \in I} \gamma_i^\vee$ as in Lemma 4.3.7;
- there exist $i_1, i_2 \in I$ with $\gamma_{i_1}^\vee = \alpha^\vee$ in $\pi_1(M)$, $\gamma_{i_2}^\vee = -\tau(\alpha)^\vee$ in $\pi_1(M)$;
- for all $i \in I \setminus \{i_1, i_2\}$, $\gamma_i^\vee = 0$ in $\pi_1(M)$.

Thus $\mu_x, s_{\gamma_{i_1}}(\mu_x) = \mu_x + \gamma_{i_1}^\vee$ and $s_{\gamma_{i_2}}(\mu_x) = \mu_x - \tau(\alpha)^\vee$ are in the same Weyl group orbit. In particular, $\mu_x + \gamma_{i_1}^\vee$ and $\mu_x - \tau(\alpha)^\vee$ are M -minuscule. So $(\mu_x + \gamma_{i_1}^\vee)_{M\text{-dom}} = \mu_{x+\alpha^\vee}$ and $(\mu_x - \tau(\alpha)^\vee)_{M\text{-dom}} = \mu_{x-\tau(\alpha)^\vee}$. It follows that $(\mu_{x+\alpha^\vee})_{G\text{-dom}} = \mu$ and $(\mu_{x-\tau(\alpha)^\vee})_{G\text{-dom}} = \mu$. The equalities $(\mu_x + \alpha^\vee)_{G\text{-dom}} = \mu = (\mu_x - w_x \tau(\alpha)^\vee)_{G\text{-dom}}$ follow directly from Lemma 4.4.5, which also implies the last equality as

$$(\mu_x + \alpha^\vee - w_x \tau(\alpha)^\vee)_{M\text{-dom}} = (\mu_{x-\tau(\alpha)^\vee} + \alpha^\vee)_{M\text{-dom}} = (\mu_{x+\alpha^\vee - \tau(\alpha)^\vee})_{M\text{-dom}}. \quad \square$$

LEMMA 4.4.7. *Suppose that α is an adapted root in N . Then for all $w \in W_M$ we have $\langle w\alpha, \mu_x \rangle \leq \langle w_x \alpha, \mu_x \rangle$ and the root $w_x \alpha$ is the unique minimal element in the set*

$$\{w\alpha \mid w \in W_M, \langle w\alpha, \mu_x \rangle = \langle w_x \alpha, \mu_x \rangle\}$$

for the order $\preceq = \preceq_M$.

Proof. Since $w_{0,M}\mu_x$ is M -antidominant, for $w \in W_M$, $w_{0,M}\mu_x \preceq w\mu$, and hence $\langle \alpha, w_{0,M}\mu_x \rangle \geq \langle \alpha, w\mu_x \rangle$, as α is adapted. By Lemma 4.4.5(i), this implies $\langle \alpha, w_x^{-1}\mu_x \rangle \geq \langle \alpha, w\mu_x \rangle$. Hence,

$$\begin{aligned} I_{w_x\alpha} &:= \{w\alpha \mid w \in W_M, \langle w\alpha, \mu_x \rangle \geq \langle w_x\alpha, \mu_x \rangle\} \\ &= \{w\alpha \mid w \in W_M, \langle w\alpha, \mu_x \rangle = \langle w_x\alpha, \mu_x \rangle\} \\ &= \{w\alpha \mid w \in W_M, \langle w\alpha, \mu_x \rangle = \sup_{w' \in W_M} \langle w'\alpha, \mu_x \rangle\}. \end{aligned}$$

We first prove that $w_x\alpha$ is a minimal element in the set $I_{w_x\alpha}$ by reduction to absurdity. Suppose that $w_x\alpha$ is not a minimal element. Then there exists $w' \in W_M$ such that $w'\alpha \preceq w_x\alpha$ with $w'\alpha \neq w_x\alpha$ and $\langle w_x\alpha, \mu_x \rangle = \langle w'\alpha, \mu_x \rangle$. As $w'\alpha$ and $w_x\alpha$ are in the same Weyl group orbit, they have the same length, so $\langle w_x\alpha - w'\alpha, w_x\alpha^\vee \rangle = 2 - \langle w'\alpha, w_x\alpha^\vee \rangle \geq 1$. Hence, there exists a positive simple root β in M such that $\langle w_x\alpha, \beta^\vee \rangle > 0$ and $w'\alpha + \beta \preceq w_x\alpha$. Moreover,

$$\langle w'\alpha, \mu_x \rangle = \langle w_x\alpha, \mu_x \rangle \geq \langle w'\alpha + \beta, \mu_x \rangle \geq \langle w'\alpha, \mu_x \rangle$$

implies that $\langle \beta, \mu_x \rangle = 0$. Then β is a root in M_x . As the groups M_x and M are both of type A, the root $-w_{0,x}(\beta)$ is simple in M_x and $w_x^{-1}(\beta) = w_{0,M}w_{0,x}(\beta)$ is a simple root in M . Therefore, $\langle \alpha, w_x^{-1}(\beta^\vee) \rangle \leq 0$ as α is M -antidominant. This is a contradiction to $\langle w_x\alpha, \beta^\vee \rangle > 0$.

We now show that $w_x\alpha$ is the unique minimal element. By Lemma 3.1.1, the Dynkin diagram of M is of type A. As we can work separately with each connected component of the Dynkin diagram of M , we may suppose without loss of generality that the Dynkin diagram of M is connected with simple roots β_1, \dots, β_m with $\langle \beta_i, \beta_{i+1}^\vee \rangle = -1$ for $1 \leq i \leq m-1$. If for all $1 \leq i \leq m$, $\langle \beta_i, \alpha^\vee \rangle = 0$, then the set $I_{w_x\alpha}$ contains a single element α and we are done. Otherwise $\langle \sum_{i=1}^m \beta_i, \alpha^\vee \rangle = -1$, and hence there exists a unique $1 \leq i_0 \leq m$ with $\langle \beta_{i_0}, \alpha^\vee \rangle = -1$. If $\langle \beta_i, \mu_x \rangle = 0$ for all $1 \leq i \leq m$, then $I_{w_x\alpha} = W_M\alpha$ and $w_x\alpha = \alpha$ is the unique minimal element as it is M -antidominant. It remains the case when there exists $1 \leq j_0 \leq m$ such that $\langle \beta_{j_0}, \mu_x \rangle = 1$. We may assume that $j_0 \leq i_0$, the other case being analogous. Then

$$I_{w_x\alpha} = \{s_{\beta_k} s_{\beta_{k+1}} \cdots s_{\beta_{i_0}} \alpha \mid 1 \leq k \leq j_0\}.$$

This is a totally ordered set and therefore has a unique minimal element. □

DEFINITION 4.4.8. Let $x_1, x_2 \in \bar{I}_{\mu,b}^{M,G}$ such that $x_2 - x_1 = \alpha^\vee - \sigma^m(\alpha^\vee)$ in $\pi_1(M)$ with α a positive root in N and $m \in \mathbb{N}$. By Lemma 4.4.1, we may assume that α is adapted. Let $\Omega := \Gamma\alpha$ and $\alpha^i := \sigma^i(\alpha)$ for $i \in \mathbb{N}$. The distance from x_1 to x_2 is called *immediate* if the following two conditions are satisfied:

- (i) if Ω is of type I (respectively II, III), we require that $0 < m < |\Omega|$ (respectively $0 < m \leq |\Omega|/2$, $0 < m < 2|\Omega|/3$).
- (ii) $x_1 + \alpha^{i\vee} - \alpha^{m\vee} \notin \bar{I}_{\mu,b}^{M,G}$ and $x_1 + \alpha^\vee - \alpha^{i\vee} \notin \bar{I}_{\mu,b}^{M,G}$ for all $0 < i < m$.

We write $x_1 \rightarrow x_2$ when the distance from x_1 to x_2 is immediate.

Remark 4.4.9. Using the same notation as in the above definition, we assume that Ω is of type III and $d < m < 2d$ with $d = |\Omega|/3$. Suppose that Γ acts transitively on the connected components of the Dynkin diagram of G_Ω . By Proposition 4.2.11, let $\{(\beta^i)_{0 \leq i \leq d-1}, (\alpha^i)_{0 \leq i \leq 3d-1}\}$ be the basis of G_Ω with β^i the common neighbor of α^i, α^{i+d} and α^{i+2d} . As $x_2 = x_1 + \alpha^\vee - \alpha^{m\vee} \in \bar{I}_{\mu,b}^{M,G}$, $\langle \alpha, \mu_{x_1} \rangle = -1$ and $\langle \alpha^m + \beta^{m-d}, \mu_{x_1} \rangle = 1$. Similarly, $x_1 + \alpha^\vee - \alpha^{i\vee} \notin \bar{I}_{\mu,b}^{M,G}$ and $x_1 + \alpha^{i\vee} - \alpha^{m\vee} \notin \bar{I}_{\mu,b}^{M,G}$ for $i = m-d, d$ imply that $\langle \alpha^d, \mu_{x_1} \rangle = 0$ and $\langle \alpha^{m-d} + \beta^{m-d}, \mu_{x_1} \rangle = 0$. Therefore, the vector $(\langle \beta^0, \mu_{x_1} \rangle, \langle \alpha^0, \mu_{x_1} \rangle, \langle \alpha^d, \mu_{x_1} \rangle)$ is equal either to $(0, -1, 0)$ or to $(1, -1, 0)$, and the vector $(\langle \beta^{m-d}, \mu_{x_1} \rangle, \langle \alpha^{m-d}, \mu_{x_1} \rangle, \langle \alpha^m, \mu_{x_1} \rangle)$ is equal either to $(1, -1, 0)$ or to $(0, 0, 1)$.

PROPOSITION 4.4.10. For $x, x' \in \bar{I}_{\mu,b}^{M,G}$, there exist $n \in \mathbb{N}$ and a series of elements $x_1, \dots, x_{n+1} \in \bar{I}_{\mu,b}^{M,G}$ such that $x_1 = x$, $x_{n+1} = x'$ and for $i = 1, \dots, n$, either $x_i \rightarrow x_{i+1}$ or $x_{i+1} \rightarrow x_i$.

Proof. By Proposition 4.1.6 and Lemma 4.4.1, we may assume that $x' - x = \alpha^\vee - \sigma^m(\alpha^\vee)$ with α an adapted, positive root in N . Then $x - x' = \alpha'^\vee - \sigma^{m'}(\alpha'^\vee)$ with $m' = |\Omega| - m$ and $\alpha' = \sigma^m(\alpha)$. We may assume that $m \leq |\Omega|/2$, as otherwise we can exchange x and x' . Then the first condition of Definition 4.4.8 is already satisfied.

We use induction on m to prove that we can achieve that the second condition of Definition 4.4.8 holds. Suppose that the condition is not satisfied for the pair (x, x') . Then there exists some $1 \leq i < m$, such that $x + \alpha^{i\vee} - \alpha^{m\vee} \in \bar{I}_{\mu,b}^{M,G}$ or $x + \alpha^\vee - \alpha^{i\vee} \in \bar{I}_{\mu,b}^{M,G}$. We may assume that $x + \alpha^{i\vee} - \alpha^{m\vee} \in \bar{I}_{\mu,b}^{M,G}$, the other case being analogous. Then we can apply the induction hypothesis to the couple $(x, x + \alpha^{i\vee} - \alpha^{m\vee})$ and the pair $(x + \alpha^{i\vee} - \alpha^{m\vee}, x')$. \square

4.5 Proof of Proposition 4.1.7

In this subsection we will construct affine lines in the immediate distance case to prove Proposition 4.1.7. For any $x \in \bar{I}_{\mu,b}^{M,G}$, let μ_x, w_x be as above. In the following, two roots in G which are in the same irreducible sub-root system corresponding to a connected component of Dynkin diagram of G will also be said to be in the same connected component of the Dynkin diagram of G . We use the analogous expression for the roots in other groups.

We need one more lemma.

LEMMA 4.5.1. Let $x \in \bar{I}_{\mu,b}^{M,G}$ and let α be a positive root in N . Suppose that

$$(\mu_{x+\alpha^\vee})_{G\text{-dom}} \neq \mu \quad \text{and} \quad (\mu_{x-\alpha^\vee})_{G\text{-dom}} \neq \mu.$$

Then $\langle \alpha, \mu_x \rangle = 0$. Furthermore, μ_x is central on each connected component of the Dynkin diagram of M satisfying that there is a simple root β in that component with $\langle \beta, \alpha^\vee \rangle \neq 0$. In particular, $w_x(\alpha) = \alpha$.

Proof. Suppose that $\langle \alpha, \mu_x \rangle \neq 0$. Then, depending on the sign of $\langle \alpha, \mu_x \rangle$, one of $\mu_x + \alpha^\vee$ and $\mu_x - \alpha^\vee$ is conjugate to μ_x in G , and, in particular, is G -minuscule. Hence, $(\mu_{x+\alpha^\vee})_{G\text{-dom}} = \mu$ or $(\mu_{x-\alpha^\vee})_{G\text{-dom}} = \mu$. This implies the first assertion.

The same argument also shows that our assumption implies $\langle \alpha, w\mu_x \rangle = 0$ for all $w \in W_M$. Fix a connected component of the Dynkin diagram of M and assume that there is a simple root β in that component such that $\langle \beta, \alpha^\vee \rangle \neq 0$. As $\langle \alpha, \mu_x \rangle = \langle \alpha, s_\beta \mu_x \rangle = 0$, we have $\langle \beta, \mu_x \rangle = 0$. Similarly, for every neighbor β' of β in the Dynkin diagram of M we have $\langle \alpha, \mu_x \rangle = \langle \alpha, s_\beta s_{\beta'} \mu_x \rangle = 0$. Thus $\langle \beta', \mu_x \rangle = 0$. By induction, we obtain $\langle \gamma, \mu_x \rangle = 0$ for every simple root γ in that connected component of the Dynkin diagram of M . Hence, μ_x is central in that connected component. The last assertion follows. \square

Remark 4.5.2. Let $x, x' \in \bar{I}_{\mu,b}^{M,G}$ and $x \rightarrow x'$. Suppose that $x' - x = \alpha^\vee - \alpha^{m\vee}$ with α adapted, and m satisfying the conditions in Definition 4.4.8. By Lemma 4.4.6, $\mu_{x+\alpha^\vee}$ and $\mu_{x-\alpha^{m\vee}}$ are G -minuscule. Hence, for any α^i not in the same connected component of the Dynkin diagram of G as α or α^m with $0 < i < m$, the conditions $x + \alpha^\vee - \alpha^{i\vee} \notin \bar{I}_{\mu,b}^{M,G}$ and $x + \alpha^{i\vee} - \alpha^{m\vee} \notin \bar{I}_{\mu,b}^{M,G}$ imply that $(\mu_{x+\alpha^{i\vee}})_{G\text{-dom}} \neq \mu$ and $(\mu_{x-\alpha^{i\vee}})_{G\text{-dom}} \neq \mu$. Hence, by Lemma 4.5.1, we have $\langle \alpha^i, \mu_x \rangle = 0$ and $w_x(\alpha^i) = \alpha^i$.

4.5.3. Let $x \in \bar{I}_{\mu,b}^{M,G}$. By Remark 4.1.8, there is a $g_x \in M(L)$ with $g_x^{-1}b\sigma(g_x) = b_x$. Then $g_x M(\mathcal{O}_L) \in X_{\mu_x}^M(b)$.

The main ingredient of the proof of Proposition 4.1.7 is the following proposition.

PROPOSITION 4.5.4. Let $x, x' \in \bar{I}_{\mu, b}^{M, G}$ and $x \rightarrow x'$. Suppose that $x - x' = \alpha^\vee - \alpha^{m^\vee}$ as in Definition 4.4.8 with α adapted. Let $g_x M(\mathcal{O}_L) \in X_{\mu_x}^M(b)$ as before. Then there exists $g' M(\mathcal{O}_L) \in X_{\mu_{x'}}^M(b)$ such that g_x and g' have the same image in $\pi_0(X_\mu^G(b))$. Moreover,

$$w_M(g_x) - w_M(g') = \sum_{i=0}^{m-1} \alpha^{i^\vee} \text{ in } \pi_1(M). \tag{4.5.5}$$

Before giving the proof of Proposition 4.5.4, we first show how to use it to prove Proposition 4.1.7.

Proof of Proposition 4.1.7. By Proposition 4.4.10, we may assume that the distance from x to x' is immediate. As $J_b^M(F)$ acts transitively on $\pi_0(X_{\mu_x}^M(b))$ by Proposition 3.3.1, for any $g \in X_{\mu_x}^M(b)$ there exists $j \in J_b^M(F)$ such that g and kg have the same image in $\pi_0(X_{\mu_x}^M(b))$. In particular, they have the same image in $\pi_0(X_\mu^G(b))$. By Proposition 4.5.4, there exists $g_1 M(\mathcal{O}_L) \in X_{\mu_{x'}}^M(b)$ such that g_x and g_1 have the same image in $\pi_0(X_\mu^G(b))$. Therefore g and kg_1 have the same image in $\pi_0(X_\mu^G(b))$. So $g' = kg_1$ is the desired element. \square

4.5.6. Now it remains to prove Proposition 4.5.4. The strategy of the proof is as follows. First we construct some ‘affine lines’ $g_{x, x'}$ and view them as part of ‘projective lines’. By an explicit computation, we will see that g and g' are both on the ‘projective lines’ corresponding to the points at 0 and ∞ respectively. The proposition then follows.

Keep the notation of Proposition 4.5.4, and let $\Omega = \Gamma\alpha$. Recall the element $b_x = \mu_x(p)\dot{w}_x$ in the σ -conjugacy class of b , defined in 4.1.8. For $i \geq 0$ we set $b_x^{(i)} = b_x \sigma(b_x) \cdots \sigma^i(b_x)$. It will be convenient to set $b_x^{(-1)} = 1$. The root subgroup $U_\alpha \subset G$ is naturally defined over \mathcal{O}_L . In the following we fix isomorphisms $\theta_\gamma : U_\alpha \xrightarrow{\sim} \mathbb{G}_a$ over \mathcal{O}_L , satisfying $\sigma^*(\theta_\gamma) = \theta_{\sigma(\gamma)}$. Then $\dot{w}_x U_\alpha(y) \dot{w}_x^{-1} = U_{w_x \alpha}(c_\alpha^x y)$ for some $c_\alpha^x \in \mathcal{O}_L^\times$ depending on \dot{w}_x and on α .

Let $R = \bar{k}[y]$ and $\mathcal{R} = \mathcal{O}_L\langle y \rangle$ equipped with the Frobenius $\sigma(y) = y^q$. We define $g_{x, x'}(y) \in G(\mathcal{R}_L)/G(\mathcal{R})$ as follows:

$$g_{x, x'}(y) := g_x(b_x^{(m-2)} \sigma^{m-1} U_\alpha(p^{-1}y)(b_x^{(m-2)})^{-1}) \cdots (b_x \sigma U_\alpha(p^{-1}y) b_x^{-1}) U_\alpha(p^{-1}y),$$

except if Ω is of type III, $d < m < 2d$, and $\langle \beta^{m-d}, \mu_x \rangle = 1$, in which case we let

$$g_{x, x'}(y) := g_{x'}(b_{x'}^{(m-2)} \sigma^{m-1} U_{-\alpha}(p^{-1}y)(b_{x'}^{(m-2)})^{-1}) \cdots (b_{x'} \sigma U_{-\alpha}(p^{-1}y) b_{x'}^{-1}) U_{-\alpha}(p^{-1}y).$$

PROPOSITION 4.5.7. With the notation above, we have

$$S_{\leq \mu}(g_{x, x'}(y)^{-1} b \sigma g_{x, x'}(y)) = \text{Spec } R.$$

Proof. We first deal with the case where Ω is of type I or II. By Lemma 4.5.1 and Remark 4.5.2, we have $b_x^{(i-1)} U_{\alpha^i}(p^{-1} \sigma^i y) (b_x^{(i-1)})^{-1} = U_{\alpha^i}(c_i p^{-1} \sigma^i y)$ for $i = 1, \dots, m-1$ with $c_i \in \mathcal{O}_L^\times$ arising from the action of the representative \dot{w}_x on the root subgroups. By Lemma 4.4.6, $\mu_x - w_x \alpha^{m^\vee}$ and $\mu_x + \alpha^\vee$ are G -minuscule, so $\langle \alpha, \mu_x \rangle = -1$ and $\langle w_x \alpha^m, \mu_x \rangle = 1$. As $U_\alpha, \dots, U_{\alpha^{m-1}}$ are in different connected components they obviously commute. Using this, together with Remark 4.5.2, and keeping in mind that $g_x^{-1} b \sigma(g_x) = b_x$, many of the factors in the definition of $g(y)^{-1} b \sigma(g(y))$ cancel and we obtain

$$\begin{aligned} A &:= g_{x, x'}(y)^{-1} b \sigma g_{x, x'}(y) = U_\alpha(-p^{-1}y)(b_x^{(m-1)} U_{\alpha^m}(p^{-1} \sigma^m y) (b_x^{(m-1)})^{-1}) b_x \\ &= U_\alpha(-p^{-1}y) U_{w_x \alpha^m}(c \sigma^m y) p^{\mu_x} \dot{w}_x \end{aligned}$$

for some $c \in \mathcal{O}_L^\times$. Here in the second equality we have used $\langle w_x \alpha^m, \mu_x \rangle = 1$.

We want to show that $A \in G(\mathcal{R})p^{\mu_x}G(\mathcal{R})$. This assertion only depends on the element $U_\alpha(-p^{-1}y)U_{w_x\alpha^m}(c\sigma^m(y))p^{\mu_x} \in G_\Omega(L)$. This element (and also every factor in the product) is contained in the standard Levi subgroup of $H_\Omega \subset G_\Omega$ corresponding to the Galois orbit of the connected component of the Dynkin diagram of G_Ω which contains Ω . Note that Γ acts transitively on the connected components of the Dynkin diagram of H_Ω .

If U_α and $U_{w_x\alpha^m}$ commute, then using $\langle \alpha, \mu_x \rangle = -1$, we obtain

$$A \in G(\mathcal{R})U_\alpha(-p^{-1}y)p^{\mu_x}G(\mathcal{R}) = G(\mathcal{R})p^{\mu_x}U_\alpha(-y)G(\mathcal{R}) = G(\mathcal{R})p^{\mu_x}G(\mathcal{R}).$$

If U_α and $U_{w_x(\alpha^m)}$ do not commute, then Ω is of type II and all the roots in H_Ω are of the same length. In this case, $\langle w_x\alpha^m, \alpha^\vee \rangle = -1$ and $\alpha + w_x\alpha^m$ is the only positive linear combination of α and $w_x\alpha^m$ which can be a root. By Lemma 4.4.6, $\mu_x + \alpha^\vee - w_x\alpha^{m\vee}$ is G -minuscule. On the other hand,

$$\langle w_x\alpha^m, \mu_x + \alpha^\vee - w_x\alpha^{m\vee} \rangle = -2,$$

so we get a contradiction.

We now deal with the case where Ω is of type III. Recall that $|\Omega| = 3d$. Suppose that either $m \leq d$ or $d < m < 2d$ with $\langle \beta^0, \mu_x \rangle = \langle \beta^{m-d}, \mu_x \rangle = 0$. Then by Lemma 4.5.1 and Remark 4.4.9, $\langle \alpha^i, \mu_x \rangle = 0$, and $w_x(\alpha^i) = \alpha^i$ $i = 1, \dots, m - 1$, and hence $b_x^{(i-1)}U_{\alpha^i}(p^{-1}\sigma^i(y))(b_x^{(i-1)})^{-1} = U_{\alpha^i}(p^{-1}c_i\sigma^i(y))$ for some $c_i \in \mathcal{O}_L^\times$. Keeping in mind that in this case U_{α^i} and $U_{\alpha^{i+d}}$ commute, and that U_α and $U_{w_x\alpha^m}$ commute, the same calculation for A as in the case above applies.

Now suppose that $d < m < 2d$. We may assume that $\langle \beta^{m-d}, \mu_x \rangle = 0$. Otherwise, $x' - x = (-\alpha)^\vee - (-\alpha)^{m\vee}$, and one checks that $x' \rightarrow x$ if we use negative roots instead of positive ones. Now $\langle \beta^{m-d}, \mu_x \rangle = 1$ implies that $\langle (-\beta)^{m-d}, \mu_{x'} \rangle = 0$. Therefore, we may reduce to the above case by exchanging x and x' , and using the opposite Borel group and negative roots.

It remains to consider the case where $d < m < 2d$, $\langle \beta^0, \mu_x \rangle = 1$ and $\langle \beta^{m-d}, \mu_x \rangle = 0$. By Remark 4.4.9, we have $\langle \alpha, \mu_x \rangle = -1$, $\langle \alpha^m, \mu_x \rangle = 1$ and $\langle \alpha^i, \mu_x \rangle = 0$ for $i = d, m - d$.

For $i = 1, \dots, m - 1$, $i \neq m - d$, α^i is not in the same connected component as α^m , so

$$\begin{aligned} A &= g_{x,x'}(y)^{-1}b\sigma g_{x,x'}(y) \\ &= U_\alpha(-p^{-1}y)(b_x^{(m-d-1)}U_{\alpha^{m-d}}(-p^{-1}\sigma^{m-d}(y))(b_x^{(m-d-1)})^{-1}) \\ &\quad \times (b_x^{(m-1)}U_{\alpha^m}(p^{-1}\sigma^m(y))(b_x^{(m-1)})^{-1}) \\ &\quad \times (b_x^{(m-d-1)}U_{\alpha^{m-d}}(p^{-1}\sigma^{m-d}(y))(b_x^{(m-d-1)})^{-1})b_x \\ &= U_\alpha(-p^{-1}y)U_{\alpha^{m-d}}(-p^{-1}c_1\sigma^{m-d}(y))U_{\alpha^{m+\beta^{m-d}}}(pc_2\sigma^m(y))U_{\alpha^{m-d}}(p^{-1}c_1\sigma^{m-d}(y))b_x \end{aligned}$$

where the last equality follows by Lemma 4.5.1 and where $c_1, c_2 \in \mathcal{O}_L^\times$ are constants arising from the action of the representative w_x on the root subgroups.

Note that α is also not in the same connected component as α^{m-d} and α^m . Thus in order to show that $A \in G(\mathcal{R})p^\mu G(\mathcal{R})$, it suffices to show that the following elements are in $G(\mathcal{R})p^\mu G(\mathcal{R})$:

$$\begin{aligned} A_1 &:= U_{\alpha^{m-d}}(-p^{-1}c_1\sigma^{m-d}(y))U_{\alpha^{m+\beta^{m-d}}}(pc_2\sigma^m(y))U_{\alpha^{m-d}}(p^{-1}c_1\sigma^{m-d}(y))p^{\mu_x}, \\ A_2 &:= U_\alpha(-p^{-1}y)p^{\mu_x}. \end{aligned}$$

But $A_2 = p^{\mu_x}U_\alpha(-y) \in G(\mathcal{R})p^\mu G(\mathcal{R})$ and

$$A_1 = U_{\alpha^{m+\beta^{m-d}}}(pc_2\sigma^m(y))U_{\alpha^{m+\beta^{m-d}+\alpha^{m-d}}}(c_3\sigma^{m-d}(y)\sigma^m(y))p^{\mu_x} \in G(\mathcal{R})p^\mu G(\mathcal{R}).$$

where $c_3 \in \mathcal{O}_L$ such that

$$[U_{\alpha^{m+\beta^{m-d}}}(pc_2\sigma^m(y)), U_{\alpha^{m-d}}(-p^{-1}c_1\sigma^{m-d}(y))] = U_{\alpha^{m+\beta^{m-d}+\alpha^{m-d}}}(c_3\sigma^{m-d}(y)\sigma^m(y)). \quad \square$$

Proof of Proposition 4.5.4. Let $\mathcal{R}' = \mathcal{O}_L\langle y, y^{-1} \rangle$, equipped with the Frobenius given by $\sigma(y) = y^q$. So the natural map $\mathcal{R} \rightarrow \mathcal{R}'$ is a morphism of frames.

Recall that for any root γ in G , we have chosen an isomorphism of \mathcal{O}_L -groups $\theta_\gamma : U_\gamma \xrightarrow{\sim} \mathbb{G}_a$, with $\sigma^*(\theta_\gamma) = \theta_{\sigma(\gamma)}$. An SL_2 -calculation shows that, given $\theta_\gamma, \theta_{-\gamma}$ may be chosen so that we have

$$U_\gamma(p^{-1}y) = U_{-\gamma}(py^{-1})p^{-\gamma^\vee}h \tag{4.5.8}$$

for some $h \in G(\mathcal{O}_L[y, y^{-1}]) \subset G(\mathcal{R}')$. Moreover, $(\theta_{\sigma\gamma}, \theta_{-\sigma\gamma}) = (\sigma^*\theta_\gamma, \sigma^*\theta_{-\gamma})$ then also satisfy the same property with respect to the root $\sigma(\gamma)$. In the following we fix such a choice for the Galois orbits of all roots γ .

If Ω is of type I or II, then $\alpha, \dots, \alpha^{m-1}$ are in different connected components of the Dynkin diagram of G_Ω . We have

$$\begin{aligned} g_{x,x'}(y) &= g_x U_{\alpha^{m-1}}(p^{-1}c_{m-1}\sigma^{m-1}(y)) \cdots U_{\alpha^1}(p^{-1}c_1\sigma(y))U_\alpha(p^{-1}y) \\ &\in g_x U_{-\alpha^{m-1}}(pc_{m-1}^{-1}\sigma^{m-1}(y^{-1})) \cdots U_{-\alpha^1}(pc_1^{-1}\sigma(y^{-1})) \\ &\quad \times U_{-\alpha}(py^{-1})p^{-\sum_{i=0}^{m-1} \alpha^{i\vee}} G(\mathcal{R}') \end{aligned} \tag{4.5.9}$$

for suitable constants $c_i \in \mathcal{O}_L^\times$.

We define a second element $f_{x,x'}(y) \in G(\mathcal{R}_L)$ by setting

$$f_{x,x'}(y) = g_x U_{-\alpha}(py)U_{-\alpha^1}(pc_1^{-1}\sigma(y)) \cdots U_{-\alpha^{m-1}}(pc_{m-1}^{-1}\sigma^{m-1}(y))p^{-\sum_{i=0}^{m-1} \alpha^{i\vee}}.$$

Then $f_{x,x'}(y) \in g_{x,x'}(y^{-1})$ in $G(\mathcal{R}'_L)/G(\mathcal{R}')$. In particular, by Proposition 4.5.7,

$$S_{\leq \mu}(f_{x,x'}^{-1}b\sigma(f_{x,x'})) \supseteq \text{Spec}(k[y]) \setminus \{0\}.$$

By Lemma 2.1.6, this set is Zariski closed. Hence, $f_{x,x'}$ defines an element of $X_{\leq \mu}(b)(\mathcal{R})$. In particular, $g_{x,x'}(0) = g_x \in X_{\mu_x}^M(b)$ and $g' := f_{x,x'}(0)$ have the same image in $\pi_0(X_{\leq \mu}(b))$.

By the definition of $f_{x,x'}$, we have

$$g' = f_{x,x'}(0) = g_x p^{-\sum_{i=0}^{m-1} \alpha^{i\vee}}$$

in $M(\mathcal{R}_L)/M(\mathcal{R})$. Therefore, $g' \in X_{\mu_{\tilde{x}}}^M(b)$ for some $\tilde{x} \in \bar{I}_{\mu,b}^{M,G}$. As $\tilde{x} = w_M(g'^{-1}b\sigma g') = x'$ in $\pi_1(M)$, we have $g' \in X_{\mu_{x'}}^M(b)$ and (4.5.5) holds.

If Ω is of type III, we apply the same construction. As in the proof of Proposition 4.5.7 we may assume that Γ acts transitively on the connected components of the Dynkin diagram of G_Ω and that $\langle \beta^{m-d}, \mu_x \rangle = 0$ (otherwise, we exchange x and x' and use negative roots instead of positive ones). Moreover, if $m \leq d$ or $d < m < 2d$ with $\langle \beta^0, \mu_x \rangle = 0$, then the definition of $f_{x,x'}$ and the computation of $g' := f_{x,x'}(0)$ are the same as above. It remains to consider the case where $d < m < 2d$, $\langle \beta^0, \mu_x \rangle = 1$ and $\langle \beta^{m-d}, \mu_x \rangle = 0$. By Remark 4.4.9,

$$\begin{aligned} g_{x,x'}(y) &= g_x U_{\alpha^{m-1}+\beta^{m-d-1}}(c_{m-1}\sigma^{m-1}(y)) \cdots U_{\alpha^{d+\beta^0}}(c_d\sigma^d(y)) \\ &\quad \times U_{\alpha^{d-1}}(p^{-1}c_{d-1}\sigma^{d-1}(y)) \cdots U_{\alpha^1}(p^{-1}c_1\sigma^1(y))U_\alpha(p^{-1}y) \end{aligned}$$

where as usual the c_i are constants in \mathcal{O}_L^\times arising from the conjugation by the representative w_x on the root subgroups. We can decompose $g_{x,x'}(y) = g_x h_0(y) \cdots h_{d-1}(y)$ into the terms corresponding to the different connect components of the Dynkin diagram of G_Ω . Here

$$h_i(y) = \begin{cases} U_{\beta^i+\alpha^{i+d}}(c_{i+d}\sigma^{i+d}(y))U_{\alpha^i}(p^{-1}c_i\sigma^i(y)), & i = 0, \dots, m-d-1, \\ U_{\alpha^i}(p^{-1}c_i\sigma^i(y)), & i = m-d, \dots, d-1. \end{cases}$$

When $0 \leq i \leq m - d - 1$, we have the following equalities in $G(\mathcal{R}'_L)/G(\mathcal{R}')$:

$$\begin{aligned} h_i(y) &\in U_{\beta^{i+\alpha^{i+d}}}(c_{i+d}\sigma^{i+d}(y))U_{-\alpha^i}(pc_i^{-1}\sigma^i(y^{-1}))p^{-\alpha^{i\nu}} \\ &= U_{-\alpha^i}(pc_i^{-1}\sigma^i(y^{-1}))p^{-\alpha^{i\nu}}U_{\beta^{i+\alpha^{i+d}}}(p^{-1}c_{i+d}\sigma^{i+d}(y)) \\ &= U_{-\alpha^i}(pc_i^{-1}\sigma^i(y^{-1}))p^{-\alpha^{i\nu}}U_{-\beta^{i-\alpha^{i+d}}}(pc_{i+d}^{-1}\sigma^{i+d}(y^{-1}))p^{-\beta^{i\nu}-\alpha^{i+d\nu}} \\ &= U_{-\alpha^i}(pc_i^{-1}\sigma^i(y^{-1}))U_{-\beta^{i-\alpha^{i+d}}}(c_{i+d}^{-1}\sigma^{i+d}(y^{-1}))p^{-\alpha^{i\nu}-\beta^{i\nu}-\alpha^{i+d\nu}}. \end{aligned}$$

Write the last of the expressions above as $f_{x,x'}^i(y^{-1})$, where $f_{x,x'}^i(y) \in G(\mathcal{R}_L)$. Then $f_{x,x'}^i(y) = h_i(y^{-1})$ in $G(\mathcal{R}'_L)/G(\mathcal{R}')$. Moreover, $f_{x,x'}^i(0) = p^{-\alpha^{i\nu}-\beta^{i\nu}-\alpha^{i+d\nu}}$.

When $i \geq m - d$ and $y \neq 0$,

$$h_i(y) \in U_{-\alpha^i}(pc_i^{-1}\sigma^i(y^{-1}))p^{-\alpha^{i\nu}}G(\mathcal{R}').$$

Defining $f_{x,x'}^i(y) = U_{-\alpha^i}(pc_i^{-1}\sigma^i(y))p^{-\alpha^{i\nu}}$, we obtain again $f_{x,x'}^i(y) = h_i(y^{-1})$ in $G(\mathcal{R}'_L)/G(\mathcal{R}')$, and $f_{x,x'}^i(0) = p^{-\alpha^{i\nu}}$. Let $f_{x,x'} = g_x f_{x,x'}^0 \cdots f_{x,x'}^{d-1}$. Then

$$g' := f_{x,x'}(0) = g_x f_{x,x'}^0(0) \cdots f_{x,x'}^{d-1}(0) = g_x p^{-\sum_{i=0}^{m-1} \alpha^{i\nu} - \sum_{j=0}^{m-d-1} \beta^{j\nu}}$$

and (4.5.5) holds. The same verification as in the type I and II cases shows that $g' = f_{x,x'}(0) \in X_{\mu_{x'}}^M(b)$. □

4.6 Proof of Proposition 4.1.9

In order to prove Proposition 4.1.9, we need the following lemma.

LEMMA 4.6.1. *Let $H \subset G$ be a standard Levi subgroup, and α a positive root of G , which is H -antidominant. If $\gamma \in W_H\alpha$, then there exists a finite set of positive roots $(\beta_i)_{i \in J}$ in H such that:*

- $\langle \beta_i, \beta_j^\vee \rangle = 0$ for all $i, j \in J$ with $i \neq j$;
- $\gamma = (\prod_{i \in J} s_{\beta_i})(\alpha)$ where the product does not depend on the order of s_{β_i} ;
- $\langle \gamma, \beta_i^\vee \rangle > 0 > \langle \alpha, \beta_i^\vee \rangle$ for $i \in J$;
- $|\gamma| = |\alpha| + \sum_{i \in J} |\langle \alpha, \beta_i^\vee \rangle| \cdot |\beta_i|$.

Proof. Case 1: α is not longer than any root in G . As $\gamma \in W_H\alpha$, γ has the same length as α . Then for any root β in G other than $\pm\alpha, \pm\gamma$,

$$|\langle \alpha, \beta^\vee \rangle|, |\langle \gamma, \beta^\vee \rangle| \in \{0, 1\}.$$

Since α is H -antidominant, we may write $\gamma - \alpha = \sum_{i \in J} \beta_i$ with β_i positive roots in H . By Lemma 4.2.5, after regrouping β_i , we may assume that $\langle \beta_i, \beta_j^\vee \rangle \geq 0$ for all $i, j \in J$. As the β_j are roots in H we have $\beta_j \neq \pm\alpha, \pm\gamma$ for every $j \in J$. Therefore,

$$2 \geq \langle \gamma, \beta_j^\vee \rangle - \langle \alpha, \beta_j^\vee \rangle = \left\langle \sum_{i \in J} \beta_i, \beta_j^\vee \right\rangle \geq 2.$$

This implies that $\langle \gamma, \beta_j^\vee \rangle = 1$, $\langle \alpha, \beta_j^\vee \rangle = -1$ and $\langle \beta_i, \beta_j^\vee \rangle = 0$ for all $i, j \in J$ with $i \neq j$. So the $(\beta_i)_{i \in J}$ have all the desired properties.

Case 2: α is a long root in G . Then α^\vee is not longer than any coroots in G . Applying the above construction using coroots instead of roots, we find a finite set of positive roots $(\beta_i)_{i \in J}$ in H

such that $\gamma^\vee = (\prod_{i \in J} s_{\beta_i})(\alpha^\vee)$, $\langle \beta_i, \gamma^\vee \rangle = 1$, $\langle \beta_i, \alpha^\vee \rangle = -1$, and $\langle \beta_i, \beta_j^\vee \rangle = 0$ for all $i, j \in J$ with $i \neq j$. Then $\gamma = (\prod_{i \in J} s_{\beta_i})(\alpha)$, $\langle \gamma, \beta_i^\vee \rangle > 0$ and $\langle \alpha, \beta_i^\vee \rangle < 0$. Therefore $(\beta_i)_{i \in J}$ is still the set of desired roots. \square

Proof of Proposition 4.1.9. Recall that we are assuming $b = p^{\mu_{x_0}} \dot{w}_{x_0}$ with $x_0 \in \bar{I}_{\mu, b}^{M, G}$.

By assumption G^{ad} is simple, so Γ acts transitively on the set of connected components of the Dynkin diagram of G . Let

$$C_1 := \{\alpha^\vee \in X_*(T) \mid \alpha \text{ is a positive root in } N, \text{ such that } \langle \alpha, \mu_{x_0} \rangle < 0\},$$

$$C_2 := \left\{ \alpha^\vee \in X_*(T) \mid \begin{array}{l} \alpha \text{ is an } M\text{-antidominant and positive root in } N, \\ \text{such that } \langle \alpha, \mu_{x_0} \rangle < 0 \end{array} \right\}.$$

Then $C \subset C_2 \subset C_1$.

Let L_C (respectively L_{C_i}) be the \mathbb{Z} -lattice generated by the elements of the Galois orbit of C (respectively C_i for $i = 1, 2$) and the coroots of M .

Let α be a simple root in N , and $\Omega = \Gamma\alpha$. Let \tilde{G}_Ω be the standard Levi subgroup of G corresponding to the set of simple roots not in Ω . We set

$$R_\alpha := W_{\tilde{G}_\Omega} \alpha,$$

$$\tilde{R}_\alpha := \{\gamma \in R_\alpha \mid \langle \gamma, \mu_{x_0} \rangle < 0\},$$

$$\tilde{R}_\Omega := \bigcup_{\alpha' \in \Omega} \tilde{R}_{\alpha'} \subset R_\Omega := \bigcup_{\alpha' \in \Omega} R_{\alpha'}.$$

Claim 1: $\tilde{R}_\Omega \neq \emptyset$.

Once Claim 1 is proved for the Galois orbit Ω , we define $\gamma(\Omega)$ to be a minimal element in \tilde{R}_Ω for the order \preceq .

We now prove this claim. Take $w \in W_{\tilde{G}_\Omega}$ with $w\mu_{x_0}$ \tilde{G}_Ω -dominant. Then $w\mu_{x_0}$ is not G -dominant, otherwise $w\mu_{x_0} = \mu$ and $\mu_{x_0} = \mu$ in $\pi_1(\tilde{G}_\Omega)$, which contradicts that (μ, b) is Hodge–Newton irreducible. So there exists $\tilde{\alpha} \in \Omega$ with $\langle \tilde{\alpha}, w\mu_{x_0} \rangle < 0$ and therefore $w^{-1}\tilde{\alpha} \in \tilde{R}_\Omega$. This shows Claim 1.

Claim 2: L_{C_1} is the coroot lattice of G .

In order to show Claim 2, it suffices to show that for any simple root α in N , there exists $\tau \in \Gamma$ such that $(\tau\alpha)^\vee \in L_{C_1}$. We may assume that $\gamma(\Omega) \in R_\alpha$, and we show that this implies $\alpha^\vee \in L_{C_1}$.

By the definition of $\gamma(\Omega)$, we have $\langle \gamma(\Omega), \mu_{x_0} \rangle < 0$. Then $\gamma(\Omega)^\vee \in C_1$. By Lemma 4.6.1, there exists a finite set of positive roots $(\beta_i)_{i \in J}$ such that $\gamma(\Omega) = (\prod_{i \in J} s_{\beta_i})\alpha$ satisfying the conditions in Lemma 4.6.1. Therefore, in order to show $\alpha^\vee \in L_{C_1}$, it suffices to show that for all $i \in J$, $\beta_i^\vee \in L_{C_1}$.

For $i \in J$, if β_i is a root in M , then $\beta_i^\vee \in L_{C_1}$ by the definition of L_{C_1} . This remains the case when β_i is a root in N . Since $\langle \gamma, \beta_i^\vee \rangle > 0$, $s_{\beta_i}(\gamma(\Omega)) \preceq \gamma(\Omega)$. Hence, by the minimality of $\gamma(\Omega)$, we have $\langle s_{\beta_i}\gamma(\Omega), \mu_{x_0} \rangle \geq 0 > \langle \gamma(\Omega), \mu_{x_0} \rangle$, and hence $\langle \beta_i, \mu_{x_0} \rangle < 0$. Therefore $\beta_i^\vee \in C_1$. This shows Claim 2.

For any $\gamma^\vee \in C_1$, let $\tilde{\gamma}$ be the M -antidominant representative of γ in $W_M\gamma$. Then $\tilde{\gamma} \in C_2$ and L_{C_2} is the coroot lattice of G by Claim 2. Hence, in order to show this proposition, it suffices to show that $C_2 \subset L_C$.

Suppose that $\gamma^\vee \in C_2 \setminus C$. Then there exists a positive root β in M such that $\langle \beta, \gamma^\vee \rangle < -1$. This implies that there is a simple root β' of M with $\beta' \preceq \beta$ and $\langle \beta', \gamma^\vee \rangle < 0$. Since M is of type

α, β and β' have the same length, so $\langle \beta', \gamma^\vee \rangle = \langle \beta, \gamma^\vee \rangle < -1$. Replacing β by β' , we may assume that β is simple. Let $\gamma_1 = s_\gamma(\beta)$. Then γ_1 is longer than γ and $\gamma_1 \in C_1$ since

$$\langle \gamma_1, \mu_{x_0} \rangle = \langle \beta - \langle \beta, \gamma^\vee \rangle \gamma, \mu_{x_0} \rangle = \langle \beta, \mu_{x_0} \rangle + \langle \beta, \gamma^\vee \rangle < 0.$$

Furthermore, as $\gamma_1^\vee = s_\gamma(\beta^\vee) = \gamma^\vee + \beta^\vee$, we have

$$\langle \beta, \gamma_1^\vee \rangle = \langle \beta, \gamma^\vee + \beta^\vee \rangle \leq 0,$$

so γ_1^\vee is M -antidominant as γ is M -antidominant. Therefore $\gamma_1^\vee \in L_C$ and then $\gamma^\vee \in L_C$. \square

4.7 Proof of Proposition 4.1.10

We continue to use the notation introduced above. Thus for $x \in \bar{I}_{\mu,b}^{M,G}$, we have the element $b_x = p^{\mu_x} \dot{w}_x \in M(L)$ defined in §4.1, so that b_x is basic in M and there is a $g_x \in G(L)$ with $g_x^{-1} b \sigma(g_x) = b_x$. Then $g_x \in X_{\mu_x}^M(b)$. Moreover, we continue to use the normalization of the root subgroups of G fixed in the proof of Proposition 4.5.4, and, as above, for any root α of G , we write $\alpha^i = \sigma^i(\alpha)$.

Let $\Omega \in \Phi_{N,\Gamma}$ be adapted and $\alpha \in \Omega$. Let $d > 0$ be the minimal positive integer such that α and α^d are in the same connected component of the Dynkin diagram of G_Ω . Then $n := |\Omega|$ is equal to $d, 2d$, or $3d$ if Ω is of type I, II, or III, respectively. If Ω is of type II or III, by Proposition 4.2.11, all the roots in Ω are simple in G_Ω . If Ω is of type II and α, α^d are not neighbors, then by Lemma 3.1.1 applied to M the two simple roots α, α^d have a common neighbor β in the Dynkin diagram of G_Ω . If Ω is of type III, let β be the common neighbor of α, α^d and α^{2d} . In all other cases let $\beta = 0$. Let

$$\tilde{\alpha} = \begin{cases} \alpha & \text{if } \Omega \text{ is of type I,} \\ \alpha + \beta + \alpha^d & \text{if } \Omega \text{ is of type II,} \\ \alpha + \alpha^d + \alpha^{2d} + \beta & \text{if } \Omega \text{ is of type III.} \end{cases}$$

Note that in all cases $\tilde{\alpha}$ is a positive root.

LEMMA 4.7.1. *Let $\Omega \in \Phi_{N,\Gamma}$. For any $x \in \bar{I}_{\mu,b}^{M,G}$, we have $x \in \bar{I}_{\mu_x,b}^{M,G_\Omega} \subseteq \bar{I}_{\mu,b}^{M,G}$. Moreover, for any $x_1 \in \bar{I}_{\mu_x,b}^{M,G_\Omega}$, if $x_2 = x_1 + \alpha^\vee - \alpha^{l\vee} \in \bar{I}_{\mu_x,b}^{M,G}$ with $\alpha, \alpha^l \in \Omega$, then $x_2 \in \bar{I}_{\mu_x,b}^{M,G_\Omega}$.*

Proof. Recall that

$$\bar{I}_{\mu_x,b}^{M,G_\Omega} = \{y \in \pi_1(M) \mid (\mu_y)_{G_\Omega\text{-dom}} = (\mu_x)_{G_\Omega\text{-dom}}, y = \kappa_M(b) \text{ in } \pi_1(M)_\Gamma\}.$$

It is obvious that $x \in \bar{I}_{\mu_x,b}^{M,G_\Omega}$. For the second assertion, let x_1, x_2 be as in the lemma. As $(\mu_{x_1})_{G\text{-dom}} = \mu = (\mu_{x_2})_{G\text{-dom}}$ and $\mu_{x_2} - \mu_{x_1}$ is a linear combination of coroots of G_Ω , we have $(\mu_{x_2})_{G_\Omega\text{-dom}} = (\mu_{x_1})_{G_\Omega\text{-dom}} = (\mu_x)_{G_\Omega\text{-dom}}$. Thus $x_2 \in \bar{I}_{\mu_x,b}^{M,G_\Omega}$. \square

LEMMA 4.7.2. *Let $\Omega \in \Phi_{N,\Gamma}$ be adapted. Let $x, x' \in \bar{I}_{\mu,b}^{M,G}$ with $x' = x + \alpha^\vee - \alpha^{l\vee}$ for some $\alpha \in \Omega$ and $0 < l < n$. We assume in addition that either Ω is of type I or $\langle \tilde{\alpha}^i, \mu_y \rangle \geq 0$ for all $i \in \mathbb{N}$ and all $y \in \bar{I}_{\mu_x,b}^{M,G_\Omega}$. Then for all $g \in X_{\mu_x}^M(b)$ there is a $g' \in X_{\mu_{x'}}^M(b)$ such that $g \sim g'$ and $w_M(g') = w_M(g) - \sum_{i=0}^{l-1} \alpha^{i\vee}$.*

Proof. We remind the reader that $g \sim g'$ means that g, g' are in the same connected component of $X_\mu(b)$.

We use induction on l . Suppose that $x'' := x + \alpha^{l_0\vee} - \alpha^{l\vee} \in \bar{I}_{\mu,b}^{M,G}$ for some $0 < l_0 < l$. Then $x'', x' \in \bar{I}_{\mu_x,b}^{M,G_\Omega}$ by Lemma 4.7.1. Applying the induction hypothesis to (x, x'') and (x'', x') , we

obtain a $g'' \in X_{\mu_{x''}}^M(b)$ such that $g \sim g''$ and $w_M(g'') = w_M(g) - \sum_{i=0}^{l_0-1} \alpha^{i\vee}$, and a $g' \in X_{\mu_{x'}}^M(b)$ such that $g'' \sim g'$ and $w_M(g'') = w_M(g') - \sum_{i=l_0}^{l-1} \alpha^{i\vee}$. Then g' is the desired element. Thus we may assume that for all $0 < i < l$, we have $x + \alpha^\vee - \alpha^{i\vee} \notin \bar{I}_{\mu,b}^{M,G}$. A similar argument shows that we may also assume that $x + \alpha^{i\vee} - \alpha^{l\vee} \notin \bar{I}_{\mu,b}^{M,G}$ for $0 < i < l$. We assume from now on that these two conditions hold.

As $J_b^M(F)$ acts transitively on the set of connected components of each $X_{\mu_x}^M(b)$ by Proposition 3.3.1, and w_M is constant on connected components by Lemma 2.1.6, it is enough to prove the lemma for the particular element $g = g_x$. If $x \rightarrow x'$ is immediate, then the desired element g' is the one constructed in Proposition 4.5.4. Thus it remains to consider the case where $x \rightarrow x'$ does not hold. In particular, by Definition 4.4.8, we only need to consider the following two cases: either Ω is of type II and $d < l < 2d$ or Ω is of type III and $2d \leq l < 3d$. For $i \in \mathbb{N}$, let

$$\tilde{U}_\alpha^i(y) = b_x^{(i-1)} \sigma^i(U_\alpha(y)) (b_x^{(i-1)})^{-1}.$$

For $i = 0$ this coincides with $U_\alpha(y)$. Let $R = \bar{k}[y]$ and \mathcal{R} be the R -frame chosen in 4.5.6. We define $g(y) \in G(\mathcal{R}_L)/G(\mathcal{R})$ by

$$g(y) = g_x \tilde{U}_\alpha^{l-1}(p^{-1}y) \cdots \tilde{U}_\alpha^0(p^{-1}y).$$

Using the same strategy as in §4.5, we want to show that $S_{\geq \mu}(g(y)^{-1} b \sigma g(y)) = \text{Spec } R$. Then we will extend this family to a ‘projective line’ and use the fact that the point $g(0)$ and the point g' ‘at infinity’ are in the same connected component of $X_{\geq \mu}^G(b)$. In order to compute \tilde{U}_α^i , to verify the above statement and to compute g' we consider the different types of Ω separately. We distinguish two cases according to the type of Ω .

LEMMA 4.7.3. *Keep the above notation and assumptions, and suppose that Ω is of type II and $d < l < 2d$. Then $\beta \neq 0$ if and only if $\langle \beta, \mu_x \rangle = 1$. Moreover, we have:*

- $w_x \alpha^d = \alpha^d + \beta$ and $\langle w_x \alpha^d, \mu_x \rangle = 1$;
- $w_x \alpha^{l-d} = \alpha^{l-d}$ and $\langle w_x \alpha^{l-d}, \mu_x \rangle = 0$;
- for $0 < i < l$ with $i \neq l - d, d$, $w_x \alpha^i = \alpha^i$, $\langle \beta^i, \mu_x \rangle = 0$ and $\langle \alpha^i, \mu_x \rangle = 0$.

Proof. As $x + \alpha^\vee - \alpha^{l\vee} \in \bar{I}_{\mu,b}^{M,G}$ we have $\langle \alpha, \mu_x \rangle = -1$ and $\langle w_x \alpha^l, \mu_x \rangle = 1$, by Lemma 4.4.6. Our assumption that $\langle \alpha + \beta + \alpha^d, \mu_x \rangle \geq 0$ and the fact that μ_x is minuscule then imply that $\langle \beta + \alpha^d, \mu_x \rangle = 1$. If $\langle \beta, \mu_x \rangle = 1$, then we have

$$1 = \langle \beta + \alpha^d, \mu_x \rangle = \langle s_\beta(\alpha^d), \mu_x \rangle,$$

and if $\langle \beta, \mu_x \rangle = 0$, then we have $1 = \langle \alpha^d, \mu_x \rangle = \langle w_x \alpha^d, \mu_x \rangle$. Therefore, by Lemma 4.4.7, we obtain $w_x \alpha^d = \alpha^d + \beta \langle \beta, \mu_x \rangle$ and $\langle w_x \alpha^d, \mu_x \rangle = 1$.

Moreover, if $\langle \alpha^d, \mu_x \rangle = 1$ and $\langle \alpha, \alpha^{d\vee} \rangle = 0$, then $x + \alpha^\vee - \alpha^{d\vee} = s_{\alpha^d}(\mu_x + \alpha^\vee)$, which contradicts $x + \alpha^\vee - \alpha^{d\vee} \notin \bar{I}_{\mu,b}^{M,G}$. Hence, $\langle \beta, \mu_x \rangle = 0$ implies that α, α^d are neighbors and $\beta = 0$. In particular, $w_x \alpha^d = \alpha^d + \beta$.

If $\langle w_x \alpha^{l-d}, \mu_x \rangle = 1$, then $s_{w_x \alpha^{l-d}}(\mu_x) = \mu_x - w_x \alpha^{l-d\vee} = \mu_{x - \alpha^{l-d}}$ by Lemma 4.4.5, which contradicts $x + \alpha^\vee - \alpha^{l-d\vee} \notin \bar{I}_{\mu,b}^{M,G}$. Hence, we obtain that

$$\langle \alpha^{l-d}, \mu_x \rangle \leq \langle w_x \alpha^{l-d}, \mu_x \rangle \leq 0. \tag{4.7.4}$$

We use an indirect proof to show $\langle \alpha^{l-d}, \mu_x \rangle = 0$, so assume that $\langle \alpha^{l-d}, \mu_x \rangle = -1$. By assumption $\langle \tilde{\alpha}^{l-d}, \mu_x \rangle = \langle \alpha^{l-d} + \beta^{l-d} + \alpha^l, \mu_x \rangle \geq 0$, hence $\langle \beta^{l-d} + \alpha^l, \mu_x \rangle = 1$ and $\langle \tilde{\alpha}^{l-d}, \mu_x \rangle = 0$. As above this implies that $w_x \alpha^l = \alpha^l$ or $w_x \alpha^l = \alpha^l + \beta^{l-d}$ by Lemma 4.4.7. Thus

$$\begin{aligned} \langle \alpha^{l-d} + \beta^{l-d} + \alpha^l, \mu_{x'} \rangle &= \langle \alpha^{l-d} + \beta^{l-d} + \alpha^l, \mu_x - w_x \alpha^{l\nu} \rangle \\ &= 0 - 1 < 0, \end{aligned}$$

which contradicts an assumption of the lemma. So $\langle \alpha^{l-d}, \mu_x \rangle = 0$. Then by (4.7.4), $0 = \langle \alpha^{l-d}, \mu_x \rangle \leq \langle w_x \alpha^{l-d}, \mu_x \rangle \leq 0$. This implies that $\alpha^{l-d} = w_x \alpha^{l-d}$ by Lemma 4.4.7.

Finally, for $0 < i < l$ with $i \neq l - d, d$, the conditions $x + \alpha^\vee - \alpha^{i\nu} \notin \bar{I}_{\mu,b}^{M,G}$ and $x + \alpha^{i\nu} - \alpha^{l\nu} \notin \bar{I}_{\mu,b}^{M,G}$ imply that $(\mu_{x+\alpha^{i\nu}})_{G\text{-dom}} \neq \mu$ and $(\mu_{x-\alpha^{i\nu}})_{G\text{-dom}} \neq \mu$. Then by Lemma 4.5.1, $w_x \alpha^i = \alpha^i$, $\langle \beta^i, \mu_x \rangle = 0$ and $\langle \alpha^i, \mu_x \rangle = 0$. \square

4.7.5. *Proof of Lemma 4.7.2 continued.* Assume that Ω is of type II and $d < l < 2d$. As $w_x \alpha^{l-d} = \alpha^{l-d}$, we have $w_x \beta^{l-d} = \beta^{l-d}$. Then

$$w_x \sigma^{l-d} w_x \alpha^d = w_x (\alpha^l + \beta^{l-d}) = w_x \alpha^l + \beta^{l-d}.$$

Using the M -dominance of μ_x , we have $\langle w_x \sigma^{l-d} w_x \alpha^d, \mu_x \rangle \geq \langle w_x \alpha^l, \mu_x \rangle = 1$.

Altogether, using Lemma 4.7.3, we obtain

$$\tilde{U}_\alpha^i(p^{-1}y) = \begin{cases} U_{\alpha^i}(p^{-1}c_i\sigma^i(y)) & \text{if } 0 \leq i < d, \\ U_{\sigma^{i-d}w_x\alpha^d}(c_i\sigma^i(y)) & \text{if } d \leq i < l, \\ U_{w_x\sigma^{l-d}w_x\alpha^d}(pc_l\sigma^l(y)) & \text{if } i = l, \end{cases}$$

with $c_i \in \mathcal{O}_L^\times$ as usual depending on \dot{w}_x and α^i , but not on y , and with $c_0 = 1$. Obviously root subgroups corresponding to roots in different connected components of the Dynkin diagram of G_Ω commute. By definition, we have

$$p^{\mu_x} \dot{w}_x \sigma(\tilde{U}_\alpha^i(y))(p^{\mu_x} \dot{w}_x)^{-1} = \tilde{U}_\alpha^{i+1}(y).$$

Using these two facts, many of the factors in the definition of $g(y)^{-1}b\sigma(g(y))$ cancel and we obtain

$$\begin{aligned} g(y)^{-1}b\sigma(g(y)) &= \tilde{U}_\alpha^0(-p^{-1}y)\tilde{U}_\alpha^{l-d}(-p^{-1}y)\tilde{U}_\alpha^l(p^{-1}y)\tilde{U}_\alpha^{l-d}(p^{-1}y)p^{\mu_x}\dot{w}_x \\ &= U_\alpha(-p^{-1}y)U_{\alpha^{l-d}}(-p^{-1}c_{l-d}\sigma^{l-d}(y))U_{w_x\sigma^{l-d}w_x\alpha^d}(pc_l\sigma^l(y)) \\ &\quad \times U_{\alpha^{l-d}}(p^{-1}c_{l-d}\sigma^{l-d}(y))p^{\mu_x}\dot{w}_x. \end{aligned}$$

If $\langle w_x \sigma^{l-d} w_x \alpha^d, \alpha^{l-d\nu} \rangle = 0$, then $U_{\alpha^{l-d}}$ and $U_{w_x \sigma^{l-d} w_x \alpha^d}$ commute. Using in addition $\langle \alpha, \mu_x \rangle = -1$, we obtain

$$g(y)^{-1}b\sigma(g(y)) = U_\alpha(-p^{-1}y)U_{w_x\sigma^{l-d}w_x\alpha^d}(pc_l\sigma^l(y))p^{\mu_x}\dot{w}_x \in G(\mathcal{R})p^{\mu_x}G(\mathcal{R}).$$

If $\langle w_x \sigma^{l-d} w_x \alpha^d, \alpha^{l-d\nu} \rangle = -1$, then $U_{\alpha^{l-d}}$ and $U_{w_x \sigma^{l-d} w_x \alpha^d}$ do not commute. We obtain

$$g(y)^{-1}b\sigma(g(y)) = U_\alpha(-p^{-1}y)U_{\alpha^{l-d}+w_x\sigma^{l-d}w_x\alpha^d}(c\sigma^l(y)\sigma^{l-d}(y))U_{w_x\sigma^{l-d}w_x\alpha^d}(pc_l\sigma^l(y))p^{\mu_x}\dot{w}_x$$

where $c \in \mathcal{O}_L$ is the product of c_l, c_{l-d} and the structure constant obtained from the commutator of the two root subgroups. Thus $g(y)^{-1}b\sigma(g(y))$ is again in $G(\mathcal{R})p^{\mu_x}G(\mathcal{R})$, hence $S_{\leq \mu}(g(y)^{-1}b\sigma g(y)) = \text{Spec } R$.

Now we compute the point g' ‘at infinity’ of the affine line $g(y)$. Let \mathcal{R}' be as in the proof of Proposition 4.5.4. Then for $0 \leq i \leq l - d - 1$ we have (using (4.5.8)) the following equalities in $G(\mathcal{R}'_L)/G(\mathcal{R}')$:

$$\begin{aligned} \tilde{U}_\alpha^{i+d}(p^{-1}y)\tilde{U}_\alpha^i(p^{-1}y) &= U_{\alpha^{i+d+\beta^i}}(c_{i+d}\sigma^{i+d}(y))U_{\alpha^i}(p^{-1}c_i\sigma^i(y)) \\ &= U_{\alpha^{i+d+\beta^i}}(c_{i+d}\sigma^{i+d}(y))U_{-\alpha^i}(pc_i^{-1}\sigma^i(y^{-1}))p^{-\alpha^{i\vee}} \\ &= U_{-\alpha^i}(pc_i^{-1}\sigma^i(y^{-1}))p^{-\alpha^{i\vee}}U_{\alpha^{i+d+\beta^i}}(p^{-1}c_{i+d}\sigma^{i+d}(y)) \\ &= U_{-\alpha^i}(pc_i^{-1}\sigma^i(y^{-1}))p^{-\alpha^{i\vee}}U_{-(\alpha^{i+d+\beta^i})}(pc_{i+d}^{-1}\sigma^{i+d}(y^{-1}))p^{-(\alpha^{i+d+\beta^i})\vee} \\ &= U_{-\alpha^i}(pc_i^{-1}\sigma^i(y^{-1}))U_{-(\alpha^{i+d+\beta^i})}(c_{i+d}^{-1}\sigma^{i+d}(y^{-1}))p^{-(\alpha+\beta^i+\alpha^{i+d})\vee}. \end{aligned}$$

We define a second element $f(y) \in G(\mathcal{R}_L)$ by setting

$$\begin{aligned} f(y) &= g_x \prod_{i=0}^{l-d-1} (U_{-\alpha^i}(pc_i^{-1}\sigma^i(y))U_{-(\alpha^{i+d+\beta^i})}(c_{i+d}^{-1}\sigma^{i+d}(y))p^{-(\alpha^i+\beta^i+\alpha^{i+d})\vee}) \\ &\quad \times \prod_{i=l-d}^{d-1} (U_{-\alpha^i}(pc_i^{-1}\sigma^i(y))p^{-\alpha^{i\vee}}) \end{aligned}$$

where the d factors of the two products correspond to different connected components of the Dynkin diagram of G_Ω and can thus be multiplied in any order. The above computation shows that for all $y \neq 0$ we have $f(y) = g(y^{-1})$ in $G(\mathcal{R}'_L)/G(\mathcal{R}')$. In particular, $S_{\leq \mu}(f^{-1}b\sigma(f)) \supseteq \text{Spec}(k[t]) \setminus \{0\}$. By Lemma 2.1.6, this set is Zariski closed. Hence, $f(y)$ defines an element of $X_{\leq \mu}(b)(\mathcal{R})$. In particular, $g(0) = g_x$ and $g' = f(0)$ have the same image in $\pi_0(X_{\leq \mu}(b))$. Furthermore, $g' = f(0) \in g_x p^{-\sum_{i=0}^{l-1} \alpha^{i\vee} - \sum_{i=0}^{l-d-1} \beta^{i\vee}}$, which proves the lemma in this case.

Next we consider the case where Ω is of type III.

LEMMA 4.7.6. *With the above assumptions and notation, suppose that Ω is of type III and $2d \leq l < 3d$.*

- If $d \nmid i, (l - i)$, then $w_x \alpha^i = \alpha^i, \langle \beta^i, \mu_x \rangle = 0$ and $\langle \alpha^i, \mu_x \rangle = 0$.
- If $l = 2d$, then $\langle \beta, \mu_x \rangle = 0, \langle \alpha, \mu_x \rangle = -1, \langle \alpha^d, \mu_x \rangle = 0$ and $\langle \alpha^{2d}, \mu_x \rangle = 1$.
- If $l > 2d$, then $\langle \beta, \mu_x \rangle = 1, \langle \beta^{l-2d}, \mu_x \rangle = 0, \langle \alpha, \mu_x \rangle = -1, \langle \alpha^i, \mu_x \rangle = 0$ for $i = d, 2d, l - d, l - 2d$, and $\langle \alpha^l, \mu_x \rangle = 1$.

Proof. The equalities when $d \nmid i, (l - i)$ follow as in the proof of Lemma 4.7.3, using Lemma 4.5.1.

If $l = 2d$, then $x + \alpha^\vee - \alpha^{l\vee} \in \bar{I}_{\mu,b}^{M,G}$ implies that $\langle \alpha, \mu_x \rangle = -1$ and $\langle \alpha^{2d}, \mu_x \rangle = 1$. Hence, $\langle \beta, \mu_x \rangle = 0$. The minimality assumption on l , and the condition $\langle \tilde{\alpha}, \mu_x \rangle \geq 0$, then imply $\langle \alpha^d, \mu_x \rangle = 0$.

Suppose that $l > 2d$. As before, we have $\langle \alpha, \mu_x \rangle = -1$ and $\langle w_x \alpha^l, \mu_x \rangle = 1$ by Lemma 4.4.6. Then the minimality assumption on l implies $\langle \alpha^i, \mu_x \rangle \leq 0$ for $i = d, 2d$, and also for $i = l - d, l - 2d$, using Lemma 4.4.5, as above. As

$$\langle \tilde{\alpha}, \mu_x \rangle = \langle \alpha + \beta + \alpha^d + \alpha^{2d}, \mu_x \rangle \geq 0,$$

we have $\langle \beta, \mu_x \rangle = 1$ and $\langle \alpha^i, \mu_x \rangle = 0$ for $i = d, 2d$.

Next we show that $\langle \beta^{l-2d}, \mu_x \rangle = 0$. Suppose that $\langle \beta^{l-2d}, \mu_x \rangle = 1$. Then one checks that $x + \alpha^\vee - \alpha^{i\vee} \notin \bar{I}_{\mu,b}^{M,G}$ for $i = l - d, l - 2d$ implies that $\langle \alpha^i, \mu_x \rangle = -1$, and hence $\langle \tilde{\alpha}^{l-2d}, \mu_x \rangle = -1 < 0$, which contradicts our standing assumptions. Therefore, $\langle \beta^{l-2d}, \mu_x \rangle = 0, \langle \alpha^l, \mu_x \rangle = 1$ and $\langle \alpha^i, \mu_x \rangle = 0$ for $i = l - d, l - 2d$. \square

4.7.7. *Proof of Lemma 4.7.2 continued.* Suppose that Ω is of type III, and $l = 2d$. Then using Lemma 4.7.6, we have

$$\tilde{U}_\alpha^i(p^{-1}y) = \begin{cases} U_{\alpha^i}(p^{-1}c_i\sigma^i(y)) & \text{if } 0 \leq i < 2d, \\ U_{\alpha^{2d}}(c_{2d}\sigma^{2d}(y)) & \text{if } i = 2d, \end{cases}$$

with $c_i \in \mathcal{O}_L^\times$. In particular, all these elements commute, and we can easily verify that $g(y)^{-1}b\sigma(g(y)) \in G(\mathcal{R})p^\mu G(\mathcal{R})$.

Now suppose that Ω is of type III, and $l > 2d$. Using Lemma 4.7.6, we obtain

$$\tilde{U}_\alpha^i(p^{-1}y) = \begin{cases} U_{\alpha^i}(p^{-1}c_i\sigma^i(y)) & \text{if } 0 \leq i < d, \\ U_{\alpha^{i+\beta i-d}}(c_i\sigma^i(y)) & \text{if } d \leq i < 2d, \\ U_{\alpha^i}(c_i\sigma^i(y)) & \text{if } 2d \leq i < l, \\ U_{\alpha^l}(pc_l\sigma^l(y)) & \text{if } i = l, \end{cases}$$

with $c_i \in \mathcal{O}_L^\times$. When computing $g(y)^{-1}b\sigma(g(y))$ many factors commute and cancel. We obtain

$$\begin{aligned} &g(y)^{-1}b\sigma(g(y)) \\ &= U_\alpha(-p^{-1}y)\tilde{U}_\alpha^{l-2d}(-p^{-1}y)\tilde{U}_\alpha^{l-d}(-p^{-1}y)\tilde{U}_\alpha^l(p^{-1}y)\tilde{U}_\alpha^{l-d}(p^{-1}y)\tilde{U}_\alpha^{l-2d}(p^{-1}y)p^{\mu_x}\dot{w}_x \\ &= U_\alpha(-p^{-1}y)U_{\alpha^{l-2d}}(-p^{-1}c_{l-2d}\sigma^{l-2d}(y))(U_{\alpha^{l-d+\beta l-2d}}(-c_{l-d}\sigma^{l-d}(y))U_{\alpha^l}(pc_l\sigma^l(y)) \\ &\quad \times U_{\alpha^{l-d+\beta l-2d}}(c_{l-d}\sigma^{l-d}(y)))U_{\alpha^{l-2d}}(p^{-1}c_{l-2d}\sigma^{l-2d}(y))p^{\mu_x}\dot{w}_x \\ &= U_\alpha(-p^{-1}y)U_{\alpha^{l-2d}}(-p^{-1}c_{l-2d}\sigma^{l-2d}(y))U_{\alpha^{l-d+\beta l-2d+\alpha^l}}(pc'\sigma^l(y)\sigma^{l-d}(y))U_{\alpha^l}(pc_l\sigma^l(y)) \\ &\quad \times U_{\alpha^{l-2d}}(p^{-1}c_{l-2d}\sigma^{l-2d}(y))p^{\mu_x}\dot{w}_x \\ &= U_\alpha(-p^{-1}y)U_{\alpha^{l-2d+\alpha^l-d+\alpha^l+\beta l-2d}}(c''\sigma^l(y)\sigma^{l-d}(y)\sigma^{l-2d}(y)) \\ &\quad \times U_{\alpha^{l-d+\beta l-2d+\alpha^l}}(pc'\sigma^l(y)\sigma^{l-d}(y))U_{\alpha^l}(pc_l\sigma^l(y))p^{\mu_x}\dot{w}_x \end{aligned}$$

with $c', c'' \in \mathcal{O}_L$. The final expression is in $G(\mathcal{R})p^\mu G(\mathcal{R})$ as $\langle \alpha, \mu_x \rangle = -1$.

The construction and computation of the ‘point at infinity’ g' are as in case of type II. \square

Before we prove Proposition 4.1.10, we need one more lemma.

LEMMA 4.7.8. *Let $\Omega \in \Phi_{N,\Gamma}$ be adapted. Then for all $x, x' \in \bar{I}_{\mu_{x_0}, b}^{M, G_\Omega}$, there exists a series of elements x_1, \dots, x_r in $\bar{I}_{\mu_{x_0}, b}^{M, G_\Omega}$ such that $x_1 = x$, $x_r = x'$ and $x_{i+1} - x_i = \alpha^\vee - \alpha'^\vee$ in $\pi_1(M)$ for some $\alpha, \alpha' \in \Omega$ (depending on i) for all $1 \leq i \leq r - 1$.*

Proof. As the problem only concerns the elements in $\pi_1(M)$, and $x = x'$ in $\pi_1(G_\Omega)$, after replacing G_Ω by the standard Levi subgroup corresponding to the Galois orbit of any connected component of the Dynkin diagram of G_Ω which contains some element in Ω , we may assume that Γ acts transitively on the set of connected components of the Dynkin diagram of G_Ω . If $v \in X_*(T)$ is a linear combination of coroots of G_Ω , let $|v|_M = \sum_{\alpha \in \Omega} |n_\alpha|$ where $v = \sum_{\alpha \in \Omega} n_\alpha \alpha^\vee$ in $\pi_1(M)$. For $x, x' \in \bar{I}_{\mu_{x_0}, b}^{M, G_\Omega}$, let $d_M(x, x') := |\mu_{x'} - \mu_x|_M$. We will prove the lemma by induction on $d_M(x, x')$.

Suppose that $x' \neq x$ in $\pi_1(M)$. Write $x' - x = \sum_{\alpha \in \Omega} n_\alpha \alpha^\vee$ in $\pi_1(M)$ with $n_\alpha \in \mathbb{Z}$. Then $\sum_{\alpha \in \Omega} n_\alpha = 0$. Write $\Omega^+ = \{\alpha \in \Omega | n_\alpha > 0\}$ and $\Omega^- = \{\alpha \in \Omega | n_\alpha < 0\}$. Let $\mu_{x'} - \mu_x = \sum_{i \in I'} \gamma_i^\vee$ be as in Lemma 4.3.7. Write $I = \{i \in I' | \gamma_i^\vee \neq 0 \text{ in } \pi_1(M)\}$, $I^+ = \{i \in I | \gamma_i \text{ is positive}\}$ and $I^- = \{i \in I | \gamma_i \text{ is negative}\}$. Then for any $i \in I^+$ (respectively $i \in I^-$), the image of γ_i^\vee in $\pi_1(M)$ is a linear combination of $(\alpha^\vee)_{\alpha \in \Omega^+}$ (respectively $(\alpha^\vee)_{\alpha \in \Omega^-}$).

If all the $(\gamma_i)_{i \in I}$ are in the same connected component of Dynkin diagram of G_Ω , then we may replace G_Ω by the standard Levi subgroup corresponding to that component, and assume

that G_Ω has connected Dynkin diagram. If Ω is of type I, this implies $x = x'$ contrary to our assumption. If Ω is of type II, then $|\Omega| = 2$, and $|\Omega^+| = |\Omega^-| = 1$. Therefore $|\gamma_i^\vee|_M = 1$ for all $i \in I$. Take any $i^+ \in I^+$ and $i^- \in I^-$, and define $x_1 = x + \gamma_{i^+}^\vee + \gamma_{i^-}^\vee \in \pi_1(M)$. Then $x_1 - x = \alpha^\vee - \alpha'^\vee$ for some $\alpha, \alpha' \in \Omega$ and $(\mu_{x_1})_{G_\Omega\text{-dom}} = (\mu_x)_{G_\Omega\text{-dom}}$, hence $x_1 \in \bar{I}_{\mu_{x_0}, b}^{M, G_\Omega}$. Moreover,

$$d_M(x_1, x') = \sum_{i \in I} |\gamma_i^\vee|_M - |\gamma_{i^+}^\vee|_M - |\gamma_{i^-}^\vee|_M = \sum_{i \in I} |\gamma_i^\vee|_M - 2 < \sum_{i \in I} |\gamma_i^\vee|_M = d_M(x, x').$$

By the induction hypothesis, we are done.

If Ω is of type III, then $|\Omega| = 3$ and the Dynkin diagram of G_Ω is of type D_4 . As $|\Omega^+| + |\Omega^-| \leq |\Omega| = 3$, we have $|\Omega^+| = 1$ or $|\Omega^-| = 1$. We may assume that $\Omega^+ = \{\alpha\}$ has only one element, the other case being analogous. Then, as before, $|\gamma_i^\vee|_M = 1$ for all $i \in I^+$. If there exists $i^- \in I^-$ such that $|\gamma_{i^-}^\vee|_M = 1$, then the choice of i^+ and i^- as before applies and we are done. Otherwise there exists $i \in I^-$ such that $|\gamma_i^\vee|_M \geq 2$. As $x' = x$ in $\pi_1(M)_\Gamma$, $\sum_{\alpha \in \Omega^+} |n_\alpha| = \sum_{\alpha \in \Omega^-} |n_\alpha| \geq 2$. Thus there exist two different elements $i_1, i_2 \in I^+$ such that $\gamma_{i_1}^\vee = \gamma_{i_2}^\vee = \alpha^\vee$ in $\pi_1(M)$. This is impossible since $\langle \gamma_{i_1}, \gamma_{i_2} \rangle = 0$.

It remains to consider the case where not all the γ_i for $i \in I$ are in the same connected component of Dynkin diagram of G_Ω . Choose $i^+ \in I^+$ and $i^- \in I^-$ such that γ_{i^+} and γ_{i^-} are not in the same connected component of G_Ω . As $\langle \gamma_{i^+}, \mu_x \rangle = -1$, there exists an $\alpha \in \Omega$ such that $\alpha^\vee \preceq \gamma_{i^+}^\vee$ and $\langle \alpha, \mu_x \rangle = -1$. On the other hand, suppose that $\gamma_{i^-}^\vee = -\alpha_1^\vee - \dots - \alpha_s^\vee$ in $\pi_1(M)$ for $\alpha_1, \dots, \alpha_s \in \Omega$. Then by Lemma 4.4.5(i),

$$1 = \langle -\gamma_{i^-}, \mu_x \rangle \leq \sum_{1 \leq j \leq s} \langle w_x \alpha_j, \mu_x \rangle.$$

Therefore, there exists $\alpha' := \alpha_i \in \Omega$, such that $\langle w_x \alpha', \mu_x \rangle = 1$. Let $x_1 = x + \alpha^\vee - \alpha'^\vee$. As α and α' are not in the same connected component of G_Ω , we have $\mu_{x_1} = s_\alpha s_{w_x \alpha'}(\mu_x)$ by Lemma 4.4.5, so $x_1 \in \bar{I}_{\mu_{x_0}, b}^{M, G}$. Hence, $x_1 \in \bar{I}_{\mu_{x_0}, b}^{M, G_\Omega}$ by Lemma 4.7.1. As $d_M(x_1, x') < d_M(x, x')$, we are done by induction. \square

In the following we will prove Proposition 4.1.10 by subdividing it into several particular cases which we prove in the form of Lemmas 4.7.9, 4.7.10 and 4.7.19.

LEMMA 4.7.9. Proposition 4.1.10 holds under the following additional hypotheses:

- the set $\bar{I}_{\mu_{x_0}, b}^{M, G_\Omega}$ has at least two elements;
- Ω is of type I or $\langle \tilde{\alpha}^i, \mu_x \rangle \geq 0$ for all $i \in \mathbb{N}$ and all $x \in \bar{I}_{\mu_{x_0}, b}^{M, G_\Omega}$.

Proof. As the set $\bar{I}_{\mu_{x_0}, b}^{M, G_\Omega}$ has at least two elements, by Lemmas 4.7.1 and 4.7.8, there exists $x_1 = x_0 + \alpha^\vee - \alpha'^\vee \in \bar{I}_{\mu_{x_0}, b}^{M, G_\Omega}$ with $\alpha \in \Omega$ and $0 < l < n = |\Omega|$. For any $g_1 \in X_{\mu_{x_0}}^M(b)(W(\bar{k}))$, by applying Lemma 4.7.2 to the pair (x_0, x_1) , we obtain a $g' \in X_{\mu_{x_1}}^M(b)$ such that $g_1 \sim g'$ and $w_M(g') = w_M(g_1) - \sum_{i=0}^{l-1} \alpha^{i\vee}$. As $\bar{I}_{\mu_{x_0}, b}^{M, G_\Omega} = \bar{I}_{\mu_{x_1}, b}^{M, G_\Omega}$, we apply again Lemma 4.7.2 to the pair (x_1, x_0) . We obtain a $g_2 \in X_{\mu_{x_0}}^M(b)(W(\bar{k}))$ such that $g' \sim g_2$ and $w_M(g_2) = w_M(g') - \sum_{i=l}^{n-1} \alpha^{i\vee}$. Then g_2 is the desired element of Proposition 4.1.10 for $x = x_0$. \square

LEMMA 4.7.10. Proposition 4.1.10 holds under the following additional hypotheses:

- the set $\bar{I}_{\mu_{x_0}, b}^{M, G_\Omega} = \{x_0\}$ contains only one element;
- Ω is of type I or $\langle \tilde{\alpha}, \mu_{x_0} \rangle \geq 0$ for all $\alpha \in \Omega$.

Example 4.7.11. Here is an example where all the hypotheses of Lemma 4.7.10 are satisfied. Let G be a unitary similitude group such that $G_L \simeq \mathrm{GL}_5 \times \mathbb{G}_{m,L}$ with standard simple roots $\beta_i = e_i - e_{i+1}$ for $i = 1, 2, 3, 4$. The group $\Gamma = \{\mathrm{Id}, \sigma\}$ acts on G_L with $\sigma\beta_i = \beta_{5-i}$ for $i = 1, \dots, 4$. The Levi subgroup M is defined by the roots β_1 and β_4 . The cocharacter μ_{x_0} is defined as follows:

$$\begin{aligned} \mu_{x_0} : \mathbb{G}_{m,L} &\rightarrow G_L \simeq \mathrm{GL}_5 \times \mathbb{G}_{m,L} \\ y &\mapsto (\mathrm{diag}(y, y, 1, y, 1), y). \end{aligned}$$

Then μ_{x_0} determines x_0 and μ . Therefore it determines w_{x_0} and $b = p^{\mu_{x_0}} \dot{w}_{x_0}$. Let $\alpha = \beta_3$. One can check that the datum $(M, G, \Gamma\alpha, b, \mu)$ satisfies all the conditions of Lemma 4.7.10.

Proof of Lemma 4.7.10. For simplicity, we write x for x_0 , and let $\alpha \in \Omega$ such that $\langle \alpha, \mu_x \rangle < 0$. Let $g = g_x \in \bar{I}_{\mu_x, b}^{M, G_\Omega}$.

Suppose that $0 < i < n$ and that α and α^i are in different components of the Dynkin diagram of G_Ω . Since $x + \alpha^\vee - \alpha^{i\vee} \notin \bar{I}_{\mu_x, b}^{M, G_\Omega}$ we have $\langle w_x \alpha^i, \mu_x \rangle \leq 0$. By assumption ν_b is G -dominant, and so $\langle \alpha, \nu_b \rangle \geq 0$. Since b is basic in M , ν_b is the W_M -average of the Galois average of μ_x . Using $\langle w_x \alpha^i, \mu_x \rangle \leq 0$ and Lemma 4.4.7, there exists $\alpha_1 \in \Gamma\alpha$ which is in the same connected component of the Dynkin diagram of G_Ω as α such that $\langle w_x \alpha_1, \mu_x \rangle > 0$. Since $x - \alpha_1^\vee + \alpha^{i\vee} \notin \bar{I}_{\mu_x, b}^{M, G_\Omega}$, we obtain that $\langle \alpha^i, \mu_x \rangle \geq 0$. Hence, by Lemma 4.4.7, $\alpha^i = w_x \alpha^i$, and for every positive root β in M , we have $-\langle \alpha^i, \beta^\vee \rangle \langle \beta, \mu_x \rangle \leq \langle s_\beta \alpha^i, \mu_x \rangle \leq 0$. In particular, if β is a maximal root in M , such that α^i and β are contained in the same component of the Dynkin diagram of G_Ω , then $\langle \alpha^i, \beta^\vee \rangle < 0$ so $\langle \beta, \mu_x \rangle = 0$, as μ_x is M -dominant. This implies $\langle \beta, \mu_x \rangle = 0$ for every positive root β in the same component as α^i . Thus μ_x and w_x are central in the connected component of G_Ω containing α^i .

Case 1: Ω is of type I. By the above, μ_x and w_x are central in the connected component of G_Ω containing α^i for $0 < i < n$, and $\alpha_1 = \alpha$. In particular, $\langle w_x \alpha^i, \mu_x \rangle = 0$, and $\langle w_x \alpha, \mu_x \rangle > 0$.

Claim: U_α and $U_{w_x(\alpha)}$ commute.

By Lemma 4.6.1, there exist positive roots $(\beta_i)_{i \in J}$ in M such that:

- $\langle \beta_i, \beta_j^\vee \rangle = 0$ for all $i \neq j \in J$;
- $w_x \alpha = (\prod_{j \in J} s_{\beta_j})(\alpha)$ and $\langle \alpha, \beta_i^\vee \rangle < 0$ for all $i \in J$;
- $|w_x \alpha| = |\alpha| + \sum_{i \in J} |\langle \alpha, \beta_i^\vee \rangle| \cdot |\beta_i|$.

By the hypothesis of Proposition 4.1.10, $\langle \beta_i, \alpha^\vee \rangle = -1$ for all $i \in J$. And by Lemma 4.4.7, $\langle \beta_i, \mu_x \rangle = 1$ for all $i \in J$ (Indeed, if $\langle \beta_i, \mu_x \rangle = 0$ for some i , then $\langle s_{\beta_i} w_x \alpha, \mu_x \rangle = \langle w_x \alpha, \mu_x \rangle$. But $s_{\beta_i} w_x \alpha \preceq w_x \alpha$, so this contradicts the minimality of $w_x \alpha$ in Lemma 4.4.7.) Therefore,

$$2 = \langle w_x \alpha, \mu_x \rangle - \langle \alpha, \mu_x \rangle = - \sum_{i \in J} \langle \alpha, \beta_i^\vee \rangle \cdot \langle \beta_i, \mu_x \rangle = - \sum_{i \in J} \langle \alpha, \beta_i^\vee \rangle.$$

In particular, the cardinality of the set J is at most 2. Furthermore, we have

$$\langle w_x \alpha, \alpha^\vee \rangle = \left\langle \alpha - \sum_{i \in J} \langle \alpha, \beta_i^\vee \rangle \beta_i, \alpha^\vee \right\rangle = 2 + \sum_{i \in J} \langle \alpha, \beta_i^\vee \rangle = 0.$$

Thus, if $\alpha + w_x \alpha$ is a root, then it is longer than α and hence longer than β_i for all $i \in J$. And so is the root $s_\alpha(\alpha + w_x \alpha)$. As

$$s_\alpha(\alpha + w_x \alpha) = w_x \alpha - \alpha = - \sum_{i \in J} \langle \alpha, \beta_i^\vee \rangle \beta_i$$

is a root in M , it should have the same length as β_i for any $i \in J$. We get a contradiction. Therefore $\alpha + w_x\alpha$ cannot be a root and this finishes the proof of the claim.

Let $R = \bar{k}[y]$ and \mathcal{R} the R -frame chosen in 4.5.6. We define $g(y) \in G(\mathcal{R}_L)/G(\mathcal{R})$ by

$$g(y) = g_x U_\alpha(p^{-1}y) U_{\alpha^1}(p^{-1}c_1\sigma(y)) \cdots U_{\alpha^{n-1}}(p^{-1}c_{n-1}\sigma^{n-1}(y))$$

where the $c_i \in \mathcal{O}_L^\times$ are such that $\dot{w}_x \sigma U_{\alpha^i}(c_i \sigma^i(y)) \dot{w}_x^{-1} = U_{\alpha^{i+1}}(c_{i+1} \sigma^{i+1}(y))$ and $c_0 = 1$. In type I, all of these root subgroups commute. Using the above equations to compute the conjugation action of $b_x = p^{\mu_x} \dot{w}_x$ on these root subgroups, we obtain

$$\begin{aligned} g^{-1} b \sigma(g) &= U_\alpha(-p^{-1}y) p^{\mu_x} \dot{w}_x U_\alpha(p^{-1}\sigma(c_{n-1})\sigma^n(y)) \\ &= U_\alpha(-p^{-1}y) p^{\mu_x} U_{w_x\alpha}(p^{-1}c_\alpha^x \sigma(c_{n-1})\sigma^n(y)) \dot{w}_x \\ &= U_{w_x\alpha}(c_\alpha^x \sigma(c_{n-1})\sigma^n(y)) p^{\mu_x} U_\alpha(-y) \dot{w}_x \end{aligned}$$

and the final expression is in $G(\mathcal{R})p^{\mu_x}G(\mathcal{R})$. Here, in the last equality, we have used that U_α and $U_{w(\alpha)}$ commute and that $\langle \alpha, \mu_x \rangle < 0$ and $\langle w_x\alpha, \mu_x \rangle > 0$. Thus

$$S_{\leq \mu}(g(y)^{-1} b \sigma(g(y))) = \text{Spec } R.$$

In the usual way (as, for example, in the proof of Lemma 4.7.2) we can extend this family to a ‘projective line’ and use the fact that the point $g(0)$ and the point g' ‘at infinity’ are in the same connected component of $X_{\leq \mu}^G(b)$. Here one obtains $g' \in g_x p^{-\sum_{i=0}^{n-1} \alpha^{i\vee}} K$, which finishes the proof in this case.

We now deal with the case where Ω is of type II or III. After replacing G_Ω by the standard Levi subgroup corresponding to the Galois orbit of any connected component of the Dynkin diagram of G_Ω containing some element of Ω , we may assume that Γ acts transitively on the Dynkin diagram of G_Ω . As we only use G_Ω to distinguish several cases, this modification does not change the following argument.

Case 2: Ω is of type II. By assumption $\langle \alpha + \beta + \alpha^d, \mu_x \rangle \geq 0$, hence $\langle \beta + \alpha^d, \mu_x \rangle = 1$. We have that μ_x and w_x are central on all connected components of the Dynkin diagram of G_Ω except for the one containing α and α^d .

LEMMA 4.7.12. *We have:*

- $\langle w_x \alpha^d, \mu_x \rangle = 1$;
- $w_x \alpha^d = \alpha^d + \beta$;
- $\beta \neq 0$ if and only if $\langle \beta, \mu_x \rangle = 1$;
- $\langle w_x \alpha^d, \alpha^\vee \rangle = -1$.

In particular, $\alpha + w_x \alpha^d$ is equal to the root $\tilde{\alpha}$, and $\alpha, w_x \alpha^d$ do not commute.

Proof. As α^d is M -antidominant, we have

$$\langle w_x \alpha^d, \mu_x \rangle = \langle \alpha^d, w_{M,0} \mu_x \rangle \geq \langle \alpha^d + \beta, \mu_x \rangle = 1.$$

If $\beta = 0$ then $\langle \alpha^d, \mu_x \rangle = 1$, and thus $\langle \alpha^d, \mu_x \rangle = \langle w_x \alpha^d, \mu_x \rangle$. Thus by Lemma 4.4.7, $w_x \alpha^d = \alpha^d = \alpha^d + \beta$. Suppose that $\beta \neq 0$. If $\langle \beta, \mu_x \rangle = 0$, then $\langle \alpha^d, \mu_x \rangle = 1$. This implies $x + \alpha^\vee - \alpha^{d\vee} \in \bar{I}_{\mu_x, b}^{M, G_\Omega}$ (use $\langle \alpha, \mu_x \rangle < 0$ and α M -antidominant), which is impossible. Thus $\langle \beta, \mu_x \rangle = 1$ and

$$\langle w_x \alpha^d, \mu_x \rangle = 1 = \langle \alpha^d + \beta, \mu_x \rangle > 0 = \langle \alpha^d, \mu_x \rangle.$$

Hence, by Lemma 4.4.7, $w_x \alpha^d = \alpha^d + \beta$. Now the formula $\langle w_x \alpha^d, \alpha^\vee \rangle = -1$ is clear, and the final claim follows. \square

Case 2.1: $w_x\tilde{\alpha} \neq \tilde{\alpha}$.

LEMMA 4.7.13. $\langle w_x\tilde{\alpha}, \mu_x \rangle = 1$ and $\langle w_x\tilde{\alpha}, \alpha^\vee \rangle \geq 0$.

Proof. We check the lemma according to the type of the Dynkin diagram of G_Ω , which can be only A_m, D_m or E_6 .

Suppose that the Dynkin diagram of G_Ω is of type A_m . By Lemma 4.7.12, we have $w_x\tilde{\alpha} = w_x(\alpha + \beta + \alpha^d) = w_x\alpha - \beta + \alpha^d + \beta$. By the assumption of case 2.1, this implies that $w_x\alpha \neq \alpha + \beta$. Thus, by Lemma 4.4.7, $\langle w_x\alpha, \mu_x \rangle > \langle \alpha + \beta, \mu_x \rangle$. Combined with the fact that if $\beta = 0$, $\langle w_x\alpha, \mu_x \rangle \leq \langle \alpha, \mu_x \rangle + 1$, we have

$$\langle w_x\alpha, \mu_x \rangle = \begin{cases} 1 & \text{if } m \text{ is odd, i.e. } \beta \neq 0, \\ 0 & \text{if } m \text{ is even, i.e. } \beta = 0. \end{cases}$$

Furthermore, as $\langle \beta, \mu_x \rangle = 1$ if $\beta \neq 0$,

$$\langle w_x\tilde{\alpha}, \mu_x \rangle = \langle w_x\alpha + \alpha^d, \mu_x \rangle = 1$$

and

$$\langle w_x\tilde{\alpha}, \alpha^\vee \rangle = \langle w_x\alpha + \alpha^d, \alpha^\vee \rangle.$$

If $\beta \neq 0$, this sum is $\geq 0 + 0$, and if $\beta = 0$, it is $\geq 1 - 1$, thus in all cases non-negative.

If the Dynkin diagram of G_Ω is of type D_m , we denote the simple roots by $\beta_r, \dots, \beta_1, \beta, \alpha, \alpha^d$ where β is the simple root with three neighbors $\beta_1, \alpha, \alpha^d$, and β_i is a neighbor of β_{i-1} for all $i > 1$. By Lemma 4.7.12, we have $\langle \beta, \mu_x \rangle = 1$. As μ_x is M -dominant and minuscule, this implies that $\langle \beta_i, \mu_x \rangle = 0$ for $i = 1, \dots, r$. Then the explicit definition of w_x implies that $w_x = s_\beta s_{\beta_1} \cdots s_{\beta_r}$. Thus, $w_x\tilde{\alpha} = w_x(\alpha + \beta + \alpha^d) = \alpha + \beta_1 + 2\beta + \alpha^d$. Hence,

$$\langle w_x\tilde{\alpha}, \mu_x \rangle = \langle \alpha + \beta_1 + 2\beta + \alpha^d, \mu_x \rangle = -1 + 0 + 2 + 0 = 1$$

and

$$\langle w_x\tilde{\alpha}, \alpha^\vee \rangle = \langle \alpha + \beta_1 + 2\beta + \alpha^d, \alpha^\vee \rangle = 2 + 0 - 2 + 0 = 0.$$

If the Dynkin diagram of G_Ω is of type E_6 , the simple root β again has three neighbors in the Dynkin diagram denoted α, α^d , and β_{-1} . Denote the other neighbors of α, α^d by γ, γ^d , respectively. As μ_x is G -minuscule and M -dominant and $\langle \beta, \mu_x \rangle = 1$, we have $\langle \beta_{-1}, \mu_x \rangle = 0$ and likewise $\langle \gamma^d, \mu_x \rangle = 0$, as $\langle \alpha^d, \mu_x \rangle = 0$. If $\langle \gamma, \mu_x \rangle = 0$, then $w_x = s_\beta$ and hence $w_x\tilde{\alpha} = \alpha + w_x\alpha^d$, which contradicts the hypothesis. Therefore, we have $\langle \gamma, \mu_x \rangle = 1$. We have that $\gamma + \alpha + 2\beta + \alpha^d + \beta_{-1}$ is a root, but $\langle \gamma + \alpha + 2\beta + \alpha^d + \beta_{-1}, \mu_x \rangle = 2$, in contradiction to the fact that μ_x is minuscule. Thus this subcase may not occur, which finishes the proof of the lemma. \square

4.7.14. *Proof of Lemma 4.7.10, case 2 continued.* We remind the reader that, by Lemma 4.7.12, we have

$$w_x\sigma^d(\alpha + w_x\alpha^d) = w_x\sigma^d(\tilde{\alpha}) = w_x\tilde{\alpha}.$$

For R as above, we define $g(y) \in G(\mathcal{R}_L)/G(\mathcal{R})$ as

$$g(y) = g_x U_\alpha(p^{-1}y) U_{\alpha^1}(p^{-1}c_1\sigma(y)) \cdots U_{\alpha^{d-1}}(p^{-1}c_{d-1}\sigma^{d-1}(y)) \\ \times U_{\alpha+w_x\alpha^d}(-p^{-1}c'_0y\sigma^d(y)) \cdots U_{\sigma^{d-1}(\alpha+w_x\alpha^d)}(-p^{-1}c'_{d-1}\sigma^{d-1}(y)\sigma^{2d-1}(y))$$

where the $c_i \in \mathcal{O}_L^\times$ for $i = 0, \dots, d$ are such that

$$\dot{w}_x\sigma U_{\alpha^i}(c_i\sigma^i(y))\dot{w}_x^{-1} = U_{w_x\alpha^{i+1}}(c_{i+1}\sigma^{i+1}(y))$$

and $c_0 = 1$. Furthermore, $c'_0 \in \mathcal{O}_L$ is such that

$$U_\alpha(y)U_{w_x\alpha^d}(z) = U_{\alpha+w_x\alpha^d}(c'_0yz)U_{w_x\alpha^d}(z)U_\alpha(y),$$

and the $c'_i \in \mathcal{O}_L$ for $i = 1, \dots, d$ are such that

$$\dot{w}_x\sigma U_{\sigma^{i-1}(\alpha+w_x\alpha^d)}(c'_{i-1}\sigma^{i-1}(y))\dot{w}_x^{-1} = U_{w_x\sigma^i(\alpha+w_x\alpha^d)}(c'_i\sigma^i(y)).$$

We remark that $c'_i \in \mathcal{O}_L^\times$ for $i = 0, \dots, d$. Indeed, it suffices to check this for $i = 0$. If c'_0 is in $p\mathcal{O}_L$, then the root groups U_α and $U_{w_x\alpha^d}$ commute in $G \otimes \bar{k}$. Since all the roots of G_Ω have the same length, this is impossible, by [SGA3, XXIII Proposition 6.5].

Now we can compute the conjugation action of $b_x = p^{\mu_x}\dot{w}_x$ on the root subgroups by using the above equations. We obtain

$$\begin{aligned} g^{-1}b\sigma(g) &= U_{\alpha+w_x\alpha^d}(p^{-1}c'_0y\sigma^d(y))U_\alpha(-p^{-1}y)U_{w_x\alpha^d}(c_d\sigma^d(y)) \\ &\quad \times U_{w_x\sigma^d(\alpha+w_x\alpha^d)}(-c'_d\sigma^d(y)\sigma^{2d}(y))p^{\mu_x}\dot{w}_x \\ &= U_{w_x\alpha^d}(c_d\sigma^d(y))U_\alpha(-p^{-1}y)U_{w_x\sigma^d(\alpha+w_x\alpha^d)}(-c'_d\sigma^d(y)\sigma^{2d}(y))p^{\mu_x}\dot{w}_x. \end{aligned}$$

As $\langle w_x\sigma^d(\alpha + w_x\alpha^d), \alpha^\vee \rangle \geq 0$, the corresponding root subgroups commute and the above expression is indeed in $Kp^{\mu_x}K$. Thus $S_{\leq \mu}(g(y)^{-1}b\sigma(g)) = \text{Spec } R$. As before, we can extend this family to a ‘projective line’ and use the fact that the point $g(0)$ and the point g' ‘at infinity’ are in the same connected component of $X_\mu^G(b)$. It remains to compute g' . Let $\mathcal{R}' = \mathcal{O}_L\langle y, y^{-1} \rangle$ be the frame introduced above. We consider each connected component of the Dynkin diagram of G_Ω separately, and for $0 \leq i \leq d - 1$ we compute in $G(\mathcal{R}'_L)/G(\mathcal{R}')$:

$$\begin{aligned} &U_{\alpha^i}(p^{-1}c_i\sigma^i(y))U_{\sigma^i(\alpha+w_x\alpha^d)}(-p^{-1}c'_i\sigma^i(y\sigma^d(y))) \\ &= U_{\alpha^i}(p^{-1}c_i\sigma^i(y))U_{-\sigma^i(\alpha+w_x\alpha^d)}(-p(c'_i)^{-1}\sigma^i(y\sigma^d(y))^{-1})p^{-\sigma^i(\alpha+w_x\alpha^d)^\vee} \\ &= U_{-\sigma^i(\alpha+w_x\alpha^d)}(-p(c'_i)^{-1}\sigma^i(y^{-1}\sigma^d(y^{-1})))U_{-\sigma^iw_x\alpha^{d+i}}(d_i\sigma^{d+i}(y^{-1}))p^{-\sigma^i(\alpha+w_x\alpha^d)^\vee} \end{aligned}$$

for some $d_i \in \mathcal{O}_L^\times$. Here in the last line we have used the fact that the root groups U_{α^i} and $U_{-\sigma^iw_x\alpha^d}$ commute, and that $\langle \alpha^i, \sigma^i(\alpha + w_x\alpha^d) \rangle = 1$. Thus we define the second family $f(y)$ as

$$f(y) = \prod_{i=0}^{d-1} U_{-\sigma^i(\alpha+w_x\alpha^d)}(-p(c'_i)^{-1}\sigma^i(y\sigma^d(y)))U_{-\sigma^iw_x\alpha^{d+i}}(d_i\sigma^{d+i}(y))p^{-\sigma^i(\alpha+w_x\alpha^d)^\vee}.$$

In particular, $g' = f(0) = g_x p^{-\sum_{i=0}^{d-1} \sigma^i(\alpha+w_x\alpha^d)^\vee}$ is as claimed.

Case 2.2: $w_x\tilde{\alpha} = \tilde{\alpha}$. Let $c_0 = c'_0 = 1$ and let c_i, c'_i be defined inductively by

$$\begin{aligned} \dot{w}_x\sigma U_{\alpha^i}(c_i\sigma^i(y))\dot{w}_x^{-1} &= U_{w_x\alpha^{i+1}}(c_{i+1}\sigma^{i+1}(y)), \\ \dot{w}_x\sigma U_{\sigma^i(\alpha+w_x\alpha^d)}(c'_i\sigma^i(y))\dot{w}_x^{-1} &= U_{\sigma^{i+1}(\alpha+w_x\alpha^d)}(c'_{i+1}\sigma^{i+1}(y)). \end{aligned}$$

Furthermore, let $\tilde{c} \in \mathcal{O}_L$ be such that

$$U_\alpha(y)U_{w_x\alpha^d}(z) = U_{\alpha+w_x\alpha^d}(\tilde{c}yz)U_{w_x\alpha^d}(z)U_\alpha(y).$$

We evidently have $c'_i \in \mathcal{O}_L^\times$ and $\tilde{c} \in \mathcal{O}_L^\times$ by the same argument as in case 2.1 above. We now define the frame we will need.

LEMMA 4.7.15. *Let $h = z - \tilde{c}y^{qd+1} - c'_d z^{qd}$, and set $A = \mathcal{O}_L[y, z]/h$. Then $\text{Spec } A$ is a dense Zariski open in a smooth, proper curve X over \mathcal{O}_L having geometrically connected fibers.*

Proof. Let $h' = z^{-qd} - \tilde{c}w^{qd+1} - c'_d z^{-1}$, and $A' = \mathcal{O}_L[w, z^{-1}]/h'$. Then $A' \subset A[z^{-1}]$ by sending w to yz^{-1} , and $\text{Spec } A, \text{Spec } A'$ glue along $\text{Spec } A[z^{-1}]$ into a proper flat curve X over \mathcal{O}_L , which admits a finite map $\eta : X \rightarrow \mathbb{P}^1$ given by the function z .

Note that $\partial h/\partial z = 1$ in $A \otimes \bar{k}$, and $\partial h'/\partial z^{-1} = -c'_d \neq 0$ in $A' \otimes \bar{k}$. Hence, X is a smooth curve. Since η is totally ramified over $z = 0$, X has geometrically irreducible fibers. \square

4.7.16. *Proof of case 2.2 continued.* Let $x_0 \in X$ be the point given by $y = z = 0$, and $x_1 \in X$ the point given by $z^{-1} = w = 0$, using the covering of X introduced in Lemma 4.7.15 above. Choose a map $\xi : X \rightarrow \mathbb{P}^1$ such that ξ is étale above $\xi(x_0)$ and $\xi(x_1)$. To see that this is possible, choose points x_2, \dots, x_r for r big enough (e.g. $r \geq 2g$ with g the genus of $X \otimes \bar{k}$) and such that the x_i are distinct in $X \otimes \bar{k}$ for $i = 0, \dots, r$. By the Riemann–Roch theorem there is a section $g_0 \in \Gamma(X \otimes \bar{k}, \mathcal{O}(\sum_j x_j))$ which does not vanish at any x_i . Lift g_0 to $g \in \Gamma(X, \mathcal{O}(\sum_j x_j))$. Then as a meromorphic function on X , g has a simple pole at each x_i with a residue which is non-zero mod p . Take ξ to be given by g^{-1} . Then ξ is étale over 0, and $\xi(x_i) = 0$ for all i .

Let \widehat{X} and $\widehat{\mathbb{P}^1}$ denote the p -adic completion of X and \mathbb{P}^1 . Let $U_0 \subset \widehat{\mathbb{P}^1} \otimes \bar{k}$ be the open subset where $\xi \otimes \bar{k}$ is étale, and let $U \subset \widehat{\mathbb{P}^1}$ denote the corresponding formal open affine, and $Y = \xi^{-1}(U) \subset \widehat{X}$. Since U_0 is stable by Frobenius on $\mathbb{P}^1 \otimes \bar{k}$, U is stable by any Frobenius lift on $\widehat{\mathbb{P}^1}$. Fix such a lift. Since $Y \rightarrow U$ is finite étale, by Lemma 2.1.4, the Frobenius lift on W lifts uniquely to a Frobenius lift on $Y = \text{Spf } \mathcal{R}$. We denote by σ the corresponding q -Frobenius on \mathcal{R} .

It will be convenient to denote by $\text{Spf } \mathcal{R}_0$ and $\text{Spf } \mathcal{R}_1$ the formal affine subsets of Y , which are the complements of the mod p reductions of x_1 and x_0 , respectively. Thus $z, y \in \mathcal{R}_0$ and $z^{-1}, w = yz^{-1} \in \mathcal{R}_1$. Likewise, we denote by $\text{Spf } \mathcal{R}'$ the complement of $\{x_0, x_1\}$ in $\text{Spf } \mathcal{R}$. Define an element $g \in G(\mathcal{R}_{0,L})$ by

$$g = g_x U_\alpha(p^{-1}c_0 y) \cdots U_{\alpha^{d-1}}(p^{-1}c_{d-1} \sigma^{d-1}(y)) U_{\alpha+w_x \alpha^d}(-p^{-1}c'_0 z) \cdots \\ \times U_{\sigma^{d-1}(\alpha+w_x \alpha^d)}(-p^{-1}c'_{d-1} \sigma^{d-1}(z)).$$

Recall that $\langle w_x \alpha^d, \mu_x \rangle = 1$ and that $w_x \sigma^d(\alpha + w_x \alpha^d) = \alpha + w_x \alpha^d$. We obtain

$$g^{-1} b \sigma(g) = U_{\alpha+w_x \alpha^d}(p^{-1}z) U_\alpha(-p^{-1}y) U_{w_x \alpha^d}(c_d \sigma^d(y)) U_{w_x \sigma^d(\alpha+w_x \alpha^d)}(-p^{-1}c'_d \sigma^d(z)) p^{\mu_x} \dot{w}_x \\ = U_{\alpha+w_x \alpha^d}(p^{-1}z - p^{-1} \tilde{c} y \sigma^d(y) - p^{-1} c'_d \sigma^d(z)) U_{w_x \alpha^d}(c_d \sigma^d(y)) U_\alpha(-p^{-1}y) p^{\mu_x} \dot{w}_x.$$

Recall that $\langle \alpha + w_x \alpha^d, \alpha^\vee \rangle = 2 - 1 = 1$, thus α and $\alpha + w_x \alpha$ commute. For the second equality above we use the fact that

$$w_x \sigma^d(\alpha + w_x \alpha^d) = w_x \tilde{\alpha} = \tilde{\alpha} = \alpha + w_x \alpha^d$$

commutes with $w_x \alpha^d$ and α , and the definition of \tilde{c} . Since $\langle \alpha, \mu_x \rangle = -1$, and

$$z - \tilde{c} y \sigma^d(y) - c'_d \sigma^d(z) = z - \tilde{c} y^{dq+1} - c'_d z^{dq} = h = 0 \quad \text{in } \mathcal{R}_0/p\mathcal{R}_0,$$

we see that $g^{-1} b \sigma(g) \in G(\mathcal{R}_0) p^{\mu_x} G(\mathcal{R}_0)$.

To define and compute a ‘point at infinity’ of the above family we first compute, for $0 \leq i < d$,

$$U_{\alpha^i}(p^{-1}c_i \sigma^i(y)) U_{\sigma^i(\alpha+w_x \alpha^d)}(-p^{-1}c'_i \sigma^i(z)) G(\mathcal{R}') \\ = U_{\alpha^i}(p^{-1}c_i \sigma^i(y)) U_{-\sigma^i(\alpha+w_x \alpha^d)}(-p(c'_i)^{-1} \sigma^i(z^{-1})) p^{-\sigma^i(\alpha+w_x \alpha^d)^\vee} G(\mathcal{R}') \\ = U_{-\sigma^i(w_x \alpha^d)}(-d_i \sigma^i(yz^{-1})) U_{-\sigma^i(\alpha+w_x \alpha^d)}(-p(c'_i)^{-1} \sigma^i(z^{-1})) p^{-\sigma^i(\alpha+w_x \alpha^d)^\vee} G(\mathcal{R}')$$

for some $d_i \in \mathcal{O}_L$, where the third line comes from moving U_{α^i} to the right.

Define an element $f \in G(\mathcal{R}_{1,L})$ by setting

$$f = g_x \prod_{i=0}^{d-1} U_{-\sigma^i(w_x \alpha^d)}(-d_i \sigma^i(w)) U_{-\sigma^i(\alpha + w_x \alpha^d)}(-p(c'_i)^{-1} \sigma^i(z)) p^{-\sigma^i(\alpha + w_x \alpha^d)^\vee}.$$

Then we have $f = g$ in $G(\mathcal{R}'_L)$. By what we saw above, $S_{\leq \mu}(f^{-1} b \sigma(f))$ contains the open and dense subset $\text{Spec } \mathcal{R}'/p\mathcal{R}'$ of $\text{Spec } \mathcal{R}_1/p\mathcal{R}_1$. By Lemma 2.1.6, $S_{\leq \mu}(f^{-1} b \sigma(f))$ is Zariski closed. Hence, f defines an element of $X_{\leq \mu}(b)(\mathcal{R}')$. In particular, $g(x_0) = g_x \in X_{\mu_x}^M(b)$ and $g' = f(x_1) = g_x p^{-\sum_{i=0}^{d-1} \sigma^i(\alpha + w_x \alpha^d)^\vee}$ are in the same connected component of $X_{\leq \mu}(b)$. \square

Case 3: Ω is of type III. The same argument as for the preceding cases shows that μ_x and thus w_x are central on all connected components of the Dynkin diagram of G_Ω except for the one containing $\alpha, \alpha^d, \alpha^{2d}$. As $x + \alpha^\vee - \alpha^{d\vee} \notin \bar{I}_{\mu_x, b}^{M, G_\Omega}$, $x + \alpha^\vee - \alpha^{2d\vee} \notin \bar{I}_{\mu_x, b}^{M, G_\Omega}$, we have $\langle \alpha^d, \mu_x \rangle \leq 0$ and $\langle \alpha^{2d}, \mu_x \rangle \leq 0$. Combined with the fact that $\langle \tilde{\alpha}, \mu_x \rangle \geq 0$, we obtain $\langle \beta, \mu_x \rangle = 1$ and $\langle \alpha^d, \mu_x \rangle = \langle \alpha^{2d}, \mu_x \rangle = 0$. Let

$$\tilde{U}_\alpha^i(y) = b_x^{(i)} \sigma^i(U_\alpha(y)) (b_x^{(i)})^{-1}.$$

Then

$$\tilde{U}_\alpha^i(p^{-1}y) = \begin{cases} U_{\alpha^i}(p^{-1}c_i \sigma^i(y)) & \text{if } 0 \leq i < d, \\ U_{\alpha^i + \beta^i}(c_i \sigma^i(y)) & \text{if } d \leq i < 2d, \\ U_{\alpha^i}(c_i \sigma^i(y)) & \text{if } 2d \leq i < 3d, \\ U_{\alpha + \beta}(c_n \sigma^n(y)) & \text{if } i = 3d = n. \end{cases}$$

Let $R = \bar{k}[y]$ and \mathcal{R} the R -frame chosen in 3.2.2. We define $g(y) \in G(\mathcal{R}_L)$ by

$$g(y) = g_x \tilde{U}_\alpha^{3d-1}(p^{-1}y) \cdots \tilde{U}_\alpha^0(p^{-1}y).$$

Then

$$\begin{aligned} g(y)^{-1} b \sigma(g(y)) &= U_\alpha(-p^{-1}y) U_{\alpha^d + \beta}(-c_d \sigma^d(y)) (U_{\alpha^{2d}}(-c_{2d} \sigma^{2d}(y)) U_{\alpha + \beta}(c_{3d} \sigma^{3d}(y))) \\ &\quad \times U_{\alpha^{2d}}(c_{2d} \sigma^{2d}(y)) U_{\alpha^d + \beta}(c_d \sigma^d(y)) p^{\mu_x} \dot{w}_x \\ &= U_\alpha(-p^{-1}y) U_{\alpha^d + \beta}(-c_d \sigma^d(y)) U_{\alpha + \beta}(c_{3d} \sigma^{3d}(y)) \\ &\quad \times U_{\alpha + \alpha^{2d} + \beta}(-c' \sigma^{2d}(y) \sigma^{3d}(y)) U_{\alpha^d + \beta}(c_d \sigma^d(y)) p^{\mu_x} \dot{w}_x \\ &= U_\alpha(-p^{-1}y) U_{\alpha + \beta}(c_{3d} \sigma^{3d}(y)) U_{\alpha + \alpha^d + \alpha^{2d} + 2\beta}(-c'' \sigma^d(y) \sigma^{2d}(y) \sigma^{3d}(y)) \\ &\quad \times U_{\alpha + \alpha^{2d} + \beta}(c' \sigma^{2d}(y) \sigma^{3d}(y)) p^{\mu_x} \dot{w}_x \end{aligned}$$

with $c', c'' \in \mathcal{O}_L$. Now U_α commutes with the other factors and can be moved to the right. We obtain that $g(y)^{-1} b \sigma(g(y)) \in K p^{\mu_x} K$. A computation analogous to the above constructs a point g' ‘at infinity’ and shows that it has the required properties, which finishes the proof of Lemma 4.7.10. \square

Remark 4.7.17. Example 4.7.11 is in case 2.1 of the proof of Lemma 4.7.10. Another interesting example is the following. Let G be a unitary similitude group such that $G_L \simeq \text{GL}_3 \times \mathbb{G}_{m,L}$ with standard simple roots $\beta_i = e_i - e_{i+1}$ for $i = 1, 2$. The group $\Gamma = \{\text{Id}, \sigma\}$ acts on G_L with $\sigma \beta_i = \beta_{3-i}$ for $i = 1, 2$. Take $M = T$, $\alpha = \beta_2$, and the cocharacter μ_{x_0} is defined as follows which determines b and μ :

$$\begin{aligned} \mu_{x_0} : \mathbb{G}_{m,L} &\rightarrow G_L \simeq \text{GL}_3 \times \mathbb{G}_{m,L} \\ y &\mapsto (\text{diag}(y, 1, y), y). \end{aligned}$$

Then the datum $(M, G, \Gamma, \alpha, b, \mu)$ still satisfies all the conditions of Lemma 4.7.10 and corresponds to case 2.2 in that proof.

COROLLARY 4.7.18. Let $\Omega \in \Phi_{N,\Gamma}$ be adapted and of type I. Let $x \in \bar{I}_{\mu,b}^{M,G}$. Suppose that there exists $\alpha \in \Omega$ such that $\langle \alpha, \mu_x \rangle < 0$. Then there exist $g, g' \in X_{\mu_x}^M(b)(W(\bar{k}))$ such that:

- g and g' are in the same connected component of $X_{\mu}^G(b)$;
- $w_M(g') - w_M(g) = \sum_{\beta \in \Omega} \beta^\vee$ in $\pi_1(M)^\Gamma$.

Proof. As Ω is of type I, by Lemmas 4.7.9 and 4.7.10, Proposition 4.1.10 holds for Ω . In particular, the corollary holds for $x = x_0$. Moreover, if we replace x_0 by x in Proposition 4.1.10, it still holds once we replace correspondingly b by b_x . This means that there exist $g_1, g'_1 \in X_{\mu_x}^M(b_x)(W(\bar{k}))$ such that:

- g and g' are in the same connected component of $X_{\mu}^G(b)$;
- $w_M(g') - w_M(g) = \sum_{\beta \in \Omega} \beta^\vee$ in $\pi_1(M)^\Gamma$.

By Remark 4.1.8, $[b_x] = [b]$ in $B(M)$. So there exists an element $h \in M(L)$ such that $g \mapsto hg$ gives an isomorphism between $X_{\mu_x}^M(b_x)$ and $X_{\mu_x}^M(b)$. Therefore, $g = hg_1, g' = hg'_1 \in X_{\mu_x}^M(b)(W(\bar{k}))$ are the desired elements. \square

LEMMA 4.7.19. Proposition 4.1.10 holds if Ω is of type II or III and $\langle \tilde{\alpha}, \mu_x \rangle < 0$ for some $\alpha \in \Omega$ and some $x \in \bar{I}_{\mu,b}^{M,G}$.

Proof. Let $\alpha \in \Omega$ be as in the lemma. Let $\Omega' := \Gamma\tilde{\alpha}$. Then Ω' is adapted and of type I. Therefore, we can apply Corollary 4.7.18 to (Ω', x) and obtain elements $g, g' \in X_{\mu_x}^M(b)(W(\bar{k}))$ such that $g \sim g'$ and $w_M(g') - w_M(g) = \sum_{\beta \in \Omega'} \beta^\vee = \sum_{\beta \in \Omega} \beta^\vee$ in $\pi_1(M)^\Gamma$. \square

Proposition 4.1.10 then follows immediately from Lemmas 4.7.9, 4.7.10, and 4.7.19.

5. Application to Rapoport–Zink spaces

In this section we apply the main results of this paper to (simple) unramified Rapoport–Zink spaces of EL type or unitary/symplectic PEL type.

5.1.1. From now on, suppose that $F = \mathbb{Q}_p$. In the previous sections we have studied the connected components of affine Deligne–Lusztig varieties $X_{\mu}^G(b)$ defined from the datum (G, b, μ) . Now we require that the datum (G, b, μ) satisfies two additional conditions.

First, G belongs to one of the following three cases:

- * *EL case:* $G = \text{Res}_{\mathcal{O}_{F_1}|\mathbb{Z}_p} \text{GL}(\Lambda_0)$ where F_1 is a finite unramified extension of \mathbb{Q}_p , and where V is a finite-dimensional F_1 -vector space with $\Lambda_0 \subset V$ a lattice.
- * *PEL symplectic case:* $G = \text{GSp}(\Lambda_0, \langle \cdot, \cdot \rangle)$ where F_1, V are as above and where $\langle \cdot, \cdot \rangle : V \times V \rightarrow \mathbb{Q}_p$ is a non-degenerate alternating \mathbb{Q}_p -bilinear form on V such that $\langle \lambda x, y \rangle = \langle x, \lambda y \rangle$ for all $x, y \in V, \lambda \in F_1$, and $\Lambda_0 \subset V$ is an autodual lattice in V for this form.
- * *PEL unitary case:* $G = \text{GU}(\Lambda_0, \langle \cdot, \cdot \rangle)$ where F_1, V are as above, $*$ is a non-trivial involution on $F_1, \langle \cdot, \cdot \rangle : V \times V \rightarrow \mathbb{Q}_p$ is a non-degenerate alternating hermitian form on V , and $\Lambda_0 \subset V$ is a autodual lattice in V for this form.

Second, the weight decomposition of μ in $V \otimes_{\mathbb{Q}_p} L$ has only slopes 0 and 1, where we consider $\mu \in X_*(T)$ as the representation

$$\mu : \mathbb{G}_{m,L} \rightarrow T_L \hookrightarrow G_L \hookrightarrow (\text{Res}_{F_1|\mathbb{Q}_p} \text{GL}(V))_L.$$

A datum (G, b, μ) satisfying the above conditions is called a (simple) unramified Rapoport–Zink datum of EL type or unitary/symplectic PEL type. To this kind of datum we can associate a Rapoport–Zink space $\check{\mathcal{M}} = \check{\mathcal{M}}(G, b, \mu)$. These spaces are formal schemes locally formally of finite type over $\mathrm{Spf}\mathcal{O}_L$, which are defined as moduli spaces parametrizing certain families of p -divisible groups in a fixed isogeny class. They are equipped with a natural action of $J_b(\mathbb{Q}_p)$. For the precise definition of these spaces we refer to [RZ96]. There exists a $J_b(\mathbb{Q}_p)$ -equivariant locally constant morphism on $\check{\mathcal{M}}$,

$$\varkappa_{\check{\mathcal{M}}} : \check{\mathcal{M}}(G, b, \mu) \rightarrow \mathrm{Hom}(X_{\mathbb{Q}_p}^*(G), \mathbb{Z}),$$

where $X_{\mathbb{Q}_p}^*(G)$ is the group of \mathbb{Q}_p -rational characters of G . The classification of p -divisible groups over $\bar{\mathbb{F}}_p$ via Dieudonné theory, induces a natural bijection $\check{\mathcal{M}}(G, b, \mu)(\bar{\mathbb{F}}_p) \simeq X_{\mu}^G(b)(W(\bar{\mathbb{F}}_p))$ compatible with the $J_b(\mathbb{Q}_p)$ -action.

PROPOSITION 5.1.2. *Suppose that (G, b, μ) is HN-indecomposable. Then the natural bijection $\theta : X_{\mu}^G(b)(W(\bar{\mathbb{F}}_p)) \simeq \check{\mathcal{M}}(G, b, \mu)(\bar{\mathbb{F}}_p)$ induces a map on the sets of connected components*

$$\pi_0(X_{\mu}^G(b)) \rightarrow \pi_0(\check{\mathcal{M}}(G, b, \mu)),$$

which is necessarily surjective.

Proof. Let R be a smooth integral \bar{k} -algebra, and \mathcal{R} a frame for R . We have to show that if $g_0, g_1 \in X_{\mu}^G(b)(W(\bar{\mathbb{F}}_p))$ are connected via a $g \in X_{\mu}^G(b)(\mathcal{R})$ then $\theta(g_0)$ and $\theta(g_1)$ are in the same connected component in $\check{\mathcal{M}}$. Let $s_0, s_1 \in \mathrm{Spec}(R)(\bar{\mathbb{F}}_p)$ with $g(s_0) = g_0$ and $g(s_1) = g_1$, as in (2.3.5).

By Proposition 2.1.11, there exists an étale covering $f : \mathrm{Spec}(R') \rightarrow \mathrm{Spec}(R)$ such that $g^{-1}b\sigma(g) \in G(\mathcal{R}')p^{\mu}G(\mathcal{R}')$ where \mathcal{R}' is the canonical frame for R' . It suffices to prove the statement with \mathcal{R} replaced by the affine ring of one of the connected components of $\mathrm{Spec}\mathcal{R}'$. (Indeed, we can find a chain of elements $(h_i)_{1 \leq i \leq n} \in X_{\mu}^G(b)(W(\bar{\mathbb{F}}_p))$ such that $h_1 = g_0$ and $h_n = g_1$ and there exist $s_i, s'_i \in \mathrm{Spec}(R')(\bar{\mathbb{F}}_p)$ in the same connected component with $g(s_i) = h_i$ and $g(s'_i) = h_{i+1}$ for $1 \leq i \leq n - 1$. We can then consider separately each pair (h_i, h_{i+1}) with the connected component of $\mathrm{Spec}(R')$ containing s_i .) Therefore, we reduce to the case where $g^{-1}b\sigma(g) \in G(\mathcal{R})p^{\mu}G(\mathcal{R})$.

We now define an element in $\check{\mathcal{M}}(R)$ corresponding to g by using Dieudonné theory. The proof is very similar to the proof of [Kis13, Lemma 1.4.6]. Here we only give a sketch. Let $\Lambda_0 \subset V$ be as in the definition of G . Let $M := g(\Lambda_0 \otimes_{\mathbb{Z}_p} \mathcal{R}) \subset V \otimes_{\mathbb{Q}_p} \mathcal{R}_L$. The Frobenius map $F = b\sigma$ acts on M . As the weight decomposition of μ on $V \otimes L$ has only slopes 0 and 1, we have $pM \subset FM \subset M$. Therefore M is stable under Frobenius and Verschiebung.

We write \mathcal{R}_n for the ring \mathcal{R} considered as an \mathcal{R} -algebra via $\sigma^n : \mathcal{R} \rightarrow \mathcal{R}$. Similarly, let $R_n := \mathcal{R}_n/p\mathcal{R}_n$. As the action of σ on $\Omega_{\mathcal{R}/\mathcal{O}_L}^1$ is topologically nilpotent, there exists $n \in \mathbb{N}$ sufficiently large such that $g^{-1}dg \in \mathrm{End}(\Lambda_0) \otimes_{\mathbb{Z}_p} \Omega_{\mathcal{R}_n/\mathcal{O}_L}^1$. Then we can check that $g(\Lambda_0 \otimes_{\mathbb{Z}_p} \mathcal{R}_n)$ is stable under the canonical connection

$$\nabla = 1 \otimes d : \Lambda_0 \otimes_{\mathbb{Z}_p} \mathcal{R}_{n,L} \rightarrow \Lambda_0 \otimes_{\mathbb{Z}_p} \Omega_{\mathcal{R}_{n,L}/L}^1.$$

Therefore, $(M \otimes_{\mathcal{R}} \mathcal{R}_n, \nabla, F, V)$ gives rise to a Dieudonné crystal on \mathcal{R}_n with G -structures. This corresponds to a point in $\check{\mathcal{M}}(G, b, \mu)(R_n)$ by [dJo95, Theorem 4.1.1]. \square

5.1.3. Recall that $G^{\text{ab}} = G/G^{\text{der}}$ is the cocenter of G . Then $X_{\mathbb{Q}_p}^*(G) = X^*(G^{\text{ab}})^\Gamma$ and $\pi_1(G) = \pi_1(G^{\text{ab}}) = X_*(G^{\text{ab}})$ by [Bor98, Lemma 1.5] since G^{der} is simply connected. Then by comparing the definition of w_G and $\varkappa_{\check{\mathcal{M}}}$, we can check that the diagram

$$\begin{CD} X_\mu^G(b)(W(\overline{\mathbb{F}}_p)) @>w_G>> c_{b,\mu}\pi_1(G)^\Gamma @= c_{b,\mu}X_*(G^{\text{ab}})^\Gamma \\ @V\sim VV @. @VVV \\ \check{\mathcal{M}}(G, b, \mu)(\overline{\mathbb{F}}_p) @>\varkappa_{\check{\mathcal{M}}}>> \text{Hom}(X_{\mathbb{Q}_p}^*(G), \mathbb{Z}) @= \text{Hom}(X^*(G^{\text{ab}})^\Gamma, \mathbb{Z}) \end{CD} \tag{5.1.4}$$

commutes, where the vertical arrow on the right is induced by the natural Γ -equivariant pairing $X_*(G^{\text{ab}}) \otimes X^*(G^{\text{ab}}) \rightarrow \mathbb{Z}$.

THEOREM 5.1.5. (i) $\theta : \pi_0(X_\mu^G(b)) \rightarrow \pi_0(\check{\mathcal{M}}(G, b, \mu))$ is a bijection.

(ii) If (μ, b) is HN-irreducible, then $\varkappa_{\check{\mathcal{M}}}$ induces an injection on the connected components

$$\varkappa_{\check{\mathcal{M}}} : \pi_0(\check{\mathcal{M}}) \rightarrow \text{Hom}(X_{\mathbb{Q}_p}^*(G), \mathbb{Z}).$$

Proof. Suppose that (μ, b) is HN-irreducible. By Proposition 5.1.2, the above diagram induces a commutative diagram on the connected components:

$$\begin{CD} \pi_0(X_\mu^G(b)) @>\varkappa_{\check{\mathcal{M}}}>> c_{b,\mu}X_*(G^{\text{ab}})^\Gamma \\ @VVV @VVV \\ \pi_0(\check{\mathcal{M}}) @>\varkappa_{\check{\mathcal{M}}}>> \text{Hom}(X^*(G^{\text{ab}})^\Gamma, \mathbb{Z}) \end{CD} \tag{5.1.6}$$

where the top horizontal morphism is a bijection by Theorem 1.1 and Corollary 2.4.3. In order to show (i) and (ii), it suffices to show that $c_{b,\mu}X_*(G^{\text{ab}})^\Gamma \rightarrow \text{Hom}(X^*(G^{\text{ab}})^\Gamma, \mathbb{Z})$ is injective. Since $X_*(G^{\text{ab}})^\Gamma$ is torsion free, it suffices to prove the statement after $\otimes \mathbb{Q}$, and then the map is an isomorphism, as Γ acts on $X_*(G^{\text{ab}})$ through a finite quotient.

We now prove (i) in the general case. If (G, b, μ) is Hodge–Newton indecomposable, by Theorem 2.5.6, we only need to deal with the case where b is σ -conjugate to p^μ with μ central. We may assume that $b = p^\mu$. For any algebraically closed extension k of $\overline{\mathbb{F}}_p$, one uses Dieudonné theory and the same computation as in Remark 2.5.10 to show that

$$\begin{aligned} \check{\mathcal{M}}(G, b, \mu)(k) &= \{g \in G(W(k)[1/p])/G(W(k)) \mid g^{-1}b\sigma(g) \in G(W(k))p^\mu G(W(k))\} \\ &= G(\mathbb{Q}_p)/G(\mathbb{Z}_p) \end{aligned}$$

where the second equality follows from Lang’s theorem $H^1(\langle \sigma \rangle, G(W(k))) = 0$. It follows that $\check{\mathcal{M}}(G, b, \mu)$ is discrete and (i) follows from Theorem 1.1. This remains the case when (G, b, μ) is Hodge–Newton decomposable. In this case there exists a standard parabolic P with Levi subgroup M containing T and a $b' \in M \cap [b]$ such that (M, b', μ) is Hodge–Newton indecomposable. We may assume that $b' = b$. With (M, b, μ) and (P, b, μ) one can also associate analogs of Rapoport–Zink spaces $\check{\mathcal{M}}(M, b, \mu)$ and $\check{\mathcal{M}}(P, b, \mu)$, respectively. They are moduli spaces of p -divisible groups with additional structure of the same type as for $\check{\mathcal{M}}(G, b, \mu)$, but which are in addition equipped with a slope decomposition (respectively a slope filtration) corresponding to M (respectively to P); see [Man08] for the precise construction. One obtains naturally defined morphisms

$$\check{\mathcal{M}}(M, b, \mu) \xrightarrow{s} \check{\mathcal{M}}(P, b, \mu) \xrightarrow{p_2} \check{\mathcal{M}}(G, b, \mu).$$

Moreover, Mantovan also constructed a morphism $p_1 : \check{\mathcal{M}}(P, b, \mu)^{\text{an}} \rightarrow \check{\mathcal{M}}(M, b, \mu)^{\text{an}}$ satisfying $p_1 \circ s^{\text{an}} = \text{Id}$ by considering the graded pieces of the filtration on the p -divisible groups, where $(-)^{\text{an}}$ always denotes the generic fiber. Then s^{an} induces an injection on the connected components. By [She14, Proposition 6.3], p_2^{an} induces an isomorphism of analytic spaces on the generic fiber, and we find that

$$\pi_0(\check{\mathcal{M}}(M, b, \mu)) \xrightarrow{\sim} \pi_0(\check{\mathcal{M}}(M, b, \mu))^{\text{an}} \hookrightarrow \pi_0(\check{\mathcal{M}}(G, b, \mu))^{\text{an}} \xrightarrow{\sim} \pi_0(\check{\mathcal{M}}(G, b, \mu)).$$

Here the two bijections follow from the fact that $\check{\mathcal{M}}(M, b, \mu)$ and $\check{\mathcal{M}}(G, b, \mu)$ are both formally smooth by Grothendieck–Messing deformation theory (cf. [dJo95, Theorem 7.4.1]). Thus $\pi_0(p_2 \circ s)$ is an injection. But we already know that θ induces a surjective map on connected components. Hence, using Proposition 2.5.4, $\pi_0(p_2 \circ s)$ is also surjective. Then (i) follows from the Hodge–Newton-indecomposable case. \square

5.1.7. Theorem 5.1.5 confirms [Che14, Conjecture 6.1.1]. As the main results in [Che14] are proved after assuming that conjecture, we can now state all these results without this hypothesis.

Let $\check{\mathcal{F}}$ be the flag variety of parabolic subgroups of type μ of G/L . Let $\check{\pi} : \check{\mathcal{M}}^{\text{an}} \rightarrow \check{\mathcal{F}}^{\text{an}}$ be the period morphism (cf. [RZ96, ch. 5]), where $\check{\mathcal{M}}^{\text{an}}$ is the generic fiber of $\check{\mathcal{M}}$ as Berkovich’s analytic space, and $\check{\mathcal{F}}^{\text{an}}$ is Berkovich’s analytic space associated to $\check{\mathcal{F}}$. Let $\check{\mathcal{F}}^a$ be the image of $\check{\pi}$.

PROPOSITION 5.1.8 (cf. [Che14, Lemma 6.1.3]). *If (μ, b) is HN-irreducible, then $\check{\mathcal{F}}^a$ is connected.*

5.1.9. Recall that $(\check{\mathcal{M}}_{\check{K}})_{\check{K} \subset G(\mathbb{Z}_p)}$ is a tower of finite étale covers over $\check{\mathcal{M}}^{\text{an}}$ parametrizing the \check{K} -level structures with $\check{K} \subset G(\mathbb{Z}_p)$ open compact. The group $J_b(\mathbb{Q}_p)$ acts on the left on each $\check{\mathcal{M}}_{\check{K}}$ and the group $G(\mathbb{Q}_p)$ acts on the right on the tower $(\check{\mathcal{M}}_{\check{K}})_{\check{K}}$ by Hecke correspondences. As in the introduction, we have the map

$$\delta = (\delta_{J_b}, \delta_G, \chi_{\delta_G, \mu}) : J_b(\mathbb{Q}_p) \times G(\mathbb{Q}_p) \times \text{Gal}(\bar{L}/L) \rightarrow G^{\text{ab}}(\mathbb{Q}_p).$$

THEOREM 5.1.10 (cf. [Che14, Theorem 6.3.1]). *If (μ, b) is HN-irreducible, then the action of $J_b(\mathbb{Q}_p) \times G(\mathbb{Q}_p) \times \text{Gal}(\bar{L}/L)$ on $\pi_0(\check{\mathcal{M}}_{\check{K}} \hat{\otimes} \mathbb{C}_p)$ factors through δ , and makes $\pi_0(\check{\mathcal{M}}_{\check{K}} \hat{\otimes} \mathbb{C}_p)$ into a $G^{\text{ab}}(\mathbb{Q}_p)/\delta_G(\check{K})$ -torsor. In particular, we have bijections*

$$\pi_0(\check{\mathcal{M}}_{\check{K}} \hat{\otimes} \mathbb{C}_p) \simeq G^{\text{ab}}(\mathbb{Q}_p)/\delta_G(\check{K})$$

which are compatible when \check{K} varies.

Remark 5.1.11. Write

$$\pi_0(\check{\mathcal{M}}_{\infty} \hat{\otimes} \mathbb{C}_p) := \varprojlim_{\check{K}} \pi_0(\check{\mathcal{M}}_{\check{K}} \hat{\otimes} \mathbb{C}_p).$$

Then the theorem above is equivalent to the statement that when (μ, b) is HN-irreducible, the action of $J_b(\mathbb{Q}_p) \times G(\mathbb{Q}_p) \times \text{Gal}(\bar{L}/L)$ on $\pi_0(\check{\mathcal{M}}_{\infty} \hat{\otimes} \mathbb{C}_p)$ makes this set a $G^{\text{ab}}(\mathbb{Q}_p)$ -torsor.

When $\check{\mathcal{M}}$ is of EL type,⁴ we can form the inverse limit $\check{\mathcal{M}}_{\infty, \mathbb{C}_p} = \varprojlim_{\check{K}} \check{\mathcal{M}}_{\check{K}} \hat{\otimes} \mathbb{C}_p$ as a perfectoid space as in [SW13]. In this case the set $\pi_0(\check{\mathcal{M}}_{\infty} \hat{\otimes} \mathbb{C}_p)$, defined formally above, coincides with the set of connected components of $\check{\mathcal{M}}_{\infty, \mathbb{C}_p}$.

ACKNOWLEDGEMENTS

The authors would like thank Robert Kottwitz, Dennis Gaitsgory and Jilong Tong for useful discussions, and Xuhua He and Rong Zhou for useful comments on a previous version of the manuscript. We thank the referee for helpful comments.

⁴This condition is presumably unnecessary.

REFERENCES

- BT65 A. Borel and J. Tits, *Groupes réductifs*, Publ. Math. Inst. Hautes Études Sci. **27** (1965), 55–150.
- Bor98 M. Borovoi, *Abelian Galois cohomology of reductive groups*, Memoirs of the American Mathematical Society 626 (American Mathematical Society, Providence, RI, 1998).
- Bou02 N. Bourbaki, *Lie groups and Lie algebras. Chapters 4–6* (Springer, Berlin, 2002), Translated from the 1968 French original by Andrew Pressley.
- Bro08 M. Broshi, *Moduli of finite flat group schemes with G -structure*, Thesis, University of Chicago (2008).
- BT72 F. Bruhat and J. Tits, *Groupes réductifs sur un corps local I*, Publ. Math. Inst. Hautes Études Sci. **41** (1972), 5–251.
- Che14 M. Chen, *Composantes connexes géométriques d’espaces de modules de groupes p -divisibles*, Ann. Sci. Éc. Norm. Supér. (4) **47** (2014), 723–764.
- CS79 J.-L. Colliot-Thélène and J.-J. Sansuc, *Fibrés quadratiques et composantes connexes réelles*, Math. Ann. **244** (1979), 105–134.
- dJo95 A. J. de Jong, *Crystalline Dieudonné module theory via formal and rigid geometry*, Publ. Math. Inst. Hautes Études Sci. **82** (1995), 5–96.
- EGA A. Grothendieck and J. Dieudonné, *Eléments de géométrie algébrique I, II, III, IV*, Publ. Math. Inst. Hautes Études Sci. **4**, **8**, **11**, **17**, **20**, **24**, **28**, **32** (1961–67).
- Gas10 Q. Gashi, *On a conjecture of Kottwitz and Rapoport*, Ann. Sci. Éc. Norm. Supér. **43** (2010), 1017–1038.
- GHKR06 U. Görtz, T. J. Haines, R. E. Kottwitz and D. C. Reuman, *Dimensions of some affine Deligne–Lusztig varieties*, Ann. Sci. Éc. Norm. Supér. **39** (2006), 467–511.
- Hab05 W. Haboush, *Infinite dimensional algebraic geometry: algebraic structures on p -adic groups and their homogeneous spaces*, Tohoku Math. J. (2) **57** (2005), 65–117.
- HR08 T. Haines and M. Rapoport, *On parahoric subgroups*, Adv. Math. **219** (2008), 188–198.
- Hum95 J. E. Humphreys, *Conjugacy classes in semisimple algebraic groups*, Mathematical Surveys and Monographs, vol. 43 (American Mathematical Society, Providence, RI, 1995).
- Kat79 N. Katz, *Slope filtrations for F -crystals*, Astérisque **63** (1979), 113–163.
- Kis13 M. Kisin, *Mod p points on Shimura varieties of abelian type*, Preprint (2013).
- Kot84 R. E. Kottwitz, *Shimura varieties and twisted orbital integrals*, Math. Ann. **269** (1984), 287–300.
- Kot85 R. E. Kottwitz, *Isocrystals with additional structure*, Compositio Math. **56** (1985), 201–220.
- Kot97 R. E. Kottwitz, *Isocrystals with additional structure. II*, Compositio Math. **109** (1997), 255–339.
- Kot03 R. E. Kottwitz, *On the Hodge–Newton decomposition for split groups*, Int. Math. Res. Not. IMRN **26** (2003), 1433–1447.
- KR03 R. E. Kottwitz and M. Rapoport, *On the existence of F -crystals*, Comment. Math. Helv. **78** (2003), 153–184.
- Kre14 M. Kreidl, *On p -adic lattices and Grassmannians*, Math. Z. **276** (2014), 859–888.
- Lus12 G. Lusztig, *Unipotent almost characters of simple p -adic groups*, Astérisque, to appear. Preprint (2012).
- Man08 E. Mantovan, *On non-basic Rapoport–Zink spaces*, Ann. Sci. Éc. Norm. Supér. (4) **41** (2008), 671–716.
- MV10 E. Mantovan and E. Viehmann, *On the Hodge–Newton filtration for p -divisible \mathcal{O} -modules*, Math. Z. **266** (2010), 193–205.

- Nor76 M. Nori, *On the representations of the fundamental group*, *Compositio Math.* **33** (1976), 29–41.
- Rap00 M. Rapoport, *A positivity property of the Satake isomorphism*, *Manuscripta Math.* **101** (2000), 153–166.
- RR96 M. Rapoport and M. Richartz, *On the classification and specialization of F -isocrystals with additional structure*, *Compositio Math.* **103** (1996), 153–181.
- RG71 M. Raynaud and L. Gruson, *Critères de platitude et de projectivité*, *Invent. Math.* **13** (1971), 1–89.
- RZ96 M. Rapoport and T. Zink, *Period spaces for p -divisible groups*, *Annals of Mathematics Studies*, vol. 141 (Princeton University Press, Princeton, NJ, 1996).
- SGA3 M. Demazure and A. Grothendieck, *Séminaire de Géométrie Algébrique du Bois Marie - 1962–64 - Schémas en groupes (SGA3), vol. 3*, *Lecture Notes in Mathematics*, vol. 153 (Springer, Berlin, 1970).
- SW13 P. Scholze and J. Weinstein, *Moduli of p -divisible groups*, *Cambridge J. Math.* **1** (2013), 145–237.
- She14 X. Shen, *On the Hodge–Newton filtration for p -divisible groups with additional structures*, *Int. Math. Res. Not.* **2014**(13) (2014), 3582–3631.
- Spr06 T. A. Springer, *Twisted conjugacy in simply connected groups*, *Trans. Groups* **11** (2006), 539–545.
- Tit79 J. Tits, *Reductive groups over local fields*, in *Automorphic forms, representations and L -functions (Corvallis 1977)*, *Proceedings of Symposia in Pure Mathematics*, vol. XXXIII (American Mathematical Society, Providence, RI, 1979), 29–69.
- Vie08 E. Viehmann, *Connected components of affine Deligne–Lusztig varieties*, *Math. Ann.* **340** (2008), 315–333.
- VW13 E. Viehmann and T. Wedhorn, *Ekedahl–Oort and Newton strata for Shmura varieties of PEL type*, *Math. Ann.* **356** (2013), 1493–1550.
- Win05 J.-P. Wintenberger, *Existence de F -cristaux avec structures supplémentaires*, *Adv. Math.* **190** (2005), 196–224.
- Zin01 T. Zink, *Windows for displays of p -divisible groups*, in *Moduli of abelian varieties (Text 1999)*, *Progress in Mathematics*, vol. 195 (Birkhäuser, Basel, 2001), 491–518.

Miaofen Chen mfchen@math.ecnu.edu.cn

Department of Mathematics, Shanghai Key Laboratory of PMMP,
East China Normal University, No. 500, Dong Chuan Road,
Shanghai, 200241, PR China

Mark Kisin kisin@math.harvard.edu

Department of Mathematics, Harvard University,
1 Oxford St, Cambridge, MA 02138, USA

Eva Viehmann viehmann@ma.tum.de

Fakultät für Mathematik der Technischen Universität München – M11,
Boltzmannstr. 3, 85748 Garching, Germany